Operators
and
Matrices
Volume 1, Number 1 (2007), 87-111

MEROMORPHIC SOLUTIONS OF LINEAR DIFFERENTIAL SYSTEMS, PAINLEVÉ TYPE FUNCTIONS

LEV SAKHNOVICH

(*communicated by Leiba Rodman*)

Abstract. We consider the $n \times n$ matrix linear differential systems in the complex plane. We find necessary and sufficient conditions under which these systems have meromorphic fundamental solutions. Using the operator identity method we construct a set of systems which have meromorphic solutions. We prove that the well known operator with the sine kernel generates a class of meromorphic Painlevé type functions. The fifth Painlevé function belongs to this class. Hence we obtain a new and simple proof that the fifth Painlevé function is meromorphic.

1. Introduction

Let us consider the $n \times n$ matrix system of the form

$$\frac{dW}{dx} = A(x)\, W, \tag{1.1}$$

where $A(x)$ is the $n \times n$ matrix function. Further we assume that the matrix function $A(x)$ is holomorphic and single valued in a punctured neighborhood of a point x_0.

Every fundamental solution $W(x)$ of system (1.1) has the form (see [2, 25])

$$W(x) = S(x)\,(x - x_0)^{\Phi}, \tag{1.2}$$

where the matrix $S(x)$ is holomorphic and single valued in the domain $0 < |x - x_0| < \rho$ and Φ is a constant matrix.

DEFINITION 1.1. (See [2, 25].) The point x_0 is called a *regular point of system* (1.1) if the corresponding matrix $S(x)$ is either holomorphic in a neighborhood of x_0 or has a pole in x_0.

DEFINITION 1.2. We shall say that the regular point x_0 of system (1.1) is *strongly regular* if $\Phi = 0$ in formula (1.2).

Mathematics subject classification (2000): 34M05, 34M55, 47B38.

Key words and phrases: Strong regularity, global solution, meromorphic solution, differential system with parameter, operator identity method, Painlevé type functions, elliptic functions.

Paper No. 01-06

We use the following transformation

$$W(x) = F(x)\,Y(x), \tag{1.3}$$

where

$$F(x) = \sum_{k=\ell}^{m} f_k (x - x_0)^k, \tag{1.4}$$

f_k are constant $n \times n$ matrices, $\det F(x) \neq 0$, $x \neq x_0$. Then system (1.1) takes the form

$$\frac{dY}{dx} = B(x)Y,$$

where

$$B(x) = F^{-1}(x)\,A(x)\,F(x) - F^{-1}(x)\frac{dF}{dx}.$$

The following theorem gives the condition of regularity .

THEOREM 1.1. (Horn's theorem (see [2]) *The point* x_0 *is regular for system* (1.1) *if and only if there exists transformation* (1.3) *such that the corresponding matrix* $B(x)$ *has the form*

$$B(x) = \frac{B_1(x)}{x - x_0},$$

where $B_1(x)$ *is holomorphic in the domain* $0 \leqslant |x - x_0| < \rho$.

The conditions of Horn's theorem are necessary conditions of the strong regularity. In the present paper we give necessary and sufficient conditions of strong regularity.

Separately we consider the case when the entries of $A(x)$ are meromorphic functions.

If x_0 is a strongly regular point, then we say that the corresponding fundamental solution is strongly regular at x_0 .

DEFINITION 1.3. We say that the fundamental solution $W(x)$ of system (1.1) is *global strongly regular* if this solution is strongly regular for all singular points of $A(x)$.

It is easy to see that the global strong solution is meromorphic. We apply the obtained results to the canonical differential systems [19] with the spectral parameter ρ :

$$\frac{dW(x,\rho)}{dx} = \big[P(x) + \rho\, Q(x)\big]\; W(x,\rho). \tag{1.5}$$

We investigate in detail the special case when $n = 2$, $P(x) = 0$, and

$$Q(x) = \begin{bmatrix} 0 & r^{-2}(x) \\ r^2(x) & 0 \end{bmatrix}. \tag{1.6}$$

Now we shall explain the connection of system (1.5), (1.6) with the classical second order equations. The solution of this system

$$U(x,\rho) = \operatorname{col}\big[u_1(x,\rho), u_2(x,\rho)\big]$$

satisfies the relations

$$\frac{du_1}{dx} = i\rho\, r^{-2}(x)\, u_2(x,\rho), \qquad \frac{du_2}{dx} = i\rho\, r^2(x)\, u_1(x,\rho).$$

System (1.5), (1.6) reduces to two equations of the second order.

$$-\frac{d}{dx}r^2(x)\frac{du_1}{dx} = \rho^2 r^2(x)\,u_1(x,\rho), \tag{1.7}$$

$$-\frac{d}{dx}r^{-2}(x)\frac{du_2}{dx} = \rho^2 r^{-2}(x)\,u_2(x,\rho). \tag{1.8}$$

Let us note that equations (1.7) and (1.8) are mutually dual [7, 11, 20] and play an important role in a number of theoretical and applied problems (prediction theory [15], vibration of a thin straight rod [5], generalized string equation [19]).

Using the operator identity method [19] we construct classes $r(x)$ such that the corresponding equations (1.7) and (1.8) have meromorphic solutions in respect to x for all ρ. In particular we construct a class of the rational functions $r(x)$ with this property.

The operator identity method allows to construct an analytic continuation of $r^2(x)$ from half-axis $(0,\infty)$ onto the complex plane. We have applied this approach to the third and the fifth Painlevé functions. In particular we have obtained a new and simple proof that the fifth Painlevé function is meromorphic (see [10]).

REMARK 1.1. The global Fuchsian theory (see [2, 25]) requires that the regularity condition be met at infinity as well. In our approach this condition can fail for $x=\infty$. Thus our theory can be applied to the important examples (see Sections 8–10) in which classical Fuchsian theory does not work.

REMARK 1.2. The meromorphic solutions of the differential systems play an important role in the spectral theory in the space with indefinite metric [17].

2. Conditions of strong regularity

Taking into account Horn's theorem we begin with the matrix function $A(x)$ which can be represented in the form

$$A(x) = \frac{a_{-1}}{x-x_0} + a_0 + a_1(x-x_0) + \dots\,, \tag{2.1}$$

where a_k are $n\times n$ matrices. We investigate the case when x_0 is either a regular point of $W(x)$ or a pole. Hence the following relation

$$W(x) = \sum_{k\geqslant m} b_k(x-x_0)^k, \quad b_m \neq 0, \tag{2.2}$$

is true. Here b_k are $n\times n$ matrices. We note that m can be negative. From formulas (1.1), (2.1) and (2.2) we deduce that

$$(k+1)\,b_{k+1} = \sum_{j+\ell=k} a_j b_\ell, \tag{2.3}$$

where $j\geqslant -1$, $\ell\geqslant m$. Relation (2.3) can be rewritten in the recurrent form

$$\left[(k+1)I_n - a_{-1}\right] b_{k+1} = \sum_{j+\ell=k} a_j b_\ell, \quad k\geqslant m, \tag{2.4}$$

where $j \geqslant 0$, $\ell \geqslant m$. When $k = m-1$ we have

$$(mI_n - a_{-1})\, b_m = 0. \tag{2.5}$$

From relation (2.5) we deduce the following assertion.

PROPOSITION 2.1. (necessary condition) *If the solution of system* (1.1) *has form* (2.2) *then* m *is an eigenvalue of* a_{-1}.

We denote by M the greatest integer eigenvalue of the matrix a_{-1}. Using relation (2.5) we obtain the assertion.

PROPOSITION 2.2. (sufficient condition) *If the matrix system*

$$\left[(k+1)I_n - a_{-1}\right] b_{k+1} = \sum_{j+\ell=k} a_j b_\ell, \tag{2.6}$$

where $m \leqslant k+1 \leqslant M$ *has a solution* $b_m, b_{m+1}, \ldots, b_M$ *and* $b_m \neq 0$ *then system* (1.1) *has a solution of form* (2.2).

