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ORDER REDUCTION OF DISCRETE–TIME ALGEBRAIC RICCATI EQUATIONS WITH SINGULAR CLOSED LOOP MATRIX

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Abstract. We study the general discrete-time algebraic Riccati equation and deal with the case where the closed loop matrix corresponding to an arbitrary solution is singular. In this case the extended symplectic pencil associated with the DARE has 0 as a characteristic root and the corresponding spectral deflating subspace gives rise to a subspace where all solutions of the DARE coincide. This allows for a reduction of the original DARE to an equation of smaller size.

1. Introduction

In this paper we consider a discrete-time algebraic Riccati equation (DARE)

$$\begin{aligned}\mathscr{D}(X) &= \mathscr{D}(X;F,G,Q,R,S) \\ &= X - F^*XF + (G^*XF+S)^*(R+G^*XG)^{-1}(G^*XF+S) - Q = 0\end{aligned} \tag{1.1}$$

where $F \in \mathbb{C}^{n\times n}$, $S \in \mathbb{C}^{m\times n}$, $G \in \mathbb{C}^{n\times m}$, $Q = Q^* \in \mathbb{C}^{n\times n}$, $R = R^* \in \mathbb{C}^{m\times m}$. It is well known (see e.g. [3], [1], [12], [10]) that algebraic Riccati equations play an important role in theories of control, filtering and estimation. Here we are concerned with hermitian solutions of (1.1), and we assume that a hermitian solution of (1.1) exists. We refer to [12], [4] for existence results on DAREs. No assumptions on the definiteness of Q or R are made. If $X = X^*$ we put

$$F_X = F - G(R+G^*XG)^{-1}(G^*XF+S). \tag{1.2}$$

If X is a solution of (1.1) then F_X is the the *closed loop matrix* associated with X. Suppose F_X is singular. Let $\mathscr{U}_0 := \operatorname{Ker}(F_X)^n$ be the generalized eigenspace corresponding to the eigenvalue 0 of F_X. This space, which can be defined independently of X, will play a crucial role in our study. It is the purpose of our paper to employ the space $\mathscr{U}_0$ to set up an equivalent DARE (E) of order $n - \dim\mathscr{U}_0$ such that each solution X of (1.1) can be obtained from a corresponding solution X_E of the reduced equation (E). That order reduction will be derived in Section 3.. It will be shown in

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Section 2. that $\mathscr{U}_0$ does not depend on a particular solution X and that all solutions of (1.1) coincide on $\mathscr{U}_0$. We determine $\mathscr{U}_0$ using the *extended symplectic pencil*

$$M - sL = \begin{bmatrix} F & 0 & G \\ Q & I & S^* \\ S & 0 & R \end{bmatrix} - s \begin{bmatrix} I & 0 & 0 \\ 0 & F^* & 0 \\ 0 & G^* & 0 \end{bmatrix} \tag{1.3}$$

associated with the DARE (1.1).

2. The space $\mathscr{U}_0$

The following result is essential for the proposed order reduction.

PROPOSITION 2.1. *Let X and Y be solutions of (1.1). Then*

$$\operatorname{Ker}(F_X)^n = \operatorname{Ker}(F_Y)^n.$$

Let $\mathscr{U}_0 = \operatorname{Ker}(F_X)^n$. *Set* $\Delta = X - Y$. *Then*

$$\mathscr{U}_0 \subseteq \operatorname{Ker}\Delta, \tag{2.1}$$

and

$$F_X = F_Y \text{ on } \mathscr{U}_0. \tag{2.2}$$

The proof of the proposition requires two lemmas.

LEMMA 2.2. *Let $X,\ Y \in \mathbb{C}^{n\times n}$ be hermitian and set $\Delta = X - Y$. Then*

$$\mathscr{D}(X) - \mathscr{D}(Y) = \Delta - F_Y^* \Delta F_X. \tag{2.3}$$

Proof. We have

$$\begin{aligned} F_Y - F_X &= G(R + G^*XG)^{-1} \cdot \\ &\qquad \left[(G^*XF + S) - (R + G^*XG)(R + G^*YG)^{-1}(G^*YF + S)\right]. \end{aligned}$$

Because of $R + G^*XG = R + G^*YG + G^*\Delta G$ we obtain

$$\begin{aligned} F_Y - F_X &= G(R + G^*XG)^{-1} \cdot \left[G^*\Delta F - G^*\Delta(R + G^*YG)^{-1}(G^*YF + S)\right] \\ &= G(R + G^*XG)^{-1} G^*\Delta F_Y. \end{aligned}$$

