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MIXED TYPE WEIGHTED INTEGRAL INEQUALITIES FOR THE HARDY–STEKLOV INTEGRAL OPERATORS

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Abstract. We characterize the weights ω,ρ,ϕ and ψ for which the integral operator of Hardy-Steklov type, $\mathscr{I}f(t)=h(t)\int_{\alpha(t)}^{\beta(t)}K(t,z)f(z)w(z)dz$ satisfies weak type mixed modular inequalities of the form

$$\mathscr{U}^{-1}\Big(\int_{\{\mathscr{I}f>\gamma\}}\mathscr{U}(\gamma\omega)\rho\Big)\leqslant\mathscr{V}^{-1}\Big(\int\mathscr{V}(Cf\phi)\psi\Big),$$

where the functions α and β are increasing and the kernel K satisfies certain monotone conditions. We also prove the following mixed integral inequalities of the extra-weak type under appropriate conditions on the weights ω,ϕ and ψ.

$$\omega\Big(\{\mathscr{I}f>\gamma\}\Big)\leqslant\mathscr{U}\circ\mathscr{V}^{-1}\Big(\int\mathscr{V}\Big(\frac{Cf\phi}{\gamma}\Big)\psi\Big).$$

Further, we discuss the above two integral inequalities for the adjoint of the integral operator of Hardy-Steklov type.

1. Introduction

We consider the Hardy-Steklov integral operator $\mathscr{I}$, for a non-negative measurable function f on $-\infty\leqslant a<b\leqslant\infty$, defined by

$$\mathscr{I}f(t)=h(t)\int_{\alpha(t)}^{\beta(t)}K(t,z)f(z)w(z)dz, \tag{1}$$

where $\alpha,\beta:(a,b)\to\mathbb{R}$ are continuous and increasing functions satisfying $\alpha(z)\leqslant\beta(z)$ for each $z\in(a,b)$, h and w are positive measurable functions, and the kernel $K(t,z)$ defined on $\{(t,z):\alpha(t)\leqslant z\leqslant\beta(t)\}$ satisfies the following conditions.

(a) $K(t,z)\geqslant 0$.

(b) $K(t,z)$ is non-decreasing in t and non-increasing in z.

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(c) There exists a constant $M \geqslant 1$ independent of t, z and τ such that

$$K(t,z) \leqslant M\Big[K(t,\beta(\tau)) + K(\tau,z)\Big], \tag{2}$$

where $\tau \leqslant t$ and $\alpha(t) \leqslant z \leqslant \beta(\tau)$.

For $K \equiv 1$, the operator (1) is reduced to the Hardy-Steklov operator defined by

$$\mathscr{S} f(t) = h(t)\int_{\alpha(t)}^{\beta(t)} f(z)w(z)dz. \tag{3}$$

From (3), it is observed that the Hardy-Steklov operator extends the notion of the Hardy operator to dynamic limits. We refer to [11, 15] and the references therein for a detailed investigation on Hardy operators. Riemann-Liouville integral operators of the form $\int_{\alpha(t)}^{\beta(t)}(t-z)^{\mu} f(z)dz$, $\mu > 0$; Steklov operators $\int_{t-\gamma}^{t+\gamma} f(z)dz$, $\gamma > 0$ are some particular cases of the Hardy-Steklov operators. Many fruitful applications of this operator mainly include the study of the abstract Cauchy problem with delay and analyzing the stock market [11].

Weighted weak and strong type estimates for the Hardy-Steklov operator and its integral version have been studied substantially by several authors [4, 5, 7, 8, 9, 19]. In [4], characterization of weights ρ and ψ has been established such that

$$\Big(\int_{\{t\in(a,b):\mathscr{I} f(t)>\gamma\}} \gamma^q \rho(y)dy\Big)^{\frac{1}{q}} \leqslant C\Big(\int_{\alpha(a)}^{\beta(b)} f(y)^p \psi(y)dy\Big)^{\frac{1}{p}} \tag{4}$$

holds for a suitable constant $C > 0$ and in the range $0 < q < p$, $1 < p < \infty$ with $w = 1$. In the case $p \leqslant q$, Gogatishvili and Lang [8] obtained the weak and strong type (p,q) estimates in Banach function spaces for the operator (1). Stepanov and Ushakova [19] proved $L_p - L_q$ boundedness of (1) considering h and w as weight functions.

Through this article, we plan to address the inequality (4) in the Orlicz space setting for the Hardy-Steklov integral operator and its adjoint $\tilde{\mathscr{I}}$ defined by

$$\tilde{\mathscr{I}} f(t) = w(t)\int_{\beta^{-1}(t)}^{\alpha^{-1}(t)} K(z,t)f(z)h(z)dz. \tag{5}$$

Among the various equivalent generalization of the estimate (4) in to the Orlicz space setting, we will consider the following form.

$$\mathscr{U}^{-1}\Bigg(\int\limits_{\{t\in(a,b):\mathscr{I} f(t)>\gamma\}} \mathscr{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Bigg) \leqslant \mathscr{V}^{-1}\Bigg(\int_{\alpha(a)}^{\beta(b)} \mathscr{V}\Big(Cf(y)\phi(y)\Big)\psi(y)dy\Bigg), \tag{6}$$

where $\gamma > 0$; ω, ρ, ϕ and ψ are weights and the conditions on $\mathscr{U}$ and $\mathscr{V}$ will be set down later.

The estimate (6) for the Hardy operators has been addressed in [6, 12, 14, 16]. Ortega Salvador and Ramírez Torreblanca [18] have established the inequality (6) with

$\omega = 1 = \phi$ for the Hardy-Steklov operators. The second objective of our article is to prove the following weaker version of (6), that is

$$\omega\Big(\{t \in (a,b) : \mathscr{I} f(t) > \gamma\}\Big) \leqslant \mathscr{U} \circ \mathscr{V}^{-1}\left(\int_{\alpha(a)}^{\beta(b)} \mathscr{V}\left(\frac{Cf(y)\phi(y)}{\gamma}\right)\psi(y)dy\right). \quad (7)$$

The estimate (7) is known as the extra-weak type mixed integral inequality as it follows from (6) but not contrariwise. It was proved in [1, 2] that the extra-weak type inequality provides exquisite bounds for the strong type integral estimates. Extra-weak type inequalities for Hilbert transform, maximal function and its one-sided version have been discussed in [3, 13, 17].

Before presenting the result, we will briefly discuss some basics associated with N-functions [10]. An N-function $\mathscr{U}$ is continuous and convex on $[0,\infty)$ such that $\mathscr{U}(0) = 0$ and $\frac{\mathscr{U}(t)}{t} \to 0$ (and ∞) when $t \to 0$ (and ∞). It is always possible to write an N-function $\mathscr{U}$ in the integral form as, $\mathscr{U}(t) = \int_0^t u(y)dy$, where u is non-decreasing and right continuous at each point and satisfies $u(0) = 0$, $u(r) > 0$ for $r > 0$ and $u(r) \to \infty$ as $r \to \infty$. The complementary function $\tilde{\mathscr{U}}$ corresponding to a given N-function $\mathscr{U}$ is defined by $\tilde{\mathscr{U}}(t) = \sup_{\tau \geqslant 0}(t\tau - \mathscr{U}(\tau))$ also verifies properties of N-functions. For $t, \tau > 0$, the pair $(\mathscr{U}, \tilde{\mathscr{U}})$ satisfies the following relations [6].

$$t\tau \leqslant \mathscr{U}(t) + \tilde{\mathscr{U}}(\tau). \quad (8)$$

$$\mathscr{U}\left(\frac{\tilde{\mathscr{U}}(t)}{t}\right) \leqslant \tilde{\mathscr{U}}(t). \quad (9)$$

$$\mathscr{U}(t) \leqslant tu(t) \leqslant \mathscr{U}(2t). \quad (10)$$

Now we state the main results of the article.