We consider the system of equations

$$\frac{dY}{dx} = -YA(x), \tag{2.7}$$

where $Y(x)$ has the form

$$Y(x) = \sum_{k \geqslant p} c_k (x - x_0)^k, \quad c_p \neq 0. \tag{2.8}$$

Formulas (2.1), (2.7) and (2.8) imply that

$$(k+1)c_{k+1} = -\sum_{j+\ell=k} c_\ell a_j, \tag{2.9}$$

where $j \geqslant -1$, $\ell \geqslant p$. We rewrite relation (2.9) in the form

$$c_{k+1}[(k+1)I_n + a_{-1}] = -\sum_{j+\ell=k} c_\ell a_j,$$

where $j \geqslant 0$, $\ell \geqslant p$. In the same way as Propositions 2.1 and 2.2 we obtain the following results.

PROPOSITION 2.3. *If the solution of system* (2.7) *has form* (2.8) *then* $-p$ *is an eigenvalue of* a_{-1}.

PROPOSITION 2.4. *Let* $-P$ *be the smallest integer eigenvalue of the matrix* a_{-1}. *If the matrix system*

$$c_{k+1}\left[(k+1)I_n + a_{-1}\right] = -\sum_{j+\ell=k} c_\ell a_j,$$

where $p \leqslant k+1 \leqslant P$ *has a solution* $c_p \neq 0, c_{p+1}, \ldots, c_P$, *then system* (1.1) *has a solution of form* (2.2).

REMARK 2.1. If $W(x)$ is a fundamental solution of system (1.1), then $Y(x) = W^{-1}(x)$ is a fundamental solution of system (2.7).

PROPOSITION 2.5. *If $W(x)$ and $W^{-1}(x)$ satisfy relations* (1.1), (2.1) *and* (2.7), (2.8), *respectively, then m and $-p$ are eigenvalues of the matrix a_{-1}. The corresponding matrix a_{-1} is either scalar or has at least two different integer eigenvalues.*

Proof. Let the matrix a_{-1} not be a scalar one. Then it follows from relation (2.5) that

$$\det b_m = 0. \tag{2.10}$$

Let us suppose that a_{-1} doesn't have integer eigenvalues different from m. In view of Propositions 2.1 and 2.3 the equality $p = -m$ is true. From relations (2.2), (2.8) and the equality

$$W(x)\, W^{-1}(x) = I_n$$

we have $b_m c_m = I_n$ which contradicts relation (2.10). This proves the proposition. □

3. Integer eigenvalues

We consider again differential system (1.1), where $A(x)$ has form (2.1). Let T be a constant matrix such that

$$T^{-1} a_{-1} T = b_{-1},$$

where b_{-1} is Jordan matrix, i. e. b_{-1} has the following structure

$$b_{-1} = \operatorname{diag}(J_1, J_2, \dots, J_s), \quad s \leqslant n\,.$$

Here $J_k = \lambda_k I_k + H_k$, $1 \leqslant k \leqslant s$, and

$$H_k = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

We reduce system (1.1) to the form

$$\frac{dV}{dx} = B(x)V,$$

where $W(x) = TV$, $B(x) = T^{-1}A(x)T$. Now we describe the "shearing" transformation (see [25], Ch. 5) which lowers the eigenvalue λ_s of the matrix b_{-1} by one, while leaving the others unchanged.

We denote by q the order of Jordan matrix J_s and represent $B(x)$ in the form

$$B(x) = \frac{1}{x - x_0} \begin{bmatrix} \widetilde{b}_{-1} & 0 \\ 0 & J_s \end{bmatrix} + \widetilde{B}(x), \tag{3.1}$$

where $\widetilde{B}(x)$ is holomorphic at x_0 and

$$\widetilde{b}_{-1} = \operatorname{diag}(J_1, J_2, \dots, J_{s-1}).$$

The "shearing" transformation is defined by the relation (see [2, 25])

$$V = S(x)U. \tag{3.2}$$

Here

$$S(x) = \begin{bmatrix} I_{n-q} & 0 \\ 0 & (x-x_0)I_q \end{bmatrix}. \tag{3.3}$$

Using (3.2) we deduce that

$$\frac{dU}{dx} = C(x)U,$$

where

$$C(x) = S^{-1}(x)\,B(x)\,S(x) - S^{-1}(x)\,\frac{d}{dx}\,S(x).$$

It follows from (3.1) and (3.3) that

$$c_{-1} = \begin{bmatrix} \widetilde{b}_{-1} & 0 \\ \Gamma & J_s - I_q \end{bmatrix}.$$

It is easy to see that the matrix c_{-1} has the same eigenvalues as b_{-1} except that the eigenvalue λ_s has been decreased by unity.

THEOREM 3.1. *If the fundamental solution of system* (1.1) *is strongly regular then all the eigenvalues of the corresponding matrix* a_{-1} *are integers.*

Proof. Using a finite number of pairs of constant and "shearing" transformations we can reduce system (1.1) to the system

$$\frac{d}{dx}\,\widetilde{W}(x) = \widetilde{A}(x)\,\widetilde{W}(x), \tag{3.4}$$

where all the integer eigenvalues of $\widetilde{a}_{-1}$ coincide with the smallest integer eigenvalue of a_{-1}, the non integer eigenvalues of a_{-1} and $\widetilde{a}_{-1}$ coincide. If the fundamental solution $W(x)$ of system (1.1) is strongly regular then the fundamental solution $\widetilde{W}(x)$ of system (3.4) is strongly regular as well. If $\widetilde{a}_{-1}$ has non integer eigenvalues then according to Proposition 2.5 the matrix $\widetilde{a}_{-1}$ has at least two different integer eigenvalues. The theorem is proved. □

REMARK 3.1. In paper [21] we consider the Knizhnik-Zamolodnikov system of linear differential equations. The coefficients of this system are rational functions. Using the results of Sections 1–3 we prove that under some conditions the solution of the KZ system is rational matrix function too. This assertion confirms partially the conjecture of Chervov-Talalaev [3].

4. Examples

EXAMPLE 4.1. V. Katsnelson and D. Volok [14] investigated the case when the point x_0 is a simple pole of $W(x)$ and a holomorphic point of the inverse matrix function $W^{-1}(x)$. They proved that in this case

$$a_{-1}^2 = -a_{-1}, \quad a_{-1}a_0a_{-1} = -a_0a_{-1}. \tag{4.1}$$

It follows from the first of the relations (4.1) that the eigenvalues of a_{-1} are equal -1 or 0. From Proposition 2.2 we deduce the assertion.

PROPOSITION 4.1. *Let conditions* (4.1) *be fulfilled. Then system* (1.1) *has a strongly regular solution, where* $m = -1$.

Proof. In the case under consideration we have $m = -1$, $M = 0$. Hence system (2.6) takes the form

$$(I_n + a_{-1})b_{-1} = 0, \quad -a_{-1}b_0 = a_0 b_{-1}. \tag{4.2}$$

Comparing the first relations of (4.1) and (4.2) we obtain the equality

$$b_{-1} = a_{-1}c,$$

where c is an arbitrary invertible matrix. It follows from the second relation of (4.1) that

$$b_0 = a_0 a_{-1} c$$

satisfies the second equality of (4.2). The proposition is proved. $\square$

EXAMPLE 4.2. Let us consider the case when $A(x)$ has a pole of the second order. We suppose that the matrix $A(x)$ has the form

$$A(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix}, \tag{4.3}$$

where

$$a_{11}(x) = \alpha_0 + \alpha_1(x - x_0) + \dots, \tag{4.4}$$

$$a_{22}(x) = \beta_0 + \beta_1(x - x_0) + \dots, \tag{4.5}$$

$$a_{12}(x) = \gamma_{-2}(x - x_0)^{-2} + \gamma_{-1}(x - x_0)^{-1} + \dots, \tag{4.6}$$

$$a_{21}(x) = \mu_2(x - x_0)^2 + \mu_3(x - x_0)^3 + \dots. \tag{4.7}$$

We introduce the matrix function

$$\widetilde{A}(x) = F^{-1}(x)\,A(x)\,F(x) - F^{-1}(x)\frac{d}{dx}F(x), \tag{4.8}$$

where

$$F(x) = \begin{bmatrix} \dfrac{1}{x - x_0} & 0 \\ 0 & 1 \end{bmatrix}$$

It is easy to see that the matrix function $V(x) = F^{-1}(x)W(x)$ satisfies the equation

$$\frac{d}{dx}V = \widetilde{A}(x)V.$$

It is important that the matrix function $\widetilde{A}(x)$ has a pole of the first order. Indeed it follows from formulas (4.3)–(4.8) that

$$\widetilde{A}(x) = \frac{\widetilde{a}_{-1}}{x - x_0} + \widetilde{a}_0 + \dots,$$

where

$$\widetilde{a}_{-1} = \begin{bmatrix} 1 & \gamma_{-2} \\ 0 & 0 \end{bmatrix}, \quad \widetilde{a}_0 = \begin{bmatrix} \alpha_0 & \gamma_{-1} \\ 0 & \beta_0 \end{bmatrix}.$$

