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INEQUALITIES FOR FUSION FRAMES IN HILBERT SPACES WITH NEW STRUCTURES

ZHONG-QI XIANG*, CHUN-XIA LIN AND XIANG-CHUN XIAO

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Abstract. This paper is concerned with inequalities for fusion frames in Hilbert spaces. With the help of two inequalities on bounded linear operators, several bilateral inequalities associated with a parameter for fusion frames are established, which, comparing to existing ones on this topic, possess new structures.

1. Introduction

Throughout this paper, $\mathbb{I}$, $\mathscr{H}$ and $\mathbb{R}$ are, respectively, a countable index set, a complex Hilbert space and the set of real numbers, $\mathscr{N}_i$ and $\mathscr{P}_i$ are closed subspaces of $\mathscr{H}$ for each $i \in \mathbb{I}$. The notation $\{\alpha_i\}_{i\in\mathbb{I}}$ is used to denote the family of weights, that is, each $\alpha_i > 0$, and the same goes for $\{\beta_i\}_{i\in\mathbb{I}}$. Also, we denote respectively by $BL(\mathscr{H})$, $\pi_{\mathscr{N}_i}$ and $\mathrm{Id}_{\mathscr{H}}$ the set of all bounded linear operators on $\mathscr{H}$, the orthogonal projection onto $\mathscr{N}_i$ and the identity operator on $\mathscr{H}$.

By extracting the basic ideas of Gabor [14] on signal processing, Duffin and Schaeffer [10] raised the notion of (Hilbert spaces) frames based on an in-depth research on nonharmonic Fourier series in 1952. More than 30 years later, frames were brought back to people's vision because of the fundamental work [9] of Daubechies et al. on wavelet. In the past two decades, frames have already been applied to many research fields (see e.g. [4, 8, 20, 22]), due to some of their nice properties.

Recall that a family $\mathscr{F} = \{f_i\}_{i\in\mathbb{I}} \subseteq \mathscr{H}$ is called a *frame* for $\mathscr{H}$, if there are constants $0 < C_{\mathscr{F}} \leqslant D_{\mathscr{F}} < +\infty$ such that the inequality

$$C_{\mathscr{F}}\|x\|^2 \leqslant \sum_{i\in\mathbb{I}} |\langle x, f_i\rangle|^2 \leqslant D_{\mathscr{F}}\|x\|^2$$

holds for each $x \in \mathscr{H}$. A new frame $\widetilde{\mathscr{F}} = \{\widetilde{f_i} : \widetilde{f_i} = S_{\mathscr{F}}^{-1} f_i\}_{i\in\mathbb{I}}$, induced by $\mathscr{F}$ and the corresponding frame operator $S_{\mathscr{F}}$, is said to be the *canonical dual frame* of $\mathscr{F}$.

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* Corresponding author.

To process some large systems, Casazza and Kutyniok [6], and Fornasier [12] independently proposed the concept of fusion frames, as a generalization of frames. Because of the complex structure, fusion frames do admit some new behaviors although they share most properties with frames, which means that the investigation of fusion frames is interesting. For the application of fusion frames, the reader can refer to the papers [5, 7].

One calls $\mathscr{N} = \{(\mathscr{N}_i, \alpha_i)\}_{i\in\mathbb{I}}$ a *fusion frame* for $\mathscr{H}$, if there exist numbers $0 < C_{\mathscr{N}} \leqslant D_{\mathscr{N}} < +\infty$ such that

$$C_{\mathscr{N}}\|x\|^2 \leqslant \sum_{i\in\mathbb{I}} \alpha_i^2 \|\pi_{\mathscr{N}_i}(x)\|^2 \leqslant D_{\mathscr{N}}\|x\|^2 \tag{1}$$

is valid for any $x \in \mathscr{H}$. The fusion frame $\mathscr{N}$ is said to be *Parseval* if $C_{\mathscr{N}} = D_{\mathscr{N}} = 1$ and, we call $\mathscr{N}$ a *Bessel fusion sequence* if the right-hand inequality of (1) is required to be satisfied.

Let $\mathscr{N} = \{(\mathscr{N}_i, \alpha_i)\}_{i\in\mathbb{I}}$ be a fusion frame for $\mathscr{H}$. Then it can naturally lead to a self-adjoint and invertible operator $S_{\mathscr{N}}$, called the *fusion frame operator* of $\mathscr{N}$, given below

$$S_{\mathscr{N}} : \mathscr{H} \to \mathscr{H}, \quad S_{\mathscr{N}}x = \sum_{i\in\mathbb{I}} \alpha_i^2 \pi_{\mathscr{N}_i}(x), \quad \forall x \in \mathscr{H}. \tag{2}$$

From [16] we know that $\mathscr{N}' = \{(S_{\mathscr{N}}^{-1}\mathscr{N}_i, \alpha_i)\}_{i\in\mathbb{I}}$ remains to be a fusion frame for $\mathscr{H}$, which is said to be the *dual fusion frame* of $\mathscr{N}$. Also, by (2) we can easily obtain the so-called *reconstruction formula*

$$\sum_{i\in\mathbb{I}} \alpha_i^2 S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x) = S_{\mathscr{N}}^{-1}S_{\mathscr{N}}x = x = S_{\mathscr{N}}S_{\mathscr{N}}^{-1}x = \sum_{i\in\mathbb{I}} \alpha_i^2 \pi_{\mathscr{N}_i}(S_{\mathscr{N}}^{-1}x) \tag{3}$$

for every $x \in \mathscr{H}$.

Suppose that $\mathscr{N} = \{(\mathscr{N}_i, \alpha_i)\}_{i\in\mathbb{I}}$ is a fusion frame for $\mathscr{H}$ with the fusion frame operator $S_{\mathscr{N}}$, and that $\mathscr{P} = \{(\mathscr{P}_i, \beta_i)\}_{i\in\mathbb{I}}$ is a Bessel fusion sequence for $\mathscr{H}$. We call the pair $\mathscr{P}$ an *alternate dual fusion frame* of $\mathscr{N}$, if

$$x = \sum_{i\in\mathbb{I}} \alpha_i\beta_i\pi_{\mathscr{P}_i}S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x), \quad \forall x \in \mathscr{H}. \tag{4}$$

Let $\mathscr{N} = \{(\mathscr{N}_i, \alpha_i)\}_{i\in\mathbb{I}}$ be a Bessel fusion sequence for $\mathscr{H}$. For any $\mathbb{J} \subset \mathbb{I}$, we let $\mathbb{J}^c = \mathbb{I}\backslash\mathbb{J}$. Then, associated with $\mathscr{N}$, $\mathbb{J}$ and $\mathbb{J}^c$ there are always two self-adjoint operators $S_{\mathscr{N}}^{\mathbb{J}}$ and $S_{\mathscr{N}}^{\mathbb{J}^c}$, defined by

$$S_{\mathscr{N}}^{\mathbb{J}}, S_{\mathscr{N}}^{\mathbb{J}^c} : \mathscr{H} \to \mathscr{H}, \quad S_{\mathscr{N}}^{\mathbb{J}}x = \sum_{i\in\mathbb{J}} \alpha_i^2 \pi_{\mathscr{N}_i}(x), \quad S_{\mathscr{N}}^{\mathbb{J}^c}x = \sum_{i\in\mathbb{J}^c} \alpha_i^2 \pi_{\mathscr{N}_i}(x). \tag{5}$$

As a derivative result of the famous Parseval frame identity arising in the process of exploring efficient algorithms for the reconstruction of signals [2], Balan et al. in [3] offered us an interesting inequality for Parseval frames, which is listed as follows.

THEOREM A. (see [3, Proposition 4.1]) *Suppose that $\mathscr{F}=\{f_i\}_{i\in\mathbb{I}}$ is a Parseval frame for $\mathscr{H}$. Then for any $\mathbb{J}\subset\mathbb{I}$ and any $x\in\mathscr{H}$, we get*

$$\sum_{i\in\mathbb{J}}|\langle x,f_i\rangle|^2+\left\|\sum_{i\in\mathbb{J}^c}\langle x,f_i\rangle f_i\right\|^2\geqslant\frac{3}{4}\|x\|^2. \tag{6}$$

By means of the operator $S_{\mathscr{F}}^{\mathbb{J}^c}:\mathscr{H}\to\mathscr{H}$ given by $S_{\mathscr{F}}^{\mathbb{J}^c}x=\sum_{i\in\mathbb{J}^c}\langle x,f_i\rangle f_i$, Găvruţa in [15] extended inequality (6) to the case of general frames.