THEOREM 1. *Let $\tilde{\mathscr{V}}$ be the complementary function corresponding to an N-function $\mathscr{V}$. Suppose that, $\mathscr{V} \circ \mathscr{U}^{-1}$ is countably subadditive, where $\mathscr{U}$ is strictly increasing and positive with $\mathscr{U}(0) = 0$. We assume that h is monotone on $\mathbb{R}$ and let the function $h(\cdot)K(\cdot,y)$ satisfies that*

$$\inf_{x\in\Omega} h(x)K(x,y) = \inf_{x\in(\inf\Omega, \sup\Omega)} h(x)K(x,y)$$

for all bounded set Ω and all y. Then the following assertions are equivalent.

(i) There exists a positive constant C such that

$$\mathscr{U}^{-1}\left(\int_{\{t\in(a,b):\mathscr{I} f(t)>\gamma\}} \mathscr{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\right) \leqslant \mathscr{V}^{-1}\left(\int_{\alpha(a)}^{\beta(b)} \mathscr{V}\Big(Cf(y)\phi(y)\Big)\psi(y)dy\right) \quad (11)$$

holds for each $\gamma > 0$ and all $f \geqslant 0$.

(ii) There exists $C>0$ such that

$$\int_{\alpha(\tau)}^{\beta(t)} \tilde{\mathscr{V}}\left[\frac{(\inf_{(t,\tau)} h)K(t,s)w(s)\eta(\gamma;t,\tau)}{C\gamma\phi(s)\psi(s)}\right]\psi(s)ds \leqslant \eta(\gamma;t,\tau) \tag{12}$$

and

$$\int_{\alpha(\tau)}^{\beta(z)} \tilde{\mathscr{V}}\left[\frac{\inf_{(t,\tau)}\big(h(y)K(y,\beta(z))\big)w(s)\eta(\gamma;t,\tau)}{C\gamma\phi(s)\psi(s)}\right]\psi(s)ds \leqslant \eta(\gamma;t,\tau) \tag{13}$$

hold, where $a<z\leqslant t<\tau<b$ with $\alpha(\tau)\leqslant\beta(z)\leqslant\beta(t)$ and

$$\eta(\gamma;t,\tau)=\left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_t^\tau \mathscr{U}\left(\gamma\omega(z)\right)\rho(z)dz\right).$$

For $K\equiv 1$, estimates (12) and (13) are equivalent and reduced to the following form:

$$\int_{\alpha(\tau)}^{\beta(t)} \tilde{\mathscr{V}}\left[\frac{(\inf_{(t,\tau)} h)w(s)\eta(\gamma;t,\tau)}{C\gamma\phi(s)\psi(s)}\right]\psi(s)ds \leqslant \eta(\gamma;t,\tau). \tag{14}$$

Thus (14) characterizes the estimate (11) for the Hardy-Steklov operators of the form $\mathscr{S}f(t)=h(t)\int_{\alpha(t)}^{\beta(t)} f(z)w(z)dz$.

THEOREM 2. *Let $\mathscr{U},\mathscr{V},\tilde{\mathscr{V}}$ and $\mathscr{V}\circ\mathscr{U}^{-1}$ satisfy all the conditions stated in Theorem 1. Suppose that w is monotone on $\mathbb{R}$ and let the function $w(\cdot)K(y,\cdot)$ satisfies that*

$$\inf_{x\in\Omega} w(x)K(y,x)=\inf_{x\in(\inf\Omega,\sup\Omega)} w(x)K(y,x)$$

for all bounded set Ω and all y. Then the following conditions are equivalent.

(i) There exists a positive constant C such that

$$\mathscr{U}^{-1}\left(\int_{\{t\in(a,b):\tilde{\mathscr{I}}f(t)>\gamma\}} \mathscr{U}\left(\gamma\omega(y)\right)\rho(y)dy\right)\leqslant\mathscr{V}^{-1}\left(\int_{\beta^{-1}(a)}^{\alpha^{-1}(b)}\mathscr{V}\left(Cf(y)\phi(y)\right)\psi(y)dy\right) \tag{15}$$

holds for each $\gamma>0$ and all $f\geqslant 0$.

(ii) There exists $C>0$ such that

$$\int_{\beta^{-1}(\tau)}^{\alpha^{-1}(t)} \tilde{\mathscr{V}}\left[\frac{(\inf_{(t,\tau)} w)K(s,\tau)h(s)\eta(\gamma;t,\tau)}{C\gamma\phi(s)\psi(s)}\right]\psi(s)ds \leqslant \eta(\gamma;t,\tau) \tag{16}$$

and

$$\int_{\beta^{-1}(z)}^{\alpha^{-1}(t)} \tilde{\mathscr{V}}\left[\frac{(\inf_{(t,\tau)} w(y)K(\beta^{-1}(z),y))h(s)\eta(\gamma;t,\tau)}{C\gamma\phi(s)\psi(s)}\right]\psi(s)ds \leqslant \eta(\gamma;t,\tau) \quad (17)$$

hold for each $a<t<\tau\leqslant z<b$ *satisfying* $\beta^{-1}(\tau)\leqslant\beta^{-1}(z)\leqslant\alpha^{-1}(t)$, *where*

$$\eta(\gamma;t,\tau)=\left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_t^\tau \mathscr{U}\left(\gamma\omega(z)\right)\rho(z)dz\right).$$

Similarly in the case of adjoint if we consider $K\equiv 1$, then estimates (16) and (17) are equivalent and reduced to the following form,

$$\int_{\beta^{-1}(\tau)}^{\alpha^{-1}(t)} \tilde{\mathscr{V}}\left[\frac{(\inf w)h(s)\eta(\gamma;t,\tau)}{C\gamma\phi(s)\psi(s)}\right]\psi(s)ds \leqslant \eta(\gamma;t,\tau). \quad (18)$$

Thus (18) characterizes the estimate (15) for the adjoint of Hardy-Steklov operators of the form $\tilde{\mathscr{S}}f(t)=w(t)\int_{\beta^{-1}(t)}^{\alpha^{-1}(t)}f(z)h(z)dz$.

We also prove extra-weak type integral inequalities and the results are as follows.

THEOREM 3. *Let* $\mathscr{U},\mathscr{V},\tilde{\mathscr{V}},\mathscr{V}\circ\mathscr{U}^{-1},h$ *and the function* $h(\cdot)K(\cdot,y)$ *satisfy all the conditions stated in Theorem 1. Then the following assertions are equivalent.*

(i) There exists a positive constant C *such that*

$$\omega\left(\left\{t\in(a,b):\mathscr{I}f(t)>\gamma\right\}\right)\leqslant\mathscr{U}\circ\mathscr{V}^{-1}\left(\int_{\alpha(a)}^{\beta(b)}\mathscr{V}\left(\frac{Cf(y)\phi(y)}{\gamma}\right)\psi(y)dy\right) \quad (19)$$

holds for each $\gamma>0$ *and all* $f\geqslant 0$.

