Operators
and
Matrices
Volume 1, Number 1 (2007), 143-152

KANTOROVICH TYPE OPERATOR INEQUALITIES FOR FURUTA INEQUALITY

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(*communicated by Chi-Kwong Li*)

Abstract. In this paper, we shall present Kantorovich type operator inequalities for Furuta inequality related to the usual order and the chaotic one in terms of a generalized Kantorovich constant, a generalized condition number and the Specht ratio, in which we use variants of the grand Furuta inequality.

1. Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator A is said to be positive (in symbol: $A \geqslant 0$) if $(Ax,x) \geqslant 0$ for all $x \in H$. The Löwner-Heinz theorem asserts that $A \geqslant B \geqslant 0$ ensures $A^p \geqslant B^p$ for all $1 \geqslant p \geqslant 0$. However $A \geqslant B$ does not always ensure $A^p \geqslant B^p$ for $p > 1$ in general. Related to this, Furuta [7] established the following ingenious operator inequality:

THEOREM F. (Furuta inequality)

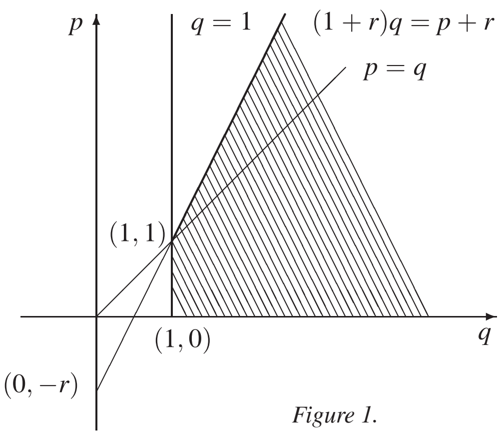

Figure 1.

Mathematics subject classification (2000): 47A63.

Key words and phrases: Furuta inequality, Grand Furuta inequality, Kantorovich inequality, generalized Kantorovich constant, generalized condition number, Specht ratio.

If $A \geqslant B \geqslant 0$*, then for each* $r \geqslant 0$
(i) $\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant \left(B^{\frac{r}{2}}B^pB^{\frac{r}{2}}\right)^{\frac{1}{q}}$ *and*
(ii) $\left(A^{\frac{r}{2}}A^pA^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}$ *hold for* $p \geqslant 0$ *and* $q \geqslant 1$ *with* $(1+r)q \geqslant p+r$.

Alternative proofs of Theorem F have been given in [3], [12], and one-page proof in [8]. It is shown in [14] that the domain of the parameters p, q and r drawn in Figure is the best possible for Theorem F.

On the other hand, the celebrated Kantorovich inequality asserts that if A is a positive operator on a Hilbert space H satisfying $M \geqslant A \geqslant m$ for some scalars $M > m > 0$, then $(A^{-1}x,x)(Ax,x) \leqslant \frac{(M+m)^2}{4Mm}$ holds for every unit vector x in H. The constant $\frac{(M+m)^2}{4Mm}$ is called the Kantorovich constant. As an application of the Kantorovich inequality, Fujii, Izumino, Nakamoto and the author [6] showed that t^2 is order preserving in the following sense:

$$A \geqslant B \geqslant 0 \quad \text{and} \quad M \geqslant A \geqslant m > 0 \quad \text{imply} \quad \frac{(M+m)^2}{4Mm}A^2 \geqslant B^2. \tag{1.1}$$

Related to this, Furuta [10] showed the following Kantorovich type operator inequality:

THEOREM A. *Let* A *and* B *be positive operators satisfying* $M \geqslant A \geqslant m$ *for some scalars* $M > m > 0$*. If* $A \geqslant B > 0$*, then*

$$\left(\frac{M}{m}\right)^{p-1} A^p \geqslant K(m,M,p)A^p \geqslant B^p \qquad \textit{for all } p \geqslant 1,$$

where a generalized Kantorovich constant $K(m,M,p)$ *([10, 11]) is defined as*

$$K(m,M,p) = \frac{mM^p - Mm^p}{(p-1)(M-m)}\left(\frac{p-1}{p}\frac{M^p-m^p}{mM^p-Mm^p}\right)^p \tag{1.2}$$

for any real number $p \in \mathbb{R}$.

