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BURKHOLDER–DAVIS–GUNDY INEQUALITY FOR g–MARTINGALES

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Abstract. In this study, we establish a Burkholder-Davis-Gundy (BDG) inequality type for certain nonlinear martingales arising from backward stochastic differential equations (BSDE) with generalized Lipschitz generator. As a consequence, we attempt to prove the equivalence between the convergence in probability of a g-martingale sequence and the associated quadratic variation sequence. Using a counterexample, we prove that BDG fails when g is quadratic.

1. Introduction

Since Peng's pioneering paper [4], nonlinear expectation theory has undergone considerable development. As its name indicates, nonlinear expectation is a nonlinear generalization of the classical expectation. It has some properties in common with the latter, but it differs from it especially by linearity property. This operator is widely used in financial mathematics, more precisely in decisions problems under model uncertainty, such as risk assessment problems under knight uncertainty situation. A major category of nonlinear expectations is the one generated by the BSDE called g-expectation. As in the case of classical expectation, a theory of nonlinear martingales has developed over the past two decades. Some generalizations of the results concerning classical martingales have been made for nonlinear martingales. One of the well-known results for classical martingales is the Burkholder-Davis-Gundy (BDG) inequality. This inequality is an important tool in the theory of stochastic processes and has applications in various fields of probability, including stochastic calculus, mathematical finance, and statistical mechanics. The BDG inequality is a refinement of the Doob's maximal inequality, and can be seen as a way of controlling the maximum of a classical martingale in terms of its local behavior. In this paper, we attempt to establish BDG inequality for the g-martingale in the case where g is generalized Lipchitz function. This paper is organized as follows: Section 2 provides the preliminaries, the necessary notations, conceptions and some properties about the g-martingales. In section 3, we further explore the main problem of this paper, namely the BDG Inequality for g-martingale when g is generalized Lipchitz generator.

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2. g-martingales

For the sake of clarity, we will consider a finite time horizon $T > 0$. However, the results presented below remain valid in the case of an infinite time horizon. Let $(B_t)_{0\leqslant t\leqslant T}$ be a standard d-dimensional Brownian motion defined on some complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$ is the augmented natural filtration of B which satisfies the usual conditions of completeness and right-continuity. Throughout this paper, we adopt the following notations:

- $L^2(\Omega, \mathscr{F}_t, \mathbb{P})$ the space of all the $\mathscr{F}_t$- measurable square integrable $\mathbb{R}$-valued random variables.
- $\mathscr{S}^2(0,T;\mathbb{R}) := \left\{ (Y_t)_{t\in[0,T]} : \begin{array}{c} Y \text{ is the RCLL } \mathbb{R}\text{-valued process,} \\ \text{such that } \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |Y_t|^2\right] < +\infty. \end{array} \right\}.$
- $\mathscr{H}^2\left(0,T;\mathbb{R}^d\right) := \left\{ (Z_t)_{t\in[0,T]} : \begin{array}{c} Z \text{ is the adapted } \mathbb{R}^d\text{-valued process,} \\ \text{with } \mathbb{E}(\int_0^T |Z_t|^2 dt) < +\infty. \end{array} \right\},$
  where $|z|$ denotes the Euclidean norm of $z \in \mathbb{R}^d$.
- $\langle . \rangle$ stands for the quadratic variation, that is, $\langle X \rangle_t = \lim_{\|\delta\|\to 0} \sum_{k=1}^n \left(X_{t_k} - X_{t_{k-1}}\right)^2$, where δ ranges over partitions of the interval $[0,t]$ and the norm of the partition δ is the mesh.
- If $X = (X_t)_{t\in[0,T]}$ is a $\mathbb{R}$-valued stochastic process, we will simply write X_T^* instead of $\sup_{t\in[0,T]} |X_t|$.

The generator $g(t,\omega,y,z) : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d \longmapsto \mathbb{R}$ is a random function which is a progressively measurable stochastic process for any (y,z). We assume that it satisfies the following assumptions:

(H1) There are two functions u and v from $[0,T]$ to $\mathbb{R}_+$, satisfying $\int_0^T \left[u(t)+v^2(t)\right] \mathrm{d}t < +\infty$, such that $\forall (t,y,y',z,z') \in [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$;

$$\left|g(t,y,z) - g\left(t,y',z'\right)\right| \leqslant u(t)\left|y-y'\right| + v(t)\left|z-z'\right|.$$

(H2) $\forall y \in \mathbb{R}$; $g(t,y,0) = 0$, $d\mathbb{P}\times dt$-a.e.

