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ON (n,p)–TH VON NEUMANN–JORDAN CONSTANTS FOR BANACH SPACES

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Abstract. In this paper, we introduce and discuss the upper, lower, upper modified and lower modified (n,p)-th von Neumann-Jordan constant for $p \geqslant 1$, which is a further generalization for von Neumann-Jordan constant in Banach spaces. We give some relationships between these constants and characterize uniformly non-l_n^1 Banach spaces by upper and upper modified (n,p)-th von Neumann-Jordan constant. Moreover, the exact value of this constant is calculated for the spaces l^p, L^p and $l_\infty - l_1$.

1. Introduction

Let $S(X)$ (resp. $B(X)$) be the unit sphere (resp. the unit closed ball) of a real Banach space X. Throughout this paper, the letter $\mathbb{N}$ stands for the set of positive integers, and we will use $ex(B(X))$ to denote the set of extreme points of $B(X)$.

In 1937, the von Neumann-Jordan constant $C_{NJ}(X)$ of a Banach space X was introduced by Clarkson [3], as the smallest constant C for which,

$$\frac{1}{C} \leqslant \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leqslant C,$$

holds for all $x, y \in X$ with $\|x\|^2 + \|y\|^2 \neq 0$. The properties of $C_{NJ}(X)$ have been investigated in many papers (for example, see [1,5,8–10,12,14–17,19,20,22,23,25,28–30]). The constant $C_{NJ}(X)$ can be used to characterize geometry properties of Banach spaces. For any Banach space X, $1 \leqslant C_{NJ}(X) \leqslant 2$ and $C_{NJ}(X) = 1$ if and only if X is a Hilbert space in [13]; X is uniformly non-square if and only if $C_{NJ}(X) < 2$ in [17].

In 1998, the n-th von Neumann-Jordan constant $C_{NJ}^{(n)}(X)$ was introduced by Kato, Takahashi and Hashimoto in [18]. In 2008, on n-th James constants of Banach spaces was discussed by Maligranda, Nikolova, Persson and Zachariades in [21]. In 2020, the

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n-th von Neumann-Jordan constant was modified by Ciesielski and Pluciennik as the upper n-th von Neumann-Jordan constant and denoted by $\overline{C}_{NJ}^{(n)}(X)$ in [2] as follows,

$$\overline{C}_{NJ}^{(n)}(X) = \sup\left\{\frac{\sum_{\theta_j=\pm 1}\|x_1+\sum_{j=2}^{n}\theta_j x_j\|^2}{2^{n-1}\sum_{j=1}^{n}\|x_j\|^2} : x_j \in X (j=1,2,\dots,n) \text{ and } \sum_{j=1}^{n}\|x_j\|^2 \neq 0\right\} \tag{1.1}$$

And by changing "sup" into "inf" in (1.1), the authors defined the lower n-th von Neumann-Jordan constant $\underline{C}_{NJ}^{(n)}(X)$. If the supremum resp. infimum in (1.1) is taken over all $x_j \in S(X)$, $(j=1,2,\dots,n, n \geqslant 2)$, then it is called upper, resp. lower modified n-th von Neumann-Jordan constant and denoted by $\overline{C}_{mNJ}^{(n)}(X)$, resp. $\underline{C}_{mNJ}^{(n)}(X)$. More details on n-th von Neumann-Jordan constant can be seen in [26]. For a Banach space X and its dual space X^*, if $n \geqslant 2$, then

(i) $1 \leqslant \overline{C}_{mNJ}^{(n)}(X) \leqslant \overline{C}_{NJ}^{(n)}(X) \leqslant n$ and $\frac{1}{n} \leqslant \underline{C}_{NJ}^{(n)}(X) \leqslant \underline{C}_{mNJ}^{(n)}(X) \leqslant 1$;

(ii) $\overline{C}_{mNJ}^{(n)}(X) \leqslant \frac{n+1}{n}\overline{C}_{mNJ}^{(n+1)}(X)$;

(iii) $\underline{C}_{NJ}^{(n)}(X^*) \geqslant \frac{1}{\overline{C}_{NJ}^{(n)}(X)}$ and $\underline{C}_{NJ}^{(n)}(X) \geqslant \frac{1}{\overline{C}_{NJ}^{(n)}(X^*)}$;

(iv) If $1 \leqslant p \leqslant 2$ and $n \leqslant m \leqslant \infty$, then $\overline{C}_{NJ}^{(n)}(l_m^p) = \overline{C}_{mNJ}^{(n)}(l_m^p) = n^{\frac{2}{p}-1}$;

(v) If $2^{n-1} \leqslant m \leqslant \infty$, then $\overline{C}_{NJ}^{(n)}(l_m^\infty) = \overline{C}_{mNJ}^{(n)}(l_m^\infty) = n^{\frac{2}{p}-1}$;

(vi) Let $X = L^p(\mu)$ or $X = l_m^p$ $(1 \leqslant m \leqslant \infty)$. Then

$$\overline{C}_{mNJ}^{(n)}(X) = \begin{cases} n^{\frac{2}{p}-1}, & \text{if } 1 \leqslant p \leqslant 2 \text{ and } m \geqslant n, \\ \frac{1}{n}\left(2^{1-n}\sum_{k=0}^{[\frac{n}{2}]} C_n^k (n-2k)^p\right)^{\frac{2}{p}}, & \text{if } 2 \leqslant p < \infty \text{ and } m \geqslant 2^{n-1}. \end{cases}$$

In 2015, the generalization von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ of X was defined in [4] for any $p \geqslant 1$ by

$$C_{NJ}^{(p)}(X) = \sup\left\{\frac{\|x+y\|^p + \|x-y\|^p}{2^{p-1}(\|x\|^p + \|y\|^p)} : x, y \in X, (x,y) \neq (0,0)\right\}.$$

It is well known that there are some properties of this constant as follows (see [4, 6, 7, 27, 31–33])

(i) $1 \leqslant C_{NJ}^{(p)}(X) \leqslant 2$;

(ii) X is uniformly non-square if and only if $C_{NJ}^{(p)}(X) < 2$,

(iii) Let X be the Banach space $L_r[0,1]$, $1 \leqslant r \leqslant 2$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Then

(1) if $1 < p \leqslant r$, then $C_{NJ}^{(p)}(L_r[0,1]) = 2^{2-p}$ and if $r < p \leqslant r'$, then $C_{NJ}^{(p)}(L_r[0,1]) = 2^{\frac{p}{r}-p+1}$;

(2) if $r' < p < \infty$, then $C_{NJ}^{(p)}(L_r[0,1]) = 1$.

Based on the above constants, we now introduce new geometric constants as follows.