For positive invertible operators A and B on a Hilbert space H, the order defined by $\log A \geqslant \log B$ is called *the chaotic order*. Since $\log t$ is an operator monotone function, the chaotic order is weaker than the usual one $A \geqslant B$. The following chaotic Furuta inequality is due to Fujii, Furuta and Kamei [4], which is a generalization of Ando's theorem [1]:

THEOREM FC. (Chaotic Furuta inequality) *If* $\log A \geqslant \log B$*, then for each* $r \geqslant 0$
(i) $\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant \left(B^{\frac{r}{2}}B^pB^{\frac{r}{2}}\right)^{\frac{1}{q}}$ *and*
(ii) $\left(A^{\frac{r}{2}}A^pA^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}$
hold for $p \geqslant 0$ *and* $q \geqslant 1$ *with* $rq \geqslant p+r$.

Yamazaki and Yanagida [16] showed the following Kantorovich type operator inequality related to the chaotic order which is parallel to Theorem A:

THEOREM B. *Let A and B be positive invertible operators satisfying $M \geqslant A \geqslant m$ for some scalars $M > m > 0$. If $\log A \geqslant \log B$, then*

$$\left(\frac{M}{m}\right)^p A^p \geqslant K(m,M,p+1)A^p \geqslant B^p \qquad \text{for all } p \geqslant 0.$$

In fact, $\log A \geqslant \log B$ does not always ensure $A \geqslant B$ in general. However, by Theorem B, it follows that

$$\log A \geqslant \log B \quad \text{and} \quad M \geqslant A \geqslant m > 0 \quad \text{imply} \quad \frac{(M+m)^2}{4Mm}A \geqslant B.$$

Also, Specht [13] estimated the upper bound of the arithmetic mean by the geometric one for positive numbers: For $x_1, \cdots, x_n \in [m,M]$ with $M \geqslant m > 0$,

$$S(h,1)\sqrt[n]{x_1 \cdots x_n} \geqslant \frac{x_1 + \cdots + x_n}{n} \geqslant \sqrt[n]{x_1 \cdots x_n},$$

where $h = \frac{M}{m} (\geqslant 1)$ is a generalized condition number in the sense of Turing [15] and a generalized Specht ratio $S(k,r)$ ([11]) is defined for $r > 0$ as

$$S(k,r) = \frac{(k^r - 1)k^{\frac{r}{k^r-1}}}{re\log k} \quad (k > 0, k \neq 1) \quad \text{and} \quad S(1,r) = 1. \tag{1.3}$$

Yamazaki and Yanagida [16] investigated analytic properties of the Specht ratio via a generalized Kantorovich constant and thereby showed a more precise characterization of the chaotic order:

THEOREM C. *Let A and B be positive invertible operators satisfying $M \geqslant A \geqslant m$ for some scalars $M > m > 0$. Then $\log A \geqslant \log B$ if and only if*

$$S(h,p)A^p \geqslant B^p \quad \text{for all } p > 0,$$

where $h = \frac{M}{m} \geqslant 1$.

Moreover, in [2] we showed the following result related to Theorem C:

THEOREM D. *Let A and B be positive invertible operators satisfying $k \geqslant A \geqslant \frac{1}{k}$ for a scalar $k > 1$. Then*

(i) $A \geqslant B$ *if and only if*

$$S(k,2(p-1)s)^{\frac{2}{s}} A^p \geqslant B^p \qquad \text{for all } p \geqslant 1,\ s \geqslant 1 \text{ with } p-1 \geqslant \tfrac{1}{s}.$$

(ii) $\log A \geqslant \log B$ *if and only if*

$$S(k,2ps)^{\frac{2}{s}} A^p \geqslant B^p \qquad \text{for all } p \geqslant 0,\ s \geqslant 1.$$

In this paper, we shall present Kantorovich type operator inequalities for Furuta inequality related to the usual order and the chaotic one in terms of a generalized Kantorovich constant, a generalized condition number and the Specht ratio, in which we use variants of the grand Furuta inequality.