Hence

$$F_X = F_Y - G(R + G^*XG)^{-1} G^*\Delta F_Y. \tag{2.4}$$

From [1, p.382] it is known that

$$\mathscr{D}(X) - \mathscr{D}(Y) = \Delta - F_Y^*\Delta F_Y + F_Y^*\Delta G(R + G^*XG)^{-1} G^*\Delta F_Y. \tag{2.5}$$

Now (2.3) follows from (2.5) and (2.4). $\square$

LEMMA 2.3. *Let $X \in \mathbb{C}^{n\times n}$ be hermitian. Then*

$$X - F_X^* X F_X = \mathscr{D}(X) + \begin{bmatrix} I & -Z^* \end{bmatrix} \begin{bmatrix} Q & S^* \\ S & R \end{bmatrix} \begin{bmatrix} I \\ -Z \end{bmatrix} \tag{2.6}$$

where

$$Z = (R + G^*XG)^{-1}(G^*XF + S). \tag{2.7}$$

Proof. Set

$$\Lambda(X) = \begin{bmatrix} -X + F^*XF + Q & S^* + F^*XG \\ S + G^*XF & R + G^*XG \end{bmatrix}.$$

Then a straightforward computation yields

$$\begin{bmatrix} I & -Z^* \end{bmatrix} \Lambda(X) \begin{bmatrix} I \\ -Z \end{bmatrix} = -\mathscr{D}(X).$$

Writing $\Lambda(X)$ as

$$\Lambda(X) = \begin{bmatrix} -X & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} F^* \\ G^* \end{bmatrix} X \begin{bmatrix} F & G \end{bmatrix} + \begin{bmatrix} Q & S^* \\ S & R \end{bmatrix}$$

and taking into account that

$$\begin{bmatrix} F & G \end{bmatrix} \begin{bmatrix} I \\ -Z \end{bmatrix} = F_X$$

we readily obtain (2.6). □

Proof of Proposition 2.1. If $T \in \mathbb{C}^{n\times n}$ is nonsingular and $\tilde{F} = T^{-1}FT$, $\tilde{G} = T^{-1}G$, $\tilde{Q} = T^*QT$, $\tilde{S} = ST$, $\tilde{R} = R$, and $\tilde{X} = T^*XT$, then $\mathscr{D}(X; F, G, Q, R, S) = 0$ is equivalent to $\mathscr{D}(\tilde{X}; \tilde{F}, \tilde{G}, \tilde{Q}, \tilde{R}, \tilde{S}) = 0$. Moreover $\tilde{F}_{\tilde{X}} = T^{-1}F_X T$, and if $\tilde{\mathscr{U}}_0 = \operatorname{Ker}(\tilde{F}_{\tilde{X}})^n$ then $\tilde{\mathscr{U}}_0 = T^{-1}\mathscr{U}_0$. Suppose $\dim \mathscr{U}_0 = t > 0$. Choose T such that

$$T^{-1}F_X T = \begin{bmatrix} N & * \\ 0 & * \end{bmatrix}$$

where $N \in \mathbb{C}^{t\times t}$ is nilpotent. Then $\mathscr{U}_0 = T \operatorname{Im} \begin{bmatrix} I_t \\ 0 \end{bmatrix}$ and $\tilde{\mathscr{U}}_0 = \operatorname{Im} \begin{bmatrix} I_t \\ 0 \end{bmatrix}$. Thus we can assume without loss of generality

$$F_X = \begin{bmatrix} N & * \\ 0 & A_X \end{bmatrix}, \; N \text{ nilpotent}, \; A_X \text{ nonsingular}, \tag{2.8}$$

and

$$\mathscr{U}_0 = \operatorname{Im} \begin{bmatrix} I_t \\ 0 \end{bmatrix}. \tag{2.9}$$

Let

$$X = \begin{bmatrix} W_1 & W_{21}^* \\ W_{21} & X_2 \end{bmatrix} \tag{2.10}$$

be partitioned in correspondence with (2.8).

