Mathematical
Inequalities
& Applications
Volume 27, Number 3 (2024), 601–612 doi:10.7153/mia-2024-27-41

HARDY–STEKLOV OPERATORS ON TOPOLOGICAL MEASURE SPACES

KAIRAT T. MYNBAEV

(*Communicated by L. E. Persson*)

Abstract. We give necessary and sufficient conditions on non-negative weights u,v and measures μ,ν in the inequality

$$\left(\int_\Omega |Tf(x)|^q u(x)d\mu(x)\right)^{1/q} \leqslant C\left(\int_\Omega |f(x)|^p v(x)d\nu(x)\right)^{1/p}.$$

Here the integral operator T is a Hardy-Steklov type operator associated with a family of open subsets $\Omega(t)$ of an open set Ω in a Hausdorff topological space X; μ,ν are σ-additive Borel measures, and $1<p<\infty$, $0<q<\infty$. The integration in T is over domains of type $\Omega(b(t))\setminus\Omega(a(t))$ where a,b are non-negative, increasing, continuous functions on $[0,\infty)$ that vanish at zero, tend to ∞ at ∞ and satisfy $a(t)<b(t)$ for $t\in(0,\infty)$. Previously such results have been known for an operator on a subset of a Euclidean space.

1. Introduction

We consider a multi-dimensional version of the Hardy-Steklov inequality

$$\left[\int_0^\infty \left|\int_{a(x)}^{b(x)} f d\nu\right|^q u(x)d\mu(x)\right]^{1/q} \leqslant C\left(\int_0^\infty |f|^p v d\nu\right)^{1/p}$$

where the functions a,b are non-negative, increasing, continuous and satisfy

$$a(0)=b(0)=0,\ a(x)<b(x) \text{ for } x\in(0,\infty),\ a(\infty)=b(\infty)=\infty.$$

Much of the history of the weighted Hardy inequality has been covered in [3]–[6]. The ideas and results developed for the Hardy inequality have been applied to study the Hardy-Steklov inequality. In the one-dimensional case necessary and sufficient conditions on the weights u,v have been obtained in [2] (see a special case in [1]). A full account of their results can also be found in [5]. [15] have developed a different approach to the same problem, giving the criterion in simpler terms. They also provided a compactness criterion. See also [9] for further developments, especially for the results in an integral form for the case $q<p$. The case of starshaped regions in the

Mathematics subject classification (2020): 26D15, 47G10, 26D10.
Keywords and phrases: Hardy-Steklov operator, weighted inequality, boundedness, compactness.
This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19676673).

Euclidean space has been considered in [13]. Here we follow [15] as their method is most amenable to extending to our situation.

We obtain a far-reaching generalization of the results just described. Our domains $\Omega(b(x))\setminus\Omega(a(x))$ are subsets of a Hausdorff topological space X where the dimension notion is generally not defined. The assumptions on the sets $\Omega(t)$ are the same as in [8] and are close to those in [14]. Our results have been made possible by theorems on the Hardy inequality in [8] and the investigation of ordered cores done in [14].

[10], [11] and [12] contain the Hardy inequality on homogeneous groups, connected Lie groups, hyperbolic spaces and Cartan-Hadamard manifolds. Our Theorems 3–4 below hold in these cases too.

ASSUMPTION 1. *(on $\Omega(t)$) Let Ω be an open set in a Hausdorff topological space X with σ-finite Borel measures μ,ν. The measures are defined on the same σ-algebra $\mathfrak{M}$ that contains Borel-measurable sets. The domains $\Omega(t)\subset\Omega$ are assumed to be parameterized by $t\geqslant 0$ and satisfy monotonicity (total orderedness)*

$$\text{for } t_1<t_2,\ \Omega(t_1) \text{ is a proper subset of } \Omega(t_2). \tag{1}$$

We assume that

$$\Omega(0)=\cap_{t>0}\Omega(t)=\varnothing,\ \ \mu\left(\Omega\setminus\cup_{t>0}\Omega(t)\right)=0.$$

Denote $\omega(t)=\overline{\Omega(t)}\cap\overline{(\Omega\setminus\Omega(t))}$ the boundary of $\Omega(t)$ in the relative topology. We require the boundaries to be disjoint and cover almost all Ω:

$$\omega(t_1)\cap\omega(t_2)=\varnothing,\ \ t_1\neq t_2,\ \mu(\Omega\setminus\cup_{t>0}\,\omega(t))=0.$$

This implies that for μ-almost each $y\in\Omega$ there exists a unique $\tau(y)>0$ such that $y\in\omega(\tau(y))$. On the set $\Omega_0\subset\Omega$ of those y for which $\tau(y)$ is not defined we can put $\tau(\Omega_0)=\emptyset$. Passing to a different parametrization, if necessary, we can assume that $\mu\left(\Omega\setminus\cup_{t\leqslant N}\,\omega(t)\right)>0$ for any $N<\infty$.

For a set Δ on R we can define a set $\Omega[\Delta]=\{y\in\Omega:\tau(y)\in\Delta\}$. In particular, with $\Delta=[a(\tau(x)),b(\tau(x))]$ the main integral operator we consider is

$$Tf(x)=\int_{\Omega[a(\tau(x)),b(\tau(x))]} f\,d\nu,\ x\in\Omega,$$

for any non-negative $\mathfrak{M}$-measurable f.