(ii) There exists $C>0$ *such that*

$$\int_{\alpha(\tau)}^{\beta(t)} \tilde{\mathscr{V}}\left[\frac{(\inf_{(t,\tau)} h)K(t,s)w(s)\theta(t,\tau)}{C\phi(s)\psi(s)}\right]\psi(s)ds \leqslant \theta(t,\tau) \quad (20)$$

and

$$\int_{\alpha(\tau)}^{\beta(z)} \tilde{\mathscr{V}}\left[\frac{\left(\inf h(y)K(y,\beta(z)\right)w(s)\theta(t,\tau)}{C\phi(s)\psi(s)}\right]\psi(s)ds \leqslant \theta(t,\tau) \quad (21)$$

hold for each $a<z\leqslant t<\tau<b$ *with* $\alpha(\tau)\leqslant\beta(z)\leqslant\beta(t)$, *where*

$$\theta(t,\tau)=\left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_t^\tau \omega(z)dz\right).$$

THEOREM 4. *Let $\mathscr{U}, \mathscr{V}, \tilde{\mathscr{V}}$ and $\mathscr{V}\circ\mathscr{U}^{-1}$ satisfy all the conditions stated in Theorem 1. Let the function w be monotone on $\mathbb{R}$ and let the function $w(\cdot)K(y,\cdot)$ satisfies that*

$$\inf_{x\in\Omega} w(x)K(y,x) = \inf_{x\in(\inf\Omega,\sup\Omega)} w(x)K(y,x)$$

for all bounded set Ω and all y. Then the following assertions are equivalent.

(i) There exists a positive constant C such that

$$\omega\Big(\big\{t\in(a,b) : \tilde{\mathscr{I}}f(t)>\gamma\big\}\Big) \leqslant \mathscr{U}\circ\mathscr{V}^{-1}\bigg(\int_{\beta^{-1}(a)}^{\alpha^{-1}(b)} \mathscr{V}\Big(\frac{Cf(y)\phi(y)}{\gamma}\Big)\psi(y)dy\bigg) \tag{22}$$

holds for each $\gamma>0$ and $f\geqslant 0$.

(ii) There exists $C>0$ such that

$$\int_{\beta^{-1}(\tau)}^{\alpha^{-1}(t)} \tilde{\mathscr{V}}\left[\frac{(\inf w)K(s,\tau)h(s)\theta(t,\tau)}{C\phi(s)\psi(s)}\right]\psi(s)ds \leqslant \theta(t,\tau) \tag{23}$$

and

$$\int_{\beta^{-1}(z)}^{\alpha^{-1}(t)} \tilde{\mathscr{V}}\left[\frac{\big(\inf w(y)K(\beta^{-1}(z),y)\big)h(s)\theta(t,\tau)}{C\phi(s)\psi(s)}\right]\psi(s)ds \leqslant \theta(t,\tau) \tag{24}$$

hold for each $a<t<\tau\leqslant z<b$ with $\beta^{-1}(\tau)\leqslant\beta^{-1}(z)\leqslant\alpha^{-1}(t)$, where

$$\theta(t,\tau) = \Big(\mathscr{V}\circ\mathscr{U}^{-1}\Big)\bigg(\int_t^{\tau}\omega(z)dz\bigg).$$

Similarly, the extra-weak type integral inequalities for the Hardy-Steklov operators and its adjoint follow directly from the above two theorems by considering $K\equiv 1$. We skip the proof of Theorem 2 and Theorem 4 as those can be obtained with some modifications from Theorem 1 and Theorem 3 respectively. Based on the methods developed in [4, 5, 18] we will prove the Theorem 1 and give a sketch of the proof of Theorem 2. Next, we state the following lemma [5], which plays a pivotal role in the remaining sections.

LEMMA 1. *Let $\Gamma=\{z\in(a,b):\alpha(z)<\beta(z)\}$. Then there exists a countable collection of open intervals $\{(a_m,b_m)\}$ such that $\Gamma=\cup_m(a_m,b_m)$ and*

(a) $\big(\alpha(a_k),\beta(b_k)\big)\cap\big(\alpha(a_m),\beta(b_m)\big)=\phi$ for $m\neq k$,

(b) for each m, there exists a sequence of real numbers $\{\xi_k^m\}$ satisfying

(i) $(a_m,b_m)=\cup_k(\xi_k^m,\xi_{k+1}^m)$ a.e. for all m,

(ii) $a_m \leqslant \xi_k^m < \xi_{k+1}^m \leqslant b_m$ *for each* m *and* k,

(iii) $\alpha(\xi_{k+1}^m) \leqslant \beta(\xi_k^m)$ *for each* m,k *and also,* $\alpha(\xi_{k+1}^m) = \beta(\xi_k^m)$ *if* $a_m < \xi_k^m < \xi_{k+1}^m < b_m$.

In section 2 we prove Theorem 1 and the proof of Theorem 3 is given in section 3. We use C to denote a positive constant not necessarily same in all cases.

2. Proof of Theorem 1

Proof. $(ii) \implies (i)$.
Let $\{\xi_k^m\}$ be the sequence given by Lemma 1, then we have

$$\begin{aligned}&\mathscr{V}\circ\mathscr{U}^{-1}\left(\int_{\{t\in(a,b):\mathscr{I}f(t)>\gamma\}}\mathscr{U}\left(\gamma\omega(y)\right)\rho(y)dy\right)\\&\leqslant\sum_{m,k}\mathscr{V}\circ\mathscr{U}^{-1}\left(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathscr{I}f(t)>\gamma\}}\mathscr{U}\left(\gamma\omega(y)\right)\rho(y)dy\right).\end{aligned}\tag{25}$$

Now for $t\in(\xi_k^m,\xi_{k+1}^m)$, we use Lemma 1 to break the integral (1) as

$$\begin{aligned}\mathscr{I}f(t)&=h(t)\int_{\alpha(t)}^{\alpha(\xi_{k+1}^m)}K(t,s)f(s)w(s)ds+h(t)\int_{\alpha(\xi_{k+1}^m)}^{\beta(\xi_k^m)}K(t,s)f(s)w(s)ds\\&\quad+h(t)\int_{\beta(\xi_k^m)}^{\beta(t)}K(t,s)f(s)w(s)ds=\mathscr{I}_1f(t)+\mathscr{I}_2f(t)+\mathscr{I}_3f(t).\end{aligned}\tag{26}$$

Thus from (26), we have

$$\begin{aligned}&\mathscr{V}\circ\mathscr{U}^{-1}\left(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathscr{I}f(t)>\gamma\}}\mathscr{U}\left(\gamma\omega(y)\right)\rho(y)dy\right)\\&\leqslant\sum_{i=1}^{3}\mathscr{V}\circ\mathscr{U}^{-1}\left(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathscr{I}_if(t)>\frac{\gamma}{3}\}}\mathscr{U}\left(\gamma\omega(y)\right)\rho(y)dy\right).\end{aligned}\tag{27}$$

We will first estimate $\mathscr{I}_1f$. Applying inequlity (2), we now break the kernel K as

$$\begin{aligned}\mathscr{I}_1f(t)&=h(t)\int_{\alpha(t)}^{\alpha(\xi_{k+1}^m)}K(t,s)f(s)w(s)ds\leqslant M\left[h(t)K(t,\beta(\xi_k^m))\int_{\alpha(t)}^{\alpha(\xi_{k+1}^m)}f(s)w(s)ds\right.\\&\quad\left.+h(t)\int_{\alpha(t)}^{\alpha(\xi_{k+1}^m)}K(\xi_k^m,s)f(s)w(s)ds\right]=M\Big[\mathscr{I}_{1,1}f(t)+\mathscr{I}_{1,2}f(t)\Big].\end{aligned}$$

Thus

$$\mathscr{V}\circ\mathscr{U}^{-1}\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathscr{I}_1 f(t)>\frac{\gamma}{3}\}}\mathscr{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big)$$

$$\leqslant\sum_{i=1}^{2}\mathscr{V}\circ\mathscr{U}^{-1}\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathscr{I}_{1,i} f(t)>\frac{\gamma}{6M}\}}\mathscr{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big). \tag{28}$$