DEFINITION 2.1. Let $n \in \mathbb{N}$ and $p \geqslant 1$, the constants

$$\overline{C}_{NJ}^{(n,p)}(X) = \sup\left\{ \frac{\sum_{\theta_j=\pm 1} \| \sum_{j=1}^{n} \theta_j x_j \|^p}{2^{n+p-2} \sum_{j=1}^{n} \|x_j\|^p} : x_j \in X (j = 1,2,\dots,n) \text{ and } \sum_{j=1}^{n} \|x_j\|^p \neq 0 \right\} \tag{1.2}$$

and

$$\underline{C}_{NJ}^{(n,p)}(X) = \inf\left\{ \frac{\sum_{\theta_j=\pm 1} \| \sum_{j=1}^{n} \theta_j x_j \|^p}{2^{n+p-2} \sum_{j=1}^{n} \|x_j\|^p} : x_j \in X (j = 1,2,\dots,n) \text{ and } \sum_{j=1}^{n} \|x_j\|^p \neq 0 \right\} \tag{1.3}$$

are called an upper (n,p)-th von Neumann-Jordan constant and lower (n,p)-th von Neumann-Jordan constant, respectively. If the supremum in (1.2) resp. infimum in (1.3) is taken over all $x_j \in S(X)$, $(j = 1,2,\dots,n,\ n \geqslant 2)$, then it is called upper, resp. lower modified (n,p)-th von Neumann-Jordan constant and denoted by $\overline{C}_{mNJ}^{(n,p)}(X)$, resp. $\underline{C}_{mNJ}^{(n,p)}(X)$. It is clear that $\overline{C}_{NJ}^{(n,2)}(X) = \overline{C}_{NJ}^{(n)}(X)$ and $\overline{C}_{NJ}^{(2,p)}(X) = C_{NJ}^{(p)}(X)$, respectively.

Recall that a Banach space X is called uniformly non-l_n^1 (see [11]) if there exists $\delta > 0$ such that for each n elements of the unit ball $B(X)$ (or $S(X)$, equivalently)

$$\min_{\theta_j=\pm 1} \|x_1 + \sum_{j=2}^{n} \theta_j x_j\| \leqslant n(1-\delta).$$

If X is uniformly non-l_n^1 for $n = 2$, X is uniformly non-square.

In this paper, we will give some relationships among these constants $\overline{C}_{mNJ}^{(n,p)}(X)$, $\overline{C}_{NJ}^{(n,p)}(X)$, $\underline{C}_{NJ}^{(n,p)}(X)$ and $\underline{C}_{mNJ}^{(n,p)}(X)$ for $p \geqslant 1$. The uniformly non-l_n^1 Banach space will be characterized by upper and upper modified (n,p)-th von Neumann-Jordan constant. Moreover, the exact value of the constant will be calculated for the spaces l^p and L^p. As an application, the exact expression of $\overline{C}_{NJ}^{(3,p)}(l_\infty - l_1)$ will be calculated. Finally, we also will obtain that the exact value of $\overline{C}_{NJ}^{(3,2)}(l_\infty - l_1)$ is equal to $\frac{3+\sqrt{5}}{4}$.

2. Main results

Before describing the main results, we give two lemmas. The first lemma can be seen in [24].

LEMMA 2.1. *If a Banach space X is uniformly non-l_n^1 and $1 < p < \infty$, then there exists a constant $\alpha \in (0,1)$ such that if $x_1, x_2, \dots, x_n$ are elements of X, then*

$$\sum \left\| \frac{x_1 \pm x_2 \pm \dots \pm x_n}{n} \right\|^p \leqslant \frac{2^{n-1}\alpha}{n} \sum_{j=1}^{n} \|x_j\|^p,$$

where the sum is taken over all 2^{n-1} choices of signs.

LEMMA 2.2. *Let $n \geqslant 2,\ p > 0$, then*

$$\sum_{k=0}^{n-1} C_{n-1}^{k}|n-2k|^{p} = \sum_{k=0}^{[\frac{n}{2}]} C_{n}^{k}|n-2k|^{p},$$

where $[\frac{n}{2}]$ is the integer part of $\frac{n}{2}$.

Proof. (1) If $n = 2m+1$, then

$$\begin{aligned}\sum_{k=0}^{n-1} C_{n-1}^{k}|n-2k|^{p} &= n^{p} + \sum_{k=1}^{m} C_{n-1}^{k}|n-2k|^{p} + \sum_{k=m+1}^{2m} C_{n-1}^{k}|n-2k|^{p} \\ &= n^{p} + \sum_{k=1}^{m} C_{n-1}^{k}|n-2k|^{p} + \sum_{k=1}^{m} C_{n-1}^{n-k}|n-2k|^{p} \\ &= n^{p} + \sum_{k=1}^{m} (C_{n-1}^{k} + C_{n-1}^{k-1})|n-2k|^{p} \\ &= \sum_{k=0}^{[\frac{n}{2}]} C_{n}^{k}|n-2k|^{p}.\end{aligned}$$

(2) If $n = 2m$, then

$$\begin{aligned}\sum_{k=0}^{n-1} C_{n-1}^{k}|n-2k|^{p} &= n^{p} + \sum_{k=1}^{m-1} C_{n-1}^{k}|n-2k|^{p} + \sum_{k=m+1}^{2m-1} C_{n-1}^{k}|n-2k|^{p} \\ &= n^{p} + \sum_{k=1}^{m-1} C_{n-1}^{k}|n-2k|^{p} + \sum_{k=1}^{m-1} C_{n-1}^{n-k}|n-2k|^{p} \\ &= n^{p} + \sum_{k=1}^{m-1} (C_{n-1}^{k} + C_{n-1}^{k-1})|n-2k|^{p} = \sum_{k=0}^{[\frac{n}{2}]} C_{n}^{k}|n-2k|^{p}. \quad \square\end{aligned}$$

Now we start discussing the relationships between $\overline{C}_{NJ}^{(n,p)}(X)$, $\underline{C}_{NJ}^{(n,p)}(X)$, $\overline{C}_{mNJ}^{(n,p)}(X)$ and $\underline{C}_{mNJ}^{(n,p)}(X)$.

THEOREM 2.3. *Let $n \geqslant 2,\ p \geqslant 1$ and X be a Banach space. Then*
(a) $\frac{1}{n2^{n+p-2}} \sum_{j=0}^{n} C_{n}^{j}|n-2j|^{p} \leqslant \overline{C}_{mNJ}^{(n,p)}(X) \leqslant \overline{C}_{NJ}^{(n,p)}(X) \leqslant 2^{2-p}n^{p-1}$;
(b) $\underline{C}_{NJ}^{(n,p)}(X) \geqslant \frac{2^{2-p}}{n}$;
(c) $\overline{C}_{mNJ}^{(n,p)}(X) \leqslant \frac{n+1}{n}\overline{C}_{mNJ}^{(n+1,p)}(X)$.

