Mathematical
Inequalities
& Applications
Volume 27, Number 3 (2024), 729–733 doi:10.7153/mia-2024-27-51

MONOTONICITY OF A TRACE RELATED TO TSALLIS RELATIVE OPERATOR ENTROPY

JIAHANG XU AND JIAN SHI*

(*Communicated by I. Perić*)

Abstract. In this paper, for each $\alpha \in [0,1]$ and two positive semidefinite matrices A and B, we show the monotonicity decreasing property on q of $-\mathrm{Tr}[A^{1-q}T_{\frac{\alpha}{q}}(A^q||B^q)]$ for $0<\alpha<q<1$, which implies an Ando-Hiai result that complements Hiai-Petz inequality as $q \downarrow 0$, where $T_\alpha(A||B) = \frac{A\sharp_\alpha B - A}{\alpha}$.

1. Introduction

Throughout this paper, a capital letter, such as T, means an $n \times n$ matrix. We denote $T \geqslant 0$ if T is a positive semidefinite matrix and $T > 0$ if T is positive definite, respectively. For $A > 0, B \geqslant 0$, $0 \leqslant \alpha \leqslant 1$, F. Kubo and T. Ando, in [7], introduce the α-power mean of A and B as follows,

$$A\sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}.$$

If $A, B \geqslant 0$, T. Ando and F. Hiai, in [2], introduce the following relationship, which is called log-majorization, denoted by $A \underset{(\log)}{\succ} B$, if

$$\prod_{i=1}^{k} \lambda_i(A) \geqslant \prod_{i=1}^{k} \lambda_i(B) \quad (k = 1, 2, \cdots, n-1)$$

and

$$\prod_{i=1}^{n} \lambda_i(A) = \prod_{i=1}^{n} \lambda_i(B) \quad (i.e. \quad detA = detB)$$

hold, where $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A)$ and $\lambda_1(B) \geqslant \lambda_2(B) \geqslant \cdots \geqslant \lambda_n(B)$ are the eigenvalues of A and B respectively arranged in decreasing order.

There are several important concepts related to relative entropy in quantum computing as follows.

Mathematics subject classification (2020): 47A63.
Keywords and phrases: Tsallis relative operator entropy, log-majorization, Grant Furuta inequality, Löwner-Heinz inequality, Hiai-Petz inequality.

* Corresponding author.

DEFINITION 1.1. ([1], Tsallis relative entropy). For two positive semidefinite matrices A and B, the Tsallis relative entropy is defined by

$$D_\alpha(A||B) = \frac{\mathrm{Tr}[A - A^{1-\alpha}B^\alpha]}{\alpha} \tag{1.1}$$

for $0 < \alpha \leqslant 1$.

DEFINITION 1.2. ([11], Tsallis relative operator entropy). For two positive semidefinite matrices A and B, the Tsallis relative operator entropy is defined by

$$T_\alpha(A||B) = \frac{A\sharp_\alpha B - A}{\alpha} \tag{1.2}$$

for $0 < \alpha \leqslant 1$.

DEFINITION 1.3. ([10], Umegaki relative entropy). For two positive semidefinite matrices A and B, the Umegaki relative entropy is defined by

$$S_U(A||B) = \mathrm{Tr}[A(\log A - \log B)]. \tag{1.3}$$

DEFINITION 1.4. ([3], Fujii-Kamei relative operator entropy). For two positive semidefinite matrices A and B, the Fujii-Kamei relative operator entropy is defined by

$$S(A||B) = A^{1/2}(\log A^{-1/2}BA^{-1/2})A^{1/2}. \tag{1.4}$$

DEFINITION 1.5. ([3], Fujii-Kamei relative entropy). For two positive semidefinite matrices A and B, the Fujii-Kamei relative entropy is defined by

$$S_{FK}(A||B) = -\mathrm{Tr}[S(A||B)]. \tag{1.5}$$

The following result is the famous Hiai-Petz inequality, which was first shown in 1993.

THEOREM 1.1. ([8], Hiai-Petz inequality and [2]). *For* $A, B \geqslant 0$

$$-\mathrm{Tr}[A^{1-q}S(A^q||B^q)] \tag{1.6}$$

decreases to $S_U(A||B)$ *as* $q \downarrow 0$.

