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LOCATION OF THE SPECTRUM OF OPERATOR MATRICES WHICH ARE ASSOCIATED TO SECOND ORDER EQUATIONS

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Abstract. In this paper, second order equations of the form $\ddot{z}(t)+A_0z(t)+D\dot{z}(t)=0$ are studied, where A_0 is a uniformly positive operator and $A_0^{-1/2}DA_0^{-1/2}$ is a bounded non-negative operator in a Hilbert space H. This equation is equivalent to the standard first-order equation $\dot{x}(t)=\mathcal{A}x(t)$, where $\mathcal{A}$ has the domain

$$\mathscr{D}(\mathcal{A})=\left\{\left[\begin{smallmatrix} z\\ w\end{smallmatrix}\right]\in\mathscr{D}(A_0^{1/2})\times\mathscr{D}(A_0^{1/2})\mid A_0z+Dw\in H\right\}$$

and is given by

$$\mathcal{A}=\begin{bmatrix} 0 & I\\ -A_0 & -D\end{bmatrix}.$$

The location of the spectrum and the essential spectrum of the semigroup generator $\mathcal{A}$ is described under various conditions on the damping operator D. By means of an example it is shown that in general the spectrum can be quite arbitrary in the closed left half plane.

1. Introduction

The aim of this paper is the study of second order equations of the form

$$\ddot{z}(t)+A_0z(t)+D\dot{z}(t)=0. \tag{1}$$

Here the stiffness operator A_0 is a possibly unbounded positive operator on a Hilbert space H and is assumed to be boundedly invertible, and D, the damping operator, is an unbounded operator such that $A_0^{-1/2}DA_0^{-1/2}$ is a bounded non-negative operator on H. This second order equation is equivalent to the standard first-order equation $\dot{x}(t)=\mathcal{A}x(t)$, where $\mathcal{A}:\mathscr{D}(\mathcal{A})\subset\mathscr{D}(A_0^{1/2})\times H\to\mathscr{D}(A_0^{1/2})\times H$, is given by

$$\mathcal{A}=\begin{bmatrix} 0 & I\\ -A_0 & -D\end{bmatrix},$$

$$\mathscr{D}(\mathcal{A})=\left\{\left[\begin{smallmatrix} z\\ w\end{smallmatrix}\right]\in\mathscr{D}(A_0^{1/2})\times\mathscr{D}(A_0^{1/2})\mid A_0z+Dw\in H\right\}.$$

We refer the reader to [25] for a discussion of solutions for equation (1).

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This block operator matrix has been studied in the literature for more than 20 years. Interest in this particular model is motivated by various problems such as stabilization, see for example [6], [22], [23], [21], solvability of the Riccati equations [11], minimum-phase property [17] and compensator problems with partial observations [12].

It is well-known that $\mathcal{A}$ generates a C_0-semigroup of contractions in $\mathscr{D}(A_0^{1/2})\times H$, where $\mathscr{D}(A_0^{1/2})$ is equipped with the norm $x\mapsto \|A_0^{1/2}x\|_H$, and thus the spectrum of $\mathcal{A}$ is located in the closed left half plane. This goes back to [3] and [20], see also [4], [8]. Several authors have proved independently of each other that the condition

$$\inf_{z\in\mathscr{D}(A_0^{1/2})\setminus\{0\}} \frac{\langle A_0^{-1/2}Dz, A_0^{1/2}z\rangle_H}{\|z\|_H^2} > 0$$

is sufficient for exponential stability of the C_0-semigroup generated by $\mathcal{A}$, see for example [3], [4], [5], [8], [14], [15], [24], [26]. Other properties of the C_0-semigroup such as analyticity have been studied in [3], [4], [8], [9], [13] and [16].

In this paper we are interested in a more detailed study of the location of the spectrum of $\mathcal{A}$ in the left half plane. Under the extra assumption that A_0 has a compact resolvent some results in this direction were obtained in [8] and [19]. In particular, in this situation $\mathcal{A}$ has no non-real essential spectrum. We do not assume that A_0 has a compact resolvent and we show that in general the (essential) spectrum of $\mathcal{A}$ can be quite arbitrary in the closed left half plane. Under various conditions on the damping operator D we describe the location of the spectrum and the essential spectrum of $\mathcal{A}$.

We will often use the fact that associated to the block operator matrix $\mathcal{A}$ there is the operator pencil

$$L(s) := s^2A_0^{-1} + sA_0^{-1/2}DA_0^{-1/2} + I, \qquad s\in\mathbb{C}.$$

That is, there is a one to one correspondence between the spectrum of $\mathcal{A}$ and the spectrum of the operator pencil $L(\cdot)$, see Proposition 2.2 for more details. Thus all results concerning the spectrum of $\mathcal{A}$ have counterparts concerning the spectrum of $L(\cdot)$.

We proceed as follows. In Section 2 we introduce the framework and we prove some preliminary results such as the relation of the spectrum of $\mathcal{A}$ to the spectrum of the corresponding operator pencil. Section 3 is devoted to the spectrum of the operator $\mathcal{A}$. It is shown that in general the spectrum can be quite general in the closed left half plane. Sufficient conditions are given to guarantee that certain regions are contained in the resolvent set of $\mathcal{A}$. The location of the essential spectrum is the subject of Section 4 and finally in Section 5 we determine intervals of the real axis which do not contain accumulation points of the non-real spectrum. In particular we show that if A_0 has a compact resolvent, then the non-real spectrum cannot accumulate to the real axis.

2. Framework and preliminary results

Throughout this paper we make the following assumptions.

(A1) The stiffness operator $A_0 : \mathscr{D}(A_0) \subset H \to H$ is a self-adjoint uniformly positive operator, i.e. $A_0 >> 0$, on a Hilbert space H. A scale of Hilbert spaces H_α is defined as follows: For $\alpha \geqslant 0$, we define $H_\alpha = \mathscr{D}(A_0^\alpha)$ equipped with the norm $\|z\|_{H_\alpha} := \|A_0^\alpha z\|_H$ and $H_{-\alpha} = H_\alpha^*$. Here the duality is taken with respect to the pivot space H, that is, equivalently $H_{-\alpha}$ is the completion of H with respect to the norm $\|z\|_{H_{-\alpha}} = \|A_0^{-\alpha} z\|_H$. Thus A_0 extends (restricts) to $A_0 : H_\alpha \to H_{\alpha-1}$ for $\alpha \in \mathbb{R}$. We use the same notation A_0 to denote this extension (restriction), but we will mention it explicitly if A_0 is considered as an operator acting between H_α and $H_{\alpha-1}$ for some $\alpha \in \mathbb{R}$.