Notation

$L_p(vd\nu,\Omega)$ denotes the space with the norm $\|f\|_{L_p(vd\nu,\Omega)}=\left(\int_\Omega|f|^p v d\nu\right)^{1/p}$ where v is a (non-negative) weight function. $\|T\|=\|T\|_{L_p(vd\nu,\Omega)\to L_q(ud\mu,\Omega)}$ is the norm of a linear operator T acting from $L_p(vd\nu,\Omega)$ to $L_q(ud\mu,\Omega)$. Our task is to estimate $\|T\|$ where the weights u,v are non-negative and finite almost everywhere. As usual, it is enough to consider non-negative f, so $\|T\|$ is the least constant C in the inequality

$$\left[\int_\Omega\left(\int_{\Omega[a(\tau(x)),b(\tau(x))]} f\,d\nu\right)^q u(x)\,d\mu(x)\right]^{1/q}\leqslant C\left(\int_\Omega f^p v\,d\nu\right)^{1/p}. \tag{2}$$

We write $A \asymp B$ to mean that $c_1 A \leqslant B \leqslant c_2 A$ with positive constants c_1, c_2 that do not depend on weights and measures. A lower case c, with or without subscripts, denotes various constants whose values do not matter.

2. Auxiliary results on Hardy inequality

For $0 \leqslant a < b \leqslant \infty$ we need results on validity of the inequalities

$$\left[\int_{\Omega[a,b]}\left(\int_{\Omega[a,\tau(x)]} f d\nu\right)^q u(x)d\mu(x)\right]^{1/q} \leqslant C\left(\int_{\Omega[a,b]} f^p v d\nu\right)^{1/p}$$

and

$$\left[\int_{\Omega[a,b]}\left(\int_{\Omega[\tau(x),b]} f d\nu\right)^q u(x)d\mu(x)\right]^{1/q} \leqslant C^*\left(\int_{\Omega[a,b]} f^p v d\nu\right)^{1/p}$$

from [8]. For segments $\Delta_1, \Delta_2 \subseteq [0,\infty)$ denote

$$\Psi(\Delta_1,\Delta_2) = \left(\int_{\Omega[\Delta_1]} u d\mu\right)^{1/q}\left(\int_{\Omega[\Delta_2]} v^{-p'/p} d\nu\right)^{1/p'}, \ p \leqslant q,$$

$$\Phi(\Delta_1,\Delta_2) = \left(\int_{\Omega[\Delta_1]} u d\mu\right)^{r/p}\left(\int_{\Omega[\Delta_2]} v^{-p'/p} d\nu\right)^{r/p'}, \ q < p.$$

When appropriate, we also include in the notation the dependence on u or both u and v, as in $\Psi(\Delta_1,\Delta_2,u)$, etc. Everywhere we assume $1 < p < \infty$. For $q < p$ we put $1/r = 1/q - 1/p$.

THEOREM 1. *a) If* $1 < p \leqslant q < \infty$, *then* $C \asymp \sup_{x\in\Omega[a,b]} \Psi([\tau(x),b],[a,\tau(x)])$.
b) If $0 < q < p,\ 1 < p < \infty$, *then we have*

$$C \asymp \left(\int_{\Omega[a,b]} \Phi([\tau(x),b],[a,\tau(x)])u(x)\,d\mu(x)\right)^{1/r}.$$

THEOREM 2. *a) If* $1 < p \leqslant q < \infty$ *then* $C^* \asymp \sup_{x\in\Omega[a,b]} \Psi([a,\tau(x)],[\tau(x),b])$.
b) If $0 < q < p,\ 1 < p < \infty$ *then*

$$C^* \asymp \left(\int_{\Omega[a,b]} \Phi([a,\tau(x)],[\tau(x),b])u(x)\,d\mu(x)\right)^{1/r}.$$

3. Main results

The sets $\Omega[a(\tau(x)), b(\tau(x))]$ do not satisfy monotonicity (1) yet the characterization of the weights can be obtained with the help of Theorems 1–2. We use the block-diagonal method from [15]. [2] do not have a statement on compactness of T which we provide. [9, 15] do have such a statement but their indirect argument (valid for Banach function spaces on a real line) does not apply in our case. We explicitly construct a finite-rank approximation to T. Note that the method in [9] is based on what they call a fairway function. The use of the fairway function requires differentiation and is not possible in our situation.

In addition to Assumption 1 we use the following condition:

ASSUMPTION 2. (on the link between a,b and μ) *a) We suppose that* $\mu(\Omega(t))<\infty$ *for all* $t>0$ *and with some* $c>0$ *we have for all* $0<s<t<\infty$

$$\mu(\Omega[s,t]) \leqslant c\mu(\Omega[a(s),a(t)]) \text{ and } \mu(\Omega[s,t]) \leqslant c\mu(\Omega[b(s),b(t)]). \tag{3}$$

b) Let Ω *be of a special type, namely: suppose* Σ *is all or a part of the unit sphere* $\{x\in R^n : |x|=1\}$ *and let* $\Omega=\{x\in R^n : x/|x|\in\Sigma,\ 0\acute{<}|x|<\infty\}$ *be a cone provided with Lebesgue measure. In this case, instead of* (3) *we assume that* a,b *are differentiable.*

Lemma 1 and Remark 1 below explain why we need this assumption. Everywhere Assumptions 1 and 2 are assumed to hold and are not explicitly mentioned.

