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A NOTE ON ESTIMATES OF THE NEWTONIAN POTENTIAL ON BOUNDED DOMAINS

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Abstract. In this paper, we investigate the regularity of the operator K on a smooth bounded domain in $\mathbb{R}^d$ given by convolution against the Newtonian potential. We show that the gain in L^p-Sobolev spaces agrees with elliptic regularity. We also establish L^p-Sobolev to L^q-Sobolev bounds as well as bounds from L^∞-Sobolev spaces to Hölder spaces.

1. Introduction

In this paper, we study convolution against the Newtonian potential on a bounded domain $\Omega \subset \mathbb{R}^d$. If f is a function on Ω, we are interested in the operator

$$Kf(x) = \int_\Omega \frac{f(y)}{|x-y|^{d-2}}\,dy. \tag{1}$$

In parallel with solutions to Laplace's equation on a smooth domain with known boundary conditions, we prove that K gains smoothness as measured by L^p-Sobolev spaces as well as in Hölder spaces and on the L^p-Sobolev to L^q-Sobolev scale. Our main results are the following theorems.

THEOREM 1. *Let $\Omega \subset \mathbb{R}^d$ be a smooth, bounded domain and let K be the integral operator defined by (1). For every nonnegative $\ell \in \mathbb{Z}$,*

1. $K : W^{\ell,p}(\Omega) \to W^{\ell+2,p}(\Omega)$, $1 < p < \infty$;
2. $K : W^{\ell,\infty}(\Omega) \to \Lambda^{\ell+1,\alpha}(\Omega)$ *for any* $0 < \alpha < 1$;
3. $K : W^{\ell,1}(\Omega) \to W^{\ell+1,1}(\Omega)$.

In addition to proving regularity in the Sobolev scale, we are also interested in the L^p-improving properties of K.

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THEOREM 2. *Let $\Omega \subset \mathbb{R}^d$ be a smooth, bounded domain and let K be the integral operator defined by* (1).

1. *If $D_{\ell+1}$ is any derivative of order ℓ, then there exists a constant $C_\ell > 0$ so that*

$$\left|\{y \in \Omega : |D_{\ell+1}Kf(y)| > t\}\right| \leqslant C_\ell \left(\frac{\|f\|_{W^{\ell,1}(\Omega)}}{t}\right)^{\frac{d}{d-1}};$$

2. *For $1 < p < d$ and q defined by $\frac{1}{q} = \frac{1}{p} - \frac{1}{d}$, then there exists $C_{\ell,p} > 0$ so that*

$$\|Kf\|_{W^{\ell+1,q}(\Omega)} \leqslant C_{\ell,p}\|f\|_{W^{\ell,p}(\Omega)}; \tag{2}$$

3. *If $p = d$, then K satisfies* (2) *for all $1 \leqslant q < \infty$;*

4. *If $p > d$, then there exists $C_{\ell,p} > 0$ so that*

$$\|Kf\|_{W^{\ell+1,\infty}(\Omega)} \leqslant C_{\ell,p}\|f\|_{W^{\ell,p}(\Omega)}.$$

One of the first theorems that students learn in a partial differential equations class is that convolution against the Newtonian potential solves Poisson's equation in $\mathbb{R}^d$. Regularity results in $\mathbb{R}^d$ follow by integration by parts to pass the derivatives off of the kernel and onto the data. It is not so simple on domains $\Omega \subset \mathbb{R}^d$. There have been thousands of papers on elliptic regularity for elliptic equations, and the setup (in the simplest form) is the following: solve $Lu = f$ on Ω subject to the boundary condition $u|_{b\Omega} = g$ and L is elliptic. The regularity of f and g determine the smoothness of u. When Ω has a smooth boundary, the results are classical and can be found in many books (e.g., Evans [2] or Gilbarg-Trudinger [5]). Recently, the interest has been to extend the classical work to the low boundary regularity setting. Often the work involves layer potential and other techniques from harmonic analysis [9].