The assumption (H1) is a generalized Lipschitz condition, whose Lipschitz constant is replaced by two deterministic functions depending on t. Note that under assumptions (H1) and (H2), we have forall $(y,z) \in \mathbb{R}\times\mathbb{R}^d$,

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^T |g(t,y,z)|dt\right)^2\right] &= \mathbb{E}\left[\left(\int_0^T |g(t,y,z)-g(t,y,0)|dt\right)^2\right] \\ &\leqslant \mathbb{E}\left[\left(\int_0^T v^2(t)|z|^2 dt\right)^2\right] < +\infty, \end{aligned}$$

and so, according to [2], the BSDE

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\, ds - \int_t^T Z_s dB_s \tag{1}$$

admits a unique solution $(Y^{\xi}, Z^{\xi}) \in \mathscr{S}^2(0,T;\mathbb{R}) \times \mathscr{H}^2\left(0,T;\mathbb{R}^d\right)$ for all $\xi \in L^2(\Omega, \mathscr{F}_T, \mathbb{P})$.

The operator $\mathscr{E}_g$ defined by:

$$\begin{aligned} \mathscr{E}_g : L^2(\Omega, \mathscr{F}_T, \mathbb{P}) &\longmapsto \mathbb{R} \\ \xi &\longmapsto Y_0^{\xi} \end{aligned}$$

is a typical example of nonlinear expectation called g-expectation. The notion of nonlinear expectation was firstly introduced by Peng [4]. It is an operator verifying certain properties, namely

(i) Strict monotonicity:

- If $X_1 \geqslant X_2$, $\mathbb{P}-$ a.s., then $\mathscr{E}[X_1] \geqslant \mathscr{E}[X_2]$,
- If $X_1 \geqslant X_2$, $\mathbb{P}-$ a.s., then $\mathscr{E}[X_1] = \mathscr{E}[X_2] \Longleftrightarrow X_1 = X_2$, $\mathbb{P}-$ a.s.

(ii) Preserving of constants: $\mathscr{E}[c] = c$, for each constant c.

DEFINITION 1. The conditional g-expectation of ξ with respect to $\mathscr{F}_t$ is defined by

$$\mathscr{E}_g[\xi \mid \mathscr{F}_t] = Y_t^{\xi},$$

where (Y^{ξ}, Z^{ξ}) is the unique solution of the BSDE (1).

If τ is a stopping time between 0 and T, we define similarly $\mathscr{E}_g[\xi \mid \mathscr{F}_\tau]$ by

$$\mathscr{E}_g[\xi \mid \mathscr{F}_\tau] = Y_\tau^{\xi}.$$

DEFINITION 2. *A* process $(Y_t)_{0 \leqslant t \leqslant T}$ such that $E\left[Y_t^2\right] < \infty$ for all $t \in [0,T]$ is a g-martingale (resp. g-supermartingale, g-submartingale) if

$$\mathscr{E}_g[Y_t \mid \mathscr{F}_s] = Y_s, \quad (\text{ resp. } \leqslant Y_s,\ \geqslant Y_s), \quad \forall \quad 0 \leqslant s \leqslant t \leqslant T.$$

3. BDG inequality for g-martingales

BDG inequality provides an upper bound on the p^{th} moment of a stochastic process in terms of its quadratic variation. Specifically, if M is a continuous local martingale, then for any $p > 0$, there exist universal positive constants c_p and C_p such that

$$c_p E[\langle M \rangle_T^{\frac{p}{2}}] \leqslant E[(M_T^*)^p] \leqslant C_p E[\langle M \rangle_T^{\frac{p}{2}}].$$

In this section, we attempt to establish BDG inequality for g-martingale when g is a generalized Lipschitz function. Note that, the quadratic variation of a g-martingale Y satisfying equation (1) is given by

$$\langle Y \rangle_t = \int_0^t |Z_s|^2 ds;\ 0 \leqslant t \leqslant T.$$

We recall the following useful lemma due to Lenglart [3].