Proof. (a) By Hölder inequality, we have

$$\|\sum_{j=1}^{n} \theta_{j}x_{j}\|^{p} \leqslant \Big(\sum_{j=1}^{n} \|x_{j}\|\Big)^{p} \leqslant n^{p-1}\sum_{j=1}^{n} \|x_{j}\|^{p}.$$

So $\overline{C}_{NJ}^{(n,p)}(X) \leqslant 2^{2-p}n^{p-1}$. Putting $x_1 \in S(X)$ and $x_i = x_1$ for $i = 2,3,\ldots,n$, we have

$$\frac{1}{n2^{n+p-2}}\sum_{j=0}^{n} C_n^j |n-2j|^p = \frac{1}{n2^{n+p-2}}\sum_{\theta_j=\pm 1}\|\sum_{j=1}^{n}\theta_j x_j\|^p \leqslant \overline{C}_{mNJ}^{(n,p)}(X).$$

(b) To prove the (b), we use the mathematical induction principle. For $n=2$, by $1 \leqslant C_{NJ}^{(p)}(X) \leqslant 2$, we have

$$\begin{aligned}\underline{C}_{NJ}^{(2,p)}(X) &= \inf\left\{\frac{\|x+y\|^p + \|x-y\|^p}{2^{p-1}(\|x\|^p+\|y\|^p)} : (x,y) \neq (0,0)\right\}\\ &= \frac{1}{2^{p-1}}\frac{1}{\sup\{\frac{\|x\|^p+\|y\|^p)}{\|x+y\|^p+\|x-y\|^p} : (x,y)\neq(0,0)\}}\\ &= \frac{2^{2-p}}{\sup\{\frac{\|u+v\|^p+\|u-v\|^p}{2^{p-1}(\|u\|^p+\|v\|^p)} : (u,v)\neq(0,0)\}}\\ &= \frac{2^{2-p}}{\overline{C}_{NJ}^{(p)}(X)} \geqslant \frac{1}{2^{p-1}}.\end{aligned}$$

Suppose that $\underline{C}_{NJ}^{(n-1,p)}(X) \geqslant \frac{2^{2-p}}{n-1}$. Notice that

$$\|x\|^p+\|y\|^p \leqslant 2^{1-p}[2\max\{\|x\|,\|y\|\}]^p \leqslant 2^{1-p}[\|x+y\|+\|x-y\|]^p \leqslant \|x+y\|^p+\|x-y\|^p \tag{2.1}$$

holds by Hölder inequality.

Hence, for any $i = 1,2,\ldots,n,$ we have

$$\begin{aligned}\frac{\sum_{\theta_j=\pm1}\|\sum_{j=1}^n\theta_j x_j\|^p}{2^{n+p-2}\sum_{j=1}^n\|x_j\|^p} &= \frac{\sum_{\theta_j=\pm1}[\|\sum_{j\neq i}\theta_j x_j + x_i\|^p + \|\sum_{j\neq i}\theta_j x_j - x_i\|^p)]}{2^{n+p-2}\sum_{j=1}^n\|x_j\|^p}\\ &\geqslant \frac{\sum_{\theta_j=\pm1}\|\sum_{j\neq i}\theta_j x_j\|^p + 2^{n-1}\|x_i\|^p}{2^{n+p-2}\sum_{j=1}^n\|x_j\|^p}\\ &\geqslant \frac{\frac{2^{n-1}}{n-1}\sum_{j\neq i}\|x_j\|^p + 2^{n-1}\|x_i\|^p}{2^{n+p-2}\sum_{j=1}^n\|x_j\|^p}\\ &= \frac{\frac{1}{n-1}\sum_{j=1}^n\|x_j\|^p + (1-\frac{1}{n-1})\|x_i\|^p}{2^{p-1}\sum_{j=1}^n\|x_j\|^p}\\ &= \frac{1}{2^{p-1}(n-1)} + \frac{(1-\frac{1}{n-1})\|x_i\|^p}{2^{p-1}\sum_{j=1}^n\|x_j\|^p}.\end{aligned}$$

It follows that

$$\frac{\sum_{\theta_j=\pm1}\|\sum_{j=1}^n\theta_j x_j\|^p}{2^{n+p-2}\sum_{j=1}^n\|x_j\|^p} \geqslant \frac{1}{n2^{p-1}}\sum_{i=1}^n\left(\frac{1}{n-1} + \frac{(n-2)\|x_i\|^p}{(n-1)\sum_{j=1}^n\|x_j\|^p}\right) = \frac{2^{2-p}}{n}.$$

and consequently (b) is valid.

(c) For any $x_1,x_2,\dots,x_{n+1}\in S(X)$, by the inequality (2.1), we have

$$\begin{aligned}\frac{\sum_{\theta_j=\pm1}\|\sum_{j=1}^{n+1}\theta_jx_j\|^p}{2^{n+p-1}\sum_{j=1}^{n+1}\|x_j\|^p}&=\frac{\sum_{\theta_j=\pm1}[\|\sum_{j=1}^{n}\theta_jx_j+x_{n+1}\|^p+\|\sum_{j=1}^{n}\theta_jx_j-x_{n+1}\|^p]}{2^{n+p-1}(n+1)}\\&\geqslant\frac{\sum_{\theta_j=\pm1}max\{\|\sum_{j=1}^{n}\theta_jx_j\|^p,\|x_{n+1}\|^p\}}{2^{n+p-2}(n+1)}\\&\geqslant\frac{\sum_{\theta_j=\pm1}\|\sum_{j=1}^{n}\theta_jx_j\|^p}{2^{n+p-2}(n+1)}\end{aligned}$$

whence $\overline{C}_{mNJ}^{(n,p)}(X)\leqslant\frac{n+1}{n}\overline{C}_{mNJ}^{(n+1,p)}(X)$. □

Inspired by the proof of Theorem 2.3 (b), we also have

THEOREM 2.4. *Let $n\geqslant2$, $p\geqslant1$ and X be a Banach space. Then*

$$\overline{C}_{NJ}^{(n,p)}(X)\leqslant\frac{1}{n}\overline{C}_{NJ}^{(2,p)}(X)+2^{p-2}\left(1-\frac{1}{n}\right)\overline{C}_{NJ}^{(2,p)}(X)\overline{C}_{NJ}^{(n-1,p)}(X).$$

Proof. Because

$$\begin{aligned}&\sum_{\theta_j=\pm1}\|\sum_{j=1}^{n}\theta_jx_j\|^p\\&=\sum_{\theta_j=\pm1}\left[\|\sum_{j\neq i}\theta_jx_j+x_i\|^p+\|\sum_{j\neq i}\theta_jx_j-x_i\|^p)\right]\\&\leqslant2^{p-1}\overline{C}_{NJ}^{(2,p)}(X)\sum_{\theta_j=\pm1}(\|\sum_{j\neq i}\theta_jx_j\|^p+\|x_i\|^p)\\&=2^{p-1}\overline{C}_{NJ}^{(2,p)}(X)(\sum_{\theta_j=\pm1}\|\sum_{j\neq i}\theta_jx_j\|^p+2^{n-1}\|x_i\|^p)\\&\leqslant2^{n+2p-4}\overline{C}_{NJ}^{(2,p)}(X)\overline{C}_{NJ}^{(n-1,p)}(X)\sum_{j\neq i}\|x_j\|^p+2^{n+p-2}\|x_i\|^p\overline{C}_{NJ}^{(2,p)}(X)\\&=2^{n+2p-4}\overline{C}_{NJ}^{(2,p)}(X)\overline{C}_{NJ}^{(n-1,p)}(X)\sum_{j=1}^{n}\|x_j\|^p\\&\quad+(2^{n+p-2}-2^{n+2p-4}\overline{C}_{NJ}^{(n-1,p)}(X))\|x_i\|^p\overline{C}_{NJ}^{(2,p)}(X).\end{aligned}$$