We denote the inner product on H by $\langle \cdot,\cdot \rangle_H$ or $\langle \cdot,\cdot \rangle$, and the duality pairing on $H_{-\alpha} \times H_\alpha$ by $\langle \cdot,\cdot \rangle_{H_{-\alpha}\times H_\alpha}$. Note that for $(z',z) \in H \times H_\alpha$, $\alpha > 0$, we have

$$\langle z',z \rangle_{H_{-\alpha}\times H_\alpha} = \langle z',z \rangle_H.$$

(A2) The damping operator $D : H_{\frac12} \to H_{-\frac12}$ is a bounded operator such that $A_0^{-1/2} D A_0^{-1/2}$ is a bounded self-adjoint operator in H and satisfies

$$\langle Dz,z \rangle_{H_{-\frac12}\times H_{\frac12}} \geqslant 0, \qquad z \in H_{\frac12}.$$

The system (1) is equivalent to the following standard first-order equation

$$\dot{x}(t) = \mathcal{A}x(t) \tag{2}$$

where $\mathcal{A} : \mathscr{D}(\mathcal{A}) \subset H_{\frac12} \times H \to H_{\frac12} \times H$, is given by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0 & -D \end{bmatrix},$$

$$\mathscr{D}(\mathcal{A}) = \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in H_{\frac12} \times H_{\frac12} \mid A_0 z + Dw \in H \right\}.$$

The operator $\mathcal{A}$ itself is not self-adjoint in the Hilbert space $H_{\frac12} \times H$. However, in [26, Proof of Lemma 4.5] it is shown that

$$\mathcal{A}^* = J\mathcal{A}J, \qquad \text{with} \quad J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

In particular, $J\mathcal{A}$ is a self-adjoint operator in $H_{\frac12} \times H$. For $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in H_{\frac12} \times H$ we define an indefinite inner product on $H_{\frac12} \times H$ by

$$\left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right] := \left\langle J \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \langle x_1,x_2 \rangle_{H_{\frac12}} - \langle y_1,y_2 \rangle. \tag{3}$$

Then $(H_{\frac12} \times H, [\cdot,\cdot])$ is a Krein space (for the basic theory of Krein spaces and operators acting therein we refer to [7] and [1]) and $\mathcal{A}$ is a self-adjoint operator with respect to

$[\cdot,\cdot]$. Moreover, see [26, Proof of Lemma 4.5], $\mathcal{A}$ has a bounded inverse in $H_{\frac{1}{2}} \times H$, with

$$\mathcal{A}^{-1} = \begin{bmatrix} -A_0^{-1}D & -A_0^{-1} \\ I & 0 \end{bmatrix}, \tag{4}$$

where $A_0^{-1}D$ is considered as an operator acting in $H_{\frac{1}{2}}$. This and the self-adjointness of $\mathcal{A}$ in the Krein space $(H_{\frac{1}{2}} \times H, [\cdot,\cdot])$ imply the following well-known proposition (cf. [25] and [26, Proof of Lemma 4.5]).

PROPOSITION 2.1. *The operator $\mathcal{A}$ has a bounded inverse and the spectrum of $\mathcal{A}$ is symmetric with respect to the real line.*

Throughout this paper we will use the following notation. For a closed densely defined linear operator S on some Banach space X we denote by $\sigma_p(S)$ the point spectrum of S. The *approximate point spectrum*, $\sigma_{ap}(S)$, consists of all λ for which there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathscr{D}(S)$ such that

$$\|x_n\| = 1 \text{ and } \|(S-\lambda I)x_n\| \to 0 \quad \text{as} \quad n \to \infty$$

(see for example [10, page 242]). We point out that the point spectrum is a subset of the approximate point spectrum. We set

$$r(S) := \mathbb{C}\backslash\sigma_{ap}(S).$$

A point $\mu \in r(S)$ is called of *regular type* for S. It follows that for every $\mu \in r(S)$ the range of $S-\mu I$ is closed. Moreover, there exists $M > 0$ such that for $x \in \mathscr{D}(S)$

$$\|Sx - \mu x\| \geqslant M\|x\| \tag{5}$$

holds. We define

$$\operatorname{nul} S := \dim\ker S \quad \text{and} \quad \operatorname{def} S := \operatorname{codim}\operatorname{ran} S,$$

these being finite numbers or ∞. The operator S is called *Fredholm* if the above quantities are finite, i.e. the dimension of the kernel of S and the codimension of the range of S are finite. The set

$$\sigma_{ess}(S) := \{\lambda \in \mathbb{C} \mid S - \lambda I \text{ is not Fredholm}\}$$

is called the *essential spectrum* of S. Moreover, by $\sigma_{p,norm}(S)$ we denote the set of all $\lambda \in \mathbb{C}$ which are isolated points of $\sigma(S)$ and normal eigenvalues of S, that is, the corresponding Riesz-Dunford projection is of finite rank. Recall that for a self-adjoint operator in a Hilbert space we have

$$\sigma_{ess}(S) = \sigma(S) \setminus \sigma_{p,norm}(S).$$

We associate with the block operator matrix $\mathcal{A}$ the operator pencil

$$L(s) := s^2A_0^{-1} + sA_0^{-1/2}DA_0^{-1/2} + I, \qquad s \in \mathbb{C}.$$

Here $L(s)$ is considered as a bounded operator acting on H.

PROPOSITION 2.2. *Let $s \in \mathbb{C}$. The range of $\mathcal{A} - sI$ is closed if and only if $L(s)$ has a closed range. Moreover, we have*

$$\begin{aligned}
\sigma(\mathcal{A}) &= \{s \in \mathbb{C} \mid 0 \in \sigma(L(s))\}, \\
\sigma_p(\mathcal{A}) &= \{s \in \mathbb{C} \mid 0 \in \sigma_p(L(s))\}, \\
\sigma_{ess}(\mathcal{A}) &= \{s \in \mathbb{C} \mid 0 \in \sigma_{ess}(L(s))\}.
\end{aligned}$$

If $s \in \mathbb{C} \setminus \sigma_{ess}(\mathcal{A})$ then

$$\operatorname{nul}(\mathcal{A} - sI) = \operatorname{nul} L(s) \quad \textit{and} \quad \operatorname{def}(\mathcal{A} - sI) = \operatorname{def} L(s).$$

Proof. The operators $\mathcal{A}$ and $L(0)$ are boundedly invertible (cf. Proposition 2.1) and the assertions of Proposition 2.2 are true for the case $s = 0$.