THEOREM B. (see [15, Theorem 2.2]) *Let $\mathscr{F}=\{f_i\}_{i\in\mathbb{I}}$ be a frame for $\mathscr{H}$ with canonical dual frame $\widetilde{\mathscr{F}}=\{\widetilde{f_i}\}_{i\in\mathbb{I}}$. Then for any $\mathbb{J}\subset\mathbb{I}$ and any $x\in\mathscr{H}$, we have*

$$\sum_{i\in\mathbb{J}}|\langle x,f_i\rangle|^2+\sum_{i\in\mathbb{I}}|\langle S_{\mathscr{F}}^{\mathbb{J}^c}x,\widetilde{f_i}\rangle|^2\geqslant\frac{3}{4}\sum_{i\in\mathbb{I}}|\langle x,f_i\rangle|^2. \tag{7}$$

Later, Guo et al. generalized inequality (7) to the setting of fusion frames (see [17, Theorem 6]). Inequalities for some other generalized frames were also obtained and in particular, Poria in [21] (noting that this work was first published in Arxiv: https://arxiv.org/abs/1602.07912v1) showed us two inequalities admitting new forms for Hilbert-Schmidt frames assisted by an operator deriving from the Hilbert-Schmidt frame itself and by the alternate dual pair respectively, which are relevant to a parameter taken from an interval.

THEOREM C. (see [21, Theorem 3.5]) *Suppose that $\{\mathscr{G}_i\}_{i\in\mathbb{I}}$ is a Hilbert-Schmidt frame for $\mathscr{H}$ with the canonical dual $\{\widetilde{\mathscr{G}_i}\}_{i\in\mathbb{I}}$. Then for each $x\in\mathscr{H}$, for all $\lambda\in[0,1]$ and all $\sigma\subset\mathbb{I}$, we get*

$$\begin{aligned}&\sum_{i\in\mathbb{I}}\|\widetilde{\mathscr{G}_i}(S^\sigma x)\|^2+\sum_{i\in\sigma^c}\|\mathscr{G}_i(x)\|^2\\&\qquad\geqslant(2\lambda-\lambda^2)\sum_{i\in\sigma}\|\mathscr{G}_i(x)\|^2+(1-\lambda^2)\sum_{i\in\sigma^c}\|\mathscr{G}_i(x)\|^2,\end{aligned}$$

where $S^\sigma:\mathscr{H}\to\mathscr{H}$ is defined by $S^\sigma x=\sum_{i\in\sigma}\mathscr{G}_i^*\mathscr{G}_i(x)$.

THEOREM D. (see [21, Theorem 3.7]) *Suppose that $\{\mathscr{G}_i\}_{i\in\mathbb{I}}$ is a Hilbert-Schmidt frame for $\mathscr{H}$ with an alternate dual $\{\mathscr{K}_i\}_{i\in\mathbb{I}}$. Then for each $x\in\mathscr{H}$, for all $\lambda\in[0,1]$ and all $\sigma\subset\mathbb{I}$, we obtain*

$$\begin{aligned}&\mathrm{Re}\sum_{i\in\sigma^c}[\mathscr{K}_i(x),\mathscr{G}_i(x)]_\tau+\left\|\sum_{i\in\sigma}\mathscr{G}_i^*\mathscr{K}_i(x)\right\|^2\\&\qquad\geqslant(2\lambda-\lambda^2)\mathrm{Re}\sum_{i\in\sigma}[\mathscr{K}_i(x),\mathscr{G}_i(x)]_\tau+(1-\lambda^2)\mathrm{Re}\sum_{i\in\sigma^c}[\mathscr{K}_i(x),\mathscr{G}_i(x)]_\tau.\end{aligned}$$

Drawing on the idea of [21], Xiang in [23], and Li et al. in [18] presented some more general inequalities (bilateral inequalities related to a parameter) for fusion frames listed below. It should be pointed out that they were proved using the idea given in [21].

THEOREM E. (see [23, Theorem 2.3]) *Let $\mathscr{N}=\{(\mathscr{N}_i,\alpha_i)\}_{i\in\mathbb{I}}$ be a fusion frame for $\mathscr{H}$ with the fusion frame operator $S_{\mathscr{N}}$, and $\mathscr{N}'=\{(S_{\mathscr{N}}^{-1}\mathscr{N}_i,\alpha_i)\}_{i\in\mathbb{I}}$ be the dual fusion frame of $\mathscr{N}$. Then for any $\lambda\in[0,1]$, for all $\mathbb{J}\subset\mathbb{I}$ and all $x\in\mathscr{H}$, we have*

$$
\begin{aligned}
\langle S_{\mathscr{N}}x,x\rangle &\geqslant \sum_{i\in\mathbb{J}}\alpha_i^2\|\pi_{\mathscr{N}_i}(x)\|^2+\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathscr{N}_i}(S_{\mathscr{N}}^{-1}S_{\mathscr{N}}^{\mathbb{J}^c}x)\|^2\\
&\geqslant (2\lambda-\lambda^2)\langle S_{\mathscr{N}}^{\mathbb{J}}x,x\rangle+(1-\lambda^2)\langle S_{\mathscr{N}}^{\mathbb{J}^c}x,x\rangle.
\end{aligned}
$$

THEOREM F. (see [18, Theorem 5]) *Suppose that $\mathscr{N}=\{(\mathscr{N}_i,\alpha_i)\}_{i\in\mathbb{I}}$ is a fusion frame for $\mathscr{H}$ with the fusion frame operator $S_{\mathscr{N}}$ and that $\mathscr{N}'=\{(S_{\mathscr{N}}^{-1}\mathscr{N}_i,\alpha_i)\}_{i\in\mathbb{I}}$ is the dual fusion frame of $\mathscr{N}$. Then for any $\lambda\in[1,2]$, for all $\mathbb{J}\subset\mathbb{I}$ and all $x\in\mathscr{H}$, we obtain*

$$
\begin{aligned}
0 &\leqslant \sum_{i\in\mathbb{J}}\alpha_i^2\|\pi_{\mathscr{N}_i}(x)\|^2-\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathscr{N}_i}(S_{\mathscr{N}}^{-1}S_{\mathscr{N}}^{\mathbb{J}}x)\|^2\\
&\leqslant (\lambda-1)\sum_{i\in\mathbb{J}^c}\alpha_i^2\|\pi_{\mathscr{N}_i}(x)\|^2+\left(1-\frac{\lambda}{2}\right)^2\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathscr{N}_i}(x)\|^2.
\end{aligned}
$$

THEOREM G. (see [18, Theorem 6]) *Let $\mathscr{N}=\{(\mathscr{N}_i,\alpha_i)\}_{i\in\mathbb{I}}$ be a fusion frame for $\mathscr{H}$ with the fusion frame operator $S_{\mathscr{N}}$, and $\mathscr{N}'=\{(S_{\mathscr{N}}^{-1}\mathscr{N}_i,\alpha_i)\}_{i\in\mathbb{I}}$ be the dual fusion frame of $\mathscr{N}$. Then for any $\lambda\in[1,2]$, for all $\mathbb{J}\subset\mathbb{I}$ and all $x\in\mathscr{H}$, we get*

$$
\begin{aligned}
&\left(2\lambda-\frac{\lambda^2}{2}-1\right)\sum_{i\in\mathbb{J}}\alpha_i^2\|\pi_{\mathscr{N}_i}(x)\|^2+\left(1-\frac{\lambda^2}{2}\right)\sum_{i\in\mathbb{J}^c}\alpha_i^2\|\pi_{\mathscr{N}_i}(x)\|^2\\
&\quad\leqslant \sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathscr{N}_i}(S_{\mathscr{N}}^{-1}S_{\mathscr{N}}^{\mathbb{J}}x)\|^2+\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathscr{N}_i}(S_{\mathscr{N}}^{-1}S_{\mathscr{N}}^{\mathbb{J}^c}x)\|^2\\
&\quad\leqslant \lambda\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathscr{N}_i}(x)\|^2.
\end{aligned}
$$

Some bilateral inequalities for other generalized versions of frames with the same structures as those in [18, 23] are also given, see [13, 25] for continuous g-frames, and [19] for continuous fusion frames. We refer to [1, 11] for more information on continuous g-frames and continuous fusion frames.

Inspired by above works, in this paper we establish several bilateral inequalities for fusion frames from the perspective of operator theory, which admit novel structures comparing to previous ones.

2. Main results

To state the main results of this paper, we need the following simple result on bounded linear operators claimed in [24], we include the proof here for the convenience of the reader.