Our approach takes a different tack because we not trying to solve a given boundary value problem; instead, we are given the operator and a domain and must determine its regularity. Our interest in K arises from the integral operators that arise in several complex variables – see §1.1. Many operators are given by integration against an integral kernel and the integral kernel (and the data) determine the regularity of the output. If the integration is on Ω and not over $\mathbb{R}^d$, then regularity of the integral operator does not follow the (typically well-known) results on $\mathbb{R}^d$ because $C_c^\infty(\Omega)$ is not dense in many function spaces (on Ω) of interest. Integration by parts is much more complicated because of the boundary and derivatives cannot simply pass to the data.

1.1. Singular integrals in several complex variables

The central partial differential equation in several complex variables is the Cauchy-Riemann equation $\bar{\partial}u = f$. Solving this equation often proceeds along one of two lines. In L^2, one can use functional analysis and develop the L^2-theory of the canonical solution along the lines of Hörmander [8]. See Straube [13] to pursue this line of reasoning. Outside of L^2, solutions are built by integrating against constructed kernels. The most well known of these kernels are the Bochner-Martinelli and Bochner-Martinelli-Koppelman kernels. Both of these kernels generalize the one-variable Cauchy kernel in the sense that they are reproducing kernels, but they are not holomorphic. We will call this kind of kernel a BM type kernel. Unlike solving the Laplacian, the geometry of $b\Omega$ plays an integral role in the gain (if any) of regularity of solutions to $\bar{\partial}u = f$, and constructing solving operators that incorporate the regularity is difficult, in general. Many of the operators, such as the Henkin operator on convex domains, have two pieces – one involving the boundary and one involving a BM type kernel. For example, for $\Omega \subset \mathbb{C}^2$, a typical term in the expansion of a BM type kernel is

$$\frac{1}{(2\pi i)^2}\int_\Omega \psi(\zeta)\frac{\overline{\zeta_2 - z_2}}{|\zeta - z|^4}\, d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 \wedge d\zeta_2 \wedge d\bar{z}_1 \wedge d\zeta_1$$

Decomposing this piece into its real and imaginary components leaves us with the operators

$$K_j f(x) = \int_\Omega f(y)\frac{x_j - y_j}{|x-y|^4}\, dy = \frac{\partial}{\partial x_j}\int_\Omega \frac{f(y)}{|x-y|^{d-2}}\, dy$$

The regularity for K_j is a consequence of Theorem 1 and Theorem 2. We plan to use the results of this paper to continue our investigation of the $\bar{\partial}$ and $\bar{\partial}_b$-problems in L^p and L^p-Sobolev spaces on convex domains of finite and infinite type in $\mathbb{C}^n$ [6, 10, 11].

2. Proof of Theorem 1

The $\ell = 0$ case is classical, though we provide the key points.

To move from $\ell = 0$ to $\ell > 0$, we need the technical tools to prevent non-integrable singularities from arising.

2.1. The non-characteristic formula for the Laplacian and tangential derivatives

Before continuing, we need to establish notation for normal and tangential derivatives. We let $\delta(x)$ be a defining function for Ω and consider a directional derivative at x to be normal at x if it is parallel to $\nabla\delta(x)$ and tangential if it is orthogonal to $\nabla\delta(x)$. Thus, we have a notion of tangential and normal derivatives at x near $b\Omega$. We denote tangential derivatives at x by T_x and normal derivatives at x by $\frac{\partial}{\partial \nu_x}$. Let $\nu = (\nu_1, \ldots, \nu_d)$ be the unit outward normal so that $\nu(x)\cdot\nabla = \frac{\partial}{\partial \nu_x}$. We will also use $T_{\ell,x}$ to denote a tangential operator of order ℓ and $X_{\ell,x}$ for a generic differential operator of order ℓ at x, though we may suppress the x subscript when working in a neighborhood of x. Operators of order ℓ will have a nontrivial order ℓ part but may also have lower order terms. Note that $\frac{\partial}{\partial \nu_x}$ is just a smooth multiple of $\nabla\delta(x)$.