LEMMA 1. (Lenglart's domination inequality) *Let* $(X_t)_{0\leqslant t\leqslant T}$ *be a positive adapted right-continuous process dominated by a predictable increasing process* $(A_t)_{0\leqslant t\leqslant T}$ *i.e for every bounded stopping time* τ, $\mathbb{E}(X_\tau) \leqslant \mathbb{E}(A_\tau)$. *Then, for every* $k \in (0,1)$,

$$\mathbb{E}\left((X_T^*)^k\right) \leqslant \frac{2-k}{1-k}\mathbb{E}\left(A_T^k\right).$$

REMARK 1. Assumptions (H1) and (H2) imply

$$\forall (t,y,z) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^d; |g(t,y,z)| \leqslant v(t)\,|z|,$$

indeed

$$|g(t,y,z)| = |g(t,y,z) - g(t,y,0)| \leqslant u(t)\,|y-y| + v(t)\,|z-0| = v(t)\,|z|.$$

THEOREM 1. *Given* g *verifying* (H1) *and* (H2)*, then for any* $0 < p < +\infty$*, there exist two positives constants* c_p^v *and* C_p^v *such that for all* g*-martingale* Y *vanishing at zero,*

$$c_p^v \mathbb{E}[\langle Y\rangle_T^{\frac{p}{2}}] \leqslant \mathbb{E}[(Y_T^*)^p] \leqslant C_p^v \mathbb{E}[\langle Y\rangle_T^{\frac{p}{2}}].$$

Proof. We start by proving the left hand side inequality.
For each integer $n \geqslant 1$, let us introduce the stopping time

$$\tau_n = \inf\left\{t \in [0,T], \int_0^t |Z_r|^2 \, \mathrm{d}r \geqslant n\right\} \wedge T.$$

Itô's formula gives us

$$\int_0^{\tau_n} |Z_s|^2 ds = |Y_{\tau_n}|^2 + \int_0^{\tau_n} 2Y_s g(s,Y_s,Z_s)\, ds - 2\int_0^{\tau_n} Y_s Z_s \, \mathrm{d}B_s.$$

From remark 1, we have $g(s,y,z) \leqslant v(s)|z|$, and so

$$2|yg(s,y,z)| \leqslant 2v^2(s)|y|^2 + \frac{1}{2}|z|^2.$$

Thus, since $\tau_n \leqslant T$, we deduce that

$$\frac{1}{2}\int_0^{\tau_n} |Z_s|^2 \, \mathrm{d}s \leqslant (Y_T^*)^2 + 2\mu (Y_T^*)^2 + 2\left|\int_0^{\tau_n} Y_s Z_s \, \mathrm{d}B_s\right|,$$

where $\mu := \int_0^T v^2(s) ds$. It follows that

$$\int_0^{\tau_n} |Z_s|^2 \, \mathrm{d}s \leqslant (2+4\mu)(Y_T^*)^2 + 4\left|\int_0^{\tau_n} Y_s Z_s \, \mathrm{d}B_s\right|.$$

Accordingly, there is a positive constant k_p such that

$$\left(\int_0^{\tau_n}|Z_s|^2\,\mathrm{d}s\right)^{p/2}\leqslant k_p\left((Y_T^*)^p+\left|\int_0^{\tau_n}Y_sZ_sdB_s\right|^{p/2}\right).$$

Therefore,

$$\mathbb{E}\left[\left(\int_0^{\tau_n}|Z_s|^2\,\mathrm{d}s\right)^{p/2}\right]\leqslant k_p\left(\mathbb{E}\left[(Y_T^*)^p\right]+\mathbb{E}\left[\left|\int_0^{\tau_n}Y_sZ_sdB_s\right|^{p/2}\right]\right). \tag{2}$$

Using BDG inequality, we get

$$\begin{aligned}k_p\mathbb{E}\left[\left|\int_0^{\tau_n}Y_sZ_sdB_s\right|^{p/2}\right]&\leqslant d_p\mathbb{E}\left[\left(\int_0^{\tau_n}|Y_s|^2|Z_s|^2ds\right)^{p/4}\right]\\&\leqslant d_p\mathbb{E}\left[(Y_T^*)^{p/2}\left(\int_0^{\tau_n}|Z_s|^2ds\right)^{p/4}\right]\\&\leqslant\frac{d_p^2}{2}\mathbb{E}\left[(Y_T^*)^p\right]+\frac{1}{2}\mathbb{E}\left[\left(\int_0^{\tau_n}|Z_s|^2ds\right)^{p/2}\right].\end{aligned}$$