LEMMA 1. *If $U,V \in BL(\mathscr{H})$ satisfy $U+V = \mathrm{Id}_{\mathscr{H}}$, then the following assertions hold.*

(1) For each $\lambda \in \mathbb{R}$ and each $x \in \mathscr{H}$, we have

$$\|Ux\|^2 + 2\lambda \mathrm{Re}\langle Vx,x\rangle = \|Vx\|^2 + 2(1-\lambda)\mathrm{Re}\langle Ux,x\rangle + (2\lambda-1)\|x\|^2 \geqslant (2\lambda-\lambda^2)\|x\|^2.$$

(2) For any $\lambda \in [0,\frac{1}{2}]$ and any $x \in \mathscr{H}$, we get

$$\|Ux\|^2 + 2\lambda \mathrm{Re}\langle Vx,x\rangle \leqslant \frac{3\lambda + 2(1-2\lambda)\|U\|^2 + \lambda\|U-V\|^2}{2}\|x\|^2.$$

Proof. (1) The proof is similar to [21, Proposition 3.6], and we omit the details.

(2) A direct calculation gives that

$$\begin{aligned}
\|Ux\|^2 + 2\lambda \mathrm{Re}\langle Vx,x\rangle &= \|Ux\|^2 + \lambda(\langle x,x\rangle - \langle Ux,x\rangle + \langle x,x\rangle - \langle x,Ux\rangle) \\
&= 2\lambda\langle x,x\rangle + (1-2\lambda)\langle Ux,Ux\rangle - \lambda(\langle Ux,x\rangle - \langle Ux,Ux\rangle) \\
&\quad - \lambda(\langle x,Ux\rangle - \langle Ux,Ux\rangle) \\
&= 2\lambda\langle x,x\rangle + (1-2\lambda)\langle Ux,Ux\rangle - \lambda\langle Ux,Vx\rangle - \lambda\langle Vx,Ux\rangle \\
&= \frac{3\lambda}{2}\langle x,x\rangle + (1-2\lambda)\langle Ux,Ux\rangle + \frac{\lambda}{2}\langle (U+V)x,(U+V)x\rangle \\
&\quad - \lambda\langle Ux,Vx\rangle - \lambda\langle Vx,Ux\rangle \\
&= \frac{3\lambda}{2}\langle x,x\rangle + (1-2\lambda)\langle Ux,Ux\rangle + \frac{\lambda}{2}\langle (U-V)x,(U-V)x\rangle \\
&\leqslant \frac{3\lambda}{2}\|x\|^2 + (1-2\lambda)\|U\|^2\|x\|^2 + \frac{\lambda}{2}\|U-V\|^2\|x\|^2 \\
&= \frac{3\lambda + 2(1-2\lambda)\|U\|^2 + \lambda\|U-V\|^2}{2}\|x\|^2
\end{aligned}$$

for each $\lambda \in [0,\frac{1}{2}]$ and each $x \in \mathscr{H}$, and we obtain the result. □

We are now ready to give our first result and, we remark that the idea of the proof is borrowed from [19, Theorem 2.2] and [21, Theorem 3.5].

THEOREM 1. *Let $\mathscr{N} = \{(\mathscr{N}_i,\alpha_i)\}_{i\in\mathbb{I}}$ be a fusion frame for $\mathscr{H}$ with the fusion frame operator $S_{\mathscr{N}}$, and $\mathscr{N}' = \{(S_{\mathscr{N}}^{-1}\mathscr{N}_i,\alpha_i)\}_{i\in\mathbb{I}}$ be the dual fusion frame of $\mathscr{N}$. Then* *for every* $\lambda \in [1,+\infty)$, *for any* $\mathbb{J} \subset \mathbb{I}$ *and any* $x \in \mathscr{H}$, *we have*

$$\begin{aligned}
&\sum_{i\in\mathbb{J}} \alpha_i^2 \|\pi_{\mathscr{N}_i}(x)\|^2 - \lambda \sum_{i\in\mathbb{J}^c} \alpha_i^2 \|\pi_{\mathscr{N}_i}(x)\|^2 \\
&\quad \leqslant \sum_{i\in\mathbb{I}} \alpha_i^2 \|\pi_{\mathscr{N}_i}(S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}} x)\|^2 - \lambda \sum_{i\in\mathbb{I}} \alpha_i^2 \|\pi_{\mathscr{N}_i}(S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}^c} x)\|^2 \\
&\quad \leqslant (\lambda^3 - \lambda^2 + 1) \sum_{i\in\mathbb{J}} \alpha_i^2 \|\pi_{\mathscr{N}_i}(x)\|^2 + (\lambda^3 - 3\lambda^2 + 2\lambda - 1) \sum_{i\in\mathbb{J}^c} \alpha_i^2 \|\pi_{\mathscr{N}_i}(x)\|^2.
\end{aligned} \tag{8}$$

Proof. For any $\mathbb{J} \subset \mathbb{I}$, combination of (2) and (5) we get $S_{\mathscr{N}}^{\mathbb{J}} + S_{\mathscr{N}}^{\mathbb{J}^c} = S_{\mathscr{N}}$. Thus

$$S_{\mathscr{N}}^{-\frac{1}{2}} S_{\mathscr{N}}^{\mathbb{J}} S_{\mathscr{N}}^{-\frac{1}{2}} + S_{\mathscr{N}}^{-\frac{1}{2}} S_{\mathscr{N}}^{\mathbb{J}^c} S_{\mathscr{N}}^{-\frac{1}{2}} = S_{\mathscr{N}}^{-\frac{1}{2}} S_{\mathscr{N}} S_{\mathscr{N}}^{-\frac{1}{2}} = \mathrm{Id}_{\mathscr{H}}.$$

Letting $W = S_{\mathscr{N}}^{-\frac{1}{2}} S_{\mathscr{N}}^{\mathbb{J}} S_{\mathscr{N}}^{-\frac{1}{2}}$ and $T = S_{\mathscr{N}}^{-\frac{1}{2}} S_{\mathscr{N}}^{\mathbb{J}^c} S_{\mathscr{N}}^{-\frac{1}{2}}$. Then it is easily seen that $\langle Wx, x\rangle \geqslant 0$ and $\langle Tx, x\rangle \geqslant 0$ for every $x \in \mathscr{H}$, and that $WT = TW$. Therefore

$$0 \leqslant WT = (\mathrm{Id}_{\mathscr{H}} - T)T = T - T^2 = S_{\mathscr{N}}^{-\frac{1}{2}} (S_{\mathscr{N}}^{\mathbb{J}^c} - S_{\mathscr{N}}^{\mathbb{J}^c} S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}^c}) S_{\mathscr{N}}^{-\frac{1}{2}},$$

from which we conclude that $S_{\mathscr{N}}^{\mathbb{J}^c} \geqslant S_{\mathscr{N}}^{\mathbb{J}^c} S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}^c}$. Now for each $x \in \mathscr{H}$ and each $\lambda \in [1,+\infty)$, we have

$$\begin{aligned}
&\sum_{i\in\mathbb{I}} \alpha_i^2 \|\pi_{\mathscr{N}_i}(S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}} x)\|^2 - \lambda \sum_{i\in\mathbb{I}} \alpha_i^2 \|\pi_{\mathscr{N}_i}(S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}^c} x)\|^2 \\
&\quad = \langle S_{\mathscr{N}} S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}} x, S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}} x\rangle - \lambda \langle S_{\mathscr{N}} S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}^c} x, S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}^c} x\rangle \\
&\quad = \langle S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}} x, S_{\mathscr{N}}^{\mathbb{J}} x\rangle - \lambda \langle S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}^c} x, S_{\mathscr{N}}^{\mathbb{J}^c} x\rangle \\
&\quad = \langle S_{\mathscr{N}}^{-1} (S_{\mathscr{N}} - S_{\mathscr{N}}^{\mathbb{J}^c}) x, (S_{\mathscr{N}} - S_{\mathscr{N}}^{\mathbb{J}^c}) x\rangle - \lambda \langle S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}^c} x, S_{\mathscr{N}}^{\mathbb{J}^c} x\rangle \\
&\quad = \langle S_{\mathscr{N}} x, x\rangle - 2\langle S_{\mathscr{N}}^{\mathbb{J}^c} x, x\rangle + \langle S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}^c} x, S_{\mathscr{N}}^{\mathbb{J}^c} x\rangle - \lambda \langle S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}^c} x, S_{\mathscr{N}}^{\mathbb{J}^c} x\rangle \\
&\quad = (\lambda + 1)\langle S_{\mathscr{N}} x, x\rangle - \lambda \langle S_{\mathscr{N}} x, x\rangle - 2\langle S_{\mathscr{N}}^{\mathbb{J}^c} x, x\rangle + (1 - \lambda)\langle S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}^c} x, S_{\mathscr{N}}^{\mathbb{J}^c} x\rangle \\
&\quad = (\lambda - 1)\langle S_{\mathscr{N}}^{\mathbb{J}^c} x, x\rangle + (\lambda + 1)\langle S_{\mathscr{N}}^{\mathbb{J}} x, x\rangle - \lambda \langle S_{\mathscr{N}} x, x\rangle + (1 - \lambda)\langle S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}^c} x, S_{\mathscr{N}}^{\mathbb{J}^c} x\rangle \\
&\quad = (\lambda - 1)(\langle S_{\mathscr{N}}^{\mathbb{J}^c} x, x\rangle - \langle S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}^c} x, S_{\mathscr{N}}^{\mathbb{J}^c} x\rangle) + \langle S_{\mathscr{N}}^{\mathbb{J}} x, x\rangle + \lambda (\langle S_{\mathscr{N}}^{\mathbb{J}} x, x\rangle - \langle S_{\mathscr{N}} x, x\rangle) \\
&\quad \geqslant \langle S_{\mathscr{N}}^{\mathbb{J}} x, x\rangle - \lambda \langle S_{\mathscr{N}}^{\mathbb{J}^c} x, x\rangle = \sum_{i\in\mathbb{J}} \alpha_i^2 \|\pi_{\mathscr{N}_i}(x)\|^2 - \lambda \sum_{i\in\mathbb{J}^c} \alpha_i^2 \|\pi_{\mathscr{N}_i}(x)\|^2.
\end{aligned} \tag{9}$$