We shall consider the case when

$$V(x) = \sum_{k \geqslant m} \widetilde{b}_k (x - x_0)^k, \quad \widetilde{b}_m \neq 0,$$

is true. Here $\widetilde{b}_k$ are $n \times n$ matrices.

PROPOSITION 4.2. *Let the matrix $A(x)$ have the form defined by relations (4.4)–(4.7). Then system (1.1) has a strongly regular solution if and only if*

$$\gamma_{-2}(\alpha_0 - \beta_0) = \gamma_{-1}. \tag{4.9}$$

Proof. In this case we have $m = 0$, $M = 1$. From equality

$$\widetilde{a}_{-1}\widetilde{b}_0 = 0$$

we deduce that $\widetilde{b}_0$ has the following form

$$\widetilde{b}_0 = \begin{bmatrix} -s\gamma_{-2} & -t\gamma_{-2} \\ s & t \end{bmatrix}.$$

In view of (2.4) we have

$$(I_m - \widetilde{a}_{-1})\widetilde{b}_1 = \widetilde{a}_0\widetilde{b}_0. \tag{4.10}$$

Equation (4.10) has a solution $\widetilde{b}_1$ if and only if relation (4.9) is fulfilled. From Proposition 2.2 we deduce the desired assertion. □

COROLLARY 4.1. *In addition to the conditions of Proposition 4.2 we suppose that $\alpha_0 = \beta_0$. System (1.1) has a strongly regular solution if and only if $\gamma_{-1} = 0$.*

5. Differential Systems with spectral parameter

We consider the differential system with the parameter ρ:

$$\frac{dW(x,\rho)}{dx} = \big[P(x) + \rho\, Q(x)\big]\, W(x,\rho), \tag{5.1}$$

where the $n \times n$ matrix functions $P(x)$ and $Q(x)$ can be represented in the forms

$$P(x) = \frac{p_{-1}}{x - x_0} + p_0 + \dots, \tag{5.2}$$

$$Q(x) = \frac{q_{-1}}{x - x_0} + q_0 + \dots. \tag{5.3}$$

Systems (5.1) play an important role in the spectral theory of the canonical differential systems with the spectral parameter ρ (see [19]). Due to Theorem 3.1 the following assertion is true.

PROPOSITION 5.1. (necessary condition) *If system (5.1)–(5.3) has a strongly regular fundamental solution $W(x,\rho)$ for all ρ then all the eigenvalues of the matrix $p_{-1} + \rho q_{-1}$ are integer and do not depend on ρ.*

EXAMPLE 5.1. Let $n=2$ and

$$p_{-1}=\begin{bmatrix}\lambda_1 & \Gamma_1\\ 0 & \lambda_2\end{bmatrix},\quad q_{-1}=\begin{bmatrix}0 & \Gamma_2\\ 0 & 0\end{bmatrix}.$$

We assume that λ_1 and λ_2 are integer numbers. The eigenvalues of the matrix $p_{-1}+\rho q_{-1}$ are equal to λ_1 and λ_2, i. e. these eigenvalues are integer and do not depend on ρ.

EXAMPLE 5.2. We consider the system

$$\frac{d}{dx}W(x,\rho)=\rho A(x)\,W(x,\rho), \tag{5.4}$$

where the matrix function $A(x)$ is defined by relations (4.4)–(4.7). We introduce the matrix

$$\widetilde{A}(x,\rho)=\rho F^{-1}(x)\,A(x)\,F(x)-F^{-1}(x)\frac{d}{dx}F(x),$$

where

$$F(x)=\begin{bmatrix}\dfrac{1}{x-x_0} & 0\\ 0 & 1\end{bmatrix}.$$

The matrix function $V(x,\rho)=F^{-1}(x)\,W(x,\rho)$ satisfies the equation

$$\frac{d}{dx}V=\big[P(x)+\rho\,Q(x)\big]V,$$

where

$$\begin{aligned}P(x)&=-F^{-1}(x)\frac{d}{dx}F(x)=\begin{bmatrix}\dfrac{1}{x-x_0} & 0\\ 0 & 1\end{bmatrix}, &(5.5)\\ Q(x)&=F^{-1}(x)\,A(x)\,F(x). &(5.6)\end{aligned}$$

It follows from (5.5) and (5.6) that

$$p_{-1}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\quad p_0=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},$$

$$q_{-1}=\rho\begin{bmatrix}0 & \gamma_{-2}\\ 0 & 0\end{bmatrix},\quad q_0=\rho\begin{bmatrix}\alpha_0 & \gamma_{-1}\\ 0 & \beta_0\end{bmatrix}.$$

Condition (4.9) takes the form

$$\rho\gamma_{-2}(\alpha_0-\beta_0)=\gamma_{-1}.$$

Using Propositions 4.2 and 5.1 we obtain the following assertion.

PROPOSITION 5.2. *Let the matrix function $A(x)$ be defined by relations* (4.4)–(4.7). *System* (5.4) *has a strongly regular fundamental solution for all ρ if and only if*

$$\alpha_0=\beta_0,\quad \gamma_{-1}=0.$$

6. Global strongly regular solutions

In Sections 1–5 we investigated the strongly regular solutions in a punctured neighborhood of the singular point x_0. Now we deduce the conditions under which the solution of system (5.1) is strongly regular for all complex $x \neq \infty$ and all complex $\rho \neq \infty$ (global strongly regular solution). It is obvious that the global strongly regular solution is meromorphic in x and entire in ρ.

Let us consider the differential system

$$\frac{dW}{dx} = \rho A(x)\, W, \tag{6.1}$$

where the 2×2 matrix function $A(x)$ has the form

$$A(x) = \begin{bmatrix} 0 & r(x)^{-2} \\ r(x)^2 & 0 \end{bmatrix}. \tag{6.2}$$

Here $r(x)$ is a meromorphic function in the complex plane. We denote by x_k, $1 \leqslant k \leqslant n \leqslant \infty$, and by y_ℓ, $1 \leqslant \ell \leqslant m \leqslant \infty$, the different roots of $r(x)$ and of $r^{-1}(x)$, respectively. Proposition 5.2 implies the following result.

THEOREM 6.1. *Let all the roots x_k and y_ℓ of $r(x)$ and of $r^{-1}(x)$, respectively, be simple. The fundamental solution $W(x,\rho)$ of system (6.1), (6.2) is strongly regular for all x and ρ if and only if*

$$r(x_k) = 0, \quad r'(x_k) \neq 0, \quad r''(x_k) = 0, \quad (1 \leqslant k \leqslant n), \tag{6.3}$$

$$q(y_\ell) = 0, \quad q'(y_\ell) \neq 0, \quad q''(y_\ell) = 0, \quad (1 \leqslant \ell \leqslant m), \quad q(x) = r^{-1}(x). \tag{6.4}$$

The following assertions can be proved by the direct calculation.