Plugging the last inequality in inequality (2), we obtain for each $n\geqslant 1$,

$$c_p^v\mathbb{E}\left[\left(\int_0^{\tau_n}|Z_s|^2ds\right)^{p/2}\right]\leqslant\mathbb{E}\left[(Y_T^*)^p\right],$$

with some positive constant c_p^v depending on v. Fatou's lemma implies that

$$c_p^v\mathbb{E}\left[\left(\int_0^{T}|Z_s|^2ds\right)^{p/2}\right]\leqslant\mathbb{E}\left[(Y_T^*)^p\right].$$

We proceed now to the proof of the right hand side inequality. Let τ a bounded stopping time between 0 and T. Using a localization procedure, it is enough to prove the result for bounded Y. Let $q>2$ and $k\in(0,1)$ such that $p=qk$. From Itô's formula we have

$$\begin{aligned}d|Y_t|^q&=q|Y_t|^{q-1}\operatorname{sign}(Y_t)\,dY_t+\frac{q(q-1)}{2}|Y_t|^{q-2}d\langle Y\rangle_t\\&=q\operatorname{sign}(Y_t)|Y_t|^{q-1}(-g(t,Y_t,Z_t)dt+Z_tdB_t)+\frac{q(q-1)}{2}|Y_t|^{q-2}|Z_t|^2dt\\&=-q\operatorname{sign}(Y_t)|Y_t|^{q-1}g(t,Y_t,Z_t)dt+\frac{q(q-1)}{2}|Y_t|^{q-2}|Z_t|^2dt\\&\quad+q\operatorname{sign}(Y_t)|Y_t|^{q-1}Z_tdB_t.\end{aligned}$$

This leads to,

$$\begin{aligned}|Y_\tau|^q=&\int_0^\tau(-q\operatorname{sign}(Y_s)|Y_s|^{q-1}g(t,Y_s,Z_s)+\frac{q(q-1)}{2}|Y_s|^{q-2}|Z_s|^2)ds\\&+\int_0^\tau q\operatorname{sign}(Y_s)|Y_s|^{q-1}Z_sdB_s.\end{aligned}$$

By taking the expectation under $\mathbb{P}$, we obtain

$$\begin{aligned}\mathbb{E}\left[|Y_\tau|^q\right] &= \mathbb{E}\left[\int_0^\tau (-q\operatorname{sign}(Y_s)\,|Y_s|^{q-1}\,g(t,Y_s,Z_s) + \frac{q(q-1)}{2}\,|Y_s|^{q-2}\,|Z_s|^2)ds\right]\\ &\leqslant \mathbb{E}\left[\int_0^\tau (qv(s)\,|Y_s|^{q-1}\,|Z_s| + \frac{q(q-1)}{2}\,|Y_s|^{q-2}\,|Z_s|^2)ds\right].\end{aligned}$$

From Lemma 1, we deduce that

$$\begin{aligned}&\mathbb{E}\left[((Y_T^*)^q)^k\right]\\ &\leqslant \frac{2-k}{1-k}\mathbb{E}\left[\left(\int_0^T (qv(s)\,|Y_s|^{q-1}\,|Z_s| + \frac{q(q-1)}{2}\,|Y_s|^{q-2}\,|Z_s|^2)ds\right)^k\right]\\ &\leqslant \frac{2-k}{1-k}\mathbb{E}\left[\left(\int_0^T q\,|Y_s|^{q-2}\,(\frac{\delta^2}{2}|Z_s|^2 + \frac{v^2(s)}{2\delta^2}|Y_s|^2) + \frac{q(q-1)}{2}\,|Y_s|^{q-2}\,|Z_s|^2\,ds\right)^k\right]\\ &= \frac{2-k}{1-k}\mathbb{E}\left[\left(\int_0^T \frac{qv^2(s)}{2\delta^2}\,|Y_s|^q + \frac{q(q-1+\delta^2)}{2}\,|Y_s|^{q-2}\,|Z_s|^2\,ds\right)^k\right]\\ &\leqslant \frac{2-k}{1-k}\mathbb{E}\left[\left(\int_0^T \frac{qv^2(s)}{2\delta^2}\,|Y_s|^q\,ds\right)^k\right] + \frac{2-k}{1-k}\mathbb{E}\left[\left(\int_0^T \frac{q(q-1+\delta^2)}{2}\,|Y_s|^{q-2}\,|Z_s|^2\,ds\right)^k\right]\\ &\leqslant \frac{2-k}{1-k}\left(\frac{q\mu}{2\delta^2}\right)^k\mathbb{E}\left[(Y_T^*)^{qk}\right] + \frac{2-k}{1-k}\frac{q^k(q-1+\delta^2)^k}{2^k}\mathbb{E}\left[(Y_T^*)^{k(q-2)}\left(\int_0^T |Z_s|^2\,ds\right)^k\right],\end{aligned}$$