Taking $U = S_{\mathscr{N}}^{-\frac{1}{2}} S_{\mathscr{N}}^{\mathbb{J}^c} S_{\mathscr{N}}^{-\frac{1}{2}}$ and $V = S_{\mathscr{N}}^{-\frac{1}{2}} S_{\mathscr{N}}^{\mathbb{J}} S_{\mathscr{N}}^{-\frac{1}{2}}$, and using $S_{\mathscr{N}}^{\frac{1}{2}} x$ in place of x in Lemma 1 yields

$$\begin{aligned}
\langle S_{\mathscr{N}}^{-1} S_{\mathscr{N}}^{\mathbb{J}^c} x, S_{\mathscr{N}}^{\mathbb{J}^c} x\rangle &= \langle S_{\mathscr{N}}^{-\frac{1}{2}} S_{\mathscr{N}}^{\mathbb{J}^c} S_{\mathscr{N}}^{-\frac{1}{2}} S_{\mathscr{N}}^{\frac{1}{2}} x, S_{\mathscr{N}}^{-\frac{1}{2}} S_{\mathscr{N}}^{\mathbb{J}^c} S_{\mathscr{N}}^{-\frac{1}{2}} S_{\mathscr{N}}^{\frac{1}{2}} x\rangle \\
&\geqslant (2\lambda - \lambda^2)\langle S_{\mathscr{N}}^{\frac{1}{2}} x, S_{\mathscr{N}}^{\frac{1}{2}} x\rangle - \lambda (\langle S_{\mathscr{N}}^{-\frac{1}{2}} S_{\mathscr{N}}^{\mathbb{J}} S_{\mathscr{N}}^{-\frac{1}{2}} S_{\mathscr{N}}^{\frac{1}{2}} x, S_{\mathscr{N}}^{\frac{1}{2}} x\rangle \\
&\qquad + \langle S_{\mathscr{N}}^{\frac{1}{2}} x, S_{\mathscr{N}}^{-\frac{1}{2}} S_{\mathscr{N}}^{\mathbb{J}} S_{\mathscr{N}}^{-\frac{1}{2}} S_{\mathscr{N}}^{\frac{1}{2}} x\rangle) \\
&= (2\lambda - \lambda^2)\langle S_{\mathscr{N}}^{\mathbb{J}^c} x, x\rangle - \lambda^2 \langle S_{\mathscr{N}}^{\mathbb{J}} x, x\rangle.
\end{aligned} \tag{10}$$

Therefore

$$\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathscr{N}_i}(S_{\mathscr{N}}^{-1}S_{\mathscr{N}}^{\mathbb{J}}x)\|^2-\lambda\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathscr{N}_i}(S_{\mathscr{N}}^{-1}S_{\mathscr{N}}^{\mathbb{J}^c}x)\|^2 \tag{11}$$
$$=\langle S_{\mathscr{N}}^{\mathbb{J}}x,x\rangle-\langle S_{\mathscr{N}}^{\mathbb{J}^c}x,x\rangle-(\lambda-1)\langle S_{\mathscr{N}}^{-1}S_{\mathscr{N}}^{\mathbb{J}^c}x,S_{\mathscr{N}}^{\mathbb{J}^c}x\rangle$$
$$\leqslant\langle S_{\mathscr{N}}^{\mathbb{J}}x,x\rangle-\langle S_{\mathscr{N}}^{\mathbb{J}^c}x,x\rangle+\lambda^2(\lambda-1)\langle S_{\mathscr{N}}^{\mathbb{J}}x,x\rangle-(2\lambda-\lambda^2)(\lambda-1)\langle S_{\mathscr{N}}^{\mathbb{J}^c}x,x\rangle$$
$$=(\lambda^3-\lambda^2+1)\langle S_{\mathscr{N}}^{\mathbb{J}}x,x\rangle+(\lambda^3-3\lambda^2+2\lambda-1)\langle S_{\mathscr{N}}^{\mathbb{J}^c}x,x\rangle$$
$$=(\lambda^3-\lambda^2+1)\sum_{i\in\mathbb{J}}\alpha_i^2\|\pi_{\mathscr{N}_i}(x)\|^2+(\lambda^3-3\lambda^2+2\lambda-1)\sum_{i\in\mathbb{J}^c}\alpha_i^2\|\pi_{\mathscr{N}_i}(x)\|^2.$$

This along with (9) leads to (8), and we arrive at the conclusion. □

Let $\mathscr{N}=\{(\mathscr{N}_i,\alpha_i)\}_{i\in\mathbb{I}}$ be a Parseval fusion frame for $\mathscr{H}$ with the fusion frame operator $S_{\mathscr{N}}$. Then $S_{\mathscr{N}}=\mathrm{Id}_{\mathscr{H}}$. For any $\mathbb{J}\subset\mathbb{I}$ and any $x\in\mathscr{H}$, we have

$$\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathscr{N}_i}(S_{\mathscr{N}}^{-1}S_{\mathscr{N}}^{\mathbb{J}}x)\|^2=\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathscr{N}_i}(S_{\mathscr{N}}^{\mathbb{J}}x)\|^2=\|S_{\mathscr{N}}^{\mathbb{J}}x\|^2=\left\|\sum_{i\in\mathbb{J}}\alpha_i^2\pi_{\mathscr{N}_i}(x)\right\|^2,$$

and similarly,

$$\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathscr{N}_i}(S_{\mathscr{N}}^{-1}S_{\mathscr{N}}^{\mathbb{J}^c}x)\|^2=\left\|\sum_{i\in\mathbb{J}^c}\alpha_i^2\pi_{\mathscr{N}_i}(x)\right\|^2.$$

Above facts together with Theorem 1 give a direct consequence as follows.

COROLLARY 1. *Let $\mathscr{N}=\{(\mathscr{N}_i,\alpha_i)\}_{i\in\mathbb{I}}$ be a Parseval fusion frame for $\mathscr{H}$. Then for every $\lambda\in[1,+\infty)$, for any $\mathbb{J}\subset\mathbb{I}$ and any $x\in\mathscr{H}$, we have*

$$\sum_{i\in\mathbb{J}}\alpha_i^2\|\pi_{\mathscr{N}_i}(x)\|^2-\lambda\sum_{i\in\mathbb{J}^c}\alpha_i^2\|\pi_{\mathscr{N}_i}(x)\|^2$$
$$\leqslant\left\|\sum_{i\in\mathbb{J}}\alpha_i^2\pi_{\mathscr{N}_i}(x)\right\|^2-\lambda\left\|\sum_{i\in\mathbb{J}^c}\alpha_i^2\pi_{\mathscr{N}_i}(x)\right\|^2$$
$$\leqslant(\lambda^3-\lambda^2+1)\sum_{i\in\mathbb{J}}\alpha_i^2\|\pi_{\mathscr{N}_i}(x)\|^2+(\lambda^3-3\lambda^2+2\lambda-1)\sum_{i\in\mathbb{J}^c}\alpha_i^2\|\pi_{\mathscr{N}_i}(x)\|^2.$$

We state the second main result as follows, and the idea of the proof comes from [18, Theorem 6].