In the sequel we assume $s \neq 0$. We consider the operator pencil

$$M(s) := -A_0^{-1}D - sI - \frac{1}{s}A_0^{-1}, \qquad s \in \mathbb{C} \setminus \{0\}.$$

Here $M(s)$ is considered as a bounded operator acting on $H_{\frac{1}{2}}$. We have

$$\mathcal{A}^{-1} - sI = \begin{bmatrix} I & \frac{1}{s}A_0^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} M(s) & 0 \\ 0 & -sI \end{bmatrix} \begin{bmatrix} I & 0 \\ -\frac{1}{s}I & I \end{bmatrix}. \tag{6}$$

The first and the third matrix of the right hand side of the above equation (6) considered as operators acting from $H_{\frac{1}{2}} \times H$ into $H_{\frac{1}{2}} \times H$ are isomorphisms. A complex number $s \in \mathbb{C}$, $s \neq 0$, belongs to the spectrum, point spectrum or essential spectrum of $\mathcal{A}$ if and only if $\frac{1}{s}$ belongs to the spectrum, point spectrum or essential spectrum of $\mathcal{A}^{-1}$, respectively. By (6), this is equivalent to the fact that zero belongs to the spectrum, point spectrum or essential spectrum of $M(\frac{1}{s})$, respectively. Moreover, we have

$$\begin{aligned}
&\operatorname{nul}(\mathcal{A} - sI) = \operatorname{nul}(\mathcal{A}^{-1} - s^{-1}I) = \operatorname{nul} M(s^{-1}) \quad \text{and} \\
&\operatorname{def}(\mathcal{A} - sI) = \operatorname{def}(\mathcal{A}^{-1} - s^{-1}I) = \operatorname{def} M(s^{-1}).
\end{aligned}$$

Observe, that $A_0^{\frac{1}{2}}$ maps $H_{\frac{1}{2}}$ isometrically onto H. Then the assertions of Proposition 2.2 follow from

$$L(s) = -sA_0^{\frac{1}{2}}M(s^{-1})A_0^{-\frac{1}{2}}, \qquad s \in \mathbb{C} \setminus \{0\}. \qquad \square$$

COROLLARY 2.3. *If $D = 0$ then $\sigma(\mathcal{A}) = \sigma_{ap}(\mathcal{A}) \subset i\mathbb{R}$. Moreover, $i\eta \in \sigma(\mathcal{A})$, $\eta \in \mathbb{R}$, if and only if $\eta^2 \in \sigma(A_0)$*

Proof. We have

$$L(s) = s^2A_0^{-1} + I \qquad s \in \mathbb{C}.$$

Hence, by Proposition 2.2, we have $\sigma(\mathcal{A}) \subset i\mathbb{R}$ and $i\eta \in \sigma(\mathcal{A})$ if and only if $\eta^2 \in \sigma(A_0)$. Thus every point of $\sigma(A)$ is an element of the boundary of $\sigma(A)$, which proves $\sigma(A) = \sigma_{ap}(A)$, see [10, page 242]. $\square$

3. Location of the spectrum of $\mathcal{A}$

The following theorem is well known, see e.g. [3], [20], [4], [8], [14] or [25].

THEOREM 3.1. *The operator $\mathcal{A}$ is the generator of a strongly continuous semi-group $(T(t))_{t\geqslant 0}$ of contractions on the state space $H_{\frac{1}{2}} \times H$.*

This guarantees that the spectrum of $\mathcal{A}$ is contained in the closed left half plane. Moreover, by Proposition 2.1, $0 \in \rho(\mathcal{A})$. However, otherwise the spectrum of $\mathcal{A}$ is quite arbitrary. In particular, it may happen that $\sigma(\mathcal{A}) = \{s \in \mathbb{C} \mid \operatorname{Re} s \leqslant 0, |s| \geqslant \varepsilon\}$, $\varepsilon > 0$, as the following example shows.

EXAMPLE 3.2. Let $H = L^2(0,\infty)$, let $\varepsilon > 0$ and let $\{q_j\}_{j\in\mathbb{N}} \subset \mathbb{R}$ be a sequence satisfying $\{q_j\}_{j\in\mathbb{N}} = \mathbb{Q}$. We define $a : (0,\infty) \to \mathbb{R}$ and $d : (0,\infty) \to [0,\infty)$ by

$$a(x) := q_j \quad \text{if} \quad j-1 < x \leqslant j, \quad j \in \mathbb{N},$$

and

$$d(x) := \begin{cases} \frac{1}{x-j+1} - 1 & j-1 < x \leqslant j \text{ and } |q_j| \geqslant \varepsilon, \\ \frac{1}{x-j+1} - 1 + \sqrt{\varepsilon^2 - q_j^2} & j-1 < x \leqslant j \text{ and } |q_j| < \varepsilon. \end{cases}$$

Further, the function $a_0 : (0,\infty) \to (0,\infty)$ is defined by $a_0(x) := a(x)^2 + d(x)^2$, $x \in (0,\infty)$. For $x \in (0,\infty)$ we have

$$a_0(x) = a(x)^2 + d(x)^2 \geqslant \varepsilon^2. \tag{7}$$

If $d(x) \geqslant 2$ we have

$$2d(x) \leqslant a(x)^2 + d(x)^2 \tag{8}$$

and if $d(x) < 2$ we have

$$2d(x) = 2d(x)\frac{\varepsilon^2}{\varepsilon^2} \leqslant \frac{2}{\varepsilon^2}(a(x)^2 + d(x)^2). \tag{9}$$

Set

$$\mathscr{D}(A_0) := \{f \in H \mid a_0 f \in H\}.$$

It follows from (7), (8) and (9) that the operators $A_0 : \mathscr{D}(A_0) \subset H \to H$ and $D : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$, defined by

$$\begin{aligned} (A_0 f)(x) &:= a_0(x) f(x), \quad x \in (0,\infty), f \in \mathscr{D}(A_0), \\ (Dg)(x) &:= 2d(x) g(x), \quad x \in (0,\infty), g \in H_{\frac{1}{2}}, \end{aligned}$$

satisfy (A1) and (A2). Since

$$((s^2 A_0^{-1} + s A_0^{-1/2} D A_0^{-1/2} + I) f)(x) = \left(\frac{s^2 + 2sd(x) + a(x)^2 + d(x)^2}{a(x)^2 + d(x)^2} \right) f(x),$$

Proposition 2.2 implies

$$\{-d(x) \pm i a(x) \mid x \in [0,\infty)\} \subset \sigma(\mathcal{A}).$$

Thus $\sigma(\mathcal{A}) = \sigma_{ess}(\mathcal{A}) = \{s \in \mathbb{C} \mid \operatorname{Re} s \leqslant 0, |s| \geqslant \varepsilon\}$.