To estimate $\mathscr{I}_{1,1}f$, we define a sequence $\{x_j\}$ as $x_0=\xi_k^m$ and for each x_{j-1} let x_j be the number given by $\int_{\alpha(x_j)}^{\alpha(\xi_{k+1}^m)} fw=\int_{\alpha(x_{j-1})}^{\alpha(x_j)} fw$. The sequence $\{x_j\}$ increases and satisfies $\int_{\alpha(x_j)}^{\alpha(\xi_{k+1}^m)} fw=4\int_{\alpha(x_{j+1})}^{\alpha(x_{j+2})} fw$. Let us consider the set

$$\Omega_{1,1}^j=\Big\{t\in(x_j,x_{j+1}):\mathscr{I}_{1,1}f(t)>\frac{\gamma}{6M}\Big\}.$$

We define $\delta_{1,1}^j=\inf\Omega_{1,1}^j$ and $\varepsilon_{1,1}^j=\sup\Omega_{1,1}^j$. For $x\in\Omega_{1,1}^j$, we have

$$\frac{\gamma}{6M}<4h(x)K(x,\beta(\xi_k^m))\int_{\alpha(x_{j+1})}^{\alpha(x_{j+2})} f(z)w(z)dz. \tag{29}$$

As the estimate (29) holds for each $x\in\Omega_{1,1}^j$, thus

$$\gamma\leqslant 24M\big(\inf_{(\delta_{1,1}^j,\varepsilon_{1,1}^j)} h(x)K(x,\beta(\xi_k^m))\big)\int_{\alpha(x_{j+1})}^{\alpha(x_{j+2})} f(z)w(z)dz. \tag{30}$$

Let us denote $\eta(\gamma;\delta_{1,1}^j,\varepsilon_{1,1}^j)=\Big(\mathscr{V}\circ\mathscr{U}^{-1}\Big)\Big(\int_{\delta_{1,1}^j}^{\varepsilon_{1,1}^j}\mathscr{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big)$, then using (8) and (30), we obtain

$$\begin{aligned}2\eta(\gamma;\delta_{1,1}^j,\varepsilon_{1,1}^j)&\leqslant\int_{\alpha(x_{j+1})}^{\alpha(x_{j+2})}\Big[48MCf(z)\phi(z)\Big]\left[\frac{\big(\inf h(x)K(x,\beta(\xi_k^m))\big)w(z)\eta}{C\gamma\phi(z)\psi(z)}\right]\psi(z)dz\\&\leqslant\int_{\alpha(x_{j+1})}^{\alpha(x_{j+2})}\mathscr{V}\Big(48MCf(z)\phi(z)\Big)\psi(z)dz\\&\quad+\int_{\alpha(x_{j+1})}^{\alpha(x_{j+2})}\tilde{\mathscr{V}}\left(\frac{\big(\inf h(x)K(x,\beta(\xi_k^m))\big)w(z)\eta}{C\gamma\phi(z)\psi(z)}\right)\psi(z)dz.\end{aligned} \tag{31}$$

As $\alpha(\varepsilon_{1,1}^j)\leqslant\alpha(x_{j+1})\leqslant\alpha(x_{j+2})\leqslant\alpha(\xi_{k+1}^m)\leqslant\beta(\xi_k^m)\leqslant\beta(x_j)\leqslant\beta(\delta_{1,1}^j)$, thus from (13), we have

$$\int_{\alpha(x_{j+1})}^{\alpha(x_{j+2})}\tilde{\mathscr{V}}\left(\frac{\big(\inf h(x)K(x,\beta(\xi_k^m))\big)w(z)\eta}{C\gamma\phi(z)\psi(z)}\right)\psi(z)dz\leqslant\eta(\gamma;\delta_{1,1}^j,\varepsilon_{1,1}^j). \tag{32}$$

Combining (31) and (32), we obtain

$$\left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_{\Omega^j_{1,1}}\mathscr{U}\left(\gamma\omega(\tau)\right)\rho(\tau)d\tau\right)\leqslant\int_{\alpha(x_{j+1})}^{\alpha(x_{j+2})}\mathscr{V}\left(48MCf(z)\phi(z)\right)\psi(z)dz. \tag{33}$$

Summing up over j and applying sub-additivity of $\mathscr{V}\circ\mathscr{U}^{-1}$, we get

$$\begin{aligned}&\left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathscr{I}_{1,1}f(t)>\frac{\gamma}{6M}\}}\mathscr{U}\left(\gamma\omega(z)\right)\rho(z)dz\right)\\ &\leqslant\int_{\alpha(\xi_k^m)}^{\alpha(\xi_{k+1}^m)}\mathscr{V}\left(48MCf(z)\phi(z)\right)\psi(z)dz.\end{aligned} \tag{34}$$

To estimate $\mathscr{I}_{1,2}$, we define a sequence $\{y_j\}$ as $y_0=\xi_k^m$ and for each y_{j-1} let y_j be the number given by $\int_{\alpha(y_j)}^{\alpha(\xi_{k+1}^m)}K(\xi_k^m,s)f(s)w(s)ds=\int_{\alpha(y_{j-1})}^{\alpha(y_j)}K(\xi_k^m,s)f(s)w(s)ds$. Then $\{y_j\}$ increases and satisfies $\int_{\alpha(y_j)}^{\alpha(\xi_{k+1}^m)}K(\xi_k^m,s)f(s)w(s)ds=4\int_{\alpha(y_{j+1})}^{\alpha(y_{j+2})}K(\xi_k^m,s)f(s)w(s)ds$. As in the previous case, we define $\Omega^j_{1,2}=\left\{t\in(y_j,y_{j+1}):\mathscr{I}_{1,2}f(t)>\frac{\gamma}{6M}\right\}$ with $\delta^j_{1,2}=\inf\Omega^j_{1,2}$ and $\varepsilon^j_{1,2}=\sup\Omega^j_{1,2}$. For $x\in\Omega^j_{1,2}$, we have

$$\frac{\gamma}{6M}<4h(x)\int_{\alpha(y_{j+1})}^{\alpha(y_{j+2})}K(\xi_k^m,z)f(z)w(z)dz. \tag{35}$$

As the estimate (35) holds for each $x\in\Omega^j_{1,2}$, thus

$$\gamma\leqslant 24M\big(\inf_{(\delta^j_{1,2},\varepsilon^j_{1,2})}h(x)\big)\int_{\alpha(y_{j+1})}^{\alpha(y_{j+2})}K(\xi_k^m,z)f(z)w(z)dz. \tag{36}$$

We denote $\eta(\gamma;\delta^j_{1,2},\varepsilon^j_{1,2})=\left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_{\delta^j_{1,2}}^{\varepsilon^j_{1,2}}\mathscr{U}\left(\gamma\omega(\tau)\right)\rho(\tau)d\tau\right)$. Applying (8) and (36), we get

$$\begin{aligned}2\eta(\gamma;\delta^j_{1,2},\varepsilon^j_{1,2})\leqslant&\int_{\alpha(y_{j+1})}^{\alpha(y_{j+2})}\mathscr{V}\left(48MCf(z)\phi(z)\right)\psi(z)dz\\ &+\int_{\alpha(y_{j+1})}^{\alpha(y_{j+2})}\tilde{\mathscr{V}}\left(\frac{(\inf h)K(\xi_k^m,z)h(z)\eta}{C\gamma\phi(z)\psi(z)}\right)\psi(z)dz.\end{aligned} \tag{37}$$

As $\alpha(\varepsilon^j_{1,2})\leqslant\alpha(y_{j+1})\leqslant\alpha(y_{j+2})\leqslant\beta(\delta^j_{1,2})$, thus (12) gives

$$\int_{\alpha(y_{j+1})}^{\alpha(y_{j+2})}\tilde{\mathscr{V}}\left(\frac{(\inf h)K(\xi_k^m,z)h(z)\eta}{C\gamma\phi(z)\psi(z)}\right)\psi(z)dz\leqslant\eta(\gamma;\delta^j_{1,2},\varepsilon^j_{1,2}). \tag{38}$$