THEOREM 2. *Let $\mathscr{N}=\{(\mathscr{N}_i,\alpha_i)\}_{i\in\mathbb{I}}$ be a fusion frame for $\mathscr{H}$ with the fusion frame operator $S_{\mathscr{N}}$, and $\mathscr{N}'=\{(S_{\mathscr{N}}^{-1}\mathscr{N}_i,\alpha_i)\}_{i\in\mathbb{I}}$ be the dual fusion frame of $\mathscr{N}$. Then for every $\lambda\in[\frac{1}{2},+\infty)$, for any $\mathbb{J}\subset\mathbb{I}$ and any $x\in\mathscr{H}$, we have*

$$(4\lambda-1)\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathscr{N}_i}(S_{\mathscr{N}}^{-1}S_{\mathscr{N}}^{\mathbb{J}^c}x)\|^2+(1-\lambda^2)\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathscr{N}_i}(x)\|^2 \tag{12}$$
$$\leqslant\sum_{i\in\mathbb{J}}\alpha_i^2\|\pi_{\mathscr{N}_i}(x)\|^2+(1+2\lambda)\sum_{i\in\mathbb{J}^c}\alpha_i^2\|\pi_{\mathscr{N}_i}(x)\|^2$$
$$\leqslant\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathscr{N}_i}(S_{\mathscr{N}}^{-1}S_{\mathscr{N}}^{\mathbb{J}^c}x)\|^2+(1+\lambda^2)\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathscr{N}_i}(x)\|^2.$$

Proof. We obtain by (10) that

$$\begin{aligned}
&\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathcal{N}_i}(S_{\mathcal{N}}^{-1}S_{\mathcal{N}}^{\mathbb{J}^c}x)\|^2-\sum_{i\in\mathbb{J}}\alpha_i^2\|\pi_{\mathcal{N}_i}(x)\|^2 \qquad (13)\\
&\quad=\langle S_{\mathcal{N}}^{-1}S_{\mathcal{N}}^{\mathbb{J}^c}x,S_{\mathcal{N}}^{\mathbb{J}^c}x\rangle-\langle S_{\mathcal{N}}^{\mathbb{J}}x,x\rangle\\
&\quad\geqslant(2\lambda-\lambda^2)\langle S_{\mathcal{N}}^{\mathbb{J}^c}x,x\rangle-\lambda^2\langle S_{\mathcal{N}}^{\mathbb{J}}x,x\rangle-\langle S_{\mathcal{N}}^{\mathbb{J}}x,x\rangle\\
&\quad=(2\lambda-\lambda^2)\langle S_{\mathcal{N}}^{\mathbb{J}^c}x,x\rangle-(1+\lambda^2)\langle S_{\mathcal{N}}x,x\rangle+(1+\lambda^2)\langle S_{\mathcal{N}}^{\mathbb{J}^c}x,x\rangle\\
&\quad=(1+2\lambda)\langle S_{\mathcal{N}}^{\mathbb{J}^c}x,x\rangle-(1+\lambda^2)\langle S_{\mathcal{N}}x,x\rangle\\
&\quad=(1+2\lambda)\sum_{i\in\mathbb{J}^c}\alpha_i^2\|\pi_{\mathcal{N}_i}(x)\|^2-(1+\lambda^2)\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathcal{N}_i}(x)\|^2
\end{aligned}$$

for each $x\in\mathscr{H}$, telling us that

$$\begin{aligned}
&\sum_{i\in\mathbb{J}}\alpha_i^2\|\pi_{\mathcal{N}_i}(x)\|^2+(1+2\lambda)\sum_{i\in\mathbb{J}^c}\alpha_i^2\|\pi_{\mathcal{N}_i}(x)\|^2 \qquad (14)\\
&\qquad\leqslant\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathcal{N}_i}(S_{\mathcal{N}}^{-1}S_{\mathcal{N}}^{\mathbb{J}^c}x)\|^2+(1+\lambda^2)\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathcal{N}_i}(x)\|^2.
\end{aligned}$$

Now by Lemma 1 (replacing U, V and x, respectively, by $S_{\mathcal{N}}^{-\frac{1}{2}}S_{\mathcal{N}}^{\mathbb{J}^c}S_{\mathcal{N}}^{-\frac{1}{2}}$, $S_{\mathcal{N}}^{-\frac{1}{2}}S_{\mathcal{N}}^{\mathbb{J}}S_{\mathcal{N}}^{-\frac{1}{2}}$ and $S_{\mathcal{N}}^{\frac{1}{2}}x$) we have

$$\begin{aligned}
\langle S_{\mathcal{N}}^{-1}S_{\mathcal{N}}^{\mathbb{J}}x,S_{\mathcal{N}}^{\mathbb{J}}x\rangle&=\langle S_{\mathcal{N}}^{-\frac{1}{2}}S_{\mathcal{N}}^{\mathbb{J}}S_{\mathcal{N}}^{-\frac{1}{2}}S_{\mathcal{N}}^{\frac{1}{2}}x,S_{\mathcal{N}}^{-\frac{1}{2}}S_{\mathcal{N}}^{\mathbb{J}}S_{\mathcal{N}}^{-\frac{1}{2}}S_{\mathcal{N}}^{\frac{1}{2}}x\rangle\\
&\geqslant((2\lambda-\lambda^2)-(2\lambda-1))\langle S_{\mathcal{N}}^{\frac{1}{2}}x,S_{\mathcal{N}}^{\frac{1}{2}}x\rangle\\
&\qquad-(1-\lambda)(\langle S_{\mathcal{N}}^{-\frac{1}{2}}S_{\mathcal{N}}^{\mathbb{J}^c}S_{\mathcal{N}}^{-\frac{1}{2}}S_{\mathcal{N}}^{\frac{1}{2}}x,S_{\mathcal{N}}^{\frac{1}{2}}x\rangle+\langle S_{\mathcal{N}}^{\frac{1}{2}}x,S_{\mathcal{N}}^{-\frac{1}{2}}S_{\mathcal{N}}^{\mathbb{J}^c}S_{\mathcal{N}}^{-\frac{1}{2}}S_{\mathcal{N}}^{\frac{1}{2}}x\rangle)\\
&=(1-\lambda^2)\langle S_{\mathcal{N}}x,x\rangle-2(1-\lambda)\langle S_{\mathcal{N}}^{\mathbb{J}^c}x,x\rangle.
\end{aligned}$$

Noting also that $S_{\mathcal{N}}^{\mathbb{J}}S_{\mathcal{N}}^{-1}S_{\mathcal{N}}^{\mathbb{J}}\leqslant S_{\mathcal{N}}^{\mathbb{J}}$ and $S_{\mathcal{N}}^{\mathbb{J}^c}S_{\mathcal{N}}^{-1}S_{\mathcal{N}}^{\mathbb{J}^c}\leqslant S_{\mathcal{N}}^{\mathbb{J}^c}$, we obtain

$$\begin{aligned}
&\langle S_{\mathcal{N}}^{-1}S_{\mathcal{N}}^{\mathbb{J}^c}x,S_{\mathcal{N}}^{\mathbb{J}^c}x\rangle-\langle S_{\mathcal{N}}^{\mathbb{J}}x,x\rangle\\
&\qquad\leqslant\langle S_{\mathcal{N}}^{\mathbb{J}^c}x,x\rangle-\langle S_{\mathcal{N}}^{-1}S_{\mathcal{N}}^{\mathbb{J}}x,S_{\mathcal{N}}^{\mathbb{J}}x\rangle\\
&\qquad\leqslant\langle S_{\mathcal{N}}^{\mathbb{J}^c}x,x\rangle-(1-\lambda^2)\langle S_{\mathcal{N}}x,x\rangle+2(1-\lambda)\langle S_{\mathcal{N}}^{\mathbb{J}^c}x,x\rangle\\
&\qquad=(1+2\lambda)\langle S_{\mathcal{N}}^{\mathbb{J}^c}x,x\rangle-(4\lambda-2)\langle S_{\mathcal{N}}^{\mathbb{J}^c}x,x\rangle-(1-\lambda^2)\langle S_{\mathcal{N}}x,x\rangle\\
&\qquad\leqslant(1+2\lambda)\langle S_{\mathcal{N}}^{\mathbb{J}^c}x,x\rangle-(4\lambda-2)\langle S_{\mathcal{N}}^{-1}S_{\mathcal{N}}^{\mathbb{J}^c}x,S_{\mathcal{N}}^{\mathbb{J}^c}x\rangle-(1-\lambda^2)\langle S_{\mathcal{N}}x,x\rangle
\end{aligned}$$

for any $x\in\mathscr{H}$ and any $\lambda\in[\frac{1}{2},+\infty)$. Hence

$$\langle S_{\mathcal{N}}^{\mathbb{J}}x,x\rangle+(1+2\lambda)\langle S_{\mathcal{N}}^{\mathbb{J}^c}x,x\rangle\geqslant(4\lambda-1)\langle S_{\mathcal{N}}^{-1}S_{\mathcal{N}}^{\mathbb{J}^c}x,S_{\mathcal{N}}^{\mathbb{J}^c}x\rangle+(1-\lambda^2)\langle S_{\mathcal{N}}x,x\rangle,$$

giving that

$$\sum_{i\in\mathbb{J}}\alpha_i^2\|\pi_{\mathcal{N}_i}(x)\|^2+(1+2\lambda)\sum_{i\in\mathbb{J}^c}\alpha_i^2\|\pi_{\mathcal{N}_i}(x)\|^2 \geqslant (4\lambda-1)\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathcal{N}_i}(S_{\mathcal{N}}^{-1}S_{\mathcal{N}}^{\mathbb{J}^c}x)\|^2+(1-\lambda^2)\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathcal{N}_i}(x)\|^2.$$