2. Usual order version

First of all, we present Kantorovich type operator inequalities for Furuta inequality related to the usual order in terms of a generalized Kantorovich constant, a generalized condition number and the Specht ratio.

THEOREM 1. *Let A and B be positive operators satisfying $A \geqslant B$ and $M \geqslant A \geqslant m$ for some scalars $M > m > 0$. Then for each $r \geqslant 0$ and $\alpha > 1$*

$$K(m^{\frac{1}{\alpha-1}(\frac{p+r}{q}-(1+r))}, M^{\frac{1}{\alpha-1}(\frac{p+r}{q}-(1+r))}, \alpha)\, A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}} \tag{2.1}$$

holds for all $p \geqslant 1$, $q \geqslant 0$ such that $p \geqslant \alpha(1+r)q - r$.

$$K(m^{\frac{p+r}{\alpha q}}, M^{\frac{p+r}{\alpha q}}, \alpha)\, A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}} \tag{2.2}$$

holds for all $p \geqslant 1$, $q \geqslant 0$ such that $\alpha(1+r)q - r \geqslant p \geqslant (1+r)q - r$, where $K(m,M,p)$ is defined as (1.2).

In particular,

$$\frac{(m^{\frac{p+r}{q}-(1+r)} + M^{\frac{p+r}{q}-(1+r)})^2}{4m^{\frac{p+r}{q}-(1+r)}M^{\frac{p+r}{q}-(1+r)}}\, A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}} \tag{2.3}$$

holds for all $p \geqslant 1$, $q \geqslant 0$ such that $p \geqslant 2(1+r)q - r$.

In order to give a proof of Theorem 1, we cite the following variant [5, Proposition 6] of the grand Furuta inequality [9].

THEOREM G'. *If $A \geqslant B \geqslant 0$, then*

$$A^{\frac{(p+r)s+r}{q}} \geqslant \{A^{\frac{r}{2}}(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1}{q}}$$

holds for all $p, r, t, s \geqslant 0$ and $q \geqslant 1$ with $(p+t+r)q \geqslant (p+r)s+r$ and $(1+t+r)q \geqslant (p+r)s+r$.

Proof of Theorem 1. For each $r \geqslant 0$ and $\alpha > 1$, it follows from Theorem G' that

$$A^{\frac{(p+r)s+t}{\alpha}} \geqslant \{A^{\frac{t}{2}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^sA^{\frac{t}{2}}\}^{\frac{1}{\alpha}} \tag{2.4}$$

holds for all $p \geqslant 1$ and $t, s \geqslant 0$ with

$$(1+t+r)\alpha \geqslant (p+r)s + t. \tag{2.5}$$

Put $A_1 = A^{\frac{(p+r)s+t}{\alpha}}$ and $B_1 = \{A^{\frac{t}{2}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^sA^{\frac{t}{2}}\}^{\frac{1}{\alpha}}$, then $A_1 \geqslant B_1 > 0$ by (2.4) and $M \geqslant A \geqslant m > 0$ assures $M^{\frac{(p+r)s+t}{\alpha}} \geqslant A_1 \geqslant m^{\frac{(p+r)s+t}{\alpha}} > 0$. By applying Theorem A to A_1 and B_1, we have

$$K(m^{\frac{(p+r)s+t}{\alpha}}, M^{\frac{(p+r)s+t}{\alpha}}, \alpha)A_1^{\alpha} \geqslant B_1^{\alpha}.$$