PROPOSITION 1. *Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain and $x \in \bar{\Omega}$ be a point sufficiently close to $b\Omega$ so that tangential and normal derivatives at x are well-defined. Let $\ell \geqslant 1$ and $X_{\ell,x}$ be a differential operator of order ℓ. Then there exist tangential* *differential operators $T_{\ell,x}$, $T'_{\ell,x}$, $T_{\ell-1,x}$, $T'_{\ell-1,x}$ of order ℓ, ℓ,, $\ell-1$, and $\ell-1$, respectively, and when $\ell=1$, we set $X_{\ell-2,x}=0$ so that*

$$\begin{aligned} X_{\ell,x} &= X_{\ell-2,x}\triangle + T_{\ell,x} + \frac{\partial}{\partial \nu_x} T_{\ell-1,x} \\ &= X_{\ell-2,x}\triangle + T'_{\ell,x} + T'_{\ell-1,x}\frac{\partial}{\partial \nu_x}. \end{aligned}$$

Proof. The $\ell=1$ case is immediate, so we assume that $\ell \geqslant 2$. Let us first examine the case $\ell=2$. Near x, we have an orthornormal basis $\{\frac{\partial}{\partial \tau_k}\}_{k=1}^{d-1}$ of tangential vectors. This means for some smooth coefficients near x,

$$\frac{\partial}{\partial x_\ell} = \sum_{k=1}^{d-1} a_{\ell k}\frac{\partial}{\partial \tau_k} + b_\ell \frac{\partial}{\partial \nu}$$

which means

$$\begin{aligned} \frac{\partial^2}{\partial x_\ell \partial x_n} &= \Big(\sum_{k=1}^{d-1} a_{\ell k}\frac{\partial}{\partial \tau_k} + b_\ell\frac{\partial}{\partial \nu}\Big)\Big(\sum_{j=1}^{d-1} a_{nj}\frac{\partial}{\partial \tau_j} + b_n\frac{\partial}{\partial \nu}\Big) \\ &= \sum_{j,k=1}^{d-1} a_{\ell k}a_{nj}\frac{\partial^2}{\partial \tau_j \partial \tau_k} + b_\ell b_n \frac{\partial^2}{\partial \nu^2} + T_1 + T_1\frac{\partial}{\partial \nu} \end{aligned}$$

and therefore

$$\frac{\partial^2}{\partial x_\ell^2} = \sum_{j,k=1}^{d-1} a_{\ell k}a_{\ell j}\frac{\partial^2}{\partial \tau_j \partial \tau_k} + b_\ell^2\frac{\partial^2}{\partial \nu^2} + T_1 + T_1'\frac{\partial}{\partial \nu}.$$

The Laplacian

$$\triangle = \sum_{\ell=1}^{d}\left[\sum_{j,k=1}^{d-1} a_{\ell k}a_{\ell j}\frac{\partial^2}{\partial \tau_j \partial \tau_k} + b_\ell^2\frac{\partial^2}{\partial \nu^2}\right] + T_1 + T_1'\frac{\partial}{\partial \nu}.$$

It must be the case that $\sum_{\ell=1}^d b_\ell^2 > 0$ for the Laplacian is not a tangential operator. Thus,

$$\frac{\partial^2}{\partial \nu^2} = \frac{1}{\sum_{\ell=1}^d b_\ell^2}\triangle + T_2 + T_1 + T_1'\frac{\partial}{\partial \nu}.$$

Plugging in our expression for $\frac{\partial^2}{\partial \nu^2}$ into $\frac{\partial^2}{\partial x_\ell \partial x_n}$ shows that $\frac{\partial^2}{\partial x_\ell \partial x_n}$ satisfies the first equality of the conclusion and hence a general second order operator will as well. The second equality follows from taking commutators.

The proof for higher operators follows by induction and the fact that a commutator of an order ℓ and an order j differential operator produces an operator of order $\ell+j-1$. □

The linchpin of the proof of the Theorem 1 is the following technical lemma.