where δ is a positive constant. Therefore,

$$\begin{aligned}&\left(1-\frac{2-k}{1-k}\left(\frac{q\mu}{2\delta^2}\right)^k\right)\mathbb{E}\left[(Y_T^*)^p\right]\\ &\leqslant \frac{2-k}{1-k}\frac{q^k(q-1+\delta^2)^k}{2^k}\mathbb{E}\left[(Y_T^*)^{k(q-2)}\left(\int_0^T |Z_s|^2\,ds\right)^k\right].\end{aligned}$$

By Hölder inequality, we obtain

$$\begin{aligned}&\left(1-\frac{2-k}{1-k}\left(\frac{q\mu}{2\delta^2}\right)^k\right)\mathbb{E}\left[(Y_T^*)^p\right]\\ &\leqslant \frac{2-k}{1-k}\frac{q^k(q-1+\delta^2)^k}{2^k}\left(\mathbb{E}[(Y_T^*)^p]\right)^{1-\frac{2}{q}} \times \left(\mathbb{E}\left[\left(\int_0^T |Z_s|^2\,ds\right)^{\frac{kq}{2}}\right]\right)^{\frac{2}{q}}.\end{aligned}$$

By choosing δ large enough such that $\kappa = 1-\frac{2-k}{1-k}(\frac{q\mu}{2\delta^2}T)^k > 0$, we get

$$\mathbb{E}\left((Y_T^*)^p\right) \leqslant \left(\frac{2-k}{\kappa(1-k)}\frac{q^k(q-1+\delta^2)^k}{2^k}\right)^{\frac{q}{2}}\mathbb{E}\left[\left(\int_0^T |Z_s|^2\,ds\right)^{\frac{p}{2}}\right].$$

This completes the proof of Theorem 1. □

COROLLARY 1. *Let g satisfying* (H1)–(H2) *and τ a stopping time between* 0 *and* T, *then for any $0 < p < +\infty$, there exist two positive constants c_p^v and C_p^v such that for all g-martingale Y vanishing at zero we have*

$$c_p^v \mathbb{E}[\langle Y \rangle_\tau^{\frac{p}{2}}] \leqslant \mathbb{E}[(Y_\tau^*)^p] \leqslant C_p^v \mathbb{E}[\langle Y \rangle_\tau^{\frac{p}{2}}].$$

Proof. The stopped process $(Y_t^\tau)_{0 \leqslant t \leqslant T} = (Y_{t \wedge \tau})_{0 \leqslant t \leqslant T}$ satisfies the following BSDE

$$\begin{aligned} Y_t^\tau &= Y_T^\tau - \int_{t \wedge \tau}^{\tau} g(s, Y_s, Z_s) ds + \int_{t \wedge \tau}^{\tau} Z_s dBs \\ &= Y_T^\tau - \int_t^T g(s, Y_s^\tau, Z_s \mathbf{1}_{s \leqslant \tau}) ds + \int_t^T Z_s \mathbf{1}_{s \leqslant \tau} dBs. \end{aligned}$$

Which proves that $(Y_t^\tau)_{0 \leqslant t \leqslant T}$ is a g_τ-martingale vanishing at zero, where $g_\tau(t, y, z) = g(t, y, z\mathbf{1}_{t \leqslant \tau})$. The function g_τ verifies hypotheses (H1) and (H2), indeed for all $y \in \mathbb{R}$; $g_\tau(t, y, 0) = 0$ $d\mathbb{P} \times dt$-a.e and $\forall (t, y, y', z, z') \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$;

$$\begin{aligned} \left| g_\tau(t, y, z) - g_\tau\left(t, y', z'\right) \right| &= \left| g(t, y, z\mathbf{1}_{s \leqslant \tau}) - g\left(t, y', z'\mathbf{1}_{s \leqslant \tau}\right) \right| \\ &\leqslant u(t) \left| y - y' \right| + v(t) \left| z - z' \right|. \end{aligned}$$