Then (2.1) and (2.2) mean that

$$Y = \begin{bmatrix} W_1 & W_{21}^* \\ W_{21} & Y_2 \end{bmatrix} \quad \text{and} \quad \mathrm{F_Y} = \begin{bmatrix} N & * \\ 0 & A_Y \end{bmatrix}, \tag{2.11}$$

where A_Y is nonsingular. Let $\Delta = X - Y = [\Delta_1 \quad \Delta_2]$ be partitioned accordingly. Then Lemma 2.2 yields $\Delta - F_Y^* \Delta F_X = 0$. Hence $\Delta_1 - F_Y^* \Delta_1 N = 0$, and we obtain

$$\Delta_1 = F_Y^* \Delta_1 N = (F_Y^*)^2 \Delta_1 N^2 = \cdots = (F_Y^*)^n \Delta_1 N^n.$$

Since N is nilpotent, it follows that $\Delta_1 = 0$. Thus we have (2.1). Note that

$$F_Y - F_X = G(R + G^* Y G)^{-1} G^* \Delta F_X.$$

Therefore (2.8) and $\Delta = [0 \quad \Delta_2]$ imply $F_X - F_Y = [0 \quad *]$. Hence F_Y has the form given by (2.11). Moreover we have

$$\operatorname{Ker}(F_Y)^n \supseteq \operatorname{Ker}(F_X)^n. \tag{2.12}$$

Interchanging the roles of X and Y yields equality in (2.12). Hence the matrix A_Y in (2.11) is nonsingular. □

We now give a description of the subspace $\mathscr{U}_0$ in terms of the extended symplectic pencil (1.3). If X is a solution then (see also Lemma 2.5) we have

$$\det(M - sL) = \det(F_X - sI)\det(I - sF_X^*)\det(R + G^* X G). \tag{2.13}$$

Hence $\det(M - sL)$ is not the zero polynomial. We call

$$\sigma(M - sL) = \{\lambda \in \mathbb{C} \mid \det(M - \lambda L) = 0\}$$

the set of *characteristic roots* of $M - sL$. Because of (2.13) the spectrum of F_X satisfies

$$\sigma(F_X) \subseteq \sigma(M - sL). \tag{2.14}$$

Clearly, the matrix F_X is nonsingular if and only if $0 \notin \sigma(M - sL)$, or equivalently, if and only if $\det M \neq 0$.

Now let γ be a simple positively oriented curve in the complex plane such that $\lambda = 0$ is the only element of $\sigma(M - sL)$ in the interior of γ. Then (2.14) implies that

$$P_0 = \frac{1}{2\pi i} \int_\gamma (sI - F_X)^{-1}\, ds. \tag{2.15}$$

is the Riesz projector ([2, p. 66], [8, p. 64]) on the generalized eigenspace $\operatorname{Ker}(F_X)^n$. Hence $\mathscr{U}_0 = \operatorname{Im} P_0$. Define

$$\Theta = \frac{1}{2\pi i} \int_\gamma (sL - M)^{-1} L\, ds.$$

From [7, p.50] we know that the map $\Theta : \mathbb{C}^{2n+m} \to \mathbb{C}^{2n+m}$ is a projection. In accordance with [12, p.25] we call $\mathscr{S}_0 = \operatorname{Im}\Theta$ the *spectral deflating subspace* of $M - sL$ associated with $s = 0$. Clearly, $\mathscr{S}_0$ is contained in the stable deflating subspace of $M - sL$. The subsequent lemma shows that

$$\mathscr{S}_0 = \operatorname{Im} \frac{1}{2\pi i} \int_\gamma (sL - M)^{-1}\, ds. \tag{2.16}$$

LEMMA 2.4. *If $\lambda = 0$ is the only element of $\sigma(M - sL)$ contained in the interior of γ then*

$$\operatorname{Im} \frac{1}{2\pi i} \int_\gamma (sL - M)^{-1} L\, ds = \operatorname{Im} \frac{1}{2\pi i} \int_\gamma (sL - M)^{-1}\, ds. \tag{2.17}$$

Proof. Note that $sL - M$ is a regular matrix pencil in the sense of [6]. Thus there exist nonsingular complex $(2n+m) \times (2n+m)$ matrices A, B such that

$$A(sL - M)B = \begin{bmatrix} sI - N & 0 \\ 0 & s\hat{L}_2 - \hat{M}_2 \end{bmatrix},$$

and $N \in \mathbb{C}^{t \times t}$ is nilpotent and $0 \notin \sigma(\hat{M}_2 - s\hat{L}_2)$. Then

$$\frac{1}{2\pi i} \int_\gamma (sI - N)^{-1}\, ds = I_t \quad \text{and} \quad \frac{1}{2\pi i} \int_\gamma (s\hat{L}_2 - \hat{M}_2)^{-1}\, ds = 0.$$