LEMMA 1. *Let $\Omega \subset \mathbb{R}^d$ be a smooth domain. Suppose $f \in W^{\ell,\infty}(\Omega)$ and $k \in C^\infty(\mathbb{R}^d \setminus \{0\})$ is homogeneous of degree $-(d-2)$. Suppose that $\operatorname{supp} f$ is such that tangential and normal derivatives with respect to the level surfaces of $\delta(x)$ are well-defined on $\operatorname{supp} f$. Let $x \in \Omega \cap \operatorname{supp} f$. If $T_{\ell,x}$ denotes a tangential derivative of order ℓ at x, then*

$$T_{\ell,x}\left\{ \int_\Omega f(y)k(x-y)\,dy \right\} = \int_\Omega T_{\ell,y} f(y)k(x-y)\,dy + \sum_j \int_\Omega T_y^j f(y) h_j(x,y) k_j(x-y)\,dy$$

where the sum is a finite sum, T_y^j is a tangential derivative of order at most $\ell-1$ at y, $h_j(x,y) \in C^\infty(\bar{\Omega}\times\bar{\Omega})$, and k_j is a function that is homogeneous of degree $-(d-2)$ and smooth away from the origin.

Proof. The issue is that a derivative that is tangential at x is unlikely to be tangential at y, and it is only derivatives that are tangential at y that we may integrate by parts and pick up no boundary term. The way to generate a tangential derivative from T_x is straight forward – we subtract the projection of T_x onto $\frac{\partial}{\partial \nu(y)}$ from T_x and we will be left with a derivative that is tangential at y. Let $a(x) = (a_1(x), \dots, a_d(x))$ be the smooth vector so that $T_x = a(x)\cdot\nabla_x$. Set

$$T_{x,y} = -a(x)\cdot\nabla_y$$

and suppose that $k \in L^1(\Omega)$. Since T_x is tangential at x, $a(x)\cdot\nu(x) = 0$, and it follows that

$$\begin{aligned} T_x k(x-y) = T_{x,y}k(x-y) &= \Big(T_{x,y} + a(x)\cdot\nu(y)\frac{\partial}{\partial \nu_y}\Big)\{k(x-y)\} - a(x)\cdot\nu(y)\frac{\partial k(x-y)}{\partial \nu_y} \\ &= \Big(T_{x,y} + a(x)\cdot\nu(y)\frac{\partial}{\partial \nu_y}\Big)\{k(x-y)\} - a(x)\cdot(\nu(y)-\nu(x))\frac{\partial k(x-y)}{\partial \nu_y}. \end{aligned}$$

Since x is fixed, the vector field $T'_y := T_{x,y} + a(x)\cdot\nu(y)\frac{\partial}{\partial \nu_y}$ is tangential at y and we can integrate it by parts without picking up a boundary term, i.e.,

$$\int_\Omega f(y) T'_y k(x-y)\,dy = \int_\Omega (T'_y)^* f(y) k(x-y)\,dy.$$

We will see that the remaining term is well-behaved. Recall that $\nu(x)$ is a smooth multiple of $\nabla\delta(x)$, and $\nabla\delta(x)$ is Lipschitz (in fact, δ is smooth up to the reach of $b\Omega$, see [7] for details). It therefore follows that

$$\nu(y) - \nu(x) = \big(b_1(x,y)(y_1-x_1), \dots, b_d(x,y)(y_d-x_d)\big)$$

where $b_1(x,y), \dots, b_d(x,y)$ are smooth functions on $\operatorname{supp}\eta$. This means

$$\begin{aligned} a(x)\cdot(\nu(y)-\nu(x))\frac{\partial k(x-y)}{\partial \nu_y} &= \sum_{j,\ell=1}^d a_j(x) b_j(x,y) \nu_\ell(y)(x_j-y_j)\frac{\partial k(x-y)}{\partial y_\ell} \\ &= \sum_{j=1}^d h_j(x,y) k_j(x-y). \end{aligned}$$

This completes the $\ell = 1$ case. The case $\ell \geqslant 2$ is handled recursively. The only difference is that the derivatives (in x) can hit either $h_j(x,y)$ or $k_j(x-y)$. If the derivative hits $h_j(x,y)$, there is nothing more to do as the term is smooth and simply absorbs the derivative. If the derivative hits $k_j(x-y)$, we repeat the argument of the $\ell = 1$ case. $\square$

2.2. The Sobolev space estimates

With Proposition 1 and Lemma 1 in hand, we are now in a position to prove parts 1 and 3 of Theorem 1. By density, we may assume that $f \in C^\infty(\Omega)$.