Take $m_0=1$ and define recursively $m_{k+1}=a^{-1}(b(m_k)),\ k\in Z$. Then $m_k<m_{k+1},\ a(m_{k+1})=b(m_k)$ for $k\in Z$, and $\lim_{k\to\infty} m_k=\infty$, $\lim_{k\to-\infty} m_k=0$. Throughout the rest of the paper we will use the notations

$$\Delta_k=(m_k,m_{k+1}],\ a_k=a(m_k),\ b_k=b(m_k),\ \gamma_k=(a_k,b_k].$$

THEOREM 3. *a) If* $1<p\leqslant q<\infty$ *then for the best constant in* (2) *we have* $C\asymp K$ *where*

$$A(x)=\sup_{\{t>0:\ a(\tau(x))\leqslant b(t)\leqslant b(\tau(x))\}} \Psi([t,\tau(x)],[a(\tau(x)),b(t)])$$

$$K=\sup_{x\in\Omega} A(x).$$

b) If $0<q<p,\ 1<p<\infty$, *then for the best constant in* (2) *we have* $C\asymp K_1+K_2$ *where*

$$K_1=\left(\sum_k \int_{\Omega[\Delta_k]} \Phi([m_k,\tau(x)],[a(\tau(x)),a(m_{k+1})])u(x)d\mu(x)\right)^{1/r},$$

$$K_2=\left(\sum_k \int_{\Omega[\Delta_k]} \Phi([\tau(x),m_{k+1}],[b(m_k),b(\tau(x))])u(x)d\mu(x)\right)^{1/r}.$$

Denote

$$l_i=\limsup_{\tau(x)\to i} A(x),\ \text{ for } i=0 \text{ or } i=\infty,\ l=\max\{l_0,l_\infty\}.$$

$\|T\|_{ess}=\inf\|T-S\|$, where S runs over the set of all finite-rank operators, denotes the essential norm of T.

THEOREM 4. *a) If $1 < p \leqslant q < \infty$, then $\|T\|_{ess} \asymp l$. In particular, T is compact if and only if $l = 0$.*

b) If $1 < q < p < \infty$ and $\|T\| < \infty$, then T is compact.

4. Proofs

The proofs of Theorems 3–4 will be preceded with auxiliary statements. The next lemma reveals the importance of the analysis of ordered cores [14] for the problem at hand.

LEMMA 1. *If condition (3) holds, then there exists a positive linear map R_a such that*

$$\int_{\Omega[s,t]} u d\mu = \int_{\Omega[a(s),a(t)]} R_a u d\mu$$

for all u that are μ-integrable on $\Omega[s,t]$ and all $0 < s < t < \infty$. The action of R_a on the weight u obviously induces a transformation of the functionals Ψ, Φ: $\Psi(\Delta_1, \Delta_2, R_a u) = \Psi(a^{-1}(\Delta_1), \Delta_2, u)$, $\Phi(\Delta_1, \Delta_2, R_a u) = \Phi(a^{-1}(\Delta_1), \Delta_2, u)$. Replacing everywhere a by b we obtain the corresponding property for R_b.

Proof. In [14, Theorem 4.6] put $(P, \mathscr{P}, \rho) = (T, \mathscr{T}, \tau) = (\Omega, \mathfrak{M}, \mu)$. The family $\mathscr{A} = \{\Omega(t) : t \geqslant 0\}$ is a σ-bounded ordered core, that is, it is totally ordered, $\cup_{t\geqslant 0}\Omega(t)$ is a subset of, say, $\cup_{n=1}^{\infty}\Omega(n)$, and $\mu(\Omega(t)) < \infty$ for all $t > 0$. Define $r(\Omega(t)) = \Omega\left(a^{-1}(t)\right)$. Since a^{-1} is monotone, r is order-preserving. It is also bounded by (3): for $0 < s < t < \infty$:

$$\mu\left(r(\Omega(t)) \setminus r(\Omega(s))\right) = \mu\left(\Omega\left(a^{-1}(t)\right) \setminus \Omega\left(a^{-1}(s)\right)\right) \leqslant c\mu\left(\Omega(t) \setminus \Omega(s)\right).$$

By Theorem 4.6 there exists a positive linear map R_a satisfying

$$\int_{\Omega[s,t]} R_a u d\mu = \int_{\Omega\left[a^{-1}(s),a^{-1}(t)\right]} u d\mu, \quad \int_{\Omega[s,t]} |R_a u| d\mu \leqslant \int_{\Omega\left[a^{-1}(s),a^{-1}(t)\right]} |u| d\mu.$$

This gives us what we need. $\square$

In simple cases the map R_a can be constructed explicitly, as the next Remark shows.

REMARK 1. Let Σ be all or a part of the unit sphere $\{x \in R^n : |x| = 1\}$ and let $\Omega = \{x \in R^n : x/|x| \in \Sigma,\ 0 \lessdot |x| < \infty\}$ be a cone provided with Lebesgue measure. Suppose a is differentiable. Using polar coordinates and replacing $r = a^{-1}(\rho)$ we can use the equation

$$\int_{\Omega[l,m]} u(x)dx = \int_l^m \int_\Sigma u(r\sigma)d\sigma r^{n-1} dr = \int_{a(l)}^{a(m)} \int_\Sigma R_a u(a^{-1}(\rho)\sigma) d\rho d\sigma$$

instead of Lemma 1. Here

$$R_a u(a^{-1}(\rho)\sigma) = u\left(a^{-1}(\rho)\sigma\right)\left(\frac{a^{-1}(\rho)}{\rho}\right)^{n-1}\frac{d}{d\rho}a^{-1}(\rho),\ \ |\sigma| = 1.$$

This R_a is not positive, which is not an obstacle for our applications.