Combining (37) and (38), we have

$$\left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_{\Omega_{1,2}^{j}}\mathscr{U}\left(\gamma\omega(z)\right)\rho(z)dz\right)\leqslant\int_{\alpha(y_{j+1})}^{\alpha(y_{j+2})}\mathscr{V}\left(48MCf(z)\phi(z)\right)\psi(z)dz.$$

Summing up in j and applying sub-additivity of $\mathscr{V}\circ\mathscr{U}^{-1}$, hence we obtain

$$\left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathscr{I}_{1,2}f(t)>\frac{\gamma}{6M}\}}\mathscr{U}\left(\gamma\omega(z)\right)\rho(z)dz\right)$$
$$\leqslant\int_{\alpha(\xi_k^m)}^{\alpha(\xi_{k+1}^m)}\mathscr{V}\left(48MCf(z)\phi(z)\right)\psi(z)dz. \tag{39}$$

Arguing similarly as in the previous case, we have

$$\left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathscr{I}_{2}f(t)>\frac{\gamma}{3}\}}\mathscr{U}\left(\gamma\omega(z)\right)\rho(z)dz\right)$$
$$\leqslant\int_{\alpha(\xi_{k+1}^m)}^{\beta(\xi_k^m)}\mathscr{V}\left(48MCf(z)\phi(z)\right)\psi(z)dz. \tag{40}$$

For $\mathscr{I}_3 f,$ we consider $z_0=\xi_{k+1}^m$ and define a decreasing sequence $\{z_j\}$ as

$$\mathscr{L}(z_j)=\int_{\beta(\xi_k^m)}^{\beta(z_j)}K(z_j,\tau)f(\tau)w(\tau)d\tau=(M+1)^{-j}\mathscr{L}(z_0).$$

Now, we have

$$\begin{aligned}
\mathscr{L}(z_j)&=(M+1)^2\mathscr{L}(z_{j+2})\\
&=(M+1)^2\int_{\beta(\xi_k^m)}^{\beta(z_{j+2})}K(z_{j+2},\tau)f(\tau)w(\tau)d\tau \qquad (41)\\
&=(M+1)^2\Big[\int_{\beta(\xi_k^m)}^{\beta(z_{j+3})}+\int_{\beta(z_{j+3})}^{\beta(z_{j+2})}\Big]K(z_{j+2},\tau)f(\tau)w(\tau)d\tau\\
&\leqslant(M+1)^2\left[\int_{\beta(\xi_k^m)}^{\beta(z_{j+3})}M\Big\{K(z_{j+2},\beta(z_{j+3}))+K(z_{j+3},\tau)\Big\}\right.\\
&\quad\left.+\int_{\beta(z_{j+3})}^{\beta(z_{j+2})}K(z_{j+2},\tau)\right]f(\tau)w(\tau)d\tau\\
&\leqslant(M+1)^3\left\{K(z_{j+2},\beta(z_{j+3}))\int_{\beta(\xi_k^m)}^{\beta(z_{j+3})}+\int_{\beta(z_{j+3})}^{\beta(z_{j+2})}K(z_{j+2},\tau)\right\}f(\tau)w(\tau)d\tau\\
&\quad+M(M+1)^2\int_{\beta(\xi_k^m)}^{\beta(z_{j+3})}K(z_{j+3},\tau)f(\tau)w(\tau)d\tau. \qquad (42)
\end{aligned}$$

From the construction of the sequence $\{z_j\}$,

$$\int_{\beta(\xi_k^m)}^{\beta(z_{j+3})}K(z_{j+3},\tau)f(\tau)w(\tau)d\tau=\mathscr{L}(z_{j+3})=(M+1)^{-(j+3)}\mathscr{L}(z_0)=(M+1)^{-3}\mathscr{L}(z_j).$$

Thus (42) implies

$$\mathscr{L}(z_j) \leqslant (M+1)^4 \left[K(z_{j+2}, \beta(z_{j+3})) \int_{\beta(\xi_k^m)}^{\beta(z_{j+3})} f(\tau)w(\tau)d\tau + \int_{\beta(z_{j+3})}^{\beta(z_{j+2})} K(z_{j+2}, \tau)f(\tau)w(\tau)d\tau \right].$$

Next, we define $\delta_{3,l}^j = \inf \Omega_{3,l}^j$ and $\varepsilon_{3,l}^j = \sup \Omega_{3,l}^j$ for $l = 1,2,$ where

$$\Omega_{3,1}^j = \left\{ y \in (z_{j+1}, z_j) : h(y)K(z_{j+2}, \beta(z_{j+3})) \int_{\beta(\xi_k^m)}^{\beta(z_{j+3})} f(\tau)w(\tau)d\tau > \frac{\gamma}{6(M+1)^4} \right\},$$

$$\Omega_{3,2}^j = \left\{ z \in (z_{j+1}, z_j) : h(z) \int_{\beta(z_{j+3})}^{\beta(z_{j+2})} K(z_{j+2}, \tau)f(\tau)w(\tau)d\tau > \frac{\gamma}{6(M+1)^4} \right\}.$$

Thus

$$\begin{aligned} &\left(\mathscr{V} \circ \mathscr{U}^{-1}\right)\left(\int_{\{t: \mathscr{I}_3 f(t) > \frac{\gamma}{3}\}} \mathscr{U}\left(\gamma\omega(y)\right)\rho(y)dy \right) \\ &\leqslant \sum_{j \geqslant 0} \left\{ \left(\mathscr{V} \circ \mathscr{U}^{-1}\right)\left(\int_{\Omega_{3,1}^j} \mathscr{U}\left(\gamma\omega(y)\right)\rho(y)dy \right) \right. \\ &\left. \quad + \left(\mathscr{V} \circ \mathscr{U}^{-1}\right)\left(\int_{\Omega_{3,2}^j} \mathscr{U}\left(\gamma\omega(y)\right)\rho(y)dy \right) \right\}. \end{aligned} \tag{43}$$

For the first part, we define a decreasing sequence $\{d_i'\}$ in (ξ_k^m, ξ_{k+1}^m) with the iteration $d_0' = \xi_{k+1}^m$ and

$$\int_{\beta(\xi_k^m)}^{\beta(d_i')} f(\tau)w(\tau)d\tau = 2^{-i} \int_{\beta(\xi_k^m)}^{\beta(\xi_{k+1}^m)} f(\tau)w(\tau)d\tau.$$

We define $d_0 = d_0'$ and if $d_i' > z_j \geqslant d_{i+1}'$ then $d_{n+1} = d_{i+1}'$, otherwise we delete the term d_{i+1}' and continue the process. Thus, we get a subsequence $\{d_n\}$ of $\{d_i'\}$. Let $\tilde{\delta}_{3,1}^n = \inf \tilde{\Omega}_{3,1}^n$ and $\tilde{\varepsilon}_{3,1}^n = \sup \tilde{\Omega}_{3,1}^n$, where $\tilde{\Omega}_{3,1}^n = \cup_{\{j: d_n > z_{j+3} \geqslant d_{n+1}\}} \Omega_{3,1}^j$. Now, if $d_{i+1}' = d_{n+1} \leqslant z_{j+3} < d_n$, then $z_{j+3} \leqslant d_i'$ and $d_{n+2} \leqslant d_{i+2}'$. We have

$$\int_{\beta(\xi_k^m)}^{\beta(z_{j+3})} \leqslant \int_{\beta(\xi_k^m)}^{\beta(d_i')} = 4 \int_{\beta(d_{i+2}')}^{\beta(d_{i+1}')} \leqslant 4 \int_{\beta(d_{n+2})}^{\beta(d_{n+1})}. \tag{44}$$