Proof by induction. Since $\nabla K(x) \in L^1(\Omega)$, the base case for Part 3 of the theorem follows by [4, Theorem 6.18]. The base case for the $1 < p < \infty$ case is established by standard Calderón-Zygmund theory. Given a function $f \in L^p(\Omega)$, extend f by the 0 function on Ω^c. Extension by 0 is continuous in L^p and it therefore suffices to prove the base case of Part 1 on $\mathbb{R}^d$. Observe that $\frac{\partial^2}{\partial x_j \partial x_k}\{\frac{1}{|x|^{d-2}}\}$ is a homogeneous function of degree $-d$ and is mean 0 on the unit sphere. Consequently, it is a standard Calderón-Zygmund kernel and convolution against it is bounded in $L^p(\mathbb{R}^d)$. This concludes the proof of the base case for Part 1 and Part 3.

Now assume that $\ell \geqslant 1$ and D_ℓ is a constant coefficient differential operator of order ℓ. Let $\eta \in C_c^\infty(\mathbb{R}^d)$ be a cutoff function so that $\operatorname{supp}\eta \subset \{t \in \mathbb{R}^d : \operatorname{dist}(t, b\Omega) \leqslant 2\delta\}$ where $\delta > 0$ is suitably small and $\eta \equiv 1$ on $\{t \in \Omega : \operatorname{dist}(t, b\Omega) \leqslant \delta\}$. Then

$$Kf(x) = \int_\Omega \frac{f(y)\eta(y)}{|x-y|^{d-2}}\,dy + \int_\Omega \frac{f(y)(1-\eta(y))}{|x-y|^{d-2}}\,dy.$$

Since $f(y)(1-\eta(y)) \in C_c^\infty(\Omega)$, it follows that by passing one derivative through at a time and integrating by parts, we have

$$D_{\ell,x}\int_\Omega \frac{f(y)(1-\eta(y))}{|x-y|^{d-2}}\,dy = \int_\Omega D_{\ell,y}\big(f(y)(1-\eta(y))\big)\frac{1}{|x-y|^{d-2}}\,dy. \tag{3}$$

Given that we can pass derivatives onto $(1-\eta)f$, the base case establishes the desired estimates for Parts 1 and 3.

We now have only to show the estimate near $b\Omega$. In fact, we can assume both x and y are near $b\Omega$. That y is near $b\Omega$ is forced on us by the domain of η. If x is far from the boundary, then $|x-y|$ is bounded away from 0, and any estimate we wish to prove follows from the smoothness of K and its integrability on bounded domains that avoid a neighborhood of the origin.

We therefore focus on x near $b\Omega$. By Proposition 1, for $j = 1$ or 2, there exist tangential operators $T_{\ell+j,x}$ and $T_{\ell+j-1,x}$ and an operator $X_{\ell+j-2,x}$ of order $\ell+j-2$

$$D_{\ell+j} = X_{\ell+j-2,x}\triangle_x + T_{\ell+j,x} + \frac{\partial}{\partial \nu_x}T_{\ell+j-1,x}.$$

We claim that for $x \in \Omega$,

$$\triangle K(\eta f)(x) = c_d \eta f(x), \tag{4}$$

where c_d is a dimensional constant. Indeed, since the Green's function for Ω is $G(x,y) = \Phi(y-x) - \phi^x(y)$ where the Newtonian potential $\Phi(y-x)$ is a multiple of the integral kernel of K and ϕ^x is a harmonic function that agrees with $\Phi(y-x)$ on $b\Omega$, (4) follows. Consequently,

$$\|X_{\ell+j-2,x}\triangle_x K(f\eta)\|_{L^p(\Omega)} = c_d\|X_{\ell+j-2,x}(f\eta)\|_{L^p(\Omega)} \leqslant C\|f\|_{W^{\ell,p}(\Omega)}, \quad 1 \leqslant p \leqslant \infty.$$

Next, by Lemma 1,

$$T_{\ell',x}\Big\{\int_\Omega (\eta f)(y)K(x-y)\,dy\Big\} = \int_\Omega T_{\ell',y}\{\eta f\}(y)K(x-y)\,dy + \sum_j T_y^j\{\eta f\}(y)h_j(x,y)k_j(x-y)\,dy.$$