This Remark explains why we call Lemma 1 a change-of-variable type result. In applications based on this example one assumes differentiability of a,b instead of (3).

We use the block-diagonal method from [15], see also [9, Lemma 2.1]. In

$$Tf(x) = \sum_k \chi_{\Omega(\Delta_k)} Tf(x)$$

for $x \in \Omega(\Delta_k)$ we have $a_k \leqslant a(\tau(x)) \leqslant a_{k+1} = b_k \leqslant b(\tau(x)) \leqslant b_{k+1}$. This implies

$$[a(\tau(x)), b(\tau(x))] = [a(\tau(x)), a_{k+1}] \cup [b_k, b(\tau(x))]$$

where $[a(\tau(x)), a_{k+1}] \subseteq \gamma_k$ and $[b_k, b(\tau(x))] \subseteq \gamma_{k+1}$. Hence, for $x \in \Omega(\Delta_k)$

$$\int_{\Omega[a(\tau(x)),b(\tau(x))]} f d\nu = \int_{\Omega[a(\tau(x)),a_{k+1}]} f\chi_{\Omega[\gamma_k]} d\nu + \int_{\Omega[b_k,b(\tau(x))]} f\chi_{\Omega[\gamma_{k+1}]} d\nu.$$

This translates to a decomposition

$$Tf(x) = \sum_k (T_k + S_k),$$

$$T_k = \chi_{\Omega(\Delta_k)} \int_{\Omega[b_k,b(\tau(x))]} f\chi_{\Omega[\gamma_{k+1}]} d\nu,\ S_k = \chi_{\Omega(\Delta_k)} \int_{\Omega[a(\tau(x)),a_{k+1}]} f\chi_{\Omega[\gamma_k]} d\nu.$$

We denote

$$\|T_k\| = \|T_k\|_{L_p(vd\nu,\Omega[\gamma_{k+1}])\to L_q(ud\mu,\Omega(\Delta_k))},\ \|S_k\| = \|S_k\|_{L_p(vd\nu,\Omega[\gamma_k])\to L_q(ud\mu,\Omega(\Delta_k))}.$$

Then the problem of estimating $\|T\|$ is reduced to the problem of estimating $\|T_k\|$ and $\|S_k\|$ because [9, Lemma 2.1]

$$\|T\| = \max\left\{\sup_k \|T_k\|, \sup_k \|S_k\|\right\},\ p \leqslant q, \tag{4}$$

$$\|T\| \asymp \left(\sum_k \|T_k\|^r + \sum_k \|S_k\|^r\right)^{1/r},\ q < p. \tag{5}$$

LEMMA 2. *a) If* $1 < p \leqslant q < \infty$ *then*

$$\|T_k\| \asymp \sup_{\tau(x)\in\Delta_k} \Psi([\tau(x), m_{k+1}], [b(m_k), b(\tau(x))]), \tag{6}$$

$$\|S_k\| \asymp \sup_{\tau(x)\in\Delta_k} \Psi([m_k, \tau(x)], [a(\tau(x)), a(m_{k+1})]). \tag{7}$$

b) If $0<q<p,\ 1<p<\infty$, *then*

$$\|T_k\| \asymp \left(\int_{\Omega[\Delta_k]} \Phi([\tau(x), m_{k+1}], [b(m_k), b(\tau(x))])u(x)d\mu(x)\right)^{1/r},$$

$$\|S_k\| \asymp \left(\int_{\Omega[\Delta_k]} \Phi([m_k, \tau(x)], [a(\tau(x)), a(m_{k+1})])u(x)d\mu(x)\right)^{1/r}.$$

Proof. We illustrate the proof for S_k, the proof for T_k being similar. By Lemma 1

$$\left[\int_{\Omega[m_k,m_{k+1}]} \left(\int_{\Omega[a(\tau(y)),a(m_{k+1})]} f d\nu\right)^q u(y)d\mu(y)\right]^{1/q}$$
$$= \left[\int_{\Omega[a(m_k),a(m_{k+1})]} \left(\int_{\Omega[\tau(x),a(m_{k+1})]} f d\nu\right)^q R_a u(x)d\mu(x)\right]^{1/q}$$

where $\tau(y) \in [m_k, m_{k+1}]$ is mapped to $\tau(x) = a(\tau(y)) \in [a(m_k), a(m_{k+1})]$. Therefore, if $p \leqslant q$ then by Theorem 2a) and Lemma 1

$$\sup_{a(m_k)\leqslant\tau(x)\leqslant a(m_{k+1})} \Psi([a(m_k), \tau(x)], [\tau(x), a(m_{k+1})], R_a u)$$
$$= \sup_{a(m_k)\leqslant\tau(x)\leqslant a(m_{k+1})} \Psi([m_k, a^{-1}(\tau(x))], [\tau(x), a(m_{k+1})], u)$$
(replacing $\tau(y) = a^{-1}(\tau(x))$)
$$= \sup_{m_k\leqslant\tau(y)\leqslant m_{k+1}} \Psi([m_k, \tau(y)], [a(\tau(y)), a(m_{k+1})], u).$$