PROPOSITION 6.1. *The functions*

I) $r_1(x) = x$,

II) $r_2(x) = \sin x$

satisfy all the conditions of Theorem 6.1. The corresponding x_k and y_ℓ are defined by the relations:

I) $n = 1$, $m = 0$, $x_1 = 0$,

II) $x_k = k\pi$, $y_\ell = \ell\pi + \pi/2$, $-\infty < k, \ell < \infty$.

REMARK 6.1. Examples close to the case $r_2(x) = \sin x$ are contained in the book by Kamke ([12], p. 408).

PROPOSITION 6.2. *If $r(x)$ is a polynomial and $\deg r(x) \geqslant 2$ then $r(x)$ does not satisfy conditions (6.3).*

PROPOSITION 6.3. *The functions*

$$r_3(x) = \frac{x - \lambda_1}{x - \lambda_2}, \quad \lambda_1 \neq \lambda_2,$$

$$r_4(x) = \frac{(x - \lambda_1)(x - \lambda_2)}{x - \mu_1}, \quad \lambda_1 \neq \lambda_2, \quad \lambda_{1,2} \neq \mu_1$$

do not satisfy conditions (6.3), (6.4).

PROPOSITION 6.4. *The elliptic functions (see* [1] *)*

$$r_5(x) = \frac{\mathrm{sn}(x)}{\mathrm{dn}(x)}, \quad r_6(x) = \frac{\mathrm{sn}(x)}{\mathrm{cn}(x)}, \quad r_7(x) = \frac{\mathrm{cn}(x)}{\mathrm{dn}(x)}$$

satisfy conditions (6.3) *and* (6.4). *Hence the the corresponding systems* (6.1), (6.2) *have strongly regular fundamental solutions.*

This fact is important by investigating the systems of linear differential equations with elliptic coefficients (see [8]).

Theorem 6.1 and Propositions 6.1–6.4 lead to the following problems.

PROBLEM 6.1. *To construct meromorphic functions* $r(x)$ *which satisfy conditions* (6.3) *and* (6.4).

PROBLEM 6.2. *To construct rational functions* $r(x)$ *which satisfy conditions* (6.3) *and* (6.4).

These problems will be investigated in the next sections.

The connection of system (6.1), (6.2) with the classical second order equations is explained in the introduction.

7. Operator Identity

To solve Problems 6.1 and 6.2 we use the operator identity method (see [19]). We introduce the operators

$$Af = i\int_0^x f(t)\,dt, \quad f(x) \in L^2(0,a),$$

and

$$Sf = f(x) + \int_0^a f(t)\,k(x-t)\,dt,$$

where the function $k(x)$, $(-a \leqslant x \leqslant a)$, is continuous and

$$k(x) = k(-x) = \overline{k(x)}. \tag{7.1}$$

We use the following operator identity

$$AS - SA^\star = i(\Phi_1\Phi_2^\star + \Phi_2\Phi_1^\star).$$

Here the operators Φ_1 and Φ_2 are defined by the relations

$$\Phi_1 g = M(x)g, \quad \Phi_2 g = g,$$

where

$$M(x) = \int_0^x k(u)\,du + \frac{1}{2}, \quad 0 \leqslant x \leqslant a. \tag{7.2}$$

Thus the operators Φ_1 and Φ_2 map the one-dimensional space of constant numbers g into $L^2(0,a)$. Let us consider the operator

$$S_\xi f = f(x) + \int_0^\xi f(t)\,k(x-t)\,dt, \quad f(x) \in L^2(0,\xi), \quad 0 \leqslant \xi \leqslant a. \tag{7.3}$$

We introduce the operator

$$P_\xi f(x) = f(x), \quad 0 < x < \xi \leqslant a,$$

and

$$P_\xi f(x) = 0, \quad \xi < x < a.$$

Let us formulate the following results (see [17]).

THEOREM 7.1. *We assume that there are points $0 < x_1 < x_2 < \dots$ having no limit points in $[0,a]$ and such that the operator S_ξ is invertible on $L^2(0,\xi)$ for each $\xi \in [0,a)/\{x_1,x_2,\dots\}$.*

Then the matrix function

$$B(\xi) = \Pi^\star S_\xi^{-1} P_\xi \Pi, \quad \Pi = [\Phi_1, \Phi_2], \tag{7.4}$$

is continuous and nondecreasing in each of the intervals (x_k, x_{k+1}). The matrix function

$$W(\xi,\rho) = I_2 + i\rho J \Pi^\star S_\xi^{-1} P_\xi (I - \rho A)^{-1} \Pi \tag{7.5}$$

is a fundamental solution for the system

$$W(\xi,\rho) = I_2 + i\rho J \int_0^\xi \left[dB(t) \right] W(t,\rho), \tag{7.6}$$

where

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{7.7}$$

(The set of the points $0 < x_1 < x_2 < \dots$ can be either finite or infinite.)

THEOREM 7.2. *Let $B(x)$ be constructed by (7.4). Then $B(x)$ is continuously differentiable in the intervals between the singularities , and in these intervals*

$$H(\xi) = B'(\xi) = \left[h_i^\star(\xi)\, h_j(\xi) \right]_1^2, \tag{7.8}$$

where

$$h_1(\xi) = M(\xi) + \int_0^\xi \Gamma_\xi(\xi,t)\, M(t)\, dt, \tag{7.9}$$

$$h_2(\xi) = 1 + \int_0^\xi \Gamma_\xi(\xi,t)\, dt. \tag{7.10}$$

We use here the formula

$$S_\xi^{-1} f = f(x) + \int_0^\xi \Gamma_\xi(x,t) f(t)\, dt.$$

We remark that $H(\xi)$ has the special form [20]

$$H(\xi) = \frac{1}{2} \begin{bmatrix} Q(\xi) & 1 \\ 1 & Q^{-1}(\xi) \end{bmatrix}. \tag{7.11}$$

It follows from relations (7.6) and (7.7) that

$$\frac{dW(x,\rho)}{dx} = i\rho JH(x)\,W(x,\rho). \tag{7.12}$$

Introducing $U(x,\rho) = W(2x,\rho)e^{-ix\rho}$ we reduce system (7.12) to the form

$$\frac{dU(x,\rho)}{dx} = i\rho JH_1(x)\,U(x,\rho), \tag{7.13}$$

where

$$H_1(x) = \begin{bmatrix} r^2(x) & 0 \\ 0 & r^{-2}(x) \end{bmatrix}, \tag{7.14}$$

$$r^2(x) = Q(2x).$$

Let us note that obtained system (7.13), (7.14) coincides with system (6.1), (6.2).

8. Rational $r(x)$

Let us consider the operator S_ξ (see(7.3)), where $k(x)$ satisfies conditions (7.1) and is a polynomial of degree $2m$. The kernel $k(x-t)$ can be represented in the form

$$k(x-t) = \sum_{s=0}^{2m} x^s p_s(t),$$

where $p_s(t)$ are the polynomials, $\deg p_s(t) \leqslant (2m-s)$. We introduce the matrix

$$A_\xi = \left[\delta_{j,s} + \big(x^s, p_j(x)\big)_\xi\right]_0^{2m} \tag{8.1}$$

and the determinant

$$\Delta_\xi = \det A_\xi.$$

In formula (8.1) we used the notation

$$(f,g)_\xi = \int_0^\xi f(t)\overline{g(t)}\,dt.$$

The solution $g(x,\xi)$ of the equation

$$S_\xi g = f(x) \tag{8.2}$$

has the form

$$g(x,\xi) = f(x) - \sum_{s=0}^{2m} c_s(\xi)\,x^s, \tag{8.3}$$

where $c_s(\xi) = (g, p_s)_\xi$. It follows from (8.2) and Cramer's rule that

$$c_s(\xi) = \frac{d_s(\xi)}{\Delta_\xi}, \tag{8.4}$$

where the determinant $d_s(\xi)$ is formed by replacing the column under number s in Δ_ξ by the column $\mathrm{col}\big[(f,p_0)_\xi,(f,p_1)_\xi,\dots,(f,p_{2m})_\xi\big]$.