For Part 3, we use $j=1$, write $T_{\ell+j,x} + \frac{\partial}{\partial\nu_x}T_{\ell+j-1,x} = D_{1,x}T_{\ell,x}$ where $D_{1,x}$ is a first order operator in x. We write

$$D_{1,x}T_{\ell,x}\Big\{\int_\Omega (\eta f)(y)K(x-y)\,dy\Big\} = \int_\Omega T_{\ell,y}\{\eta f\}(y)D_{1,x}K(x-y)\,dy + \sum_j T_y^j\{\eta f\}(y)h_j(x,y)D_{1,x}k_j(x-y)\,dy.$$

Since $D_{1,x}K(x-y)$ and $D_{1,x}k_j(x-y)$ are integrable, we can again use [4, Theorem 6.18] to finish the proof of Part 3.

To establish Part 1, we need to bound the terms with $T_{\ell+2,x}$ and $\frac{\partial}{\partial\nu}T_{\ell+1,x}$. We handle these terms together by establishing

$$\Big\|\int_\Omega T_{\ell,y}\{\eta f\}(y)K(x-y)\,dy\Big\|_{W^{2,p}(\Omega)} \leqslant C\|f\|_{W^{\ell,p}(\Omega)} \tag{5}$$

and

$$\Big\|\int_\Omega T_{\ell,y}\{\eta f\}h(x,y)k(x-y)\,dy\Big\|_{W^{2,p}(\Omega)} \leqslant C\|f\|_{W^{\ell,p}(\Omega)}. \tag{6}$$

The bound for (5) follows immediately from the base case. We cannot directly apply Calderón-Zygmund theorem to (6) because the kernel is not homogeneous. However, by writing

$$h(x,y) = h(y,y) + \big(h(x,y) - h(y,y)\big),$$

we see that the Calderón-Zygmund theory used in the base case does allow us to establish

$$\Big\|\int_\Omega T_{\ell,y}\{\eta f\}h(y,y)k(x-y)\,dy\Big\|_{W^{2,p}(\Omega)} \leqslant C\|T_{\ell,y}\{\eta f\}h(y,y)\|_{L^p(\Omega)} \leqslant C\|f\|_{W^{\ell,p}(\Omega)}.$$

Also,

$$\big|\nabla^2\big(h(x,y) - h(y,y)\big)k(x-y)\big| = O(|x-y|^{-(d-1)}),$$

which is integrable. Thus,

$$\Big\| \int_\Omega T_{\ell,y}\{\eta f\}\big(h(x,y)-h(y,y)\big)k(x-y)\,dy\Big\|_{W^{2,p}(\Omega)} \leqslant C\|T_{\ell,y}\{\eta f\}h(y,y)\|_{L^p(\Omega)} \leqslant C\|f\|_{W^{\ell,p}(\Omega)}.$$

This completes the proof of Parts 1 and 3.

2.3. Proof of Hölder bounds

We now prove Part 2 of Theorem 1.

Suppose that $g \in L^\infty(\Omega)$, $h(x,y)$ is a smooth function on $\bar\Omega \times \bar\Omega$ and $\Theta(x)$ is a homogeneous function of degree $-(d-1)$. We start by proving a slight generalization of Range [12, Lemma IV.1.15, p. 157], namely, that the operator $\Theta g(x) = \int_\Omega g(y)h(x,y)k(x-y)\,dy$ satisfies

$$\|\Theta g\|_{\Lambda^{1+\alpha}(\Omega)} \leqslant C_\alpha \|g\|_{L^\infty(\Omega)} \tag{7}$$

for any $0<\alpha<1$. Our argument follows Range's.