Next we give sufficient conditions guaranteeing that $\sigma(\mathcal{A})$ is contained in a smaller subset of $\mathbb{C}$ than in the above example. We define the following constants:

$$\begin{aligned}
\beta &:= \inf_{z\in H_{\frac{1}{2}}\setminus\{0\}} \frac{\langle Dz,z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}}{\|z\|_H^2},\\
\gamma &:= \sup_{z\in H_{\frac{1}{2}}\setminus\{0\}} \frac{\langle Dz,z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}}{\|z\|_H^2},\\
\delta &:= \inf_{z\in H_{\frac{1}{2}}\setminus\{0\}} \frac{\langle Dz,z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}}{\|z\|_{H_{\frac{1}{2}}}^2}.
\end{aligned}$$

By definition we have $\beta,\delta\in[0,\infty)$, $\gamma\in[0,\infty]$, and it is easy to see that $\delta\leqslant\beta\leqslant\gamma$. It is well-known (see e.g. [3], [4], [5], [8], [14], [15], [24], [26]), that if $\beta>0$ then $\mathcal{A}$ generates an exponentially stable semigroup on $H_{\frac{1}{2}}\times H$. In particular, there exists a constant $\omega<0$ such that $\sigma(\mathcal{A})\subset\{s\in\mathbb{C}\mid \operatorname{Re}s\leqslant\omega\}$. For the constant ω there are quite a few upper estimates available. For example, in [8] it is shown that

$$\omega\leqslant\max\left\{-\frac{\beta}{2},\max\{\operatorname{Re}s\mid s\in\sigma(\mathcal{A})\}\right\},$$

and [5] proves

$$\omega\leqslant\max\left\{-\frac{\beta}{2},-\|\mathcal{A}^{-1}\|^{-1}\right\}.$$

We improve these estimates as follows.

THEOREM 3.3. *We have*

1. *If* $\beta>0$ *then*
$$\left\{\lambda\in\mathbb{C}\mid \operatorname{Re}\lambda>-\frac{\beta}{2},\operatorname{Im}\lambda\neq0\right\}\subset\rho(\mathcal{A}).$$
2. *If* $\gamma<\infty$ *then*
$$\left\{\lambda\in\mathbb{C}\mid \operatorname{Re}\lambda<-\frac{\gamma}{2},\operatorname{Im}\lambda\neq0\right\}\subset\rho(\mathcal{A}).$$
3. *If* $\delta>0$ *then*
$$\sigma(\mathcal{A})\subset\left\{\lambda\in\mathbb{C}\mid |\lambda+\frac{1}{\delta}|\leqslant\frac{1}{\delta}\right\}\cup(-\infty,0).$$
4. *If* $\langle Dz,z\rangle^2_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}\geqslant 4\|z\|_H^2\|z\|^2_{H_{\frac{1}{2}}}$, $z\in H_{\frac{1}{2}}$, *then*
$$\sigma(\mathcal{A})\subset(-\infty,0).$$

Under the extra assumption that A_0 has a compact resolvent and that the operator D is A_0-compact, Part 1 and Part 2 of Theorem 3.3 can be found in [19]. The proof of Theorem 3.3 will be given at the end of this section. We show first that the first two statement of the theorem in general cannot be improved.

EXAMPLE 3.4. Let $0 < \beta \leqslant \gamma < \infty$ be arbitrarily. We define the function $d : (0,\infty) \to [0,\infty)$ by

$$d(x) := \frac{1}{2}(\beta + (\gamma - \beta)(x - j + 1)) \ \text{ if } j - 1 < x \leqslant j, j \in \mathbb{N}.$$

Let H, a, a_0, A_0 and D be defined as in Example 3.2. Again, (A1) and (A2) hold. An easy calculation shows that

$$\beta = \inf_{z \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{\|z\|_H^2} \quad \text{and} \quad \gamma = \sup_{z \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{\|z\|_H^2}.$$

Then, as in Example 3.2, we have

$$\left\{ \lambda \in \mathbb{C} \mid \operatorname{Im} \lambda \neq 0, -\frac{1}{2}\gamma \leqslant \operatorname{Re} \lambda \leqslant -\frac{1}{2}\beta \right\} \subset \sigma(\mathcal{A}).$$

The following lemma is needed for the proof of Theorem 3.3.

LEMMA 3.5. *Let $\lambda = \mu + i\sigma$ with $\sigma \in \mathbb{R}$, $\mu \leqslant 0$ and $\lambda \neq 0$. Assume that there exists a sequence $\left\{ \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right\}_{n \in \mathbb{N}}$ in $\mathscr{D}(\mathcal{A})$ with*

$$\left\| \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right\|_{H_{\frac{1}{2}} \times H} = 1 \quad \text{and} \quad \lim_{n \to \infty} \left\| (\lambda I - \mathcal{A}) \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right\|_{H_{\frac{1}{2}} \times H} = 0. \tag{10}$$

Then we have

1. $\|y_n - \lambda x_n\|_{H_{\frac{1}{2}}} \to 0$ *as* $n \to \infty$.
2. $\liminf_{n \to \infty} \|x_n\|_{H_{\frac{1}{2}}} > 0$.
3. *If $\sigma \neq 0$, then we have*

$$\langle A_0^{-1} D x_n, x_n \rangle_{H_{\frac{1}{2}}} + \frac{2\mu}{\mu^2 + \sigma^2} \|x_n\|_{H_{\frac{1}{2}}}^2 \to 0, \quad n \to \infty, \tag{11}$$

$$\langle A_0^{-1} D x_n, x_n \rangle_{H_{\frac{1}{2}}} + 2\mu \|x_n\|^2 \to 0, \quad n \to \infty, \tag{12}$$

and

$$\langle A_0 x_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} - (\mu^2 + \sigma^2) \|x_n\|^2 \to 0, \quad n \to \infty, \tag{13}$$