Recently, M. Fujii and Y. Seo obtained the following result.

THEOREM 1.2. ([4], Fujii-Seo type Tsallis relative entropy inequality).

$$D_\alpha(A||B) \leqslant -\mathrm{Tr}\left[\frac{A^{1-q}}{q}T_{\frac{\alpha}{q}}(A^q||B^q)\right] \tag{1.7}$$

holds for $q \geqslant \alpha > 0$ *and* $0 < \alpha \leqslant 1$.

In this paper, we shall show the monotonically decreasing property of

$$-\mathrm{Tr}[A^{1-q}T_{\frac{\alpha}{q}}(A^q||B^q)],$$

as a complement of Fujii-Seo type Tsallis relative entropy inequality, which implies an Ando-Hiai result that complements Hiai-Petz inequality.

In order to prove the results, we list two lemmas first.

LEMMA 1.1. ([6, 9], Löwner-Heinz inequality). *If* $A \geqslant B \geqslant 0$, *then*

$$A^p \geqslant B^p \tag{1.8}$$

holds for all $0 \leqslant p \leqslant 1$.

LEMMA 1.2. ([5], Grant Furuta inequality). *If* $A \geqslant B \geqslant 0$ *and* $A > 0$, *then*

$$A^{1-t+r} \geqslant (A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}})^{\frac{1-t+r}{(p-t)s+r}} \tag{1.9}$$

holds for $t \in [0,1], p \geqslant 1, s \geqslant 1$ *and* $r \geqslant t$.

2. Main result

In this section, we shall obtain the monotonically decreasing property of

$$-\mathrm{Tr}[A^{1-q}T_{\frac{\alpha}{q}}(A^q||B^q)].$$

THEOREM 2.1. *For* $A, B \geqslant 0$ *and each* $0 < \alpha \leqslant 1$ *and* $1 \geqslant q \geqslant \alpha$,

$$-\mathrm{Tr}[A^{1-q}T_{\frac{\alpha}{q}}(A^q||B^q)]$$

decreases to $D_\alpha(A||B)$ *as* $q \downarrow 0\ (>0)$.

Proof. To show it, it suffices to show

$$\mathrm{Tr}\Big[\frac{A^{1-p}}{p}T_{\frac{\alpha}{p}}(A^p||B^p)\Big] \leqslant \mathrm{Tr}\Big[\frac{A^{1-q}}{q}T_{\frac{\alpha}{q}}(A^q||B^q)\Big] \tag{2.1}$$

holds for $0 \leqslant \alpha \leqslant q \leqslant p \leqslant 1$.

By the definition of Tsallis relative operator entropy, we only need to prove that

$$\mathrm{Tr}\Big[A^{\frac{1-p}{2}}(A^p\sharp_{\frac{\alpha}{p}}B^p)A^{\frac{1-p}{2}}\Big] \leqslant \mathrm{Tr}\Big[A^{\frac{1-q}{2}}(A^q\sharp_{\frac{\alpha}{q}}B^q)A^{\frac{1-q}{2}}\Big], \tag{2.2}$$

which can be derived from

$$A^{\frac{1-p}{2}}(A^p\sharp_{\frac{\alpha}{p}}B^p)A^{\frac{1-p}{2}} \underset{(\log)}{\prec} A^{\frac{1-q}{2}}(A^q\sharp_{\frac{\alpha}{q}}B^q)A^{\frac{1-q}{2}}. \tag{2.3}$$

Therefore, we only need to prove that

$$A^{\frac{1-q}{2}}(A^q\sharp_{\frac{\alpha}{q}}B^q)A^{\frac{1-q}{2}} \leqslant I \tag{2.4}$$

ensures that

$$A^{\frac{1-p}{2}}(A^p\sharp_{\frac{\alpha}{p}}B^p)A^{\frac{1-p}{2}} \leqslant I. \tag{2.5}$$

(2.4) is equivalent to $(A^{-\frac{q}{2}}B^qA^{-\frac{q}{2}})^{\frac{\alpha}{q}} \leqslant A^{-1}$. Let $A_1 = A^{-1}$ and $B_1 = (A^{-\frac{q}{2}}B^qA^{-\frac{q}{2}})^{\frac{\alpha}{q}}$, then we have $B_1 \leqslant A_1$, $A = A_1^{-1}$ and $B = (A_1^{-\frac{q}{2}}B_1^{\frac{q}{\alpha}}A_1^{-\frac{q}{2}})^{\frac{1}{q}}$.