Hence

$$\begin{aligned}\sum_{\theta_j=\pm1}\|\sum_{j=1}^{n}\theta_jx_j\|^p\leqslant2^{n+p-2}\overline{C}_{NJ}^{(2,p)}(X)\Big\{&2^{p-2}\overline{C}_{NJ}^{(n-1,p)}(X)\sum_{j=1}^{n}\|x_j\|^p\\&+\frac{1}{n}(1-2^{p-2}\overline{C}_{NJ}^{(n-1,p)}(X))\sum_{i=1}^{n}\|x_i\|^p\Big\}\end{aligned}$$

and consequently

$$\overline{C}_{NJ}^{(n,p)}(X) \leqslant \frac{1}{n}\overline{C}_{NJ}^{(2,p)}(X) + 2^{p-2}\left(1-\frac{1}{n}\right)\overline{C}_{NJ}^{(2,p)}(X)\overline{C}_{NJ}^{(n-1,p)}(X). \quad \square$$

Let $n \geqslant 2$. Define

$$A_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}_{2\times 2}$$

and for each integers $n > 2$,

$$A_n = \begin{bmatrix} A_{n-1} & \mathbf{1} \\ A_{n-1} & -\mathbf{1} \end{bmatrix}_{2^{n-1}\times n},$$

where $\mathbf{1}$ denotes the 2^{n-2}-by-1 column vector in which all the elements are equal to 1.

Define a linear operator $T : l_n^p(X) \to l_{2^{n-1}}^p(X)$ by $T(x) = A_n x$ for any $x \in l_n^p(X)$. Denote $x = (x_1, x_2, \cdots, x_n)$. We can see that

$$\overline{C}_{NJ}^{(n,p)}(X) = \sup\left\{ \frac{\sum_{\theta_j=\pm 1} \|x_1 + \sum_{j=2}^n \theta_j x_j\|^p}{2^{n+p-3}\sum_{j=1}^n \|x_i\|^p} : x \in S(l_n^p(X)) \right\} = \frac{\|T\|^p}{2^{n+p-3}} \tag{2.2}$$

Using (2.2), we have

THEOREM 2.5. *Let X^* be the dual space of the Banach space X. Then*

(a) $[\underline{C}_{NJ}^{(n,q)}(X^*)]^p \geqslant \frac{1}{[\overline{C}_{NJ}^{(n,p)}(X)]^q}$,

(b) $[\underline{C}_{NJ}^{(n,p)}(X)]^q \geqslant \frac{1}{[\overline{C}_{NJ}^{(n,q)}(X^*)]^p}$,

where $p, q \geqslant 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let T defined as above. Obviously, $T^* : l_{2^{n-1}}^q(X^*) \to l_n^q(X^*)$ is generated by the matrix $A_n^* = A_n^T$. Then

$$\frac{1}{\overline{C}_{NJ}^{(n,p)}(X)} = \frac{2^{n+p-3}}{\|T\|^p} = \frac{2^{n+p-3}}{\|T^*\|^p} \leqslant \frac{2^{n+p-3}\|y^*\|_{l_{2^{n-1}}^q(X^*)}^p}{\|T^* y^*\|_{l_n^q(X^*)}^p} \tag{2.3}$$

for any $y^* = (y_1^*, y_2^*, \cdots, y_{2^{n-1}}^*) \in l_{2^{n-1}}^q(X^*) \setminus \{\mathbf{0}\}$. Let $S : l_n^q(X^*) \to l_{2^{n-1}}^q(X^*)$ by $S(x^*) = \frac{1}{2^{n-1}} A_n x^*$ for any $x^* \in l_n^q(X^*)$. Since $T^*(Sx^*) = x^*$, let $y^* = S(x^*)$ in (2.3) for any

$x^* = (x_1^*, x_2^*, \cdots, x_n^*) \in (X^*)^n \setminus \{\mathbf{0}\}$, it follows that

$$\begin{aligned}
\frac{1}{\overline{C}_{NJ}^{(n,p)}(X)} &\leqslant \frac{2^{n+p-3}\|Sx^*\|^p_{l^q_{2^{n-1}}(X^*)}}{\|T^*Sx^*\|^p_{l^q_n(X^*)}} \\
&= \frac{2^{n+p-3}2^{p-np}\|A_n x^*\|^p_{l^q_{2^{n-1}}(X^*)}}{\|x^*\|^p_{l^q_n(X^*)}} \\
&= \frac{2^{n+p-3}2^{p-np}(\sum_{\theta_j=\pm 1}\|x_1^* + \sum_{j=2}^n \theta_j x_j^*\|^q)^{\frac{p}{q}}}{(\sum_{j=1}^n \|x_j^*\|^q)^{\frac{p}{q}}} \\
&= 2^{n+2p-np-3+\frac{p}{q}(n+q-3)}\Big(\frac{\sum_{j=\pm 1}\|x_1^* + \sum_{j=2}^n \theta_j x_j^*\|^q}{2^{n+q-3}\sum_{j=1}^n \|x_j^*\|^q}\Big)^{\frac{p}{q}} \\
&= \Big(\frac{\sum_{j=\pm 1}\|x_1^* + \sum_{j=2}^n \theta_j x_j^*\|^q}{2^{n+q-3}\sum_{j=1}^n \|x_j^*\|^q}\Big)^{\frac{p}{q}}.
\end{aligned}$$

By the definition of lower (n,p)-th von Neumann-Jordan constant, we see that (a) is valid.

(b) Since X can be isometrically embedded into X^{**}, it follows that

$$[\underline{C}_{NJ}^{(n,p)}(X)]^q \geqslant [\underline{C}_{NJ}^{(n,p)}(X^{**})]^q \geqslant \frac{1}{[\overline{C}_{NJ}^{(n,q)}(X^*)]^p}. \quad \square$$

The following result is a characterization of uniformly non-l_n^1 Banach spaces by use of $\overline{C}_{NJ}^{(n,p)}(X)$ and $\overline{C}_{mNJ}^{(n,p)}(X)$.

THEOREM 2.6. *Let X be a Banach space, $n \in \mathbb{N}$ and $n \geqslant 2$, $1 < p < \infty$. Then the following conditions are equivalent:*

(a) X is uniformly non-l_n^1;

(b) $\overline{C}_{NJ}^{(n,p)}(X) < 2^{2-p}n^{p-1}$;

(c) $\overline{C}_{mNJ}^{(n,p)}(X) < 2^{2-p}n^{p-1}$.

Proof. $(a) \Rightarrow (b)$. Suppose that X is uniformly non-l_n^1, by Lemma 2.1, we have that there exists a constant $\alpha \in (0,1)$ such that if $x_1, x_2, \ldots, x_n$ are elements of X, then

$$\sum \Big\|\frac{x_1 \pm x_2 \pm \ldots \pm x_n}{n}\Big\|^p \leqslant \frac{2^{n-1}\alpha}{n}\sum_{j=1}^n \|x_j\|^p$$

where the sum is taken over all 2^{n-1} choices of signs, that is

$$\frac{1}{2^{n+p-3}}\sum_{\theta_j=\pm 1}\|x_1 + \theta_2 x_2 + \ldots + \theta_n x_n\|^p \leqslant 2^{2-p}\alpha n^{p-1}\sum_{j=1}^n \|x_j\|^p$$

Hence, $\overline{C}_{NJ}^{(n,p)}(X) < 2^{2-p}n^{p-1}$.

$(b) \Rightarrow (c)$. It is obvious.