4. *If $\sigma = 0$, then we have*

$$\|x_n\|_{H_{\frac{1}{2}}}^2 + \mu \langle A_0^{-1} D x_n, x_n \rangle_{H_{\frac{1}{2}}} + \mu^2 \|x_n\|^2 \to 0, \quad n \to \infty.$$

Proof. (10) implies

$$\|y_n - \lambda x_n\|_{H_{\frac{1}{2}}} \to 0 \text{ and} \tag{14}$$

$$\|A_0 x_n + D y_n + \lambda y_n\| \to 0 \text{ as } n \to \infty. \tag{15}$$

It follows from (14) that $\{x_n\}$ has no subsequence which converges to zero in $H_{\frac{1}{2}}$. Thus Part 1 and Part 2 are shown. Combining (14) and (15) we get

$$\langle A_0x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + (\mu+i\sigma)\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + (\mu+i\sigma)^2\langle x_n, x_n\rangle \to 0, \quad (16)$$

as $n\to\infty$. This implies the result for $\sigma=0$. It remains to show Part 3. Let $\sigma\neq 0$. Then the imaginary part of (16) tends to zero, i.e.

$$\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + 2\mu\langle x_n, x_n\rangle = \langle A_0^{-1}Dx_n, x_n\rangle_{H_{\frac{1}{2}}} + 2\mu\|x_n\|^2 \to 0, \quad (17)$$

as $n\to\infty$, which proves (12). Further, the real part tends to zero, i.e.

$$\langle A_0x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \mu\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + (\mu^2-\sigma^2)\langle x_n, x_n\rangle \to 0, \quad (18)$$

as $n\to\infty$. Combining (17) and (18), we obtain (13). Finally, (17) together with (13) implies (11). $\square$

Proof of Theorem 3.3. Let $\lambda=\mu+i\sigma$ with $\mu\leqslant 0$ and $\sigma\neq 0$. Assume that λ belongs to the spectrum of $\mathcal{A}$. Then, by Proposition 2.1, $\overline{\lambda}\in\sigma(\mathcal{A})$.

Since $\mathcal{A}$ is a self-adjoint operator in the Krein space $(H_{\frac{1}{2}}\times H, [\cdot,\cdot])$, see (3), it follows from [7, Theorem VI.6.1] that at least one of the points $\lambda,\overline{\lambda}$ belongs to $\sigma_{ap}(\mathcal{A})$. Let us assume $\lambda\in\sigma_{ap}(\mathcal{A})$. Then there exists a sequence $\left\{\begin{pmatrix} x_n \\ y_n\end{pmatrix}\right\}_{n\in\mathbb{N}}$ in $\mathscr{D}(\mathcal{A})$ which satisfies (10). Lemma 3.5 implies $\liminf_{n\to\infty}\|x_n\|_{H_{\frac{1}{2}}} > 0$ and

$$\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + 2\mu\langle x_n, x_n\rangle \to 0, \quad \text{as } n\to\infty.$$

1. Let $\mu > -\frac{\beta}{2}$. Then we have
$$\lim_{n\to\infty}\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = \lim_{n\to\infty}\langle x_n, x_n\rangle = 0.$$
Then Lemma 3.5, Part 3, implies
$$\lim_{n\to\infty}\langle A_0x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = \lim_{n\to\infty}\|x_n\|_{H_{\frac{1}{2}}} = 0,$$
a contradiction.
2. Let $\mu < -\frac{\gamma}{2}$. Then we have
$$\lim_{n\to\infty}\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = \lim_{n\to\infty}\langle x_n, x_n\rangle = 0.$$
As in Part 1 this leads to a contradiction.
3. Lemma 3.5, Part 3, implies
$$-2\mu \geqslant \delta(\mu^2+\sigma^2),$$
and thus the statement follows.

4. If $\langle Dz, z\rangle^2_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} \geqslant 4\|z\|^2_H\|z\|^2_{H_{\frac{1}{2}}}$, $z \in H_{\frac{1}{2}}$ and $\sigma \neq 0$, it follows from Lemma 3.5, Part 2 and Part 3, that $\mu < 0$ holds. Moreover, by (13), we have $\liminf_{n\to\infty} \|x_n\| > 0$ and Lemma 3.5, Part 3, implies

$$\begin{aligned} 0 &\leqslant 4\|x_n\|^2_H\|x_n\|^2_{H_{\frac{1}{2}}} \leqslant \\ &\leqslant \liminf_{n\to\infty}\left(\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + (2\mu - 2\mu)\|x_n\|^2\right)\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = \\ &= \liminf_{n\to\infty} 2\mu\|x_n\|^2 \cdot \frac{2\mu}{\mu^2+\sigma^2}\|x_n\|^2_{H_{\frac{1}{2}}}. \end{aligned}$$

Hence,

$$1 \leqslant \frac{\mu^2}{\mu^2+\sigma^2},$$

a contradiction to $\sigma \neq 0$. □

4. Location of the essential spectrum of $\mathcal{A}$

In this section we consider the operator $A_0^{-1}D$ as an operator acting in $H_{\frac{1}{2}}$, that is, $A_0^{-1}D$ is a bounded self-adjoint operator acting in $H_{\frac{1}{2}}$. If the operator A_0 has a compact resolvent, then we have the following description of the essential spectrum of $\mathcal{A}$.

THEOREM 4.1. *If the operator A_0^{-1} is compact, then*

$$\sigma_{ess}(\mathcal{A}) = \left\{\lambda \in \mathbb{C}\backslash\{0\} \mid \frac{1}{\lambda} \in \sigma_{ess}(-A_0^{-1}D)\right\} \subset (-\infty, 0).$$

Here $A_0^{-1}D$ is considered as an operator acting in $H_{\frac{1}{2}}$.

For a special choice of the damping operator this theorem can be found in [19].