Using (8.3) and (8.4) we have

$$g(x) = f(x) + \int_0^\xi \Gamma_\xi(x,t)f(t)\,dt, \tag{8.5}$$

where

$$\Gamma_\xi(x,t) = \frac{1}{\Delta_\xi}\sum_{s=0}^{2m} D_s(\xi,t)\,x^s. \tag{8.6}$$

Here the determinant $D_s(\xi,t)$ is formed by replacing the column under number s in Δ_ξ by the column $\mathrm{col}\big[p_0(t),p_1(t),\dots,p_{2m}(t)\big]$.

The expression

$$\Gamma_\xi(0,\xi) = \frac{D_0(\xi,\xi)}{\Delta_\xi}$$

plays an important role in our theory.

From (8.5) and (8.6) we deduce the following assertion.

THEOREM 8.1. *If $k(x)$ satisfies conditions (7.1) and is a polynomial then the corresponding function $Q(\xi)$ (see (7.11)) is rational.*

Proof. Let $f(x) = 1$. In this case formula (8.3) gives

$$g(x,\xi) = 1 + \sum_{s=0}^{2m} x^s R_s(\xi),$$

where the functions $R_s(\xi)$ are rational. Hence the function $g(\xi,\xi) = h_2(\xi)$ is rational too. The assertion of the theorem follows directly from the equality

$$Q^{-1}(x) = 2h_2^2(x),$$

which can be obtained from (7.8) and (7.11). $\square$

We denote by $\xi_1,\xi_2,\dots,\xi_n$ the roots of the polynomial Δ_ξ.

THEOREM 8.2. *If $k(x)$ satisfies conditions (7.1) and is a polynomial then the corresponding matrix function $W(\xi,\rho)$ defined by relation (7.5) is entire in respect to ρ and meromorphic in respect to ξ with the poles in the points $\xi_1,\xi_2,\dots,\xi_n$.*

Proof. According to (7.2) the function $M(x)$ is a polynomial. Hence the function $(I-A\rho)^{-1}M(x)$ is an entire function of ρ and x. Using (7.5) and (8.6) we deduce the assertion of the theorem. $\square$

REMARK 8.1. We consider $Q(\xi)$ and $W(\xi,\rho)$ for all the complex $\xi \neq \xi_k$, $(1 \leqslant k \leqslant n)$, and for all the complex ρ.

Due to analytic continuation the equality

$$\frac{dW}{d\xi} = i\rho JH(\xi)\,W(\xi,\rho)$$

is true for all complex $\xi \neq \xi_k$, $(1 \leqslant k \leqslant n)$, and for all the complex ρ. Here the matrix function $H(\xi)$ is defined by formula (7.11). Theorems 7.2 and 8.2 imply the following assertion.

COROLLARY 8.1. *The function*

$$r(x) = \frac{1}{\left[\sqrt{2}\, h_2(2x)\right]}$$

is a rational function. The union of the sets of the roots and the poles of $r(x)$ coincides with the set

$$\frac{1}{2}\,\xi_1, \frac{1}{2}\,\xi_2, \ldots, \frac{1}{2}\,\xi_n.$$

We shall use the relation (see [9], Ch. 4.)

$$\frac{dg(\xi,\xi)}{d\xi} = \Gamma_\xi(0,\xi)\, g(\xi,\xi), \quad g(\xi,\xi) = h_2(\xi). \tag{8.7}$$

From relation (8.7) and Corollary 8.1 we deduce the assertion.

COROLLARY 8.2. *If all roots of Δ_ξ are simple then the corresponding function $r(x)$ satisfies the conditions* (6.3), (6.4).

EXAMPLE 8.1. Let us consider the case when

$$k(x) = x^2.$$

In this case we have

$$p_0(t) = t^2, \quad p_1(t) = -2t, \quad p_2(t) = 1.$$

Hence the determinants Δ_ξ and $d_0(\xi)$ are defined by the relations

$$\Delta_\xi = \begin{vmatrix} 1+\xi^3/3 & \xi^4/4 & \xi^5/5 \\ -\xi^2 & 1-2\xi^3/3 & -\xi^4/4 \\ \xi & \xi^2 & 1+\xi^3/3 \end{vmatrix}, \tag{8.8}$$

$$d_0(\xi) = \begin{vmatrix} -\xi^2 & \xi^4/4 & \xi^5/5 \\ 2xi & 1-2\xi^3/3 & -\xi^4/4 \\ -1 & \xi^2 & 1+\xi^3/3 \end{vmatrix}. \tag{8.9}$$

It follows from (8.8) and (8.9) that

$$\Delta_\xi = \frac{1}{1080}\,\xi^9 - \frac{1}{30}\,\xi^6 + 1, \tag{8.10}$$

$$d_0(\xi) = -\xi^2\left[\left(1-\frac{\xi^3}{6}\right)^2 + \frac{\xi^3}{2}\left(1-\frac{\xi^3}{15}\right) - \frac{\xi^3}{5}\left(1-\frac{\xi^3}{24}\right)\right].$$

The polynomial Δ_ξ has nine different roots

$$x_k^3 = \begin{cases} 6, & 1 \leqslant k \leqslant 3, \\ 15+9\sqrt{5}, & 4 \leqslant k \leqslant 6, \\ 15-9\sqrt{5}, & 7 \leqslant k \leqslant 9. \end{cases}$$

By the direct calculation we prove the following assertion.

PROPOSITION 8.1. *The poles of* $\Gamma_\xi(0,\xi)$ *coincide with* x_k, $(1 \leqslant k \leqslant 9)$. *These poles are simple and the residues in the points* x_k, $(1 \leqslant k \leqslant 3)$, *are equal to 1 and in the points* x_k, $(4 \leqslant k \leqslant 9)$, *are equal to* -1.

From relation (8.10) and Proposition 8.1 we deduce that

$$h_2(x) = \frac{\frac{1}{6}x^3 - 1}{\frac{1}{180}x^6 - \frac{1}{6}x^3 - 1}.$$

Hence the corresponding function $r(x) = 1/\left[\sqrt{2}\,h_2(2x)\right]$ is rational and satisfies conditions (6.3) and (6.4) (see Problem 6.2).

9. Exponential $r(x)$

The following example was considered in the paper [17].

EXAMPLE 9.1. Let the operator S_ξ have the form

$$S_\xi f = f(x) + \beta \int_0^\xi \left[e^{i\lambda(x-t)} + e^{-i\lambda(x-t)}\right] f(t)\,dt,$$

where $\beta = \overline{\beta} \neq 0$, $\lambda > 0$. We find

$$S_\xi^{-1} f = f(x) - K(x)\,T^{-1}(\xi) \int_0^\xi K^\star(t) f(t)\,dt,$$

where $K(x) = [e^{i\lambda x}, e^{-i\lambda x}]$ and

$$T(\xi) = \begin{bmatrix} \xi + \beta^{-1} & \lambda^{-1}e^{-i\lambda\xi}\sin\lambda\xi \\ \lambda^{-1}e^{i\lambda\xi}\sin\lambda\xi & \xi + \beta^{-1} \end{bmatrix}.$$

By direct calculation we have

$$h_1(x) = \frac{1}{2h_2(x)}$$

and

$$h_2(x) = \frac{u(x)}{v(x)},$$

where

$$\begin{aligned} u(x) &= x + \beta^{-1} - \lambda^{-1}\sin\lambda x, \\ v(x) &= x + \beta^{-1} + \lambda^{-1}\sin\lambda x. \end{aligned}$$

It is easy to see that all the roots and the poles of $h_2(x)$ are simple. In the same way as Corollary 8.2 we deduce the following assertion.

PROPOSITION 9.1. *The corresponding function*

$$r(x) = \frac{2x + \beta^{-1} + \lambda^{-1} \sin 2\lambda\, x}{\sqrt{2}\,(2x + \beta^{-1} + \lambda^{-1} \sin 2\lambda\, x)}$$

is rational and satisfies conditions (6.3) *and* (6.4).