Using Theorem 1, we obtain the required result. □

COROLLARY 2. *Let g satisfying* (H1) *and* (H2) *and τ a stopping time between* 0 *and T, then for any $0 < p < +\infty$, there exist two positive constants c_p^v and C_p^v such that for all g-martingale Y*

$$c_p^v \mathbb{E}[\langle Y \rangle_\tau^{\frac{p}{2}}] \leqslant \mathbb{E}[(Y - Y_0)_\tau^*)^p] \leqslant C_p^v \mathbb{E}[\langle Y \rangle_\tau^{\frac{p}{2}}].$$

Proof. We have

$$Y_t = Y_T - \int_t^T g(s, Y_s, Z_s) ds + \int_t^T Z_s dBs.$$

The stochastic process $L = (L_t)_{0 \leqslant t \leqslant T}$ defined by

$$L_t := Y_t - Y_0 \ \ \forall\, 0 \leqslant t \leqslant T,$$

satisfies the following BSDE

$$L_t = L_T - \int_t^T \tilde{g}(s, L_s, Z_s) ds + \int_t^T Z_s dBs,$$

where $\tilde{g}(s, y, z) = g(s, y + Y_0, z) ds$. So L is a $\tilde{g}$-martingale. It's clear that $\tilde{g}$ satisfies the hypotheses (H1) and (H2) with the same u and v as the generator g. The result is immediately obtained from the Corollary 1. □

COROLLARY 3. *Let g satisfying* (H1)–(H2) *and* $(Y^n)_n$ *is sequence of g-martingales vanishing at* 0*.The following assertions are equivalent:*

(i) $(\langle Y^n\rangle_T)_n$ *converges in probability to* 0.

(ii) $(Y_T^{n,*})_n$ *converges in probability to* 0.

Proof. Suppose that the sequence $(\langle Y^n\rangle_T)_n$ converges in probability to 0. It also converges in law to 0. Using Corollary 1 and Lemma (4.6) in [6], we have for all $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}\{Y_T^{n,*} > \varepsilon, \langle Y^n\rangle_T < y\} &\leqslant \frac{C_1^\nu}{\varepsilon}\mathbb{E}[\langle Y^n\rangle_T^{\frac{1}{2}} \wedge y] \\ &\leqslant \frac{C_1^\nu}{\varepsilon}\mathbb{E}[\langle Y^n\rangle_T^{\frac{1}{2}}].\end{aligned}$$

Using Fatou Lemma, we obtain

$$\begin{aligned}\mathbb{P}\{Y_T^{n,*} > \varepsilon\} &\leqslant \liminf_{y\longmapsto\infty}\mathbb{P}\{Y_T^{n,*} > \varepsilon, \langle Y^n\rangle_T < y\} \\ &\leqslant \frac{C_1^\nu}{\varepsilon}\mathbb{E}[\langle Y^n\rangle_T^{\frac{1}{2}}] \underset{n\longmapsto\infty}{\longrightarrow} 0.\end{aligned}$$

Similarly, we prove the converse implication. □

REMARK 2. (Quadratic generator case) The inequality established in Theorem 1 is no longer valid in the quadratic generator case. Indeed, for $n \in \mathbb{N}$, let Y^n the stochastic processes defined by

$$Y_t^n = nB_t - n^2 t; 0 \leqslant t \leqslant T.$$

It's clear that, for all $n \in \mathbb{N}$, the pair (Y^n, n) is solution of the quadratic BSDE

$$dY_t = -Z_t^2 dt + Z_t dB_t; Y_T = nB_T - n^2 T.$$

Therefore, for all $n \in \mathbb{N}$, Y^n is a g-martingale with $g(z) = z^2$.

If the BDG inequality holds for Y^n, then we will have

$$|\mathbb{E}(Y_T^n)| \leqslant \mathbb{E}[\sup_{0\leqslant t\leqslant T}|Y_t^n|] \leqslant C(T)\mathbb{E}[\langle Y^n\rangle_T^{\frac{1}{2}}].$$

That's means, for all $n \in \mathbb{N}$

$$n^2 T \leqslant nC(T)\sqrt{T}.$$

Which leads to a contraduction.

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