Proof. By Lemma 2.1 we have $0 \in \rho(\mathcal{A})$, hence

$$\sigma_{ess}(\mathcal{A}) = \{\lambda \in \mathbb{C}\backslash\{0\} \mid 1/\lambda \in \sigma_{ess}(\mathcal{A}^{-1})\}.$$

It remains to show $\sigma_{ess}(\mathcal{A}^{-1})\backslash\{0\} = \sigma_{ess}(-A_0^{-1}D)\backslash\{0\}$. The operator $\mathcal{A}^{-1}$ is given by (4). The operator I is a compact linear operator from $H_{\frac{1}{2}}$ to H, and $-A_0^{-1}$ is a compact operator from H to $H_{\frac{1}{2}}$. Since the essential spectrum of an operator remains unchanged under compact perturbations, we have $\sigma_{ess}(\mathcal{A}^{-1})\backslash\{0\} = \sigma_{ess}(-A_0^{-1}D)\backslash\{0\}$. □

The theorem above implies a criterion for the emptiness of the essential spectrum of $\mathcal{A}$.

COROLLARY 4.2. *Assume that A_0^{-1} is a compact operator and that the operator D is a compact operator acting from $H_{\frac{1}{2}}$ into $H_{-\frac{1}{2}}$. Then*

$$\sigma_{ess}(\mathcal{A}) = \emptyset.$$

In particular, if the operator D is a bounded operator acting from $H_{\frac{1}{2}}$ into H_α for some $\alpha > -\frac{1}{2}$, then $\sigma_{ess}(\mathcal{A}) = \emptyset$.

REMARK 4.3. Note that $\mathcal{A}$ does not necessarily have a compact resolvent if A_0 has a compact resolvent. Indeed, if we choose $A_0 = D$ then, by Theorem 4.1, $\sigma_{ess}(\mathcal{A}) = \{-1\}$, hence $-1 \in \sigma_{ess}(\mathcal{A}^{-1})$ and $\mathcal{A}^{-1}$ is not a compact operator.

Without the assumption that A_0 has a compact resolvent, it may happen that the essential spectrum of $\mathcal{A}$ is quite arbitrary in the closed left half plane, see Example 3.2. The following theorem shows under weaker assumptions that the non-real essential spectrum of $\mathcal{A}$ is located in a certain strip parallel to the imaginary axis.

THEOREM 4.4. *If $\sigma_{ess}(A_0^{-1}D) = \emptyset$ then we set $\alpha_1 := \infty$ and $\gamma_1 := 0$. If $0 \in \sigma_{ess}(A_0^{-1}D)$ we set $\gamma_1 := \infty$. Otherwise, let*

$$\alpha_1 := \frac{1}{2\|A_0^{-1}\|} \min\{s \in \mathbb{R} \mid s \in \sigma_{ess}(A_0^{-1}D)\},$$

and

$$\gamma_1 := 2\frac{1}{\min\{s \in \mathbb{R} \mid s \in \sigma_{ess}(A_0^{-1}D)\}}.$$

Here $\|A_0^{-1}\|$ is the operator norm of A_0^{-1} considered as an operator acting in H and $A_0^{-1}D$ is considered as an operator acting in $H_{\frac{1}{2}}$. Then we have:

1. *$\sigma_{ess}(\mathcal{A}) \subset (-\infty, 0) \cup \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leqslant -\alpha_1\}$.*
2. *If $\rho(\mathcal{A}) \cap \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < -\gamma_1, \operatorname{Im}\lambda \neq 0\} \neq \emptyset$, then*

$$\sigma_{ess}(\mathcal{A}) \subset (-\infty, 0) \cup \{\lambda \in \mathbb{C} \mid 0 \geqslant \operatorname{Re}\lambda \geqslant -\gamma_1\}.$$

Proof. We have, by Theorem 3.1, $\sigma_{ess}(\mathcal{A}) \subset \sigma(\mathcal{A}) \subset \{s \in \mathbb{C} \mid \operatorname{Re} s \leqslant 0\}$.

1. We set

$$\mathscr{U} := \left\{\lambda \in \mathbb{C} \mid 0 \geqslant \operatorname{Re}\lambda > -\frac{\min\{s \in \mathbb{R} \mid s \in \sigma_{ess}(A_0^{-1}D)\}}{2\|A_0^{-1}\|}, \operatorname{Im}\lambda \neq 0\right\}.$$

Let $\lambda \in \mathscr{U}$, $\lambda = \mu + i\sigma$. Then $-2\mu\|A_0^{-1}\| < \min\{s \in \mathbb{R} \mid s \in \sigma_{ess}(A_0^{-1}D)\}$ and

$$G_\lambda := \operatorname{span}\{x \in H_{\frac{1}{2}} \mid A_0^{-1}Dx = \nu x, \nu \leqslant -2\mu\|A_0^{-1}\|\} \tag{19}$$

is a finite dimensional subspace of $H_{\frac{1}{2}}$. Assume that there exists a sequence $\left\{\binom{x_n}{y_n}\right\}_{n\in\mathbb{N}}$ in $\mathscr{D}(\mathcal{A}) \cap (G_\lambda \times G_\lambda)^\perp$ which satisfies (10). Then by Lemma 3.5, Part 3,

$$\langle A_0^{-1}Dx_n, x_n\rangle_{H_{\frac{1}{2}}} + 2\mu\langle x_n, x_n\rangle \to 0, \tag{20}$$

as $n \to \infty$. We have $\{x_n\} \subset G_\lambda^\perp$, hence there exists a $\delta > 0$ with

$$\begin{aligned}\langle A_0^{-1}Dx_n, x_n\rangle_{H_{\frac{1}{2}}} &\geqslant (-2\mu + \delta)\|A_0^{-1}\|\|x_n\|_{H_{\frac{1}{2}}}^2 \\ &\geqslant -2\mu\langle x_n, x_n\rangle + \delta\|A_0^{-1}\|\|x_n\|_{H_{\frac{1}{2}}}^2.\end{aligned} \tag{21}$$

This, together with $\liminf_{n\to\infty} \|x_n\|_{H_{\frac{1}{2}}} > 0$ (see Lemma 3.5), contradicts (20). Therefore for every $\lambda \in \mathscr{U}$ there exists a finite dimensional subspace G_λ and a constant $c_\lambda > 0$ such that for all $\binom{x}{y}$ in $\mathscr{D}(\mathcal{A}) \cap (G_\lambda \times G_\lambda)^\perp$ we have

$$\left\|(\mathcal{A}-\lambda I)\binom{x}{y}\right\|_{H_{\frac{1}{2}}\times H} \geqslant c_\lambda \left\|\binom{x}{y}\right\|_{H_{\frac{1}{2}}\times H}. \tag{22}$$

Then by Proposition 2.1 and [18, IV.§5.6] it follows that $\mathcal{A}-\lambda I$ is a Fredholm operator of index 0 for all $\lambda \in \mathscr{U}$ and that there exists a discrete set Ξ in $\mathscr{U}$, $\Xi \subset \sigma_{p,norm}(\mathcal{A})$, with

$$\mathscr{U} \setminus \Xi \subset \rho(\mathcal{A}).$$

Hence, the first assertion of Theorem 4.4 is proved.