10. Analytic continuation, Painlevé transcendents

Let us consider the operator

$$(S_\xi f)(x) = f(x) + \int_0^\xi k(x,t) f(t)\, dt \tag{10.1}$$

on $L^2(0,\xi)$.

THEOREM 10.1. *Let the kernel* $k(x,t)$, $0 < x,t < \infty$, *have an extension to a function* $k(z,w)$ *which is analytic as function of* z *and* w *in a region* G *such that* G *contains the set* $(0,\infty)$ *and* $zt \in G$ *whenever* $z \in G$, $0 < t < 1$. *Then the function*

$$\sigma(\xi, f, g) = (S_\xi^{-1} f, g)_\xi$$

where $f(x)$ *and* $g(x)$ *are entire functions of* x, *has an extension to a function* $\sigma(z,f,g)$ *which is analytic in* G *except at isolated points. All finite singular points of* $\sigma(z,f,g)$ *are poles.*

Proof. For small ξ, the operator S_ξ differs from the identity operator by an operator of norm less than one. Therefore S_ξ is invertible for $0 \leqslant \xi < \varepsilon$ for some $\varepsilon > 0$. For each ξ in $(0,\infty)$, define U_ξ from $L^2(0,1)$ to $L^2(0,\xi)$ by

$$(U_\xi f)(t) = \sqrt{\frac{1}{\xi}}\, f\left(\frac{x}{\xi}\right), \quad 0 < x < \xi.$$

Then U_ξ maps $L^2(0,1)$ isometrically onto $L^2(0,\xi)$, and

$$(U_\xi^{-1} g)(x) = \sqrt{\xi}\, g(t\xi), \quad 0 < t < 1.$$

Hence $\widetilde{S_\xi} = U_\xi^{-1} S_\xi U_\xi$ is a bounded operator on $L^2(0,1)$ given by

$$\widetilde{S_\xi} f(x) = f(x) + \xi \int_0^1 k(\xi x, \xi t) f(t)\, dt. \tag{10.2}$$

Clearly S_ξ is invertible if and only if $\widetilde{S_\xi}$ is invertible. Write

$$\widetilde{S_\xi} = I + T_\xi. \tag{10.3}$$

The assumptions of the theorem allow us to define an operator $T(z)$ on $L^2(0,1)$ by

$$T(z) f = z \int_0^1 k(zx, zt) f(t)\, dt. \tag{10.4}$$

The operator $T(z)$ is compact and depends holomorphically on z, and $T(z)$ agrees with the operator $T(\xi)$ defined by (10.3) when $z=\xi$ is a point of $(0,\infty)$. Since $I+T(\xi)$ is invertible for small positive ξ, $I+T(z)$ is invertible except at isolated points of G (see Kato [13] Theorem 1.9 on p. 370) in which $\left[I+T(z)\right]^{-1}$ has the poles.

Hence the function $\left(\left[I+T(z)\right]^{-1}x^m,x^n\right)_1$ is meromorphic if m and n are non-negative integers. The assertion of the theorem follows from the relation

$$(S_\xi^{-1}x^m,x^n)_\xi=\xi^{m+n+1}(\widetilde{S}_\xi^{-1}x^m,x^n)_1. \qquad \square$$

REMARK 10.1. Arguments close to Theorem 10.1 are contained in the article [17].

The following kernels satisfy the conditions of Theorem 10.1:

$$k_1(x,t) = \gamma\,\frac{\sin\pi(x-t)}{\pi(x-t)}, \quad \gamma=\overline{\gamma}, \tag{10.5}$$

$$k_2(x,t) = \gamma\,\frac{Ai(x)\,Ai'(t)-Ai(t)\,Ai'(t)}{x-t}, \quad \gamma=\overline{\gamma}, \tag{10.6}$$

where $Ai(x)$ is the Airy function.

Let us introduce the functions

$$\phi(x)=J_\alpha(\sqrt{x}), \quad \psi(x)=x\phi', \quad x\geqslant 0,$$

and the kernel

$$k_3(x,t)=\gamma\,\frac{\phi(x)\,\psi(t)-\phi(t)\,\psi(x)}{x-t}, \quad \gamma=\overline{\gamma}, \tag{10.7}$$

where $J_\alpha(x)$ is the Bessel function of order α, $(\alpha>-1)$.

REMARK 10.2. The sine-kernel $k_1(x,t)$, the Airy-kernel $k_2(x,t)$ and the Bessel-kernel $k_3(x,t)$ play an important role in the random matrix theory (see [6, 16], [22]–[24]).

REMARK 10.3. The region G in the cases $k_1(x,t)$ and $k_2(x,t)$ is the complex plane. The region G in the case $k_3(x,t)$ is the complex plane cut by the half-axis $[0,\infty)$.

EXAMPLE 10.1. (fifth Painlevé transcendent) Let us consider the operator

$$S_tf=f(x)+\gamma\int_{-t}^{t}k(x-u)f(u)\,du, \quad f(u)\in L^2(-a,a), \tag{10.8}$$

where $|t|\leqslant a$, $\gamma=\overline{\gamma}$ and

$$k(x)=\frac{\sin x\pi}{x\pi}.$$

The operator S_t is invertible (see [6], p. 167), when $|\gamma|\leqslant 1$. Hence we have

$$S_t^{-1}f=f(x)+\int_{-t}^{t}\Gamma_t(x,u,\gamma)f(u)\,du, \quad f(u)\in L^2(-t,t),$$

where the kernel $\Gamma_t(x,u,\gamma)$ is jointly continuous to the variables x,t,u,γ. Together with the operator S_t we shall consider the operator

$$\widetilde{S}_{2t}f = f(x) + \gamma \int_0^{2t} k(x-v)f(v)\,dv, \quad f(u) \in L^2(0,2t).$$

The operator

$$U_t f(x) = f(u+t)$$

maps unitarily the space $L^2(0,2t)$ onto $L^2(-t,t)$. It is easy to see that

$$U_t^{-1} S_t U_t f = \widetilde{S}_{2t} f.$$

We have

$$\widetilde{S}_{2t}^{-1} f = f(x) + \int_0^{2t} \widetilde{\Gamma}_{2t}(x,u,\gamma) f(u)\,du, \quad f(u) \in L^2(0,2t),$$

where

$$\widetilde{\Gamma}_{2t}(x,y,\gamma) = \Gamma_t(x-t, y-t, \gamma). \tag{10.9}$$

It follows from (10.9) that

$$\widetilde{\Gamma}_{2t}(2t,2t,\gamma) = \Gamma_t(t,t,\gamma), \quad \widetilde{\Gamma}_{2t}(2t,0,\gamma) = \Gamma_t(t,-t,\gamma).$$

Now we consider the case when $\gamma = -1$. For brevity we omit the parameter $\gamma = -1$ in the notation $\Gamma_t(x,u,-1)$. Following C. Trace and H. Widom [22] we introduce the functions

$$r(t) = e^{it\pi} + \int_{-t}^{t} \Gamma_t(t,u)\, e^{iu\pi} du \tag{10.10}$$

and

$$q(t) = e^{it\pi} + \int_0^{t} \widetilde{\Gamma}_t(t,u)\, e^{iu\pi} du. \tag{10.11}$$

Relations (10.9) and (10.10), (10.11) imply that

$$q(2t) = r(t) e^{it\pi}. \tag{10.12}$$

We use the following relation (see [22])

$$\frac{d}{dt}\left[tR(t,t)\right] = \left|r(t)\right|^2, \tag{10.13}$$

where $R(t,t) = \Gamma_t(t,t)$. From (10.12) and (10.13) we have

$$tR(t,t) = \frac{1}{2}\int_0^{2t} \left|q(v)\right|^2 dv.$$

To prove the relation

$$tR(t,t) = \frac{1}{2}\left(\widetilde{S}_{2t}^{-1} e^{u\pi}, e^{u\pi}\right)_{2t} \tag{10.14}$$

we use the notion of triangular factorization (see [9], Ch. 4; [18]).