This along with (14) leads to (12). $\square$

It has been presented in the proof of Theorem 2 that

$$\langle S_{\mathcal{N}}^{-1}S_{\mathcal{N}}^{\mathbb{J}^c}x,S_{\mathcal{N}}^{\mathbb{J}^c}x\rangle-\langle S_{\mathcal{N}}^{\mathbb{J}}x,x\rangle \leqslant \langle S_{\mathcal{N}}^{\mathbb{J}^c}x,x\rangle-(1-\lambda^2)\langle S_{\mathcal{N}}x,x\rangle+2(1-\lambda)\langle S_{\mathcal{N}}^{\mathbb{J}^c}x,x\rangle = (3-2\lambda)\langle S_{\mathcal{N}}^{\mathbb{J}^c}x,x\rangle-(1-\lambda^2)\langle S_{\mathcal{N}}x,x\rangle$$

for each $\lambda\in\mathbb{R}$, for any $\mathbb{J}\subset\mathbb{I}$ and any $x\in\mathcal{H}$.
That is,

$$\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathcal{N}_i}(S_{\mathcal{N}}^{-1}S_{\mathcal{N}}^{\mathbb{J}^c}x)\|^2-\sum_{i\in\mathbb{J}}\alpha_i^2\|\pi_{\mathcal{N}_i}(x)\|^2 \leqslant (3-2\lambda)\sum_{i\in\mathbb{J}^c}\alpha_i^2\|\pi_{\mathcal{N}_i}(x)\|^2+(\lambda^2-1)\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathcal{N}_i}(x)\|^2.$$

From this fact, and taking into account (13), we immediately obtain our next result. We can, actually, prove it directly following the idea of [18, Theorem 6].

THEOREM 3. *Let $\mathcal{N}=\{(\mathcal{N}_i,\alpha_i)\}_{i\in\mathbb{I}}$ be a fusion frame for $\mathcal{H}$ with the fusion frame operator $S_{\mathcal{N}}$, and $\mathcal{N}'=\{(S_{\mathcal{N}}^{-1}\mathcal{N}_i,\alpha_i)\}_{i\in\mathbb{I}}$ be the dual fusion frame of $\mathcal{N}$. Then for every $\lambda\in\mathbb{R}$, for any $\mathbb{J}\subset\mathbb{I}$ and any $x\in\mathcal{H}$, we have*

$$(1+2\lambda)\sum_{i\in\mathbb{J}^c}\alpha_i^2\|\pi_{\mathcal{N}_i}(x)\|^2-(1+\lambda^2)\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathcal{N}_i}(x)\|^2 \leqslant \sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathcal{N}_i}(S_{\mathcal{N}}^{-1}S_{\mathcal{N}}^{\mathbb{J}^c}x)\|^2-\sum_{i\in\mathbb{J}}\alpha_i^2\|\pi_{\mathcal{N}_i}(x)\|^2 \leqslant (3-2\lambda)\sum_{i\in\mathbb{J}^c}\alpha_i^2\|\pi_{\mathcal{N}_i}(x)\|^2+(\lambda^2-1)\sum_{i\in\mathbb{I}}\alpha_i^2\|\pi_{\mathcal{N}_i}(x)\|^2.$$

Let $\mathcal{N}=\{(\mathcal{N}_i,\alpha_i)\}_{i\in\mathbb{I}}$ be a fusion frame for $\mathcal{H}$, and $\mathcal{P}=\{(\mathcal{P}_i,\beta_i)\}_{i\in\mathbb{I}}$ be an alternate dual fusion frame of $\mathcal{N}$. Then for any $\mathbb{J}\subset\mathbb{I}$, we can always define two bounded linear operators $W^{\mathbb{J}},W^{\mathbb{J}^c}:\mathcal{H}\to\mathcal{H}$ induced by $\mathcal{N}$ and $\mathcal{P}$ as follows

$$W^{\mathbb{J}}x=\sum_{i\in\mathbb{J}}\alpha_i\beta_i\pi_{\mathcal{P}_i}S_{\mathcal{N}}^{-1}\pi_{\mathcal{N}_i}(x),\quad W^{\mathbb{J}^c}x=\sum_{i\in\mathbb{J}^c}\alpha_i\beta_i\pi_{\mathcal{P}_i}S_{\mathcal{N}}^{-1}\pi_{\mathcal{N}_i}(x),\quad \forall x\in\mathcal{H}. \tag{15}$$

In what follows we will establish several bilateral inequalities for fusion frames by means of $W^{\mathbb{J}}$ and $W^{\mathbb{J}^c}$ defined by (15). The proof of the following result draws on the idea of [18, Theorem 4], [21, Theorem 3.7] and [23, Theorem 2.8].

THEOREM 4. *Suppose that $\mathscr{N}=\{(\mathscr{N}_i,\alpha_i)\}_{i\in\mathbb{I}}$ is a fusion frame for $\mathscr{H}$ and that $\mathscr{P}=\{(\mathscr{P}_i,\beta_i)\}_{i\in\mathbb{I}}$ is an alternate dual fusion frame of $\mathscr{N}$. Then for each $\lambda\in[0,1]$, for any $\mathbb{J}\subset\mathbb{I}$ and any $x\in\mathscr{H}$, we have*

$$\begin{aligned}
(\lambda-\lambda^2)&\Big\|\sum_{i\in\mathbb{I}}\alpha_i\beta_i\pi_{\mathscr{P}_i}S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x)\Big\|^2-\lambda\,\mathrm{Re}\sum_{i\in\mathbb{J}^c}\alpha_i\beta_i\langle S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x),\pi_{\mathscr{P}_i}(x)\rangle\\
&\leqslant\Big\|\sum_{i\in\mathbb{J}}\alpha_i\beta_i\pi_{\mathscr{P}_i}S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x)\Big\|^2-\lambda\,\mathrm{Re}\sum_{i\in\mathbb{J}}\alpha_i\beta_i\langle S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x),\pi_{\mathscr{P}_i}(x)\rangle\\
&\leqslant\frac{\lambda(\|W^{\mathbb{J}}-W^{\mathbb{J}^c}\|^2-1)+4(1-\lambda)\|W^{\mathbb{J}}\|^2}{4}\|x\|^2.
\end{aligned}$$

Proof. It is obvious that $W^{\mathbb{J}}+W^{\mathbb{J}^c}=\mathrm{Id}_{\mathscr{H}}$. Thus

$$\begin{aligned}
&\Big\|\sum_{i\in\mathbb{J}}\alpha_i\beta_i\pi_{\mathscr{P}_i}S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x)\Big\|^2-\lambda\,\mathrm{Re}\sum_{i\in\mathbb{J}}\alpha_i\beta_i\langle S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x),\pi_{\mathscr{P}_i}(x)\rangle\\
&\quad=\langle W^{\mathbb{J}}x,W^{\mathbb{J}}x\rangle-\frac{\lambda}{2}(\langle W^{\mathbb{J}}x,x\rangle+\langle x,W^{\mathbb{J}}x\rangle)\\
&\quad=\langle W^{\mathbb{J}}x,W^{\mathbb{J}}x\rangle-\frac{\lambda}{2}(\langle W^{\mathbb{J}}x,(W^{\mathbb{J}}+W^{\mathbb{J}^c})x\rangle+\langle (W^{\mathbb{J}}+W^{\mathbb{J}^c})x,W^{\mathbb{J}}x\rangle)\\
&\quad=\langle W^{\mathbb{J}}x,W^{\mathbb{J}}x\rangle-\frac{\lambda}{2}\langle W^{\mathbb{J}}x,W^{\mathbb{J}}x\rangle-\frac{\lambda}{2}\langle W^{\mathbb{J}}x,W^{\mathbb{J}^c}x\rangle\\
&\qquad-\frac{\lambda}{2}\langle W^{\mathbb{J}}x,W^{\mathbb{J}}x\rangle-\frac{\lambda}{2}\langle W^{\mathbb{J}^c}x,W^{\mathbb{J}}x\rangle\\
&\quad=(1-\lambda)\langle W^{\mathbb{J}}x,W^{\mathbb{J}}x\rangle-\frac{\lambda}{2}\langle W^{\mathbb{J}}x,W^{\mathbb{J}^c}x\rangle-\frac{\lambda}{2}\langle W^{\mathbb{J}^c}x,W^{\mathbb{J}}x\rangle\\
&\quad=-\frac{\lambda}{4}\langle x,x\rangle+(1-\lambda)\langle W^{\mathbb{J}}x,W^{\mathbb{J}}x\rangle+\frac{\lambda}{4}\langle (W^{\mathbb{J}}+W^{\mathbb{J}^c})x,(W^{\mathbb{J}}+W^{\mathbb{J}^c})x\rangle\\
&\qquad-\frac{\lambda}{2}\langle W^{\mathbb{J}}x,W^{\mathbb{J}^c}x\rangle-\frac{\lambda}{2}\langle W^{\mathbb{J}^c}x,W^{\mathbb{J}}x\rangle\\
&\quad=-\frac{\lambda}{4}\|x\|^2+(1-\lambda)\langle W^{\mathbb{J}}x,W^{\mathbb{J}}x\rangle+\frac{\lambda}{4}\langle (W^{\mathbb{J}}-W^{\mathbb{J}^c})x,(W^{\mathbb{J}}-W^{\mathbb{J}^c})x\rangle\\
&\quad\leqslant-\frac{\lambda}{4}\|x\|^2+(1-\lambda)\|W^{\mathbb{J}}\|^2\|x\|^2+\frac{\lambda}{4}\|W^{\mathbb{J}}-W^{\mathbb{J}^c}\|^2\|x\|^2\\
&\quad=\frac{\lambda(\|W^{\mathbb{J}}-W^{\mathbb{J}^c}\|^2-1)+4(1-\lambda)\|W^{\mathbb{J}}\|^2}{4}\|x\|^2
\end{aligned}$$