Hence

$$\begin{aligned} \frac{1}{2\pi i} \int_\gamma (sL - M)^{-1} L\, ds &= \frac{1}{2\pi i} \int_\gamma B \begin{bmatrix} sI - N & 0 \\ 0 & s\hat{L}_2 - \hat{M}_2 \end{bmatrix}^{-1} A \cdot A^{-1} \begin{bmatrix} I & 0 \\ 0 & \hat{L}_2 \end{bmatrix} B^{-1} ds \\ &= B \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} B^{-1}, \end{aligned}$$

and

$$\frac{1}{2\pi i} \int_\gamma (sL - M)^{-1}\, ds = B \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} A,$$

which implies (2.17). $\square$

The following result [9] will be used to establish the link between $\mathscr{U}_0$ to $\mathscr{S}_0$.

LEMMA 2.5. *Let X be a solution of (1.1). Define Z as in (2.7), and $D = G(R + G^*XG)^{-1}$, and*

$$K = \begin{bmatrix} I & 0 & -G(R + G^*XG)^{-1} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & -Z^* \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ F^*X & I & 0 \\ G^*X & 0 & I \end{bmatrix},$$

and

$$\Sigma = \begin{bmatrix} I & 0 & 0 \\ -X & I & 0 \\ -Z & 0 & I \end{bmatrix}. \tag{2.18}$$

Then

$$K(M - sL)\Sigma = \begin{bmatrix} F_X - sI & +sD & 0 \\ 0 & I - sF_X^* & 0 \\ 0 & -sG^* & R + G^*XG \end{bmatrix}. \tag{2.19}$$

The next theorem shows that we can obtain $\mathscr{U}_0$ from $\mathscr{S}_0$ by projecting each vector $x = [v^T, z^T, w^T]^T \in \mathscr{S}_0$ onto its first component $v \in \mathbb{C}^n$.

THEOREM 2.6. *Let X be a solution of (1.1) and let P_0 and Σ be defined by (2.15) and (2.18). Set* $\Pi = [I_n \quad 0_{n\times n} \quad 0_{n\times m}]$. *Then*

$$\mathscr{S}_0 = \operatorname{Im} \begin{bmatrix} I \\ -X \\ -Z \end{bmatrix} P_0 \tag{2.20}$$

and

$$\mathscr{U}_0 = \Pi\,\mathscr{S}_0. \tag{2.21}$$

Proof. From (2.16) and (2.19) we obtain

$$\begin{aligned} \mathscr{S}_0 &= \operatorname{Im} \frac{1}{2\pi i}\int_\gamma (M - sL)^{-1}K^{-1}\,ds \\ &= \operatorname{Im}\Sigma\,\frac{1}{2\pi i}\int_\gamma \begin{bmatrix} F_X - sI & +sD & 0 \\ 0 & I - sF_X^* & 0 \\ 0 & -sG^* & R + G^*XG \end{bmatrix}^{-1} ds. \end{aligned} \tag{2.22}$$

Note that

$$\begin{bmatrix} F_X - sI & +sD & 0 \\ 0 & I - sF_X^* & 0 \\ 0 & -sG^* & R + G^*XG \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (F_X - sI)^{-1} & (F_X - sI)^{-1}(-sD)(I - sF_X^*)^{-1} & 0 \\ 0 & (I - sF_X^*)^{-1} & 0 \\ 0 & (R + G^*XG)^{-1}\,sG^*\,(I - sF_X^*)^{-1} & (R + G^*XG)^{-1} \end{bmatrix}.$$

Since $s = 0$ is not a pole of $(I - sF_X^*)^{-1}$ we have

$$\frac{1}{2\pi i}\int_\gamma (I - sF_X^*)^{-1}ds = \frac{1}{2\pi i}\int_\gamma s(I - sF_X^*)^{-1}ds = 0. \tag{2.23}$$

We now focus on $H(s) = (-sD)(I - sF_X^*)^{-1}$ and consider an expansion

$$H(s) = \sum_{\nu=0}^{\infty} s^\nu H_\nu. \tag{2.24}$$

Because of

$$\sigma(I - sF_X^*) = \{\lambda \in \mathbb{C} \mid \lambda \neq 0\,, \overline{\lambda}^{-1} \in \sigma(F_X)\}$$

the series (2.24) converges if $|s| < 1/\rho(F_X)$. Hence, for s in the interior of γ we have

$$H(s) = \sum_{\nu=0}^{\infty}(F_X - sI)^\nu G_\nu \quad \text{with} \quad G_\nu = (-1)^\nu \sum_{\kappa=0}^{\infty}\binom{\nu+\kappa}{\kappa} F_X^\kappa H_{\nu+\kappa}.$$