2. The assertion is true if $\gamma_1 = \infty$. Let $\gamma_1 < \infty$ and define then

$$\mathscr{U} := \left\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < -\frac{2}{\min\{s \in \mathbb{R} \mid s \in \sigma_{ess}(A_0^{-1}D)\}}, \operatorname{Im}\lambda \neq 0\right\}.$$

Let $\lambda \in \mathscr{U}$, $\lambda = \mu + i\sigma$. Then

$$-\frac{2\mu}{\mu^2+\sigma^2} < -\frac{2}{\mu} < \min\{s \in \mathbb{R} \mid s \in \sigma_{ess}(A_0^{-1}D)\}$$

and thus

$$G_\lambda := \operatorname{span}\left\{x \in H_{\frac{1}{2}} \mid A_0^{-1}Dx = \nu x, \nu \leqslant -2\frac{\mu}{\mu^2+\sigma^2}\right\} \tag{23}$$

is a finite dimensional subspace of $H_{\frac{1}{2}}$. Assume that there exists a sequence $\left\{\binom{x_n}{y_n}\right\}_{n\in\mathbb{N}}$ in $\mathscr{D}(\mathcal{A}) \cap (G_\lambda \times G_\lambda)^\perp$ which satisfies (10). Then Lemma 3.5, Part 3, implies

$$\langle A_0^{-1}Dx_n, x_n\rangle_{H_{\frac{1}{2}}} + \frac{2\mu}{\mu^2+\sigma^2}\|x_n\|^2_{H_{\frac{1}{2}}} \to 0, \tag{24}$$

as $n \to \infty$. We have $\{x_n\} \subset G_\lambda^\perp$, hence there exists a $\delta > 0$ with

$$\langle A_0^{-1}Dx_n, x_n\rangle_{H_{\frac{1}{2}}} \geqslant \left(-2\frac{\mu}{\mu^2+\sigma^2} + \delta\right)\|x_n\|^2_{H_{\frac{1}{2}}}. \tag{25}$$

This, together with $\liminf_{n\to\infty} \|x_n\|_{H_{\frac{1}{2}}} > 0$ (see Lemma 3.5, Part 2), contradicts (24). Therefore for every $\lambda \in \mathscr{U}$ there exists a finite dimensional subspace G_λ and a constant $c_\lambda > 0$ such that for all $\binom{x}{y}$ in $\mathscr{D}(\mathcal{A}) \cap (G_\lambda \times G_\lambda)^\perp$ we have

$$\left\|(\mathcal{A}-\lambda I)\binom{x}{y}\right\|_{H_{\frac{1}{2}}\times H} \geqslant c_\lambda \left\|\binom{x}{y}\right\|_{H_{\frac{1}{2}}\times H}. \tag{26}$$

By the fact that $\mathscr{U} \cap \rho(\mathcal{A}) \neq \emptyset$ and [18, IV.§5.6] it follows that $\mathcal{A}-\lambda I$ is a Fredholm operator of index 0 for all $\lambda \in \mathscr{U}$ and that there exists a discrete set Ξ in $\mathscr{U}$, $\Xi \subset \sigma_{p,norm}(\mathcal{A})$, with

$$\mathscr{U} \setminus \Xi \subset \rho(\mathcal{A}). \qquad \square$$

COROLLARY 4.5. *Let* α_1 *and* γ_1 *be defined as in Theorem* 4.4. *If* $\alpha_1 > \gamma_1$ *then*

$$\sigma_{ess}(\mathcal{A}) \subset (-\infty, 0).$$

An example for $\alpha_1 > \gamma_1$ can be found in Example 5.4.

5. Accumulation points of the non-real spectrum of $\mathcal{A}$

In this section we give sufficient conditions guaranteeing that certain intervals do not contain any accumulation point of the non-real spectrum of $\mathcal{A}$.

THEOREM 5.1. *Set* $\alpha_2 := \infty$ *if* $\sigma_{ess}(A_0^{\frac{1}{2}}) = \emptyset$ *and*

$$\alpha_2 := \min\{s \in \mathbb{R} \mid s \in \sigma_{ess}(A_0^{\frac{1}{2}})\}, \text{ otherwise.}$$

Moreover, let α_1 *be defined as in Theorem* 4.4. *Set* $\alpha := \max\{\alpha_1, \alpha_2\}$.

Then no point of the interval $(-\alpha, 0)$ *is an accumulation point of the non-real spectrum of* $\mathcal{A}$.

As a corollary of Theorem 5.1 we have

COROLLARY 5.2. *Assume that* A_0^{-1} *is a compact operator. Then no point from* $\sigma_{ess}(\mathcal{A})$ *is an accumulation point of the non-real spectrum of* $\mathcal{A}$.

Proof of Theorem 5.1. First of all we show that no point of the interval $(-\alpha_1, 0)$ is an accumulation point of non-real spectrum. Let $[\cdot,\cdot]$ be defined as in (3).

Let $\lambda \in (-\alpha_1, 0)$ and choose G_λ as in (19). For every sequence $\left\{\binom{x_n}{y_n}\right\}_{n\in\mathbb{N}}$ in $\mathscr{D}(\mathcal{A}) \cap (G_\lambda \times G_\lambda)^\perp$ which satisfies (10) it follows from Lemma 3.5, Part 1 and Part 4, that

$$\begin{aligned}
\liminf_{n\to\infty}\left[\binom{x_n}{y_n},\binom{x_n}{y_n}\right] &= \liminf_{n\to\infty}\Big(\langle x_n, x_n\rangle_{H_{\frac{1}{2}}} - \langle y_n, y_n\rangle\Big)\\
&= \liminf_{n\to\infty}\Big(\langle x_n, x_n\rangle_{H_{\frac{1}{2}}} - \lambda^2\langle x_n, x_n\rangle\Big)\\
&= -\lambda\liminf_{n\to\infty}\Big(\langle A_0^{-1}Dx_n, x_n\rangle_{H_{\frac{1}{2}}} + 2\lambda\|x_n\|^2\Big).
\end{aligned}$$

Using (21) we get

$$\liminf_{n\to\infty}\left[\binom{x_n}{y_n},\binom{x_n}{y_n}\right] > 0.$$

This gives $(-\alpha_1, 0] \cap \sigma(\mathcal{A}) \subset \sigma_{\pi_+}(\mathcal{A})$ (for a definition of $\sigma_{\pi_+}(\mathcal{A})$ we refer to [2]) and, by [2, Theorem 18], it follows that no point of the interval $(-\alpha_1, 0)$ is an accumulation point of non-real spectrum.