Notice that (2.5) is equivalent to

$$(A_1^{\frac{p}{2}}(A_1^{-\frac{q}{2}}B_1^{\frac{q}{\alpha}}A_1^{-\frac{q}{2}})^{\frac{p}{q}}A_1^{\frac{p}{2}})^{\frac{\alpha}{p}} \leqslant A_1. \tag{2.6}$$

For $q \in [0,1]$, $\frac{q}{\alpha} \geqslant 1$, $\frac{p}{q} \geqslant 1$ and $p \geqslant q$, by Grant Furuta inequality, we have

$$(A_1^{\frac{p}{2}}(A_1^{-\frac{q}{2}}B_1^{\frac{q}{\alpha}}A_1^{-\frac{q}{2}})^{\frac{p}{q}}A_1^{\frac{p}{2}})^{\frac{\alpha(1-q+p)}{p}} \leqslant A_1^{1-q+p}. \tag{2.7}$$

Let $\theta_1 = \frac{1}{1-q+p}$, notice that $0 \leqslant \theta \leqslant 1$, then by Löwner-Heinz inequality,

$$(A_1^{\frac{p}{2}}(A_1^{-\frac{q}{2}}B_1^{\frac{q}{\alpha}}A_1^{-\frac{q}{2}})^{\frac{p}{q}}A_1^{\frac{p}{2}})^{\frac{\theta_1\alpha(1-q+p)}{p}} \leqslant A_1^{\theta_1(1-q+p)} \tag{2.8}$$

holds, which is just (2.6).

Hence, the proof of Theorem 2.1 is completed. □

REMARK. Obviously, Theorem 2.1 is a complement of Fujii-Seo type Tsallis relative entropy inequality which complements Hiai-Petz inequality as $q \downarrow 0$.

Acknowledgements. The authors thank anonymous reviewers for their helpful comments on an earlier draft of this paper. The authors are funded by Science Research Project of Hebei Education Department (QN2023057).

REFERENCES

[1] S. ABE, *Monotone decrease of the quantum nonadditive divergence by projective measurements*, Phys. Lett. A, **312** (2003), 336–338.

[2] T. ANDO AND F. HIAI, *Log majorization and complementary Golden-Thompson type inequalities*, Linear Algebra Appl., **197/198** (1994), 113–131.

[3] J. I. FUJII AND E. KAMEI, *Relative operator entropy in noncommutative information theory*, Math. Japon., **34** (1989), 341–348.

[4] M. FUJII AND Y. SEO, *Matrix trace inequalities related to the Tsallies relative entropies of real order*, J. Math. Anal. Appl., **498** (2021), 1–15.

[5] T. FURUTA, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Algebra Appl., **219** (1995), 139–155.

[6] E. HEINZ, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann., **123** (1951), 415–438.

[7] F. KUBO AND T. ANDO, *Means of positive linear operators*, Math. Ann., **246** (1980), 205–224.

[8] F. HIAI AND D. PETZ, *The Golden-Thompson trace inequality is complemented*, Linear Algebra Appl., **181** (1993), 153–185.

[9] K. LÖWNER, *Über monotone Matrixfunktionen*, Math. Z., **38** (1934), 177–216.

[10] H. UMEGAKI, *Conditional expectation in an operator algebra IV, (entropy and information)*, Kodai Math. Sem. Rep., **14** (1962), 59–85.

[11] K. YANAGI, K. KURIYAMA AND S. FURUICHI, *Generalized Shannon inequalities based on Tsallis relative operator entropy*, Linear Algebra Appl., **394** (2005), 109–118.

(Received December 16, 2023)

Jiahang Xu
Hebei Key Laboratory of Machine Learning and Computational Intelligence
College of Mathematics and Information Science, Hebei University
Baoding, 071002, P.R. China
e-mail: xvjiahang0220@qq.com

Jian Shi
Hebei Key Laboratory of Machine Learning and Computational Intelligence
College of Mathematics and Information Science, Hebei University
Baoding, 071002, P.R. China
e-mail: mathematic@126.com