Therefore

$$\frac{1}{2\pi i}\int_\gamma (F_X - sI)^{-1}(-sD)(I - sF_X^*)^{-1}\,ds = \frac{1}{2\pi i}\int_\gamma (F_X - sI)^{-1}G_0\,ds.$$

From (2.22) and (2.23) we obtain

$$\begin{aligned}\mathscr{S}_0 &= \operatorname{Im} \Sigma \frac{1}{2\pi i} \int_\gamma \begin{bmatrix} (F_X - sI)^{-1} & (F_X - sI)^{-1} G_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ds \\ &= \operatorname{Im} \begin{bmatrix} I \\ -X \\ -Z \end{bmatrix} \frac{1}{2\pi i} \int_\gamma (F_X - sI)^{-1} ds,\end{aligned}$$

which yields (2.20). It is obvious that (2.20) implies (2.21). □

COROLLARY 2.7. *Assume* $\mathscr{U}_0 = \operatorname{Im}\, [I_t \quad 0]^T$. *Let*

$$X = \begin{bmatrix} W_1 & W_{21}^* \\ W_{21} & X_2 \end{bmatrix}$$

be a solution of (1.1) such that $W_1 \in \mathbb{C}^{t \times t}$ *and* $W_{21} \in \mathbb{C}^{(n-t) \times t}$, *and let* $Z = (R + G^* X G)^{-1} (G^* X F + S) = [Z_1 \quad Z_2]$ *be partitioned accordingly. Then*

$$\mathscr{S}_0 = \operatorname{Im} \left[\begin{array}{c} -I_t \\ 0 \\ \hline W_1 \\ W_{21} \\ \hline Z_1 \end{array} \right]. \tag{2.25}$$

Based on Proposition 2.1 one can solve the DARE (1.1) in two steps. Assuming (2.9), the blocks W_1, W_{21} in (2.10) can be computed from the spectral deflating subspace of the pencil $M - sL$ associated with $\lambda = 0$. The second step concerns X_2. Using W_1, W_{21} one can set up a DARE (E) of reduced order where $\lambda = 0$ is not a characteristic root of the associated pencil. Each solution of (E) will then yield an admissible block X_2 in (2.10).

3. The reduced DARE

Let X be a solution of (1.1). Assume $\mathscr{U}_0$ as in (2.9) such that

$$X = \begin{bmatrix} W_1 & W_{21}^* \\ W_{21} & X_2 \end{bmatrix}, \tag{3.1}$$

and let the blocks W_1 and W_{21} be computed from the deflating subspace $\mathscr{S}_0$ and (2.25). We focus on the matrix X_2. We set up a DARE $\mathscr{D}_2(X_2) = 0$ for X_2 and consider an associated pencil $M_2 - sL_2$.

Set

$$\tilde{W} = \begin{bmatrix} W_1 & W_{21}^* \\ W_{21} & 0 \end{bmatrix}$$

such that

$$X = \tilde{W} + \begin{bmatrix} 0 \\ I \end{bmatrix} X_2 \, [0 \quad I].$$

Let

$$F = \begin{bmatrix} F_1 & F_{12} \\ F_{21} & F_2 \end{bmatrix}, \; G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \; Q = \begin{bmatrix} Q_1 & Q_{21}^* \\ Q_{21} & Q_2 \end{bmatrix}, \; S^* = \begin{bmatrix} S_1^* \\ S_2^* \end{bmatrix}$$

be partitioned conformably. Define

$$R_2 = R + G^* \tilde{W} G, \quad \hat{S}_2 = G^* \tilde{W} \begin{bmatrix} F_{12} \\ F_2 \end{bmatrix} + S_2, \quad \hat{Q}_2 = [F_{12}^* \quad F_2^*] \, \tilde{W} \begin{bmatrix} F_{12} \\ F_2 \end{bmatrix} + Q_2.$$

Recall

$$F_X = \begin{bmatrix} N & * \\ 0 & A_X \end{bmatrix} \tag{3.2}$$

with N is nilpotent and A_X nonsingular.