for each $x\in\mathscr{H}$ and each $\lambda\in[0,1]$. For the opposite inequality, we obtain

$$\|W^{\mathbb{J}}x\|^2\geqslant(2\lambda-\lambda^2)\|x\|^2-2\lambda\,\mathrm{Re}\langle W^{\mathbb{J}^c}x,x\rangle,$$

by Lemma 1. Therefore

$$\begin{aligned}
&\Big\|\sum_{i\in\mathbb{J}}\alpha_i\beta_i\pi_{\mathscr{P}_i}S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x)\Big\|^2-\lambda\mathrm{Re}\sum_{i\in\mathbb{J}}\alpha_i\beta_i\langle S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x),\pi_{\mathscr{P}_i}(x)\rangle\\
&\quad=\|W^{\mathbb{J}}x\|^2-\lambda\mathrm{Re}\langle W^{\mathbb{J}}x,x\rangle\\
&\quad\geqslant(2\lambda-\lambda^2)\|x\|^2-2\lambda\mathrm{Re}\langle W^{\mathbb{J}^c}x,x\rangle-\lambda\mathrm{Re}\langle W^{\mathbb{J}}x,x\rangle\\
&\quad=(\lambda-\lambda^2)\|x\|^2+\frac{\lambda}{2}\langle(W^{\mathbb{J}}+W^{\mathbb{J}^c})x,x\rangle+\frac{\lambda}{2}\langle x,(W^{\mathbb{J}}+W^{\mathbb{J}^c})x\rangle\\
&\qquad\quad-2\lambda\mathrm{Re}\langle W^{\mathbb{J}^c}x,x\rangle-\frac{\lambda}{2}\langle W^{\mathbb{J}}x,x\rangle-\frac{\lambda}{2}\langle x,W^{\mathbb{J}}x\rangle\\
&\quad=(\lambda-\lambda^2)\|x\|^2-2\lambda\mathrm{Re}\langle W^{\mathbb{J}^c}x,x\rangle+\frac{\lambda}{2}\langle W^{\mathbb{J}^c}x,x\rangle+\frac{\lambda}{2}\langle x,W^{\mathbb{J}^c}x\rangle\\
&\quad=(\lambda-\lambda^2)\|x\|^2-2\lambda\mathrm{Re}\langle W^{\mathbb{J}^c}x,x\rangle+\lambda\mathrm{Re}\langle W^{\mathbb{J}^c}x,x\rangle\\
&\quad=(\lambda-\lambda^2)\|x\|^2-\lambda\mathrm{Re}\langle W^{\mathbb{J}^c}x,x\rangle\\
&\quad=(\lambda-\lambda^2)\Big\|\sum_{i\in\mathbb{I}}\alpha_i\beta_i\pi_{\mathscr{P}_i}S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x)\Big\|^2-\lambda\mathrm{Re}\sum_{i\in\mathbb{J}^c}\alpha_i\beta_i\langle S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x),\pi_{\mathscr{P}_i}(x)\rangle,
\end{aligned}$$

and we are done. □

The following theorem is an immediate consequence of Lemma 1, if we take $W^{\mathbb{J}}$ and $W^{\mathbb{J}^c}$ instead of U and V, respectively. It is worth pointing out that the original idea of the proof of the inequality on the left-hand side is extracted from [21, Proposition 3.6].

THEOREM 5. *Suppose that $\mathscr{N}=\{(\mathscr{N}_i,\alpha_i)\}_{i\in\mathbb{I}}$ is a fusion frame for $\mathscr{H}$ and that $\mathscr{P}=\{(\mathscr{P}_i,\beta_i)\}_{i\in\mathbb{I}}$ is an alternate dual fusion frame of $\mathscr{N}$. Then for each $\lambda\in[0,\frac{1}{2}]$, for any $\mathbb{J}\subset\mathbb{I}$ and any $x\in\mathscr{H}$, we have*

$$\begin{aligned}
&(2\lambda-\lambda^2)\Big\|\sum_{i\in\mathbb{I}}\alpha_i\beta_i\pi_{\mathscr{P}_i}S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x)\Big\|^2\\
&\qquad\leqslant\Big\|\sum_{i\in\mathbb{J}}\alpha_i\beta_i\pi_{\mathscr{P}_i}S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x)\Big\|^2+2\lambda\mathrm{Re}\sum_{i\in\mathbb{J}^c}\alpha_i\beta_i\langle S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x),\pi_{\mathscr{P}_i}(x)\rangle\\
&\qquad\leqslant\frac{3\lambda+2(1-2\lambda)\|W^{\mathbb{J}}\|^2+\lambda\|W^{\mathbb{J}}-W^{\mathbb{J}^c}\|^2}{2}\|x\|^2.
\end{aligned}$$

REMARK 1. We can obtain [23, Theorem 2.8], when taking $\lambda=\frac{1}{2}$ in Theorem 5.

We conclude the paper with an alternative inequality involving the operators $W^{\mathbb{J}}$ and $W^{\mathbb{J}^c}$, and the idea of the proof derives from a careful consideration of [18, Theorem 4], [21, Theorem 3.7] and [23, Theorem 2.8].

THEOREM 6. *Suppose that $\mathscr{N}=\{(\mathscr{N}_i,\alpha_i)\}_{i\in\mathbb{I}}$ is a fusion frame for $\mathscr{H}$ and that $\mathscr{P}=\{(\mathscr{P}_i,\beta_i)\}_{i\in\mathbb{I}}$ is an alternate dual fusion frame of $\mathscr{N}$. Then for each $\lambda\geqslant 0$, for any $\mathbb{J}\subset\mathbb{I}$ and any $x\in\mathscr{H}$, we have*

$$\begin{aligned}
&4\lambda^2\mathrm{Re}\sum_{i\in\mathbb{J}}\alpha_i\beta_i\langle S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x),\pi_{\mathscr{P}_i}(x)\rangle-\lambda^2(1+2\lambda)\left\|\sum_{i\in\mathbb{I}}\alpha_i\beta_i\pi_{\mathscr{P}_i}S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x)\right\|^2\\
&\quad\leqslant\left\|\sum_{i\in\mathbb{J}}\alpha_i\beta_i\pi_{\mathscr{P}_i}S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x)\right\|^2-2\lambda\mathrm{Re}\sum_{i\in\mathbb{J}^c}\alpha_i\beta_i\langle S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x),\pi_{\mathscr{P}_i}(x)\rangle\\
&\qquad+2\lambda\left\|\sum_{i\in\mathbb{J}^c}\alpha_i\beta_i\pi_{\mathscr{P}_i}S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x)\right\|^2\\
&\quad\leqslant\frac{-\lambda+2\|W^{\mathbb{J}}\|^2+\lambda\|W^{\mathbb{J}}-W^{\mathbb{J}^c}\|^2}{2}\|x\|^2.
\end{aligned}$$

Proof. The proof of the right-hand inequality is similar to Theorem 4. For the left-hand inequality, we have, by Lemma 1, that

$$\begin{aligned}
&\|W^{\mathbb{J}^c}x\|^2-2\mathrm{Re}\langle W^{\mathbb{J}^c}x,x\rangle\\
&\quad\geqslant(1-\lambda^2)\|x\|^2-(2-2\lambda)\mathrm{Re}\langle W^{\mathbb{J}}x,x\rangle-2\mathrm{Re}\langle W^{\mathbb{J}^c}x,x\rangle\\
&\quad=(1-\lambda^2)\|x\|^2+2\lambda\mathrm{Re}\langle W^{\mathbb{J}}x,x\rangle-2(\mathrm{Re}\langle W^{\mathbb{J}}x,x\rangle+\mathrm{Re}\langle W^{\mathbb{J}^c}x,x\rangle)\\
&\quad=(1-\lambda^2)\|x\|^2-2\|x\|^2+2\lambda\mathrm{Re}\langle W^{\mathbb{J}}x,x\rangle\\
&\quad=2\lambda\mathrm{Re}\langle W^{\mathbb{J}}x,x\rangle-(1+\lambda^2)\|x\|^2
\end{aligned}$$

for each $x\in\mathscr{H}$ and each $\lambda\geqslant 0$.