$(c) \Rightarrow (a)$. For any $x_1, x_2, \ldots, x_n \in S(X)$, we have

$$\frac{1}{2^{n+p-2}n} \sum_{\theta_j=\pm 1} \| \sum_{j=1}^{n} \theta_j x_j \|^p \leqslant \overline{C}_{mNJ}^{(n,p)}(X) < 2^{2-p} n^{p-1}.$$

So

$$\min_{\theta_j=\pm 1} \| \sum_{j=1}^{n} \theta_j x_j \| \leqslant n(2^{p-2} n^{1-p} \overline{C}_{mNJ}^{(n,p)}(X))^{\frac{1}{p}} < n,$$

which completes the proof. □

THEOREM 2.7. *Let $n \geqslant 2$, $1 \leqslant p \leqslant 2$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\dim l^p \geqslant n$. Then*

(a) If $p \leqslant p_1 \leqslant q$, then $\overline{C}_{NJ}^{(n,p_1)}(l^p) = \overline{C}_{mNJ}^{(n,p_1)}(l^p) = 2^{2-p_1} n^{\frac{p_1}{p}-1}$;

(b) If $1 \leqslant p_1 \leqslant p$, then $\overline{C}_{NJ}^{(n,p_1)}(l^p) = 2^{2-p_1}$;

(c) If $p_1 > q$, then $\overline{C}_{NJ}^{(n,p_1)}(l^p) \leqslant 2^{2-p_1} n^{p_1-2}$.

Proof. (a) By Theorem A in [18] we have for any $x_1, x_2, \ldots, x_n$ in l^p

$$\Big\{ \frac{1}{2^n} \sum_{\theta_j=\pm 1} \| \sum_{j=1}^{n} \theta_j x_j \|^q \Big\}^{\frac{1}{q}} \leqslant \Big\{ \sum_{j=1}^{n} \|x_j\|^p \Big\}^{\frac{1}{p}}. \tag{2.4}$$

Applying (2.4) and Hölder inequality, we have

$$\begin{aligned} \sum_{\theta_j=\pm 1} \| \sum_{j=1}^{n} \theta_j x_j \|^{p_1} &\leqslant 2^{n-\frac{np_1}{q}} \Big(\sum_{\theta_j=\pm 1} \| \sum_{j=1}^{n} \theta_j x_j \|^q \Big)^{\frac{p_1}{q}} \\ &\leqslant 2^n \Big(\sum_{j=1}^{n} \|x_j\|^p \Big)^{\frac{p_1}{p}} \\ &\leqslant 2^n n^{\frac{p_1}{p}-1} \sum_{j=1}^{n} \|x_j\|^{p_1}. \end{aligned}$$

Hence $\overline{C}_{NJ}^{(n,p_1)}(l^p) \leqslant 2^{2-p_1} n^{\frac{p_1}{p}-1}$.

Taking the canonical basis $(e_i)_{i=1}^k$ $(n \leqslant k = \dim l^p)$, we get

$$\overline{C}_{mNJ}^{(n,p_1)}(l^p) \geqslant \frac{\sum_{\theta_j=\pm 1} \| \sum_{j=1}^{n} \theta_j e_j \|^{p_1}}{2^{n+p_1-2} \sum_{j=1}^{n} \|e_j\|^{p_1}} = 2^{2-p_1} n^{\frac{p_1}{p}-1}.$$

Therefore, we get (a).

(b) From (2.4),

$$\begin{aligned} \sum_{\theta_j=\pm 1} \| \sum_{j=1}^{n} \theta_j x_j \|^{p_1} &\leqslant 2^{n-\frac{np_1}{q}} \Big(\sum_{\theta_j=\pm 1} \| \sum_{j=1}^{n} \theta_j x_j \|^q \Big)^{\frac{p_1}{q}} \\ &\leqslant 2^n \Big(\sum_{j=1}^{n} \|x_j\|^p \Big)^{\frac{p_1}{p}} \\ &\leqslant 2^n \sum_{j=1}^{n} \|x_j\|^{p_1}. \end{aligned}$$

So $\overline{C}_{NJ}^{(n,p_1)}(l^p) \leqslant 2^{2-p_1}$. Taking $x_1 = e_1, x_2 = \ldots = x_n = 0$, we have $\overline{C}_{NJ}^{(n,p_1)}(l^p) \geqslant 2^{2-p_1}$.

(c) By $p_1 > q$ and (2.4), we have

$$\begin{aligned}
\sum_{\theta_j=\pm1} \|\sum_{j=1}^{n} \theta_j x_j\|^{p_1} &\leqslant (\sum_{j=1}^{n} \|x_j\|)^{p_1-q} \sum_{\theta_j=\pm1} \|\sum_{j=1}^{n} \theta_j x_j\|^{q} \\
&\leqslant 2^n (\sum_{j=1}^{n} \|x_j\|)^{p_1-q} (\sum_{j=1}^{n} \|x_j\|^{p})^{\frac{q}{p}} \\
&\leqslant 2^n (\sum_{j=1}^{n} \|x_j\|^{p_1})^{\frac{p_1-q}{p_1}} n^{(1-\frac{1}{p_1})(p_1-q)} (\sum_{j=1}^{n} \|x_j\|^{p_1})^{\frac{q}{p_1}} n^{\frac{q}{p}-\frac{q}{p_1}} \\
&= 2^n n^{p_1-2} \sum_{j=1}^{n} \|x_j\|^{p_1}.
\end{aligned}$$

Hence $\overline{C}_{NJ}^{(n,p_1)}(l^p) \leqslant 2^{2-p_1} n^{p_1-2}$. □

COROLLARY 2.8. *Let* $\dim l^p \geqslant n \geqslant 2$.

(a) *If* $2 \leqslant p \leqslant p_1$ *then* $\underline{C}_{NJ}^{(n,p_1)}(l^p) = 2^{2-p_1}$;

(b) *If* $p \geqslant 2$ *and* $q \leqslant p_1 \leqslant p$, *where* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\underline{C}_{NJ}^{(n,p_1)}(l^p) = \underline{C}_{mNJ}^{(n,p_1)}(l^p) = 2^{2-p_1} n^{\frac{p_1}{p}-1};$$

(c) *If* $p \geqslant 2$, *and* $p_1 < q$, *where* $\frac{1}{p}+\frac{1}{q}=1$, *then* $\underline{C}_{NJ}^{(n,p_1)}(l^p) \geqslant 2^{2-p_1} n^{p_1-2}$.

Proof. Let q_1 such that $\frac{1}{p_1}+\frac{1}{q_1}=1$. From Theorem 2.6 and Theorem 2.7, we have

(a)

$$[\underline{C}_{NJ}^{(n,p_1)}(l^p)]^{\frac{q_1}{p_1}} \geqslant \frac{1}{\overline{C}_{NJ}^{(n,q_1)}(l^q)} = 2^{q_1-2},$$

so $\underline{C}_{NJ}^{(n,p_1)}(l^p) \geqslant 2^{2-p_1}$.

By taking $x_1 = e_1,\ x_2 = x_3 = \ldots = x_n = 0$ and the definition of $\underline{C}_{NJ}^{(n,p_1)}(l^p)$, we can see $\underline{C}_{NJ}^{(n,p_1)}(l^p) \leqslant 2^{2-p_1}$.