Now for $x \in \tilde{\Omega}_{3,1}^n$, we obtain

$$\frac{\gamma}{6(M+1)^4} < 4h(x)K(x, \beta(z_{j+3})) \int_{\beta(d_{n+2})}^{\beta(d_{n+1})} f(\tau)w(\tau)d\tau. \tag{45}$$

As (45) holds for each $x \in \tilde{\Omega}^n_{3,1}$, thus

$$\gamma \leqslant 24(M+1)^4 \inf_{(\tilde{\delta}^n_{3,1},\tilde{\varepsilon}^n_{3,1})} \big(h(x)K(x,\beta(z_{j+3}))\big) \int_{\beta(d_{n+2})}^{\beta(d_{n+1})} f(\tau)w(\tau)d\tau. \tag{46}$$

We denote $\eta(\gamma;\tilde{\delta}^n_{3,1},\tilde{\varepsilon}^n_{3,1}) = \Big(\mathscr{V}\circ\mathscr{U}^{-1}\Big)\Big(\int_{\tilde{\delta}^n_{3,1}}^{\tilde{\varepsilon}^n_{3,1}} \mathscr{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big)$. From (8) and (13) we obtain

$$\begin{aligned} 2\eta(\gamma;\tilde{\delta}^n_{3,1},\tilde{\varepsilon}^n_{3,1}) &\leqslant \int_{\beta(d_{n+2})}^{\beta(d_{n+1})} \mathscr{V}\Big(48(M+1)^4 Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau \\ &\quad + \int_{\beta(d_{n+2})}^{\beta(d_{n+1})} \tilde{\mathscr{V}}\left[\frac{\inf\big(h(x)K(x,\beta(z_{j+3}))\big)w(\tau)\eta}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \\ &\leqslant \int_{\beta(d_{n+2})}^{\beta(d_{n+1})} \mathscr{V}\Big(48(M+1)^4 Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau + \eta(\gamma;\tilde{\delta}^n_{3,1},\tilde{\varepsilon}^n_{3,1}). \end{aligned} \tag{47}$$

Thus

$$\begin{aligned} &\Big(\mathscr{V}\circ\mathscr{U}^{-1}\Big)\Big(\int_{\tilde{\delta}^n_{3,1}}^{\tilde{\varepsilon}^n_{3,1}} \mathscr{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big) \\ \leqslant\ & \int_{\beta(d_{n+2})}^{\beta(d_{n+1})} \mathscr{V}\Big(48(M+1)^4 Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau. \end{aligned}$$

This implies

$$\begin{aligned} &\Big(\mathscr{V}\circ\mathscr{U}^{-1}\Big)\Big(\int_{\tilde{\Omega}^n_{3,1}} \mathscr{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big) \\ \leqslant\ & \int_{\beta(d_{n+2})}^{\beta(d_{n+1})} \mathscr{V}\Big(48(M+1)^4 Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau. \end{aligned}$$

Summing up over n and then applying sub-additivity of $\mathscr{V}\circ\mathscr{U}^{-1}$, we obtain

$$\begin{aligned} &\sum_{n\geqslant 0}\Big(\mathscr{V}\circ\mathscr{U}^{-1}\Big)\Big(\int_{\tilde{\Omega}^n_{3,1}} \mathscr{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big) \\ \leqslant\ & \int_{\beta(\xi^m_k)}^{\beta(\xi^m_{k+1})} \mathscr{V}\Big(48(M+1)^4 Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau. \end{aligned}$$

This implies

$$\begin{aligned} &\sum_{j\geqslant 0}\Big(\mathscr{V}\circ\mathscr{U}^{-1}\Big)\Big(\int_{\Omega^j_{3,1}} \mathscr{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big) \\ \leqslant\ & \int_{\beta(\xi^m_k)}^{\beta(\xi^m_{k+1})} \mathscr{V}\Big(48(M+1)^4 Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau. \end{aligned} \tag{48}$$

Next, for $\Omega^j_{3,2}$ working as similar to the previous cases and thus we have

$$\sum_{j\geqslant 0}\left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_{\Omega^j_{3,2}}\mathscr{U}\left(\gamma\omega(z)\right)\rho(z)dz\right)$$
$$\leqslant\int_{\beta(\xi^m_k)}^{\beta(\xi^m_{k+1})}\mathscr{V}\left(12(M+1)^4Cf(z)\phi(z)\right)\psi(z)dz. \tag{49}$$

Combining (27), (28), (34), (39), (40), (43), (48) and (49) we obtain

$$\left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_{\{t\in(\xi^m_k,\xi^m_{k+1}):\mathscr{I}f(t)>\gamma\}}\mathscr{U}\left(\gamma\omega(z)\right)\rho(z)dz\right)$$
$$\leqslant\int_{\alpha(\xi^m_k)}^{\beta(\xi^m_{k+1})}\mathscr{V}\left(96(M+1)^4Cf(z)\phi(z)\right)\psi(z)dz. \tag{50}$$

Summing up (50) over m and k we obtain the estimate (11) with constant $96(M+1)^4C$.

$(i)\implies(ii)$.

Conversely, let us assume $t<\tau$ such that $\alpha(\tau)<\beta(t)$. For each $N\in\mathbb{N}$ we consider the set $E_N=\left\{\alpha(\tau)<s<\beta(t):\frac{1}{N}\leqslant K(t,s),w(s)<N\right\}$ has finite measure. We have

$$\int_{E_N}\tilde{\mathscr{V}}\left(\frac{\lambda\left(\inf h\right)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\frac{\psi(y)+1/k}{\lambda}\right)dy\leqslant lN^2|E_N|\left(\inf h\right)\tilde{v}(\lambda lkN^2\inf h)$$
$$<\infty$$

for each $l,k\in\mathbb{N}$ and $\lambda>0$. Thus for each $\mu>0$ we can choose λ such that

$$\int_{E_N}\tilde{\mathscr{V}}\left(\frac{\lambda\left(\inf h\right)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\frac{\psi(y)+1/k}{\lambda}\right)dy=(1+\mu)C\gamma,$$

where C is the constant in (11). We consider

$$f(y)=\frac{1}{C}\tilde{\mathscr{V}}\left(\frac{\lambda\left(\inf h\right)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda\left(\inf h\right)K(t,y)w(y)}\chi_{E_N}(y).$$

For $t\leqslant x<\tau$, we have

$$\mathscr{I}f(x)=h(x)\int_{E_N}K(x,y)\frac{1}{C}\tilde{\mathscr{V}}\left(\frac{\lambda\left(\inf h\right)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda\left(\inf h\right)K(t,y)w(y)}w(y)dy$$
$$\geqslant\int_{E_N}\frac{1}{C\lambda}\tilde{\mathscr{V}}\left(\frac{\lambda\left(\inf h\right)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\psi(y)+1/k\right)dy$$
$$=(1+\mu)\gamma>\gamma.$$

This implies

$$[t,\tau)\subset\{x:\mathscr{I}f(x)>\gamma\}.$$

Thus using (9) and (11) we obtain

$$\begin{aligned}
\eta(\gamma;t,\tau) &= \Big(\mathscr{V}\circ\mathscr{U}^{-1}\Big)\left(\int_t^\tau \mathscr{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\right)\\
&\leqslant \Big(\mathscr{V}\circ\mathscr{U}^{-1}\Big)\left(\int_{\{\mathscr{I}f>\gamma\}} \mathscr{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\right)\\
&\leqslant \int_{E_N}\mathscr{V}\left(\tilde{\mathscr{V}}\left(\frac{\lambda\big(\inf h\big)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda\big(\inf h\big)K(t,y)w(y)}\phi(y)\right)\psi(y)dy\\
&\leqslant \int_{E_N}\tilde{\mathscr{V}}\left(\frac{\lambda\big(\inf h\big)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\psi(y)dy\\
&\leqslant (1+\mu)C\lambda\gamma.
\end{aligned}$$