Multiplying $A^{-\frac{t}{2}}$ on both sides, we have

$$K(m^{\frac{(p+r)s+t}{\alpha}}, M^{\frac{(p+r)s+t}{\alpha}}, \alpha)A^{(p+r)s} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^s.$$

Put $t = \frac{(p+r)s-(1+r)\alpha}{\alpha-1}$ and $s = \frac{1}{q}$. Since $p \geqslant \alpha(1+r)q - r$ and $q > 0$, then it follows that $t \geqslant 0$, $s \geqslant 0$ and the condition (2.5) is satisfied. Therefore, we have

$$K(m^{\frac{1}{\alpha-1}(\frac{p+r}{q}-(1+r))}, M^{\frac{1}{\alpha-1}(\frac{p+r}{q}-(1+r))}, \alpha)\, A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

for all $p \geqslant 1$, $q \geqslant 0$ such that $p \geqslant \alpha(1+r)q-r$, so that we have the desired inequality (2.1).

Also, putting $t = 0$ and $s = \frac{1}{q}$ in (2.4) and (2.5), we have (2.2) by the same discussion above.

For (2.3), we have only to put $\alpha = 2$ in (2.1).

Hence the proof of Theorem 1 is complete. □

By Theorem A and Theorem 1, we have the following corollary.

COROLLARY 2. *Let A and B be positive operators satisfying $A \geqslant B$ and $M \geqslant A \geqslant m$ for some scalars $M > m > 0$. Then for each $r \geqslant 0$*

$$\left(\frac{M}{m}\right)^{\frac{p+r}{q}-(1+r)} A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}} \tag{2.6}$$

holds for all $p \geqslant 1$, $q \geqslant 0$ such that $p \geqslant (1+r)q - r$.

Proof. By using Theorem A and Theorem 1, for each $r \geqslant 0$ and $\alpha \geqslant 1$

$$\left(\frac{M}{m}\right)^{\frac{p+r}{q}-(1+r)} A^{\frac{p+r}{q}} = \left(\frac{M}{m}\right)^{(\frac{1}{\alpha-1}(\frac{p+r}{q}-(1+r)))(\alpha-1)} A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for all $p \geqslant 1$, $q \geqslant 0$ such that $p \geqslant \alpha(1+r)q - r$. If we put $\alpha = 1$, then we have Corollary 2. □

REMARK 3. Putting $r = 0$, $q = 1$ and $p = \alpha \geqslant 1$ in (2.1) of Theorem 1 and $r = 0$, $q = 1$ in (2.6) of Corollary 2, we have Theorem A. Hence Theorem 1 and Corollary 2 can be considered as an extension of Theorem A.

Next, we present Kantorovich type operator inequalities for Furuta inequality related to the usual order in terms of the Specht ratio.

THEOREM 4. *Let A and B be positive operators satisfying $A \geqslant B$ and $k \geqslant A \geqslant \frac{1}{k}$ for a scalar $k > 1$. Then for each $r \geqslant 0$ and $\alpha > 1$*

$$S(k^{\frac{p+r}{q}-(1+r)}, 2s)^{\frac{2}{s}} A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \tag{2.7}$$

holds for all $p \geqslant 1$, $q \geqslant 0$, $s \geqslant 1$ such that $p \geqslant \alpha(1+r)q - r$ and $\alpha - 1 \geqslant \frac{1}{s}$.

$$S(k^{\frac{\alpha-1}{\alpha}\frac{p+r}{q}}, 2s)^{\frac{2}{s}} A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \tag{2.8}$$

holds for all $p \geqslant 1$, $q \geqslant 0$, $s \geqslant 1$ such that $\alpha - 1 \geqslant \frac{1}{s}$ and $\alpha(1+r)q - r \geqslant p \geqslant (1+r)q - r$, where $S(k,p)$ is defined as (1.3).