(b) Because $q \leqslant q_1 \leqslant p$, then

$$[\underline{C}_{NJ}^{(n,p_1)}(l^p)]^{\frac{q_1}{p_1}} \geqslant \frac{1}{\overline{C}_{NJ}^{(n,q_1)}(l^q)} = 2^{q_1-2} n^{1-\frac{q_1}{q}},$$

so $\underline{C}_{NJ}^{(n,p_1)}(l^p) \geqslant 2^{2-p_1} n^{\frac{p_1}{p}-1}$. By taking $x_1 = e_1,\ x_2 = e_2, \ldots, x_n = e_n$, we can see that $\underline{C}_{NJ}^{(n,p_1)}(l^p) \leqslant 2^{2-p_1} n^{\frac{p_1}{p}-1}$.

(c) By $q_1 \geqslant p$, we also have

$$[\underline{C}_{NJ}^{(n,p_1)}(l^p)]^{\frac{q_1}{p_1}} \geqslant \frac{1}{\overline{C}_{NJ}^{(n,q_1)}(l^q)} \geqslant 2^{q_1-2} n^{2-q_1}. \quad \square$$

LEMMA 2.9. ([2]) *Let* $2 \leqslant p < \infty$ *and* $X = L^p(\mu)$ *or* $X = l^p$. *Then*

$$\sum_{\theta_j=\pm 1} \|x_1 + \sum_{j=2}^{n} \theta_j x_j\|^p \leqslant \frac{1}{n} \sum_{k=0}^{[\frac{n}{2}]} C_n^k (n-2k)^p \sum_{j=1}^{n} \|x_j\|^p$$

for any $x_1, x_2, \dots, x_n \in X$ *and any integer* $n \geqslant 1$.

THEOREM 2.10. *Let* $p \geqslant 2$, $p_1 < p$, *then for* $X = l^p$ *or* $L^p(\mu)$,

$$\overline{C}_{mNJ}^{(n,p_1)}(X) = \frac{2^{2-p_1}}{n} \Big(2^{1-n} \sum_{k=0}^{[\frac{n}{2}]} C_n^k (n-2k)^p\Big)^{\frac{p_1}{p}}.$$

Proof. Let $x_1, x_2, \dots, x_n \in S(X)$. By applying Lemma 2.9, we have

$$\begin{aligned}\sum_{\theta_j=\pm 1} \|x_1 + \sum_{j=2}^{n} \theta_j x_j\|^{p_1} &\leqslant 2^{(n-1)(1-\frac{p_1}{p})} \Big(\sum_{\theta_j=\pm 1} \|x_1 + \sum_{j=2}^{n} \theta_j x_j\|^p\Big)^{\frac{p_1}{p}} \\ &\leqslant 2^{(n-1)(1-\frac{p_1}{p})} \Big(\frac{1}{n} \sum_{k=0}^{[\frac{n}{2}]} C_n^k (n-2k)^p \sum_{j=1}^{n} \|x_j\|^p\Big)^{\frac{p_1}{p}} \\ &= 2^{(n-1)(1-\frac{p_1}{p})} \Big(\sum_{k=0}^{[\frac{n}{2}]} C_n^k (n-2k)^p\Big)^{\frac{p_1}{p}}.\end{aligned}$$

Hence

$$\overline{C}_{mNJ}^{(n,p_1)}(l^p) \leqslant \frac{2^{2-p_1}}{n} \Big(2^{1-n} \sum_{k=0}^{[\frac{n}{2}]} C_n^k (n-2k)^p\Big)^{\frac{p_1}{p}}. \tag{2.5}$$

When $X = l^p$, let $(\theta_2, \theta_3, \dots, \theta_n)$ be $(a_{i2}, a_{i3}, \dots, a_{in}), i = 1, 2, \dots, 2^{n-1}$ for all $\theta_j = \pm 1$, where $a_{12} = a_{13} = \dots = a_{1n} = 1$.

Taking

$$z_1 = 2^{\frac{1-n}{p}}(1, 1, \dots, 1), \ \ z_2 = 2^{\frac{1-n}{p}}(1, a_{22}, a_{32}, \dots, a_{2^{n-1}2}) \dots,$$

$$z_n = 2^{\frac{1-n}{p}}(1, a_{2n}, a_{3n}, \dots, a_{2^{n-1}n}).$$

Because $(1, \theta_2 a_{k2}, \theta_3 a_{k3}, \dots, \theta_n a_{kn})$ $(k = 1, 2, \dots, 2^{n-1})$ just takes every $(1, \vartheta_2, \dots, \vartheta_n)$ for $\vartheta_j = \pm 1$, by Lemma 2.2, we can get

$$\begin{aligned}\sum_{\theta_j=\pm 1} \|z_1 + \sum_{j=2}^{n} \theta_j z_j\|^{p_1} &= 2^{(n-1)(1-\frac{p_1}{p})} \Big(\sum_{k=0}^{n-1} C_{n-1}^k |n-2k|^p\Big)^{\frac{p_1}{p}} \\ &= 2^{(n-1)(1-\frac{p_1}{p})} \Big(\sum_{k=0}^{[\frac{n}{2}]} C_n^k (n-2k)^p\Big)^{\frac{p_1}{p}}.\end{aligned}$$

So

$$\overline{C}_{mNJ}^{(n,p_1)}(l^p) \geqslant \frac{2^{2-p_1}}{n} \Big(2^{1-n} \sum_{k=0}^{[\frac{n}{2}]} C_n^k (n-2k)^p\Big)^{\frac{p_1}{p}}.$$

Since $L^p(\mu)$ contains an isometric copy of l^p, by inequality (2.5), we obtain the thesis for $X = L^p(\mu)$. $\square$

3. Examples

In this section, an expression of $\overline{C}_{NJ}^{(n,p)}(X)$ for $x_j \in ex(S(X))$ will be given. As an application, the exact expression of $\overline{C}_{NJ}^{(3,p)}(l_\infty - l_1)$ will be calculated and also the exact value of $\overline{C}_{NJ}^{(3,2)}(l_\infty - l_1)$ will be obtained.

LEMMA 3.1. *Let X be a Banach space. If $x_i = \alpha_i y_i + (1-\alpha_i) z_i$ with $\alpha_i \in [0,1]$ and $y_i, z_i \in X$, $(i = 1,2,\ldots,n)$, then*

$$\sum_{\theta_j=\pm 1} \| \sum_{j=1}^{n} \theta_j x_j \|^p \leqslant \max\Big\{ \sum_{\theta_j=\pm 1} \| \sum_{j=1}^{n} \theta_j u_j \|^p : u_j \in \{y_j, z_j\}, j = 1,2,\ldots,n \Big\}.$$