Let D_x be a generic first order, constant coefficient derivative in x.

$$Ag(x) = \int_\Omega g(y) D_x h(x,y) k(x-y)\,dy$$

and

$$Bg(x) = \int_\Omega g(y) h(x,y) D_x k(x-y)\,dy,$$

where $k(x)$ is homogeneous of degree $-(d-2)$ and smooth away from 0. The function h is smooth on $\bar\Omega\times\bar\Omega$, and extend it to a smooth function on $\mathbb{R}^d\times\mathbb{R}^d$ with bounded C^j norms, all j. The estimates are handled similarly (though A will have a better estimate). We show the estimate for B. Let $x,x'\in\Omega$, $p=\frac{x+x'}{2}$, and $\tau=|x-x'|$. Choose $R>0$ large enough that $\Omega\subset B(0,R)$. Let $\beta(x,y)=h(x,y)D_xk(x-y)$ and observe that

$$|Bg(x)-Bg(x')| \leqslant \|g\|_{L^\infty(\Omega)} \int_{B(0,R)} |\beta(x,y)-\beta(x',y)|\,dy.$$

Write

$$\int_{B(0,R)} |\beta(x,y)-\beta(x',y)|\,dy = \int_{B(0,R)\cap B(p,2\tau)} |\beta(x,y)-\beta(x',y)|\,dy + \int_{B(0,R)\setminus B(p,2\tau)} |\beta(x,y)-\beta(x',y)|\,dy := I + II.$$

Since $B(x,3\tau)\supset B(p,2\tau)$ and $B(x',3\tau)\supset B(p,2\tau)$, it follows from the boundedness of h that

$$I_1 \leqslant \int_{\beta(x,3\tau)} |\beta(x,y)|\,dy + \int_{\beta(x',3\tau)} |\beta(x',y)|\,dy \leqslant C_h \int_{B(0,3\tau)} \frac{1}{|y|^{d-1}}\,dy \leqslant C_h|x-x'|.$$

For I_2, we use the Mean Value Theorem and estimate

$$|\beta(x,y)-\beta(x',y)| \leqslant C|x-x'| \sup_{t\in[x,x']} |\nabla\beta(t,y)| \leqslant C|x-x'| \sup_{t\in[x,x']} \left(|y-t|^{-d}+|y-t|^{-(d-1)}\right).$$

Since $t \in [x,x']$, $B(0,R)\setminus B(p,2\tau) \subset B(t,2R)\setminus B(t,\tau)$, and consequently,

$$I_2 \leqslant C|x-x'| \int_{\tau\leqslant|y-t|\leqslant 2R} \frac{1}{|y-t|^d} + \frac{1}{|y-t|^{d-1}}\, dy \leqslant C|x-x'|\left(1+\log|x-x'|+R\right).$$

The proof of (7) follows immediately.

Our goal is now to show

$$\|Kf\|_{\Lambda^{\ell+\alpha}(\Omega)} \leqslant C_{\ell,\alpha}\left(\|f\|_{W^{\ell,\infty}(\Omega)} + \|\nabla^{\ell} f\|_{\Lambda^{\alpha}(\Omega)}\right). \tag{8}$$

The $\ell = 0$ case is already proved by (7). We now assume that the ℓ case holds and will shows that the $\ell+1$ case holds.

From (3) and (7), we see that the interior estimates pose no problem, and we need only to show (8) with f replaced by ηf. Because $\operatorname{supp}\eta$ is sufficiently close to $\mathrm{b}\Omega$, we may use Proposition 1 with $\ell+1$ replacing ℓ (note that $\ell \geqslant 1$ in this case). The terms $X_{\ell-1}\nabla$ and $X_{\ell,x}$ term are benign and handled by Part 3, respectively. For each other remaining terms, we integrate the first ℓ derivatives (all of which are tangential) according to Lemma 1. The terms that are generated by the integration by parts and projections are described by Θ, and the result follows from (7). This concludes the proof of Part 2, and hence of Theorem 1.

3. Proof of Theorem 2

DEFINITION 1. Let (X,μ) be a measure space. A measurable function f is *weak type* λ, $1 \leqslant \lambda < \infty$ if there exists $C > 0$ so that

$$\mu\left(\{x \in X : |f(x)| > t\}\right) \leqslant \frac{C}{t^{\lambda}}$$

for all $t > 0$.

The argument to prove Theorem 2 is a combination of the non-characteristic formula for the Laplacian, the integration by parts formula provided by Proposition 1, and the following lemma by Folland and Stein [3, Lemma 15.3], by way of Chen, Krantz, and Ma [1, Lemma 1].