Proof. For each $r \geqslant 0$ and $\alpha > 1$, it follows from Theorem G' that

$$A^{\frac{(p+r)u+t}{\alpha}} \geqslant \{A^{\frac{t}{2}}(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^u A^{\frac{t}{2}}\}^{\frac{1}{\alpha}} \tag{2.9}$$

holds for all $p \geqslant 1$ and $t, u \geqslant 0$ with

$$(1+t+r)\alpha \geqslant (p+r)u + t \tag{2.10}$$

Put $A_1 = A^{\frac{(p+r)u+t}{\alpha}}$ and $B_1 = \{A^{\frac{t}{2}}(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^u A^{\frac{t}{2}}\}^{\frac{1}{\alpha}}$, then $A_1 \geqslant B_1 > 0$ by (2.9) and $k \geqslant A \geqslant \frac{1}{k} > 0$ assures $k^{\frac{(p+r)u+t}{\alpha}} \geqslant A_1 \geqslant k^{-\frac{(p+r)u+t}{\alpha}} > 0$. By applying (i) of Theorem D to A_1 and B_1, we have

$$S(k^{\frac{(p+r)s+t}{\alpha}}, 2(\alpha-1)s)^{\frac{2}{s}} A_1^{\alpha} \geqslant B_1^{\alpha}.$$

Multiplying $A^{-\frac{t}{2}}$ on both sides, we have

$$S(k^{\frac{(p+r)u+t}{\alpha}}, 2(\alpha-1)s)^{\frac{2}{s}} A^{(p+r)u} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^u$$

holds for all $p \geqslant 1$, $u, t \geqslant 0$ and $s \geqslant 1$ such that $\alpha - 1 \geqslant \frac{1}{s}$ and the condition (2.10).

Put $t = \frac{(p+r)u-(1+r)\alpha}{\alpha-1}$ and $u = \frac{1}{q}$. Since $p \geqslant \alpha(1+r)q - r$ and $q > 0$, then it follows that $t \geqslant 0$, $u \geqslant 0$ and the condition (2.10) is satisfied. Therefore, we have

$$S(k^{\frac{(p+r)u+t}{\alpha}}, 2(\alpha-1)s)^{\frac{2}{s}} A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

for all $p \geqslant 1$, $q \geqslant 0$ and $s \geqslant 1$ such that $p \geqslant \alpha(1+r)q - r$ and $\alpha - 1 \geqslant \frac{1}{s}$, so that we have the desired inequality (2.7).

Also, putting $t = 0$ and $u = \frac{1}{q}$ in (2.9) and (2.10), we have (2.8) by the same discussion above.

Hence the proof of Theorem 4 is complete. □

REMARK 5. Putting $r = 0$, $q = 1$ and $p = \alpha > 1$ in (2.7) of Theorem 4, we have (i) of Theorem D because

$$S(k^{p-1}, 2s)^{\frac{2}{s}} = S(k, 2(p-1)s)^{\frac{2}{s}}.$$

Hence Theorem 4 can be considered as an extension of (i) of Theorem D.

COROLLARY 6. *Let A and B be positive operators satisfying $A \geqslant B$ and $k \geqslant A \geqslant \frac{1}{k}$ for a scalar $k > 1$. Then for each $r \geqslant 0$*

$$(k^4)^{\frac{p+r}{q}-(1+r)} A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \tag{2.11}$$

holds for all $p \geqslant 1$, $q \geqslant 0$ such that $p \geqslant (1+r)q - r$.

Proof. Since it follows from [11, Lemma 2.46] that

$$\lim_{s\to\infty} S(k,s)^{\frac{1}{s}} = k,$$

we have this corollary by using Theorem 4. □

3. Chaotic order version

In this section, we present Kantorovich type operator inequalities for Chaotic Furuta inequality related to the chaotic order in terms of a generalized Kantorovich constant, a generalized condition number and the Specht ratio.