PROPOSITION 3.1. *The block X_2 in (3.1) is a solution of*

$$\begin{aligned} \mathscr{D}_2(X_2) \;&=\; X_2 - F_2^* X_2 F_2 (G_2^* X_2 F_2 + \hat{S}_2)^* \\ &\qquad + (R_2 + G_2^* X_2 G_2)^{-1} (G_2^* X_2 F_2 + \hat{S}_2) - \hat{Q}_2 = 0. \end{aligned} \tag{3.3}$$

Let

$$F_{2_{X_2}} = F_2 - G_2 (R_2 + G_2^* X_2 G_2)^{-1} (G_2^* X_2 F_2 + \hat{S}_2),$$

be the associated closed loop matrix. If A_X is given as in (3.2) then $F_{2_{X_2}} = A_X$ and $F_{2_{X_2}}$ is nonsingular. Let $M_2 - sL_2$ be the pencil associated with (3.3). Then

$$0 \notin \sigma(M_2 - sL_2). \tag{3.4}$$

Moreover, if

$$\begin{bmatrix} Q & S^* \\ S & R \end{bmatrix} \geqslant 0 \tag{3.5}$$

then $W_1 \geqslant 0$.

Proof. The matrix X_2 is a solution of $[0 \;\; I] \, \mathscr{D}(X) [0 \;\; I]^T = 0$. We have

$$\begin{aligned} [0 \;\; I] F^* X F \begin{bmatrix} 0 \\ I \end{bmatrix} \;&=\; [0 \;\; I] F^* \tilde{W} F \begin{bmatrix} 0 \\ I \end{bmatrix} + [0 \;\; I] F^* \begin{bmatrix} 0 \\ I \end{bmatrix} X_2 [0 \;\; I] F \begin{bmatrix} 0 \\ I \end{bmatrix} \\ &=\; [F_{12}^* \quad F_2^*] \, \tilde{W} \begin{bmatrix} F_{12} \\ F_2 \end{bmatrix} + F_2^* X_2 F_2, \end{aligned}$$

and

$$(G^* X F + S) \begin{bmatrix} 0 \\ I \end{bmatrix} = G^* \tilde{W} \begin{bmatrix} F_{12} \\ F_2 \end{bmatrix} + G_2^* X_2 F_2 + S_2 = \hat{S}_2 + G_2^* X_2 F_2, \tag{3.6}$$

and

$$R + G^* X G = (R + G^* \tilde{W} G) + G_2^* X_2 G_2 = R_2 + G_2^* X_2 G_2.$$

Thus it is obvious that X_2 satisfies (3.3). From

$$F_X = F - \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} (R_2 + G_2^* X_2 G_2)^{-1} (G X F^* + S)$$

and (3.6) we obtain

$$A_X = [0 \;\; I]F_X \begin{bmatrix} 0 \\ I \end{bmatrix} = F_{2_{X_2}}.$$

Therefore $F_{2_{X_2}}$ is nonsingular and we have (3.4).

Now assume (3.5). Then Lemma 2.3 implies

$$X - F_X^* X F_X = \begin{bmatrix} I & -Z^* \end{bmatrix} \begin{bmatrix} Q & S^* \\ S & R \end{bmatrix} \begin{bmatrix} I \\ -Z \end{bmatrix}.$$

Therefore W_1 satisfies a Stein equation $W_1 - N^* W_1 N = C_1$ with $C_1 \geqslant 0$. Hence $W_1 \geqslant 0$. $\square$

4. Related earlier work

The starting point for our investigation is the order reduction the DARE

$$X - F^* X F + F^* X G (R + G^* X G)^{-1} G^* X F - Q = 0 \tag{4.1}$$

in [5]. Suppose the matrix F is singular and

$$\operatorname{Ker} F = \operatorname{Im} \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{4.2}$$

and let

$$Q = \begin{bmatrix} Q_1 & Q_{21}^* \\ Q_{21} & Q_2 \end{bmatrix}$$

be partitioned conformably. Then, according to [5], each solution X of (4.1) is of the form

$$X = \begin{bmatrix} Q_1 & Q_{21}^* \\ Q_{21} & X_2 \end{bmatrix},$$

and X_2 satisfies a reduced equation of type (1.1). In the case of the DARE (4.1) the use of symplectic matrix pencils and deflating subspaces for the DARE (4.1) can be traced back to [13]. The pencil approach was extended in [14] and [11] to the general DARE (1.1).

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