LEMMA 2. *Let (X,μ) and (Y,ν) be measure spaces and $k(x,y)$ be a measurable function on $X\times Y$. If there exists $\lambda \in (1,\infty)$ such that $k(x,\cdot)$ is weak type λ uniformly in x and $k(\cdot,y)$ is weak type λ uniformly in y, then the linear operator T defined by $Tf(x) = \int_X f(x)k(x,y)\,d\mu(x)$ satisfies the following estimates:*

i. T is weak type $(1,\lambda)$, that is, there exists a constant $C > 0$ so that

$$\nu\left(\{y \in Y : |Tf(y)| > t\}\right) \leqslant C\left(\frac{\|f\|_{L^1(X)}}{t}\right)^{\lambda};$$

ii. For $1 < p < \frac{\lambda}{\lambda-1}$ *and* q *defined by* $\frac{1}{q} = \frac{1}{p} + \frac{1}{\lambda} - 1$*,* T *is strong type* (p,q)*, i.e., there exists* $C_p > 0$ *so that*

$$\|Tf\|_{L^q(Y)} \leqslant C_p \|f\|_{L^p(X)};$$

iii. If $p = \frac{\lambda}{\lambda-1}$*, then* T *is strong type* (p,q) *for all* $1 \leqslant q < \infty$*;*

iv. If $p > \frac{\lambda}{\lambda-1}$*, then* T *is strong type* (∞,q)*, that is, there exists* $C > 0$ *so that*

$$\|Tf\|_{L^\infty(Y)} \leqslant C\|f\|_{L^p(X)}.$$

From Proposition 1, we can write a derivative of order $\ell+1$, $X_{\ell+1,x}$, as

$$X_{\ell+1,x} = X_{\ell-1,x}\triangle + T_{\ell+1,x} + \frac{\partial}{\partial \nu_x} T_{\ell,x}, \tag{9}$$

assuming x is sufficiently close to $\mathrm{b}\Omega$.

The proof of Theorem 2 follows the same outline as the proof of Theorem 1. In particular, letting η be the same function as above,

$$D_{\ell+1,x}\int_\Omega K(x-y)f(y)(1-\eta(y))\,dy = \int_\Omega D_{1,x}K(x-y)D_{\ell,y}\big(f(y)(1-\eta(y)\big)\,dy.$$

The function $D_{1,x}K(x-y)$ is a homogeneous function of degree $-(d-1)$. The domain Ω is bounded. Also, $|x|^{-(d-1)} = t$ means that $|x| = t^{-1/(d-1)}$, and consequently

$$\big|\{x\in\Omega : |\nabla K(x-y)| > t\}\big| = C\int_0^{t^{-1/(d-1)}} \frac{1}{|x|^{d-1}}\,dx = \frac{C}{t^{\frac{d}{d-1}}}.$$

The function $D_{1,x}K(x-y)$ is therefore weak type $\frac{d}{d-1}$, and applying Lemma 2 establishes the correct estimates.

We may now focus on $K\{\eta f\}(x)$ for x near $\mathrm{b}\Omega$. Indeed, for x away from $\mathrm{b}\Omega$, η forces y to be near $\mathrm{b}\Omega$, so $|x-y|$ is bounded away from 0, and any estimate we wish to show follows from the smoothness of K away from 0. Examining the terms in (9), we note that $X_{\ell-1,x}\triangle Kf = X_{\ell-1,x}f$ and use the Sobolev Embedding Theorem to bound this term. As before, the estimate reduced to the bounds on $T_{\ell+1,x}$ and $\frac{\partial}{\partial\nu_x}T_{\ell,x}$. In both cases, we will apply Lemma 1 and observe that

$$\begin{aligned} D_x T_{\ell,x}&\Big\{\int_\Omega f(y)k(x-y)\,dy\Big\} \\ &= \int_\Omega T_{\ell,y}f(y)D_x k(x-y)\,dy + \sum_j \int_\Omega T_y^j f(y) D_x\{h_j(x,y)k_j(x-y)\}\,dy. \end{aligned}$$

All of the kernels on the right-hand side of the above equality are functions of weak type $\frac{d}{d-1}$, and the Theorem 2 follows from Lemma 2.

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