THEOREM 7. *Let A and B be positive invertible operators satisfying $\log A \geqslant \log B$ and $M \geqslant A \geqslant m$ for some scalars $M > m > 0$. Then for each $r \geqslant 0$ and $\alpha > 1$*

$$K(m^{\frac{1}{\alpha-1}(\frac{p+r}{q}-r)}, M^{\frac{1}{\alpha-1}(\frac{p+r}{q}-r)}, \alpha)\, A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \tag{3.1}$$

holds for all $p \geqslant 0$, $q \geqslant 0$ such that $p \geqslant \alpha rq - r$.

$$K(m^{\frac{p+r}{\alpha q}}, M^{\frac{p+r}{\alpha q}}, \alpha)\, A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \tag{3.2}$$

holds for all $p \geqslant 0$, $q \geqslant 0$ such that $\alpha rq - r \geqslant p \geqslant rq - r$, where $K(m,M,p)$ is defined as (1.2).

In particular,

$$\frac{(m^{\frac{p+r}{q}-r} + M^{\frac{p+r}{q}-r})^2}{4m^{\frac{p+r}{q}-r} M^{\frac{p+r}{q}-r}}\, A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \tag{3.3}$$

holds for all $p \geqslant 0$, $q \geqslant 0$ such that $p \geqslant 2rq - r$.

In order to give a proof of Theorem 7, we cite the following variant [5, Proposition 7] of the grand Furuta inequality [9].

THEOREM H . *Let A and B be positive invertible operators. If $\log A \geqslant \log B$, then*

$$A^{\frac{(p+t)s+r}{q}} \geqslant \{A^{\frac{r}{2}} \left(A^{\frac{t}{2}} B^p A^{\frac{t}{2}}\right)^s A^{\frac{r}{2}}\}^{\frac{1}{q}}$$

holds for all $p,t,s,r \geqslant 0$ and $q \geqslant 1$ with $(t+r)q \geqslant (p+t)s + r$.

Proof of Theorem 7. We can prove this theorem by a similar method as Theorem 1 by using Theorem H instead of Theorem G'. □

By Theorem 7 and Theorem B, we have the following corollary.

COROLLARY 8. *Let A and B be positive invertible operators satisfying $\log A \geqslant \log B$ and $M \geqslant A \geqslant m$ for some scalars $M > m > 0$. Then for each $r \geqslant 0$*

$$\left(\frac{M}{m}\right)^{\frac{p+r}{q}-r} A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}} \tag{3.4}$$

holds for all $p \geqslant 0$, $q \geqslant 0$ such that $p \geqslant rq - r$.

REMARK 9. Putting $r = 0$, $q = 1$ and $p = \alpha - 1 > 0$ in (3.1) of Theorem 7 and $r = 0$, $q = 1$ in (3.4) of Corollary 8, we have Theorem B. Hence Theorem 7 and Corollary 8 can be considered as an extension of Theorem B.

Similarly, we have the following result which is considered as an extension of (ii) of Theorem D.

THEOREM 10. *Let A and B be positive invertible operators satisfying $\log A \geqslant \log B$ and $k \geqslant A \geqslant \frac{1}{k}$ for a scalar $k > 1$. Then for each $r \geqslant 0$ and $\alpha > 1$*

$$S(k^{\frac{p+r}{q}-r}, 2s)^{\frac{2}{s}} A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}} \tag{3.5}$$

holds for all $p \geqslant 0$, $q \geqslant 0$, $s \geqslant 1$ such that $p \geqslant \alpha rq - r$ and $\alpha - 1 \geqslant \frac{1}{s}$.

$$S(k^{\frac{\alpha-1}{\alpha}\frac{p+r}{q}}, 2s)^{\frac{2}{s}} A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}} \tag{3.6}$$

holds for all $p \geqslant 0$, $q \geqslant 0$, $s \geqslant 1$ such that $\alpha - 1 \geqslant \frac{1}{s}$ and $\alpha rq - r \geqslant p \geqslant rq - r$, where $S(k,p)$ is defined as (1.3).