DEFINITION 10.1. The positive operator S acting in $L^2(0,a)$ admits a *triangular factorization* if it can be represented in the form

$$S = S_- S_-^{\star}. \tag{10.15}$$

Here

$$Q_\xi S_-^{\pm 1} = Q_\xi S_-^{\pm 1} Q_\xi \,,$$

where

$$Q_\xi = I - P_\xi, \quad P_\xi f = f(x), \quad 0 \leqslant x < \xi,$$

and

$$P_\xi f = 0, \quad \xi \leqslant x \leqslant a, \quad f(x) \in L_k^2(0,a).$$

Using M. G. Krein result (see [9], Ch. 4) on the triangular factorization of the operator S with continuous kernel we obtain the assertion.

THEOREM 10.2. *The operator*

$$Sf = f(x) - \frac{1}{\pi}\int_0^a \frac{\sin \pi(x-t)}{x-t} f(t)\,dt$$

admits triangular factorization (10.15) *and*

$$S_-^{-1} f = f(v) + \int_0^v \widetilde{\Gamma}_v(v,u) f(u)\,du.$$

Hence formula (10.11) can be written in the form

$$q(x) = S_-^{-1} e^{iu\pi}. \tag{10.16}$$

REMARK 10.4. Representation (10.16) of $q(x)$ which contains the factorizing operator S_- plays an essential role in our approach.

We use the notations

$$D(\xi) = \det \widetilde{S}_\xi, \tag{10.17}$$

$$\sigma(x) = \frac{x}{\pi} D'\left(\frac{x}{\pi}\right) / D\left(\frac{x}{\pi}\right).$$

It is known (see [22]) that

$$\sigma(x) = -2tR(t,t), \quad x = 2\pi t. \tag{10.18}$$

Relations (10.14) and (10.18) imply that

$$\sigma(x) = -(\widetilde{S}_{2t}^{-1} e^{iu\pi}, e^{iu\pi})_{2t}, \quad x = 2\pi t.$$

We note that the function $\sigma(x)$ is the fifth Painlevé transcendent (see [22]). Using Proposition 10.1 and relation (10.18) we have obtained the new proof of the following well-known fact (see [10]).

COROLLARY 10.1. *The fifth Painlevé transcendent* $\sigma(\xi)$ *can be extended to the meromorphic function* $\sigma(z)$.

The function $\sigma(\xi)$ is a solution of the Painlevé equation (P_5 in the sigma form, see [22])

$$(\xi\sigma'')^2+4(\xi\sigma'-\sigma)(\xi\sigma'-\sigma+\sigma'^2)=0. \tag{10.19}$$

PROPOSITION 10.1. *All the poles z_k of $\sigma(z)$ are simple with residues z_k.*

Proof. Looking at the Laurent expansion of $\sigma(z)$ at the poles z_k we observe by (10.19) that the principal term of $\sigma(z)$ has to be $z_k/(z-z_k)$. The proposition is proved. □

According to (10.3) and (10.4) the function $D(\xi)$ can be extended to the entire function $D(z)$. From Proposition 10.1 and relation (10.17) we obtain the assertions.

COROLLARY 10.2. *All the zeroes of $D(z)$ are simple.*

COROLLARY 10.3. *All the eigenvalues of $T(z)$ are simple.*

EXAMPLE 10.2. (Painlevé type functions) Let us consider the operator S_ξ of form (10.1), where $k(x,t)=k_1(x,t)$. We introduce the functions

$$\sigma_1(\xi,\gamma,\lambda)=(S_\xi^{-1}f,g)_\xi, \tag{10.20}$$

where $f(x)=g(x)=e^{ix\lambda}$, $\lambda=\overline{\lambda}$. Using Theorem10.1 and Corollary 10.3 we obtain the following assertion.

PROPOSITION 10.2. *The function $\sigma_1(\xi,\gamma,\lambda)$ can be extended to the meromorphic function $\sigma_1(z,\gamma,\lambda)$, all the poles of $\sigma_1(z,\gamma,\lambda)$ are simple.*

DEFINITION 10.2. We call the functions $\sigma_1(z,\gamma,\lambda)$ *the Painlevé type functions.*

We note that the fifth Painlevé transcendent $\sigma(z)$ is connected with the functions of form (10.34) by the relation

$$\sigma(z)=-\sigma_1(z/\pi,-1,\pi).$$

We separately consider the function

$$\sigma_2(z,\gamma)=\sigma_1(z,\gamma,0). \tag{10.21}$$

It follows from (10.20) and (10.21) that

$$\sigma_2(z)=(S_\xi^{-1}1,1)_\xi, \quad \xi>0,$$

where the operator S_ξ and the kernel $k(x,y)$ are defined by relations (10.1) and (10.5), respectively. We introduce the operators of form (10.8) with the kernels $k(x,y)$ and

$$k_\pm(x,y)=\frac{1}{2}\left[k(x,y)\pm k(-x,y)\right].$$

Let us denote the Fredholm determinants corresponding to $k(x,y)$, $k_+(x,y)$ and $k_-(x,y)$ by $D(\gamma,t)$, $D_+(\gamma,t)$ and $D_-(\gamma,t)$, respectively. We use the following relations (see [16], Ch. 21.)

$$D(\gamma,t)=D_+(\gamma,t)D_-(\gamma,t),$$

$$\frac{D_-(\gamma,t)}{D_+(\gamma,t)} = 1 + \int_{-t}^{t} \Gamma_t(t,y,\gamma)\,dy. \tag{10.22}$$

It follows from (10.9) and (10.22) that

$$\frac{D_-(\gamma,t)}{D_+(\gamma,t)} = 1 + \int_{0}^{2t} \widetilde{\Gamma}_{2t}(t,y,\gamma)\,dy.$$

In view of (7.10) and (10.22) the relation

$$\frac{D_-(\gamma,t)}{D_+(\gamma,t)} = h_2(2t)$$

is true. Using formulas (10.2)–(10.4) we deduce the assertion.

PROPOSITION 10.3. *The functions $D_-(x,t)$ and $D_-(x,t)$ can be extended to the entire functions $D_-(z,t)$ and $D_-(z,t)$, respectively.*

Hence we have

COROLLARY 10.4. *The function $h_2(2t)$ can be extended to the meromorphic function*

$$h_2(2z) = \frac{D_-(\gamma,z)}{D_+(\gamma,z)}.$$

According to representation (7.5) and Theorem 10.1 equations (1.7) and (1.8) have the strongly regular solutions $u_1(x,\rho)$ and $u_2(x,\rho)$, respectively, when

$$r^{-2}(x) = 2h_2^2(2x). \tag{10.23}$$

From Theorem 6.1, Corollary10.4 and relation (10.23) we obtain

COROLLARY 10.5. *The function*

$$r(z) = \frac{D_-(\gamma,z)}{D_+(\gamma,z)}$$

satisfies conditions (6.3), (6.4).

EXAMPLE 10.3. (third Painlevé function) Let us consider the operator (10.1), where $k(x,t)$ is the Bessel kernel.