If $q<p$, then Theorem 2b) and a double application of Lemma 1 show that

$$\left(\int_{\Omega[a(m_k),a(m_{k+1})]} \Phi([a(m_k), \tau(x)], [\tau(x), a(m_{k+1})], R_a u) R_a u(x)\, d\mu(x)\right)^{1/r}$$
$$= \left(\int_{\Omega[a(m_k),a(m_{k+1})]} \Phi([m_k, a^{-1}(\tau(x))], [\tau(x), a(m_{k+1})], u) R_a u(x)\, d\mu(x)\right)^{1/r}$$
$$= \left(\int_{\Omega[m_k,m_{k+1}]} \Phi([m_k, \tau(y)], [a(\tau(y)), a(m_{k+1})], u) u(y)\, d\mu(y)\right)^{1/r}. \qquad \square$$

Proof of Theorem 3. The upper bound immediately follows from (4) and Lemma 2 if we note that both quantities (6) and (7) do not exceed K.

To prove the lower bound, suppose that $t \leqslant \tau(x)$ and $a(\tau(x)) \leqslant b(t)$. Take $u_0 \leqslant u,\ v_0 \geqslant v$ such that $u_0, v_0^{-p'/p}$ are integrable and put $f(y) = \chi_{\Omega[a(\tau(x)),b(t)]}(y) v_0^{-p'/p}(y)$. Then using the fact that $t \leqslant \tau(s) \leqslant \tau(x)$ implies $[a(\tau(x)), b(t)] \subset [a(\tau(s)), b(\tau(s))]$

we see that

$$\begin{aligned}
&\left(\int_{\Omega[t,\tau(x)]} u_0 d\mu\right)^{1/q}\left(\int_{\Omega[a(\tau(x)),b(t)]} v_0^{-p'/p} d\nu\right)\\
&=\left[\int_{\Omega[t,\tau(x)]}\left(\int_{\Omega[a(\tau(x)),b(t)]} f d\nu\right)^q u_0(s) d\mu(s)\right]^{1/q}\\
&\leqslant\left[\int_{\Omega}\left(\int_{\Omega[a(\tau(s)),b(\tau(s))]} f d\nu\right)^q u(s) d\mu(s)\right]^{1/q}\\
&\leqslant C\left(\int_{\Omega} f^p v d\nu\right)^{1/p}=C\left(\int_{\Omega[a(\tau(x)),b(t)]} v_0^{-p'/p} d\nu\right)^{1/p}.
\end{aligned}$$

Since $v_0^{-p'/p}$ is integrable, this leads to $\Psi([t,\tau(x)],[a(\tau(x)),b(t)],u_0,v_0)\leqslant C$. Letting $u_0\uparrow u,\ v_0\downarrow v$ we obtain $K\leqslant C$.

If $q<p,$ the statement follows directly from (5) and Lemma 2. □

For the proof of Theorem 4 we need the following proposition.

LEMMA 3. *Let* $1<p\leqslant q<\infty$. *If* $l>\varepsilon>0$ *then there exists a sequence* $\{f_n\}$ *such that*

$$\|T(f_n-f_m)\|_{L_q(ud\mu,\Omega)}>2^{1/q}\varepsilon,\ \ \|f_n-f_m\|_{L_p(vd\nu,\Omega)}=2^{1/p}.$$

Proof. Suppose $l_0>\varepsilon>0$. Then there exist sequences $\{x_n\},\ \{t_n\}$ such that $\tau(x_n)\to 0,\ t_n\in\left[b^{-1}(a(\tau(x_n))),\tau(x_n)\right]$ and $A(x_n)>\varepsilon$. Denote

$$U_n=[t_n,\tau(x_n)],\ V_n=[a(\tau(x_n)),b(t_n)],\ W_n=[a(t_n),b(\tau(x_n))].$$

$\tau(s)\in U_n$ implies $a(t_n)\leqslant a(\tau(s))\leqslant a(\tau(x_n)),\ \ b(t_n)\leqslant b(\tau(s))\leqslant b(\tau(x_n))$ which gives

$$\tau(s)\in U_n\Rightarrow V_n\subseteq[a(\tau(s)),b(\tau(s))]\subseteq W_n. \tag{8}$$

If n is fixed, by increasing m we can achieve $W_n\cap W_m=\emptyset$ and $U_n\cap U_m=\emptyset$. Put $f_n(y)=\left(\int_{\Omega[V_n]} v^{-p'/p} d\nu\right)^{-1/p}\chi_{\Omega[V_n]}(y)\,v^{-p'/p}(y)$. Then $\|f_n-f_m\|_{L_p(vd\nu,\Omega)}=2^{1/p}$,

$$\begin{aligned}
\|T(f_n-f_m)\|^q_{L_q(ud\mu,\Omega)}\geqslant&\int_{\Omega[U_n]}\left|\int_{\Omega[a(\tau(s)),b(\tau(s))]}(f_n-f_m)\,d\nu\right|^q u(s)d\mu(s)\\
&+\int_{\Omega[U_m]}\left|\int_{\Omega[a(\tau(s)),b(\tau(s))]}(f_n-f_m)\,d\nu\right|^q u(s)d\mu(s).
\end{aligned}$$