Since $\tilde{\mathscr{V}}(r)/r$ increases as r increases, thus we have

$$\begin{aligned}
&\int_{E_N}\tilde{\mathscr{V}}\left(\frac{\big(\inf h\big)K(t,y)w(y)\eta(\gamma;t,\tau)}{(1+\mu)C\gamma(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\eta(\gamma;t,\tau)}dy\\
\leqslant\ &\int_{E_N}\tilde{\mathscr{V}}\left(\frac{\lambda\big(\inf h\big)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{(1+\mu)C\lambda\gamma}dy=1.
\end{aligned}$$

By the Monotone convergence theorem

$$\int_{E_N}\tilde{\mathscr{V}}\left(\frac{\big(\inf h\big)K(t,y)w(y)\eta(\gamma;t,\tau)}{(1+\mu)C\gamma(\phi(y)+1/l)\psi(y)}\right)\frac{\psi(y)}{\eta(\gamma;t,\tau)}dy\leqslant 1.$$

Letting $l,N\to\infty$ and $\mu\to 0^+$, we thus obtain

$$\int_{\alpha(\tau)}^{\beta(t)}\tilde{\mathscr{V}}\left(\frac{\big(\inf h\big)K(t,y)w(y)\eta\big(\gamma;t,\tau\big)}{C\gamma\phi(y)\psi(y)}\right)\psi(y)dy\leqslant\eta\big(\gamma;t,\tau\big).$$

In a similar way we can prove the estimate (13). Let $a<z\leqslant t<\tau<b$ satisfying $\alpha(\tau)\leqslant\beta(z)\leqslant\beta(t)$. For $N\in\mathbb{N}$ we consider the set $E_N=\big\{\alpha(\tau)<s<\beta(z):\frac{1}{N}\leqslant w(s)<N\big\}$ has finite measure and we define

$$f(y)=\frac{1}{C}\tilde{\mathscr{V}}\left(\frac{\lambda\Big(\inf h(s)K\big(s,\beta(z)\big)\Big)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda\big(\inf h\big)K(t,y)w(y)}\chi_{E_N}(y).$$

And the rest of the proof proceeds similarly. Hence the proof is complete. □

3. Proof of Theorem 3

Proof. $(ii) \implies (i)$.

Let $\{\xi_k^m\}$ be the sequence given by Lemma 1, then using the identity (26) and applying subadditivity of $\mathscr{V}\circ\mathscr{U}^{-1}$ we obtain

$$\left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathscr{I}f(t)>\gamma\}}\omega(y)dy\right)$$
$$\leqslant \sum_{i=1}^{3}\left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathscr{I}_i f(t)>\frac{\gamma}{3}\}}\omega(y)dy\right). \tag{51}$$

We will first estimate $\mathscr{I}_1 f$. Applying inequlity (2), we now break the kernel K as

$$\begin{aligned}\mathscr{I}_1 f(t) &= h(t)\int_{\alpha(t)}^{\alpha(\xi_{k+1}^m)} K(t,z)f(z)w(z)dz\\ &\leqslant M\left[h(t)\left\{K(t,\beta(\xi_k^m))\int_{\alpha(t)}^{\alpha(\xi_{k+1}^m)} + \int_{\alpha(t)}^{\alpha(\xi_{k+1}^m)} K(\xi_k^m,z)\right\}f(z)w(z)dz\right]\\ &= M\left[\mathscr{I}_{1,1}f(t)+\mathscr{I}_{1,2}f(t)\right].\end{aligned}$$

Thus

$$\left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathscr{I}_1 f(t)>\frac{\gamma}{3}\}}\omega(y)dy\right)$$
$$\leqslant \sum_{i=1}^{2}\left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathscr{I}_{1,i} f(t)>\frac{\gamma}{6M}\}}\omega(y)dy\right). \tag{52}$$

To estimate $\mathscr{I}_{1,1}f$, we consider the sequence $\{x_j\}$ as defined in section 2. Let us define $\delta_{1,1}^j = \inf\Omega_{1,1}^j$ and $\varepsilon_{1,1}^j = \sup\Omega_{1,1}^j$, where

$$\Omega_{1,1}^j = \left\{t\in(x_j,x_{j+1}) : \mathscr{I}_{1,1}f(t) > \frac{\gamma}{6M}\right\}.$$

Let us denote $\theta(\delta_{1,1}^j,\varepsilon_{1,1}^j) = \left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_{\delta_{1,1}^j}^{\varepsilon_{1,1}^j}\omega(z)dz\right)$, then using (8) and (30), we obtain

$$\begin{aligned}2\theta(\delta_{1,1}^j,\varepsilon_{1,1}^j) &\leqslant \int_{\alpha(x_{j+1})}^{\alpha(x_{j+2})}\left[\frac{48MCf(z)\phi(z)}{\gamma}\right]\left[\frac{\left(\inf h(x)K(x,\beta(\xi_k^m))\right)w(z)\theta}{C\phi(z)\psi(z)}\right]\psi(z)dz\\ &\leqslant \int_{\alpha(x_{j+1})}^{\alpha(x_{j+2})}\mathscr{V}\left(\frac{48MCf(z)\phi(z)}{\gamma}\right)\psi(z)dz\\ &\quad+\int_{\alpha(x_{j+1})}^{\alpha(x_{j+2})}\tilde{\mathscr{V}}\left(\frac{\left(\inf h(x)K(x,\beta(\xi_k^m))\right)w(z)\theta}{C\phi(z)\psi(z)}\right)\psi(z)dz.\end{aligned} \tag{53}$$

As $\alpha(\varepsilon^j_{1,1}) \leqslant \alpha(x_{j+1}) \leqslant \alpha(x_{j+2}) \leqslant \alpha(\xi^m_{k+1}) \leqslant \beta(\xi^m_k) \leqslant \beta(x_j) \leqslant \beta(\delta^j_{1,1})$, thus from (21), we have

$$\int_{\alpha(x_{j+1})}^{\alpha(x_{j+2})} \tilde{\mathscr{V}}\left(\frac{\left(\inf h(x)K(x,\beta(\xi^m_k))\right)w(z)\theta(\delta^j_{1,1},\varepsilon^j_{1,1})}{C\phi(z)\psi(z)}\right)\psi(z)dz$$
$$\leqslant \theta(\delta^j_{1,1},\varepsilon^j_{1,1}). \tag{54}$$

Combining (53) and (54), we obtain

$$\left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_{\Omega^j_{1,1}} \omega(z)dz\right) \leqslant \int_{\alpha(x_{j+1})}^{\alpha(x_{j+2})} \mathscr{V}\left(\frac{48MCf(z)\phi(z)}{\gamma}\right)\psi(z)dz. \tag{55}$$

Summing up over j and applying subadditivity of $\mathscr{V}\circ\mathscr{U}^{-1}$, we get

$$\left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_{\{t\in(\xi^m_k,\xi^m_{k+1}):\mathscr{I}_{1,1}f(t)>\frac{\gamma}{6M}\}} \omega(z)dz\right)$$
$$\leqslant \int_{\alpha(\xi^m_k)}^{\alpha(\xi^m_{k+1})} \mathscr{V}\left(\frac{48MCf(z)\phi(z)}{\gamma}\right)\psi(z)dz. \tag{56}$$

Estimation of the integrals $\mathscr{I}_{1,2}f, \mathscr{I}_2 f$ and $\mathscr{I}_3 f$ also follows the similar pattern used to obtain (56) and in the proof of Theorem 1. Thus we obtain (19) with constant $96(M+1)^4C$.

$(i) \implies (ii)$.