Proof. We can prove this theorem by a similar method as Theorem 4 by using Theorem D (ii) and Theorem H instead of Theorem G'. □

COROLLARY 11. *Let A and B be positive invertible operators satisfying $\log A \geqslant \log B$ and $k \geqslant A \geqslant \frac{1}{k}$ for a scalar $k > 1$. Then for each $r \geqslant 0$*

$$(k^4)^{\frac{p+r}{q}-r} A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}} \tag{3.7}$$

holds for all $p \geqslant 0$ and $q \geqslant 0$ such that $p \geqslant rq - r$.

Let A and B be positive invertible operators. We consider an order $A^\delta \geqslant B^\delta$ for $\delta \in (0,1]$ which interpolates the usual order $A \geqslant B$ and the chaotic one $\log A \geqslant \log B$ continuously, where the case of $\delta = 0$ means the chaotic order. The following corollaries are easily obtained by Theorem 1 and Theorem 4 respectively:

COROLLARY 12. *Let A and B be positive invertible operators satisfying $M \geqslant A \geqslant m$ for some scalars $M > m > 0$. If $A^\delta \geqslant B^\delta$ for $\delta \in (0,1]$, then for each $r \geqslant 0$ and $\alpha > 1$*

$$K(m^{\frac{1}{\alpha-1}(\frac{p+r}{q}-(\delta+r))}, M^{\frac{1}{\alpha-1}(\frac{p+r}{q}-(\delta+r))}, \alpha)\, A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \tag{3.8}$$

holds for all $p \geqslant \delta$, $q \geqslant 0$ such that $p \geqslant \alpha(\delta+r)q - r$, where $K(m,M,p)$ is defined as (1.2).

COROLLARY 13. *Let A and B be positive invertible operators satisfying $k \geqslant A \geqslant \frac{1}{k}$ for a scalar $k > 1$. If $A^\delta \geqslant B^\delta$ for $\delta \in (0,1]$, then for each $r \geqslant 0$ and $\alpha > 1$*

$$S(k^{\frac{p+r}{q}-(\delta+r)}, 2s)^{\frac{2}{s}}\, A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \tag{3.9}$$

holds for all $p \geqslant \delta$, $q \geqslant 0$ such that $p \geqslant \alpha(\delta+r)q - r$, where $S(k,p)$ is defined as (1.3).

REMARK 14. Corollary 12 interpolates (2.1) of Theorem 1 and (3.1) of Theorem 7 by means of a generalized Kantorovich constant. Let A and B be positive invertible operators satisfying $M \geqslant A \geqslant m$ for some scalars $M \geqslant m > 0$. Then the following assertions holds:

(i) $A \geqslant B$ implies $K(m^{\frac{1}{\alpha-1}(\frac{p+r}{q}-(1+r))}, M^{\frac{1}{\alpha-1}(\frac{p+r}{q}-(1+r))}, \alpha)\, A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$ for all $p \geqslant 1$, $q \geqslant 0$ such that $p \geqslant \alpha(1+r)q - r$.

(ii) $A^\delta \geqslant B^\delta$ implies $K(m^{\frac{1}{\alpha-1}(\frac{p+r}{q}-(\delta+r))}, M^{\frac{1}{\alpha-1}(\frac{p+r}{q}-(\delta+r))}, \alpha) A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$ for all $p \geqslant \delta$, $q \geqslant 0$ such that $p \geqslant \alpha(\delta+r)q - r$.

(iii) $\log A \geqslant \log B$ implies $K(m^{\frac{1}{\alpha-1}(\frac{p+r}{q}-r)}, M^{\frac{1}{\alpha-1}(\frac{p+r}{q}-r)}, \alpha)\, A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$ for all $p \geqslant 0$, $q \geqslant 0$ with $p > \alpha r q - r$.

It follows that a generalized Kantorovich constant of (ii) interpolates the scalar of (i) and (iii) continuously. In fact, if we put $\delta = 1$ in (ii), then we have (i). Also, if we put $\delta \to 0$ in (ii), then we have (iii).

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(Received October 7, 2006)

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