Proof. By Hölder inequality, we have

$$\begin{aligned}
\sum_{\theta_j=\pm 1} \| \sum_{j=1}^{n} \theta_j x_j \|^p &= \sum_{\theta_j=\pm 1} \|\alpha_1(\theta_1 y_1 + \sum_{j=2}^{n} \theta_j x_j) + (1-\alpha_1)(\theta_1 z_1 + \sum_{j=2}^{n} \theta_j x_j)\|^p \\
&\leqslant \sum_{\theta_j=\pm 1} (\alpha_1 \|\theta_1 y_1 + \sum_{j=2}^{n} \theta_j x_j\| + (1-\alpha_1)\|\theta_1 z_1 + \sum_{j=2}^{n} \theta_j x_j\|)^p \\
&\leqslant \sum_{\theta_j=\pm 1} (\alpha_1 \|\theta_1 y_1 + \sum_{j=2}^{n} \theta_j x_j\|^p + (1-\alpha_1)\|\theta_1 z_1 + \sum_{j=2}^{n} \theta_j x_j\|^p) \\
&\leqslant \max\{ \sum_{\theta_j=\pm 1} \|\theta_1 u_1 + \sum_{j=2}^{n} \theta_j x_j\|^p : u_1 \in \{y_1, z_1\}\} \\
&\leqslant \max\{ \sum_{\theta_j=\pm 1} \|\theta_1 u_1 + \theta_2 u_2 + \sum_{j=3}^{n} \theta_j x_j\|^p : u_j \in \{y_j, z_j\}, j = 1,2\} \\
&\leqslant \ldots \\
&\leqslant \max\{ \sum_{\theta_j=\pm 1} \| \sum_{j=1}^{n} \theta_j u_j\|^p : u_j \in \{y_j, z_j\}, j = 1,2,\ldots,n\}. \quad \square
\end{aligned}$$

Hence by applying Lemma 3.1, for the finite dimensional Banach space X, we have

THEOREM 3.2. *Let X be a finite dimensional Banach space and $p \geqslant 1$. Then*

$$\overline{C}_{NJ}^{(n,p)}(X) = \sup\left\{ \frac{\sum_{\theta_j=\pm 1} \|x_1 + \sum_{j=2}^{n} \theta_j t_j x_j\|^p}{2^{n+p-3}(1 + t_2^p + \ldots + t_n^p)} : x_j \in ex(S(X)), t_j \in [0,1] \right\}.$$

As an application of Theorem 3.2, we can compute $\overline{C}_{NJ}^{(3,p)}(l_\infty - l_1)$. Firstly, we need the following lemma.

LEMMA 3.3. *Let* $p \geqslant 1$ *and* $t_j \in [0,1]$ *for* $j = 1,2,3$. *Then*
(a) $2(1+t_3)^p \leqslant (1+t_2+t_3)^p + (1-t_2+t_3)^p$;
(b) $(1+t_2)^p + t_3^p \geqslant 1 + (t_2+t_3)^p$.

Proof. (a) By the convexity of x^p, we have

$$2(1+t_3)^p = 2\Big(\frac{1+t_2+t_3+1-t_2+t_3}{2}\Big)^p \leqslant (1+t_2+t_3)^p + (1-t_2+t_3)^p.$$

(b) Because the function $h(t_2) = (1+t_2)^p - (t_2+t_3)^p$ is increasing, we have $h(t_2) \geqslant h(0) = 1 - t_3^p$. □

THEOREM 3.4. *If* $p \geqslant 1$, *and* X *is* $l_\infty - l_1$ *space, i.e,* $\mathbb{R}^2$ *endowed with the norm*

$$\|(x_1,x_2)\| = \begin{cases} \|(x_1,x_2)\|_\infty, & \text{if } x_1x_2 \geqslant 0, \\ (\|(x_1,x_2)\|_1, & \text{if } x_1x_2 < 0. \end{cases}$$

Then

$$\overline{C}_{NJ}^{(3,p)}(X) = \sup\left\{\frac{(1+t_2+t_3)^p + (1-t_2+t_3)^p + (1+t_2)^p + \max\{(1-t_2)^p, t_3^p\}}{2^p(1+t_2^p+t_3^p)} : t_2,t_3 \in [0,1]\right\}. \tag{3.1}$$

Proof. Let

$$g(t_2,t_3) = (1+t_2+t_3)^p + (1-t_2+t_3)^p + (1+t_2)^p + \max\{(1-t_2)^p, t_3^p\}$$

for $t_2,t_3 \in [0,1]$.

Now we prove that

$$\sum_{\theta_j=\pm 1} \|z_1 + \theta_2 t_2 z_2 + \theta_3 t_3 z_3\|^p \leqslant \max\{g(t_2,t_3), g(t_3,t_2)\} \tag{3.2}$$

for any $t_2,t_3 \in [0,1]$ and $z_i \in ex(B(l_\infty - l_1))$.

In fact, by the symmetry, we only need to suppose that $0 \leqslant t_2 \leqslant t_3$ and just consider $z_1 = (1,0)$ or $(1,1)$. That means we just have to think about the following five cases:

(a) If $z_1 = z_2 = z_3 = (1,0)$ or $z_1 = z_2 = z_3 = (1,1)$, then

$$\sum_{\theta_j=\pm 1} \|z_1 + \theta_2 t_2 z_2 + \theta_3 t_3 z_3\|^p = (1+t_2+t_3)^p + (1+t_2-t_3)^p + (1-t_2+t_3)^p + |1-t_2-t_3|^p \leqslant g(t_2,t_3);$$

(b) If $z_1 = z_2 = (1,0)$, $z_3 = (1,1)$ or $z_1 = z_2 = (1,0)$, $z_3 = (0,1)$ or $z_1 = z_2 = (1,1)$, $z_3 = (1,0)$ or $z_1 = z_2 = (1,1)$, $z_3 = (0,1)$ then

$$\begin{aligned}
&\sum_{\theta_j=\pm 1} \|z_1 + \theta_2 t_2 z_2 + \theta_3 t_3 z_3\|^p \\
&= \sum_{\theta_j=\pm 1} \|(1 + \theta_2 t_2 + \theta_3 t_3, \theta_3 t_3)\|^p \\
&= \sum_{\theta_j=\pm 1} \|(1 + \theta_2 t_2, \theta_3 t_3)\|^p \\
&= \sum_{\theta_j=\pm 1} \|(1 + \theta_2 t_2 + \theta_3 t_3, 1 + \theta_2 t_2)\|^p \\
&= (1+t_2+t_3)^p + (1+t_2)^p + (1-t_2+t_3)^p + \max\{(1-t_2)^p, t_3^p\} = g(t_2,t_3).
\end{aligned}$$

(c) If $z_1 = (1,0)$, $z_2 = z_3 = (0,1)$ or $z_1 = (1,0)$, $z_2 = z_3 = (1,1)$ or $z_1 = (1,1)$, $z_2 = z_3 = (0,1)$ or $z_1 = (1,1)$, $z_2 = z_3 = (1,0)$, then

$$\begin{aligned}
&\sum_{\theta_j=\pm 1} \|z_1 + \theta_2 t_2 z_2 + \theta_3 t_3 z_3\|^p \\
&= \sum_{\theta_j=\pm 1} \|(1, \theta_2 t_2 + \theta_3 t_3)\|^p \\
&= \sum_{\theta_j=\pm 1} \|(1 + \theta_2 t_2 + \theta_3 t_3, \theta_2 t_2 + \theta_3 t_3)\|^p \\
&= \sum_{\theta_j=\pm 1} \|(1, 1 + \theta_2 t_2 + \theta_3 t_3)\|^p \\
&= (1+t_2+t_3)^p + (1+t_3-t_2)^p + 1 + \max\{1, (t_2+t_3)^p\} \\
&= \begin{cases} (1+t_2+t_3)^p + (1+t_3-t_2)^p + 2, & \text{if } t_2+t_3 \leqslant 1, \\ (1+t_2+t_3)^p + (1+t_3-t_2)^p + 1 + (t_2+t_3)^p, & \text{if } t_2+t_3 \geqslant 1. \end{cases} \\
&\leqslant g(t_2,t_3).
\end{aligned}$$

holds by Lemma 3.3.