By (8) in the first integral we have $\Omega[a(\tau(s)), b(\tau(s))\,] \cap W_m = \emptyset$ and in the second one $\Omega[a(\tau(s)), b(\tau(s))\,] \cap W_n = \emptyset$. Hence,

$$\begin{aligned}\|T(f_n - f_m)\|^q_{L_q(ud\mu,\Omega)} &\geqslant \int_{\Omega[U_n]} \left(\int_{\Omega[V_n]} f_n d\nu\right)^q ud\mu + \int_{\Omega[U_m]} \left(\int_{\Omega[V_m]} f_m d\nu\right)^q ud\mu \\ &= \left(\int_{\Omega[U_n]} ud\mu\right)\left(\int_{\Omega[V_n]} \nu^{-p'/p} d\nu\right)^{q/p'} \\ &\quad + \left(\int_{\Omega[U_m]} ud\mu\right)\left(\int_{\Omega[V_m]} \nu^{-p'/p} d\nu\right)^{q/p'} > 2\varepsilon^q.\end{aligned}$$

The case $l_\infty > \varepsilon$ is handled in the same way. □

Proof of Theorem 4. Part a). Lower bound. When proving $\|T\|_{ess} \geqslant cl$ we can assume that $\|T\|_{ess} < \infty$, implying $\|T\| < \infty$ and, by Theorem 3, $K < \infty$. Without loss of generality we can also assume that $l > 0$. Let $\varepsilon = l/2$ and suppose that S is any finite-rank operator. Passing to a subsequence, if necessary, we can assume that $\{Sf_n\}$ converges for the sequence from Lemma 3. By Lemma 3

$$\begin{aligned}\|(T-S)(f_n - f_m)\|_{L_q(ud\mu,\Omega)} &\geqslant \|T(f_n - f_m)\|_{L_q(ud\mu,\Omega)} \\ &\quad - \|S(f_n - f_m)\|_{L_q(ud\mu,\Omega)} > 2^{1/q-1} l\end{aligned}$$

for large n,m. Since $\|f_n - f_m\|_{L_p(\nu d\nu,\Omega)} = 2^{1/p}$, this implies $\|T-S\| \geqslant cl$ and $\|T\|_{ess} \geqslant cl$.

Upper bound. In the proof we can assume that $l < \infty$ and we have to produce a finite-rank approximation to

$$Tf(y) = \int_{\Omega(b(\tau(y)))} f d\nu - \int_{\Omega(a(\tau(y)))} f d\nu \equiv T^+ f(y) - T^- f(y).$$

Such approximations will be developed for $T^+,\ T^-$. With the partition $(0,\infty) = \cup_k \Delta_k$ used in the proof of Theorem 3 we have $\tau(y) \in \Delta_k \Rightarrow a(\tau(y)) \in \gamma_k,\ b(\tau(y)) \in \gamma_{k+1}$. This means that we need to approximate T^+ on Δ_{k+1} and T^- on Δ_k. Let $k_1 \leqslant k \leqslant k_2$ for some fixed integers $k_1, k_2 \in Z,\ k_1 < k_2$.

Approximation for T^+. The points $t_{kj} = m_{k+1} + j(m_{k+2} - m_{k+1})/n,\ j = 0, \ldots, n$, lead to partitions of Δ_{k+1} and $\Omega[\Delta_{k+1}]$, consisting of sets

$$\Delta^+_{kj} = (t_{kj}, t_{k,j+1}],\ \Omega^+_{kj} = \Omega\left[\Delta^+_{kj}\right],\ j = 0, \ldots, n-1,$$

resp. Putting $\kappa^+_n(t) = \sum_{j=0}^{n-1} b(t_{kj})\, \chi_{\Delta^+_{kj}}(t)$ we have

$$\kappa^+_n(\tau(x)) = b(t_{kj}) \leqslant b(\tau(x)) \leqslant b(t_{k,j+1}),\ x \in \Omega^+_{kj}.$$

Define

$$T^+_n f(y) = \int_{\Omega\left(\kappa^+_n(\tau(y))\right)} f d\nu = \sum_{j=0}^{n-1} \int_{\Omega\left(b(t_{kj})\right)} f d\nu \chi_{\Omega^+_{kj}}(y),\ \tau(y) \in \Delta_{k+1}.$$

Then for the restriction to Δ_{k+1} we have

$$T^{+}f(y)-T_{n}^{+}f(y)=\sum_{j=0}^{n-1}\int_{\Omega\left[b\left(t_{kj}\right),b(\tau(y))\right]}fd\nu\chi_{\Omega_{kj}^{+}}(y).$$

By the argument used in Lemma 2 for one term in this sum we have

$$\left[\int_{\Omega_{kj}^{+}}\left(\int_{\Omega\left[b\left(t_{kj}\right),b(\tau(y))\right]}fd\nu\right)^{q}u(y)d\mu(y)\right]^{1/q}\leqslant cC_{bkj}\left(\int_{\Omega\left[b\left(t_{kj}\right),b\left(t_{k,j+1}\right)\right]}f^{p}vd\nu\right)^{1/p} \tag{9}$$

where

$$C_{bkj}=\sup_{t_{kj}\leqslant\tau(x)\leqslant t_{k,j+1}}\Psi\left(\left[\tau(x),t_{k,j+1}\right],\left[b\left(t_{kj}\right),b(\tau(x))\right]\right).$$