PROPOSITION 10.4. *The operator S_ξ defined by relations (10.1) and (10.7) is invertible on $L^2(0,\xi)$, when $|\gamma| \leqslant 1$.*

Proof. The kernel $k(x,t)$ has the form $k(x,t) = \gamma K(x,t)$, where

$$K(x,t) = \frac{1}{4}\int_0^1 \phi(xs)\,\phi(ts)\,ds, \quad \phi(x) = J_\alpha(\sqrt{x}\,). \tag{10.24}$$

The operator $T_\xi = S_\xi - I$ has the kernel $k(x,t)$ and is self-adjoint. It follows from (10.1) and (10.24) that

$$(T_\xi f,f) = \frac{\gamma}{4}\int_0^1 \left|\int_0^\xi J_\alpha(\sqrt{xs}\,)f(x)\,dx\right|^2 ds.$$

The last relation can be written in the form

$$(T_\xi f, f) = \gamma \int_0^1 \left|F(s)\right|^2 ds, \tag{10.25}$$

where

$$F(s) = \int_0^{\sqrt{\xi}} \sqrt{xs}\, J_\alpha(sx) f(x^2) \sqrt{2x}\, dx. \tag{10.26}$$

The Hankel transformation (10.26) is unitary. So we have

$$\left|(T_\xi f, f)\right| \leqslant |\gamma| \int_0^\infty |F(s)|^2 ds = |\gamma| \int_0^\xi \left|f(x)\right|^2 dx. \tag{10.27}$$

Hence $||T_\xi|| \leqslant |\gamma|$. If $|\gamma| < 1$ then the operator S_ξ is invertible. We shall consider separately the case when $\gamma = \pm 1$. Let us assume that $||T_\xi|| = 1$. In this case we have for some f the equality

$$T_\xi f = \pm f, \quad ||f|| \neq 0. \tag{10.28}$$

From relations (10.25), (10.27) and (10.28) we deduce that

$$F(s) = 0, \quad s > 1.$$

But the function $F(s)$ is analytic when $\mathrm{Re}\, s > 0$. Hence the equality $F(s) = 0$, $(s > 0)$, is true. It means that $||f(x)|| = 0$. We have obtained a contradiction, i. e. $||T_\xi|| < 1$. The proposition is proved. □

The operator S_ξ^{-1} has the form

$$S_\xi^{-1} f = f(x) + \int_0^\xi \Gamma_\xi(x,t) f(t) dt.$$

We consider the functions

$$q(\xi) = \phi(\xi) + \int_0^\xi \Gamma_\xi(\xi,t)\, \phi(t)\, dt, \tag{10.29}$$

We shall use the following relations (see [23])

$$\left[sR(s)\right]' = \frac{1}{4} q^2(s) \tag{10.30}$$

$$R(t) = -\frac{d}{dt} \log\det S_t.$$

Using M. G. Krein result (see [9], Ch. 4) on the triangular factorization of the operator S with continuous kernel we obtain the assertion.

THEOREM 10.3. *The operator S_a defined by* (10.1), (10.7) *when $\alpha \geqslant 0$ admits triangular factorization* (10.15) *and*

$$S_-^{-1} f = f(v) + \int_0^v \Gamma_v(v,u) f(u)\, du.$$

Formula (10.29) can be written in the form

$$q(x) = S_{-}^{-1}\phi. \tag{10.31}$$

We introduce the notation

$$\sigma(s) = sR(s,s).$$

Relations (10.30) and (10.31) imply that

$$\sigma(x) = \frac{1}{4}\left(S_{\xi}^{-1}\phi, \phi\right)_{\xi}. \tag{10.32}$$

Further we consider only the important case when $\gamma = -1$, $\alpha \geqslant 0$.

We note that in this case the function $\sigma(x)$ is the third Painlevé transcendent (see [22]). Using Proposition 10.1 and relation (10.32) we obtain the following fact (see [10]).

COROLLARY 10.6. *The third Painlevé transcendent $\sigma(\xi)$ can be extended to the function $\sigma(z)$ which is analytic in G except at isolated points. All finite singular points in G are poles. (The domain G is defined in Remark 10.3.)*

The function $\sigma(\xi)$ is a solution of the Painlevé equation

$$(\xi\sigma'')^2 + \sigma'(\sigma - \xi\sigma')(4\sigma' - 1) - \alpha^2\sigma'^2 = 0. \tag{10.33}$$

PROPOSITION 10.5. *All the poles z_k of $\sigma(z)$ are simple with residues z_k.*

Proof. Looking at the Laurent expansion of $\sigma(z)$ at the poles z_k we observe by (10.33) that the principal term of $\sigma(z)$ has to be $z_k/(z-z_k)$. The proposition is proved. □

REFERENCES

[1] N. I. AKHIEZER, *Elements of the Theory of Elliptic Functions, Amer. Math. Soc.*, Providence, 1990.
[2] L. BIEBERBACH, *Theorie der Gewönliche Differentialgleichungen*, Springer-Verlag, Berlin, 1965.
[3] A. CHERVOV AND D. TALALAEV, *KZ equation, G-opers and quantum Drinfeld-Sokolov reduction*, arXiv:hep-th/0607250, 2006.
[4] E. A. CODDINGTON AND N. LEVINSON, *Theory of Ordinary Differential Equations*, McGraw-Hill Book Company, New York, 1955.
[5] C. F. COLEMAN AND J. R. MCLAUGHLIN, Solution of the Inverse Spectral Problem for an impedance with integrable derivative, *Comm. Pure Appl. Math.*, **56** (1993), 145–184.
[6] P. A. DEIFT, A. R. ITS AND X. ZHOU, A Riemann-Hilbert Approach to Asymptotic Problems Arising in the Theory of Random Matrix Models, and also in the Theory of Integrable Statistical Mechanics, *Annals of Math.*, **146** (1997), 149–235.
[7] H. DYM AND L. SAKHNOVICH, On dual Canonical Systems and Dual Matrix String Equations, *Operator Theory*, **123** (1997), 207–228.
[8] F. GESZTESY AND W. STCKA, On Theorem of Picard, *Proceeding of the Amer. Math. Soc.*, **126** (1998), No. 4, 1089–1099.
[9] I. GOHBERG AND M. G. KREIN, *Theory and Applications of Volterra Operators in Hilbert Space*, Amer. Math. Soc. Providence, 1970.
[10] V. I. GROMAK, I. LAINE AND S. SHIMOMURA, Painlevé Differential Equations in the Complex Plane, *de Gruyter Studies in Mathematics*, **28** (2002), Berlin, New York.
[11] I. S. KAC AND M. G. KREIN, On the Spectral Function of the String, *Amer. Math. Soc. Translation*, **103** (1974),1–18.
[12] E. KAMKE, *Differentialgleichungen: Losungsmethoden und Losungen*, Bd. 2, Teubner, Germany, 1983.

[13] T. KATO, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1976.
[14] V. KATSNELSON AND D. VOLOK, Rational Solutions of the Schlesinger System and Isoprincipal Deformations of Rational Matrix Functions II. *Operator Theory: Advances ana Applications*, **157** (2005), 165–203.
[15] M. G. KREIN, On Main Approximation Problem of Extrapolation Theory and Filtration of Stationary Stochastic Processes, *Dokl. Akad. Nauk SSSR*, **94** (1954), No. 1, 13–16.
[16] M. L. MEHTA, *Random Matrices*, Academic press, Boston, 1991.
[17] J. ROVNYAK AND L. A. SAKHNOVICH, Inverse Problems for Canonical Differential Systems with Singulariries, *Operator Theory: Advances and Applications*, to appear.
[18] L. A. SAKHNOVICH, Factorization of Operators in $L^2(a,b)$, *Functional Anal. Appl.*, **13** (1979), 187–192 (Russian).
[19] L. A. SAKHNOVICH, *Spectral Theory of Canonical Differential Systems. Method of Operator Identities*, **107**, Operator Theory Advances and Appl., Birkhäuser, 1999.
[20] L. A. SAKHNOVICH, On Reducing the Canonical System to the Two Dual Differential Systems, *J. Math. Anal. Appl.*, **255** (2001), No. 2, 499–509.
[21] L. A. SAKHNOVICH, *Rational solutions of Knizhnik-Zamolodchikov system*, arXiv:math-ph/0609067, 2006.
[22] C. A. TRACY AND H. WIDOM, Introduction to Random Matrices, *Springer Lecture Notes in Physics*, **424** (1993), 103–130.
[23] C. A. TRACY AND H. WIDOM, Level spacing distribution and the Bessel kernel, *Common. Math. Phys.*, **161** (1994), 289–309.
[24] C. A. TRACY AND H. WIDOM, Level spacing distribution and the Airy kernel, *Common. Math. Phys.*, **159** (1994), 151–174.
[25] W. WASOW, Asymptotic Expansions for Ordinary Differential Equations, *Pure and Appl. Math.*, **14** (1965).

(Received July 23, 2006)

Lev Sakhnovich
735 Crawford ave.
Brooklyn, 11223, New York
USA
e-mail: lev.sakhnovich@verizon.net