(d) If $z_1 = z_3 = (1,0)$, $z_2 = (0,1)$ or $z_1 = z_3 = (1,0)$, $z_2 = (1,1)$ or $z_1 = z_3 = (1,1)$, $z_2 = (1,0)$ or $z_1 = z_3 = (1,1)$, $z_2 = (0,1)$, then

$$\begin{aligned}
&\sum_{\theta_j=\pm 1} \|z_1 + \theta_2 t_2 z_2 + \theta_3 t_3 z_3\|^p \\
&= \sum_{\theta_j=\pm 1} \|(1 + \theta_3 t_3, \theta_2 t_2)\|^p \\
&= \sum_{\theta_j=\pm 1} \|(1 + \theta_2 t_2 + \theta_3 t_3, \theta_2 t_2)\|^p \\
&= \sum_{\theta_j=\pm 1} \|(1 + \theta_3 t_3, 1 + \theta_2 t_2 + \theta_3 t_3)\|^p \\
&= (1+t_2+t_3)^p + (1+t_3)^p + (1+t_2-t_3)^p + \max\{(1-t_3)^p, t_2^p\} = g(t_3,t_2).
\end{aligned}$$

(e) If $z_1=(1,0)$, $z_2=(0,1)$, $z_3=(1,1)$ or $z_1=(1,0)$, $z_2=(1,1)$, $z_3=(0,1)$ or $z_1=(1,1)$, $z_2=(1,0)$, $z_3=(0,1)$ or $z_1=(1,1)$, $z_2=(0,1)$, $z_3=(1,0)$, then

$$\begin{aligned}
&\sum_{\theta_j=\pm 1}\|z_1+\theta_2t_2z_2+\theta_3t_3z_3\|^p\\
&=\sum_{\theta_j=\pm 1}\|(1+\theta_2t_2,\theta_2t_2+\theta_3t_3)\|^p\\
&=\sum_{\theta_j=\pm 1}\|(1+\theta_3t_3,\theta_2t_2+\theta_3t_3)\|^p\\
&=\sum_{\theta_j=\pm 1}\|(1+\theta_3t_3,1+\theta_2t_2)\|^p\\
&=(1-t_2)^p+(1+t_2)^p+2(1+t_3)^p\\
&\leqslant g(t_2,t_3)
\end{aligned}$$

is also valid by Lemma 3.3.

Hence, by (3.2) and Theorem 3.2 we have

$$\begin{aligned}
&\overline{C}_{NJ}^{(3,p)}(l_\infty-l_1)\\
&\leqslant \sup\left\{\frac{(1+t_2+t_3)^p+(1-t_2+t_3)^p+(1+t_2)^p+\max\{(1-t_2)^p,t_3^p\}}{2^p(1+t_2^p+t_3^p)}:t_2,t_3\in[0,1]\right\}.
\end{aligned}$$

Now let the $\frac{g(\mu_2,\mu_3)}{2^p(1+\mu_2^p+\mu_3^p)}$ for some $\mu_2.\mu_3\in[0,1]$ be the supremum of the above formula.

If $0\leqslant\mu_2\leqslant\mu_3\leqslant 1$, by taking $z_1=z_2=(1,0)$ and $z_3=(0,1)$ then

$$\overline{C}_{NJ}^{(3,p)}(l_\infty-l_1)\geqslant\frac{\sum_{\theta_j=\pm 1}\|z_1+\mu_2\theta_2z_2+\mu_3\theta_3z_3\|^p}{2^p(1+\mu_3^p+\mu_2^p)}=\frac{g(\mu_2,\mu_3)}{2^p(1+\mu_2^p+\mu_3^p)}.$$

If $0\leqslant\mu_3\leqslant\mu_2\leqslant 1$, by taking $z_2=(0,1)$, $z_3=z_1=(1,0)$, we also have

$$\overline{C}_{NJ}^{(3,p)}(l_\infty-l_1)\geqslant\frac{\sum_{\theta_j=\pm 1}\|z_1+\mu_3\theta_2z_2+\mu_2\theta_3z_3\|^p}{2^p(1+\mu_3^p+\mu_2^p)}=\frac{g(\mu_2,\mu_3)}{2^p(1+\mu_2^p+\mu_3^p)}.$$

Therefore we get (3.1). □

COROLLARY 3.5. *For $l_\infty-l_1$ space, we have*

$$\overline{C}_{NJ}^{(3,2)}(l_\infty-l_1)=\frac{3+\sqrt{5}}{4}.$$

Proof. By Theorem 3.4, we have

$$\begin{aligned}
&\overline{C}_{NJ}^{(3,2)}(l_\infty - l_1)\\
&= \sup\left\{\frac{(1+t_2+t_3)^2+(1-t_2+t_3)^2+(1+t_2)^2+\max\{(1-t_2)^2,t_3^2\}}{4(1+t_2^2+t_3^2)} : t_2,t_3\in[0,1]\right\}\\
&= \begin{cases} \frac{3}{4}+\frac{t_2+2t_3}{2(1+t_2^2+t_3^2)}, & \text{if } t_2+t_3 \geqslant 1,\ t_2,t_3\in[0,1]\\ 1+\frac{2t_3-t_3^2}{2(1+t_2^2+t_3^2)}, & \text{if } t_2+t_3\leqslant 1,\ t_2,t_3\in[0,1]. \end{cases}\\
&= \frac{3+\sqrt{5}}{4}.
\end{aligned}$$

By $1+\frac{2t_3-t_3^2}{2(1+t_2^2+t_3^2)} = \frac{3+\sqrt{5}}{4}$ at $t_2 = 0,\ t_3 = \frac{\sqrt{5}-1}{2}$ and

(1) If $t_2+t_3 \geqslant 1$ and $t_2,t_3 \in [0,1]$, then

$$\begin{aligned}
\frac{3}{4}+\frac{t_2+2t_3}{2(1+t_2^2+t_3^2)} &\leqslant \frac{3}{4}+\frac{2\sqrt{5}\sqrt{t_2^2+t_3^2}}{4(1+t_2^2+t_3^2)}\\
&\leqslant \frac{3+\sqrt{5}}{4}.
\end{aligned}$$

(2) If $t_2+t_3 \leqslant 1$ and $t_2,t_3 \in [0,1]$, then

$$\begin{aligned}
1+\frac{2t_3-t_3^2}{2(1+t_2^2+t_3^2)} \leqslant 1+\frac{2t_3-t_3^2}{2(1+t_3^2)} &\leqslant \max_{t_3\in[0,1]}\left\{1+\frac{2t_3-t_3^2}{2(1+t_3^2)}\right\}\\
&= \frac{3+\sqrt{5}}{4}. \quad \square
\end{aligned}$$

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