Summation of these bounds gives

$$\left(\int_{\Omega[\Delta_{k+1}]}\left|T^{+}f-T_{n}^{+}f\right|^{q}ud\mu\right)^{1/q}\leqslant c\sup_{0\leqslant j\leqslant n-1}C_{bkj}\left(\int_{\Omega[\gamma_{k+1}]}f^{p}vd\nu\right)^{1/p}. \tag{10}$$

Approximation for T^{-}. The points $s_{kj}=m_k+j(m_{k+1}-m_k)/n,\ j=0,\dots,n,$ give rise to partitions of Δ_k and $\Omega[\Delta_k]$, consisting of sets

$$\Delta_{kj}^{-}=(s_{kj},s_{k,j+1}],\ \Omega_{kj}^{-}=\Omega\left[\Delta_{kj}^{-}\right],\ j=0,\dots,n-1,$$

resp. Putting $\kappa_n^{-}(t)=\sum_{j=0}^{n-1}a\left(s_{k,j+1}\right)\chi_{\Delta_{kj}^{-}}(t)$ we have

$$a\left(s_{kj}\right)\leqslant a(\tau(x))\leqslant a\left(s_{k,j+1}\right)=\kappa_n^{-}(\tau(x)),\ x\in\Omega_{kj}^{-}.$$

For T^{+}, b was approximated from below; here, for T^{-}, a is approximated from above. Define

$$T_n^{-}f(y)=\int_{\Omega\left(\kappa_n^{-}(\tau(y))\right)}fd\nu=\sum_{j=0}^{n-1}\int_{\Omega\left(a\left(s_{k,j+1}\right)\right)}fd\nu\chi_{\Omega_{kj}^{-}}(y).$$

Then for the restriction to Δ_k we have

$$T_n^{-}f(y)-T^{-}f(y)=\sum_{j=0}^{n-1}\int_{\Omega\left[a(\tau(y)),a\left(s_{k,j+1}\right)\right]}fd\nu\chi_{\Omega_{kj}^{-}}(y).$$

By a statement similar to Lemma 2 for one term in this sum we have

$$\left[\int_{\Omega_{kj}^{-}}\left(\int_{\Omega\left[a(\tau(y)),a\left(s_{k,j+1}\right)\right]}fd\nu\right)^{q}u(y)d\mu(y)\right]^{1/q}\leqslant cC_{akj}\left(\int_{\Omega\left[a\left(s_{kj}\right),a\left(s_{k,j+1}\right)\right]}f^{p}vd\nu\right)^{1/p} \tag{11}$$

where

$$C_{akj}=\sup_{s_{kj}\leqslant\tau(x)\leqslant s_{k,j+1}}\Psi\left(\left[s_{kj},\tau(x)\right],\left[a(\tau(x)),a\left(s_{k,j+1}\right)\right]\right).$$

By summing these bounds we get

$$\left(\int_{\Omega[\Delta_k]}\left|T_n^- f - T^- f\right|^q u d\mu\right)^{1/q} \leqslant c \sup_{0\leqslant j\leqslant n-1} C_{akj}\left(\int_{\Omega[\gamma_k]} f^p v d\nu\right)^{1/p}. \tag{12}$$

Approximation for T. Denote

$$\Omega_1=\bigcup_{k<k_1}\Omega[\Delta_k],\ \Omega_2=\bigcup_{k_1\leqslant k\leqslant k_2}\Omega[\Delta_k],\ \Omega_3=\bigcup_{k>k_2}\Omega[\Delta_k].$$

Repeating calculations based on (6) and (7) we obtain

$$\left(\int_{\Omega_i}(Tf)^q u d\mu\right)^{1/q} \leqslant cK(\Omega_i)\left(\int_{\Omega_i} f^p v d\nu\right)^{1/p},\ K(\Omega_i)\equiv\sup_{x\in\Omega_i}A(x),\ i=1 \text{ or } i=3. \tag{13}$$

We can select k_1 and k_2 to satisfy $\max\{K(\Omega_1),K(\Omega_3)\}<2l$. On Ω_2

$$\sup_{k_1\leqslant k\leqslant k_2}\ \sup_{0\leqslant j\leqslant n-1}\left(C_{akj}+C_{bkj}\right)<l$$

if n is large enough. Then (10), (12), (13) imply $\|T\|_{ess}\leqslant\|T^+-T_n^+\|+\|T^--T_n^-\|\leqslant cl$.