Conversely, let us assume $t<\tau$ for which $\alpha(\tau)<\beta(t)$. Corresponding to each $N\in\mathbb{N}$ we consider the set $E_N = \left\{\alpha(\tau)<s<\beta(t) : \frac{1}{N} \leqslant K(t,s), w(s)<N\right\}$ has finite measure. We have

$$\int_{E_N} \tilde{\mathscr{V}}\left(\frac{\lambda\left(\inf h\right)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\frac{\psi(y)+1/k}{\lambda}\right)dy \leqslant lN^2|E_N|\left(\inf h\right)\tilde{v}(\lambda lkN^2 \inf h)$$
$$< \infty$$

for each $l,k\in\mathbb{N}$ and $\lambda>0$. Thus for each $\mu>0$ we choose λ such that

$$\int_{E_N} \tilde{\mathscr{V}}\left(\frac{\lambda\left(\inf h\right)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\frac{\psi(y)+1/k}{\lambda}\right)dy = (1+\mu)C,$$

where C is the constant in (19). For each $\gamma>0$ we consider

$$f_\gamma(y) = \frac{\gamma}{C}\tilde{\mathscr{V}}\left(\frac{\lambda\left(\inf h\right)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda\left(\inf h\right)K(t,y)w(y)}\chi_{E_N}(y).$$

If $t \leqslant x < \tau$, then

$$\begin{aligned}\mathscr{I} f_\gamma(x) &= h(x)\int_{E_N} K(x,y)\frac{\gamma}{C}\tilde{\mathscr{V}}\left(\frac{\lambda\left(\inf h\right)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\\ &\quad\times\frac{\psi(y)+1/k}{\lambda\left(\inf h\right)K(t,y)w(y)}w(y)dy\\ &\geqslant \int_{E_N}\frac{\gamma}{C\lambda}\tilde{\mathscr{V}}\left(\frac{\lambda\left(\inf h\right)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\psi(y)+1/k\right)dy\\ &= (1+\mu)\gamma > \gamma.\end{aligned}$$

This implies

$$[t,\tau) \subset \{x : \mathscr{I} f_\gamma(x) > \gamma\}.$$

Thus using (9) and (19) we obtain

$$\begin{aligned}\theta(t,\tau) &= \left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_t^\tau \omega(y)dy\right)\\ &\leqslant \left(\mathscr{V}\circ\mathscr{U}^{-1}\right)\left(\int_{\{\mathscr{I} f_\gamma>\gamma\}}\omega(y)dy\right)\\ &\leqslant \int_{E_N}\mathscr{V}\left(\tilde{\mathscr{V}}\left(\frac{\lambda\left(\inf h\right)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda\left(\inf h\right)K(t,y)w(y)}\phi(y)\right)\psi(y)dy\\ &\leqslant \int_{E_N}\tilde{\mathscr{V}}\left(\frac{\lambda\left(\inf h\right)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\psi(y)dy\\ &\leqslant (1+\mu)C\lambda.\end{aligned}$$

Since $\tilde{\mathscr{V}}(r)/r$ increases as r increases, thus we have

$$\begin{aligned}&\int_{E_N}\tilde{\mathscr{V}}\left(\frac{\left(\inf h\right)K(t,y)w(y)\theta(t,\tau)}{(1+\mu)C(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\theta(t,\tau)}dy\\ \leqslant\ &\int_{E_N}\tilde{\mathscr{V}}\left(\frac{\lambda\left(\inf h\right)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{(1+\mu)C\lambda}dy = 1.\end{aligned}$$

By the Monotone convergence theorem, we obtain

$$\int_{E_N}\tilde{\mathscr{V}}\left(\frac{\left(\inf h\right)K(t,y)w(y)\theta(t,\tau)}{(1+\mu)C(\phi(y)+1/l)\psi(y)}\right)\frac{\psi(y)}{\theta(t,\tau)}dy \leqslant 1.$$

Letting $l,N\to\infty$ and $\mu\to 0^+$, thus we obtain

$$\int_{\alpha(\tau)}^{\beta(t)}\tilde{\mathscr{V}}\left(\frac{\left(\inf h\right)K(t,y)w(y)\theta\left(t,\tau\right)}{C\phi(y)\psi(y)}\right)\psi(y)dy \leqslant \theta\left(t,\tau\right).$$

In a similar way we can prove the estimate (21). Hence the proof is complete. □

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REFERENCES

[1] R. J. BAGBY, *Weak bounds for the maximal function in weighted Orlicz spaces*, Studia Math., **95**, 3 (1990), 195–204.

[2] R. J. BAGBY AND J. D. PARSONS, *Orlicz spaces and rearranged maximal functions*, Math. Nachr., **132**, 1987, 15–27.

[3] A. GOGATISHVILI AND L. PICK, *Weak and extra-weak type inequalities for the maximal operator and the Hilber transform*, Czechoslovak Math. J., **43**, 3 (1993), 547–566.

[4] A. L. BERNARDIS, F. J. MARTÍN-REYES AND P. ORTEGA SALVADOR, *Weighted weak type inequalities for generalized Hardy operators*, J. Inequal. Appl., (2006), Art. ID 62426, 1–10.

[5] A. L. BERNARDIS, F. J. MARTÍN-REYES AND P. ORTEGA SALVADOR, *Weighted inequalities for Hardy-Steklov operators*, Can. J. Math., **59**, 2 (2007), 276–295.

[6] S. BLOOM AND R. KERMAN, *Weighted L_ϕ integral inequalities for operators of Hardy type*, Studia Math., **110**, 2 (1994), 149–167.

[7] T. CHEN AND G. SINNAMON, *Generalized Hardy operators and normalizing measures*, J. Inequal. Appl., **7**, 6 (2002), 829–866.

[8] A. GOGATISHVILI AND J. LANG, *The generelized Hardy operator with kernel and variable integral limits in Banach function spaces*, J. Inequal. Appl., **4**, 1 (1999), 1–16.

[9] H. P. HEINIG AND G. SINNAMON, *Mapping properties of integral averaging operators*, Studia Math., **129**, 2 (1998), 157–177.

[10] V. KOKILASHVILI AND M. KRBEC, *Weighted Inequalities in Lorentz and Orlicz Spaces*, World Scientific Publishing Co., Inc., River Edge, NJ, 1991.

[11] A. KUFNER, L. E. PERSSON AND N. SAMKO, *Weighted Inequalities of Hardy Type*, second edition, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017.

[12] Q. LAI, *Weighted integral inequalities for the Hardy type operator and the fractional maximal operator*, J. London Math. Soc., **49**, 2 (1994), 244–266.

[13] Q. LAI, *Two weight mixed ϕ-inequalities for the one-sided maximal function*, Studia Math., **115**, 1 (1995), 1–22.

[14] Q. LAI, *Weighted modular inequalities for Hardy type operators*, Proc. London Math. Soc., **79**, 3 (1999), 649–672.

[15] F. J. MARTÍN-REYES AND P. ORTEGA SALVADOR, *On weighted weak type inequalities for modified Hardy operators*, Proc. Amer. Math. Soc., **126**, 6 (1998), 1739–1746.

[16] P. ORTEGA SALVADOR, *Weighted generalized weak type inequalities for modified Hardy operators*, Collect. Math., **51**, 2 (2000), 149–155.

[17] P. ORTEGA SALVADOR AND L. PICK, *Two-weight weak and extra-weak type inequalities for the one-sided maximal operator*, Proc. Roy. Soc. Edinburgh Sect. A, **123**, 1993, 1109–1118.

[18] P. ORTEGA SALVADOR, C. RAMÍREZ TORREBLANCA, *Weighted modular inequalities for Hardy-Steklov operators*, J. Math. Anal. Appl., **322**, 2006, 803–814.
[19] V. D. STEPANOV AND E. P. USHAKOVA, *On boundedness of a certain class of Hardy-Steklov type operators in Lebesgue Spaces*, Banach J. Math. Anal., **4**, 1 (2010), 28–52.

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