We now choose $\mu \in (-\alpha_2, 0)$ and set $\lambda = \mu + i\sigma$ for some $\sigma \neq 0$. Then

$$G_\lambda := \operatorname{span}\{x \in H_{\frac{1}{2}} \mid A_0^{\frac{1}{2}}x = \nu x, \nu \leqslant -\mu\} \tag{27}$$

is a finite dimensional subspace of $H_{\frac{1}{2}}$. Assume that there exists a sequence $\left\{\binom{x_n}{y_n}\right\}_{n\in\mathbb{N}}$ in $\mathscr{D}(\mathcal{A})\cap(G_\lambda\times G_\lambda)^\perp$ which satisfies (10). Lemma 3.5 implies $\liminf_{n\to\infty}\|x_n\|_{H_{\frac{1}{2}}} > 0$ and

$$\langle A_0x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} - (\mu^2+\sigma^2)\langle x_n, x_n\rangle \to 0, \quad \text{as } n\to\infty.$$

As x_n belongs to $G_\lambda^\perp$, $n\in\mathbb{N}$, there exists a $\delta>0$ with

$$\langle A_0x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = \|A_0^{\frac{1}{2}}x_n\|^2 \geqslant (\mu^2+\delta^2)\|x_n\|^2$$

and, if $|\sigma|<\delta$, it follows

$$\lim_{n\to\infty}\langle A_0x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = \lim_{n\to\infty}\|x_n\|_{H_{\frac{1}{2}}} = 0,$$

a contradiction. Therefore there exists an open neighbourhood $\mathscr{V}$ in $\mathbb{C}$ of $(-\alpha_2, 0)$ such that for every $\lambda\in\mathscr{V}\setminus(-\alpha_2,0)$ there exists a finite dimensional subspace G_λ and a constant $c_\lambda>0$ such that for all $\binom{x}{y}$ in $\mathscr{D}(\mathcal{A})\cap(G_\lambda\times G_\lambda)^\perp$ relation (22) holds. Then by Lemma 2.1 and [18, IV.§5.6] there exists a discrete set $\widetilde{\Xi}$ in $\mathscr{V}\setminus(-\alpha_2,0)$ with

$$\mathscr{V}\setminus\widetilde{\Xi}\subset\rho(\mathcal{A})\cup(-\alpha_2,0).$$

We now show that no point of the interval $(-\alpha_2,0)$ is an accumulation point of points from $\widetilde{\Xi}$. Again, we consider the Krein space $(H_{\frac{1}{2}}\times H, [\cdot,\cdot])$ defined in (3).

Let $\lambda\in(-\alpha_2,0)$ and choose G_λ as in (27). For every sequence $\left\{\binom{x_n}{y_n}\right\}_{n\in\mathbb{N}}$ in $\mathscr{D}(\mathcal{A})\cap(G_\lambda\times G_\lambda)^\perp$ which satisfies (10) it follows from Lemma 3.5, Part 1,

$$\begin{aligned}\liminf_{n\to\infty}\left[\binom{x_n}{y_n},\binom{x_n}{y_n}\right] &= \liminf_{n\to\infty}\left(\langle x_n, x_n\rangle_{H_{\frac{1}{2}}} - \langle y_n, y_n\rangle\right)\\ &= \liminf_{n\to\infty}\left(\langle A_0x_n, x_n\rangle - \lambda^2\langle x_n, x_n\rangle\right)\\ &= \liminf_{n\to\infty}\left(\|A_0^{\frac{1}{2}}x_n\|^2 - \lambda^2\|x_n\|^2\right) > 0,\end{aligned}$$

as x_n belongs to $G_\lambda^\perp$, $n\in\mathbb{N}$. This implies $(-\alpha_2,0]\cap\sigma(\mathcal{A})\subset\sigma_{\pi_+}(\mathcal{A})$ and, by [2, Theorem 18], no point of the interval $(-\alpha_2,0)$ is an accumulation point of non-real spectrum. □

In Section 3. we considered the numbers β,γ and δ. For these quantities we have

$$\delta\leqslant\beta\leqslant\gamma.$$

The following examples show that there is no such relationship between α_1,α_2, β and γ_1.

EXAMPLE 5.3. *Let H be an infinite-dimensional Hilbert space, $D=0$ and $A_0=I$. Then we have*

$$\alpha_1 = 0, \quad \alpha_2 = 1, \quad \frac{\beta}{2} = 0 \quad \textit{and} \quad \gamma_1 = \infty.$$

EXAMPLE 5.4. *Let H be an infinite-dimensional Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. We define the operators A_0 and D in $\mathscr{L}(H)$ by*

$$A_0 z := 9\sum_{n=1}^{\infty}(1+n^{-1})\langle z, e_n\rangle e_n$$

$$Dz := 9\sum_{n=2}^{\infty}(1+n^{-1})\langle z, e_n\rangle e_n$$

Then we have

$$\alpha_1 = \frac{9}{2}, \quad \alpha_2 = 3, \quad \frac{\beta}{2} = 0 \quad \text{and} \quad \gamma_1 = 2.$$

EXAMPLE 5.5. *Let H and $\{e_n\}$ be as in Example* 5.4. *We define the operators A_0 and D in $\mathscr{L}(H)$ by*

$$A_0 z := \langle z, e_1\rangle e_1 + 9\sum_{n=2}^{\infty}(1+n^{-1})\langle z, e_n\rangle e_n$$

$$Dz := 9\sum_{n=1}^{\infty}(1+n^{-1})\langle z, e_n\rangle e_n$$

Then we have

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = 3, \quad \frac{\beta}{2} = \frac{9}{2} \quad \text{and} \quad \gamma_1 = 2.$$

Moreover, it turns out that the non-real spectrum of $\mathcal{A}$ can only accumulate to ∞.

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