Part b). If $\|T\|<\infty$ then by Theorem 3 $\max\{K_1,K_2\}<\infty$. Therefore for any $\varepsilon>0$ we can select $-\infty<k_1<k_2<\infty$ so that

$$\left(\sum_{\{k<k_1\}\cup\{k>k_2\}}\|T_k\|^r+\sum_{\{k<k_1\}\cup\{k>k_2\}}\|S_k\|^r\right)^{1/r}<\varepsilon.$$

Then by Theorem 3 we have

$$\left(\int_{\widetilde{\Omega}}(Tf)^q u d\mu\right)^{1/q}\leqslant c\varepsilon\left(\int_{\Omega} f^p v d\nu\right)^{1/p}$$

where $\widetilde{\Omega}=\bigcup_{\{k<k_1\}\cup\{k>k_2\}}\Omega[\Delta_k]$. As in Lemma 2, (9) and (11) are true with

$$C_{akj}=\left(\int_{\Omega[s_{kj},s_{k,j+1}]}\Phi([s_{kj},\tau(x)],\left[a(\tau(x)),a\left(s_{k,j+1}\right)\right])u(x)d\mu(x)\right)^{1/r},$$

$$C_{bkj}=\left(\int_{\Omega[t_{kj},t_{k,j+1}]}\Phi([\tau(x),t_{k,j+1}],\left[b\left(t_{kj}\right),b(\tau(x))\right])u(x)d\mu(x)\right)^{1/r}.$$

Define $\mu_{bkj}=C_{bkj}/\|T_k\|$, if the denominator is not zero and $\mu_{bkj}=0$ otherwise. The bound

$$\int_{\Omega[t_{kj},t_{k,j+1}]}\Phi([\tau(x),t_{k,j+1}],\left[b\left(t_{k,j}\right),b(\tau(x))\right])u(x)d\mu(x)$$
$$\leqslant\Phi\left([t_{kj},t_{k,j+1}],\left[b\left(t_{k,j}\right),b\left(t_{k,j+1}\right)\right]\right)\int_{\Omega[t_{kj},t_{k,j+1}]}u(x)d\mu(x)$$

and continuity of b imply that

$$\mu_n \equiv \sup_{k_1 \leqslant k \leqslant k_2} \sup_{0 \leqslant j \leqslant n-1} \mu_{bkj} \to 0, \text{ as } n \to \infty.$$

Besides, $C_{bkj} \leqslant \mu_n \|T_k\|$. This bound and (9) lead to the estimate

$$\left(\int_{\Omega_2} \left|T^+ f - T_n^+ f\right|^q u d\mu\right)^{1/q} \leqslant c\mu_n \left(\sum_{k=k_1}^{k_2} \|T_k\|^r\right)^{1/r} \left(\int_{\Omega_2} f^p v d\nu\right)^{1/p}.$$

This inequality and a similar bound for $T_n^- f - T^- f$ show that $\left(\int_{\Omega_2} |Tf - T_n f|^q u d\mu\right)^{1/q}$ can be made as small as desired by selecting a sufficiently large n. The conclusion is that T is compact as a limit of finite-rank operators. □

REFERENCES

[1] E. N. BATUEV AND V. D. STEPANOV, *Weighted inequalities of Hardy type*, Siberian Math. J., **30**, (1989), 8–16.

[2] H. P. HEINIG AND G. SINNAMON, *Mapping properties of integral averaging operators*, Stud. Math., **129**, (1998), 157–177.

[3] A. KUFNER, L.-E. PERSSON AND N. SAMKO, *Weighted inequalities of Hardy type. Second edition*, World Scientific Publishing, Hackensack, NJ, 2017.

[4] A. KUFNER AND B. OPIC, *Hardy-type inequalities*, Longman Scientific and Technical, Harlow, 1990.

[5] A. KUFNER AND L.-E. PERSSON, *Weighted Inequalities of Hardy Type*, World Sci., River Edge, NJ, 2003.

[6] A. KUFNER, L. MALIGRANDA AND L.-E. PERSSON, *The Hardy inequality. About Its History and Some Related Results*, Vydavatelský Servis, Pilsen, 2007.

[7] V. G. MAZ'YA, *Sobolev spaces*, Springer-Verlag, Berlin, 1985.

[8] K. MYNBAEV, *Three weight Hardy inequality on measure topological spaces*, Eurasian Math. J., **14**, 1 (2023), 58–78.

[9] D. V. PROKHOROV, V. D. STEPANOV AND E. P. USHAKOVA, *Hardy–Steklov Integral Operators: Part I*, Proc. Steklov Inst. Math., **300**, (2018), 1–112.

[10] M. RUZHANSKY AND D. VERMA, *Hardy inequalities on metric measure spaces*, Proc. R. Soc. A, **475**, (2019), 20180310, 15 pp.

[11] M. RUZHANSKY AND D. VERMA, *Hardy inequalities on metric measure spaces, II: The case $p > q$*, Proc. R. Soc. A, **477**, (2021), 20210136.

[12] M. RUZHANSKY AND N. YESSIRKEGENOV, *Hardy, Hardy-Sobolev and Caffarelli-Kohn-Nirenberg inequalities on general Lie groups*, 21 Feb 2019, arXiv:1810.08845v2 [math.FA].

[13] G. SINNAMON, *Hardy-type inequalities for a new class of integral operators*, in: Analysis of divergence (Orono, ME, 1997), Appl. Numer. Harmon. Anal., Birkhäuser, Boston, 1999, 297–307.

[14] G. SINNAMON, *Hardy inequalities in normal form*, Trans. Amer. Math. Soc. **375**, (2022), 961–995.

[15] V. D. STEPANOV AND E. P. USHAKOVA, *On integral operators with variable limits of integration*, Proc. Steklov Inst. Math., **232**, (2001), 290–309.

(Received August 31, 2023)

Kairat T. Mynbaev
International School of Economics
Kazakh-British Technical University
Tolebi 59, Almaty 050000, Kazakhstan
e-mail: k_mynbayev@ise.ac