Again by Lemma 1 we get

$$\begin{aligned}
&\left\|\sum_{i\in\mathbb{J}}\alpha_i\beta_i\pi_{\mathscr{P}_i}S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x)\right\|^2-2\lambda\mathrm{Re}\sum_{i\in\mathbb{J}^c}\alpha_i\beta_i\langle S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x),\pi_{\mathscr{P}_i}(x)\rangle\\
&\qquad+2\lambda\left\|\sum_{i\in\mathbb{J}^c}\alpha_i\beta_i\pi_{\mathscr{P}_i}S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x)\right\|^2\\
&\quad=\|W^{\mathbb{J}}x\|^2-2\lambda\mathrm{Re}\langle W^{\mathbb{J}^c}x,x\rangle+2\lambda\|W^{\mathbb{J}^c}x\|^2\\
&\quad\geqslant(2\lambda-\lambda^2)\|x\|^2-2\lambda\mathrm{Re}\langle W^{\mathbb{J}^c}x,x\rangle-2\lambda\mathrm{Re}\langle W^{\mathbb{J}^c}x,x\rangle+2\lambda\|W^{\mathbb{J}^c}x\|^2\\
&\quad=(2\lambda-\lambda^2)\|x\|^2+2\lambda(\|W^{\mathbb{J}^c}x\|^2-2\mathrm{Re}\langle W^{\mathbb{J}^c}x,x\rangle)\\
&\quad\geqslant(2\lambda-\lambda^2)\|x\|^2+2\lambda(2\lambda\mathrm{Re}\langle W^{\mathbb{J}}x,x\rangle-(1+\lambda^2)\|x\|^2)\\
&\quad=4\lambda^2\mathrm{Re}\langle W^{\mathbb{J}}x,x\rangle-\lambda^2(1+2\lambda)\|x\|^2\\
&\quad=4\lambda^2\mathrm{Re}\sum_{i\in\mathbb{J}}\alpha_i\beta_i\langle S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x),\pi_{\mathscr{P}_i}(x)\rangle-\lambda^2(1+2\lambda)\left\|\sum_{i\in\mathbb{I}}\alpha_i\beta_i\pi_{\mathscr{P}_i}S_{\mathscr{N}}^{-1}\pi_{\mathscr{N}_i}(x)\right\|^2,
\end{aligned}$$

and the proof is completed. □

REFERENCES

[1] M. R. ABDOLLAHPOUR AND M. H. FAROUGHI, *Continuous g-frames in Hilbert Spaces*, Southeast Asian Bull. Math., **32**, 1 (2008), 1–19.

[2] R. BALAN, P. G. CASAZZA AND D. EDIDIN, *On signal reconstruction without phase*, Appl. Comput. Harmon. Anal., **20**, 3 (2006), 345–356.

[3] R. BALAN, P. G. CASAZZA, D. EDIDIN AND G. KUTYNIOK, *A new identity for Parseval frames*, Proc. Amer. Math. Soc., **135**, 4 (2007), 1007–1015.

[4] P. BALAZS, N. HOLIGHAUS, T. NECCIARI AND D. T. STOEVA, *Frame theory for signal processing in psychoacoustics*, in: R. Balan, J. Benedetto, W. Czaja, M. Dellatorre and K. Okoudjou (Eds.), Excursions in Harmonic Analysis, vol. 5, Applied and Numerical Harmonic Analysis, Birkhäuser, Cham, 2017, pp. 225–268.

[5] B. G. BODMANN, *Optimal linear transmission by loss-insensitive packet encoding*, Appl. Comput. Harmon. Anal., **22**, 3 (2007), 274–285.

[6] P. G. CASAZZA AND G. KUTYNIOK, *Frames of subspaces*, in: Wavelets, Frames and Operator Theory, in: Contemp. Math., vol. 345, Amer. Math. Soc., Providence, RI, 2004, pp. 87–113.

[7] A. CHEBIRA, M. FICKUS AND D. G. MIXON, *Filter bank fusion frames*, IEEE Trans. Signal Process., **59**, 3 (2011), 953–963.

[8] F. DAI, *Characterizations of function spaces on the sphere using frames*, Trans. Amer. Math. Soc., **359**, 2 (2006), 567–589.

[9] I. DAUBECHIES, A. GROSSMANN AND Y. MEYER, *Painless nonorthogonal expansions*, J. Math. Phys., **27**, 5 (1986), 1271–1283.

[10] R. J. DUFFIN AND A. C. SCHAEFFER, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., **72**, 2 (1952), 341–366.

[11] M. H. FAROUGHI AND R. AHMADI, *Some properties of C-fusion frames*, Turk. J. Math., **34**, 3 (2010), 393–415.

[12] M. FORNASIER, *Decompositions of Hilbert spaces: local construction of global frames*, Proc. Int. Conf. on Constructive Function Theory, Varna, 2002, B. Bojanov Ed., DARBA, Sofia, 2003, pp. 275–281.

[13] Y. L. FU AND W. ZHANG, *Some new inequalities for dual continuous g-frames*, Mathematics, **7**, 8 (2019), Article 662, 15 pp.

[14] D. GABOR, *Theory of communication*, J. Inst. Elect. Engr., **93**, 26 (1946), 429–457.

[15] P. GĂVRUŢA, *On some identities and inequalities for frames in Hilbert spaces*, J. Math. Anal. Appl., **321**, 1 (2006), 469–478.

[16] P. GĂVRUŢA, *On the duality of fusion frames*, J. Math. Anal. Appl., **333**, 2 (2007), 871–879.

[17] Q. P. GUO, J. S. LENG AND H. B. LI, *Some equalities and inequalities for fusion frames*, SpringerPlus, **5**, 1 (2016), Article 121, 10 pp.

[18] D. W. LI AND J. S. LENG, *On some new inequalities for fusion frames in Hilbert spaces*, Math. Inequal. Appl., **20**, 3 (2017), 889–900.

[19] D. W. LI AND J. S. LENG, *On some new inequalities for continuous fusion frames in Hilbert spaces*, Mediterr. J. Math., **15**, 4 (2018), Article 173, 15 pp.

[20] C. POON, *A consistent and stable approach to generalized sampling*, J. Fourier Anal. Appl., **20**, 5 (2014), 985–1019.

[21] A. PORIA, *Some identities and inequalities for Hilbert-Schmidt frames*, Mediterr. J. Math., **14**, 2 (2017), Article 59, 14 pp.

[22] W. SUN, *Asymptotic properties of Gabor frame operators as sampling density tends to infinity*, J. Funct. Anal., **258**, 3 (2010), 913–932.

[23] Z. Q. XIANG, *New types of inequalities for fusion frames*, J. Math. Inequal., **11**, 1 (2017), 291–299.

[24] Z. Q. XIANG, C. X. LIN AND X. C. XIAO, *Frame inequalities in Hilbert spaces: two-sided inequalities with new structures*, J. Math. Inequal., **18**, 2 (2024), 477–488.

[25] W. ZHANG AND Y. Z. LI, *New inequalities and erasures for continuous g-frames*, Math. Rep., **20**, 3 (2018), 263–278.

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Zhong-Qi Xiang
School of Mathematics and Computer
Xinyu University
Xinyu, Jiangxi 338004, P.R. China
e-mail: lxsy20110927@163.com

Chun-Xia Lin
Academic Affairs Office
Xinyu University
Xinyu, Jiangxi 338004, P.R. China
e-mail: 2449886300@qq.com

Xiang-Chun Xiao
School of Mathematics and Statistics
Xiamen University of Technology
Xiamen, Fujian 361024, P.R. China
e-mail: xxc570@163.com