Operators and Matrices
Volume 1, Number 2 (2007), 153–163

EXTREMAL MARGINAL TRACIAL STATES IN COUPLED SYSTEMS

GEOFFREY L. PRICE* AND SHÔICHIRÔ SAKAI

Dedicated to
Professor Robert T. Powers
on the occasion of his sixty-fifth birthday

(*communicated by David Larson*)

Abstract. Let Γ be the convex set consisting of all states ϕ on the tensor product $B\otimes B$ of the algebra $B = M_n(\mathbb{C})$ of all $n\times n$ matrices over the complex numbers $\mathbb{C}$ with the property that the restrictions $\phi_{\upharpoonright B\otimes I}$ and $\phi_{\upharpoonright I\otimes B}$ are the unique tracial states on $B\otimes I$ and $I\otimes B$. We find necessary and sufficient conditions for such a state, called a marginal tracial state, to be extremal in Γ. We also give a characterization of those extreme points in Γ which are pure states. We conjecture that all extremal marginal tracial states are pure states.

1. Introduction

For $n \geqslant 2$ let B be the type I_n-factor consisting of the $n\times n$ matrices $M_n(\mathbb{C})$ over the complex numbers $\mathbb{C}$. Let $B\otimes B$ be the tensor product of B with itself. Then $B\otimes B$ is isomorphic to the full matrix algebra of $n^2\times n^2$ matrices over $\mathbb{C}$. The mapping $x \to x\otimes I$ (respectively, $x\to I\otimes x$), for $x\in B$ gives a unital embedding of B into the subalgebra $B\otimes I$ (respectively, $I\otimes B$) of $B\otimes B$.

Let τ be the unique tracial state on $B\otimes B$. By restriction to $B\otimes I$ (respectively, to $I\otimes B$), $\tau(x\otimes I) = \tau_B(x)$ (respectively, $\tau(I\otimes x) = \tau_B(x)$). Under these identifications we will abuse notation somewhat by using τ interchangeably to refer to the tracial state on $B\otimes B$ as well as on B.

A state ϕ on $B\otimes B$ is called a marginal tracial state if the restrictions of ϕ to $B\otimes I$ and $I\otimes B$ are the tracial states on $B\otimes I$ and $I\otimes B$ respectively. Note that the set Γ consisting of all marginal tracial states on $B\otimes B$ is non-empty (since, of course, $\tau\in\Gamma$) and convex. Moreover, Γ is $\sigma(T\otimes T, B\otimes B)$-compact, where T is the dual of B.

In [P1] K. R. Parthasarathy has shown for the case $n = 2$ that any extremal marginal tracial state is a pure state. Our work in the present paper is directed towards determining whether or not one can extend Parthasarathy's result to all n. Although we cannot solve the problem we have obtained some partial results which we hope will

Mathematics subject classification (2000): 46L30, 46L06.
Key words and phrases: Pure state, trace, marginal tracial state, Schmidt decomposition.
* Research supported by NSF grant DMS-0400841.

© Ele-Math, Zagreb
Paper No. 01-10

prove useful. Our results suggest that Parthasarathy's result may indeed extend to all n (see Lemma 2.7 and the remark following).

Given these considerations it is obviously of interest to determine those pure states on $B \otimes B$ which are marginal tracial states. The proof of the theorem below shows that there is a one-to-one correspondence between such states and $SU(n)$. The proof, which we include for the sake of completeness, is achieved by constructing the Schmidt decomposition of a unit vector corresponding to the pure state, [Sc], [VW], [EK].

THEOREM 1.1. *A marginal tracial state ϕ on $B \otimes B$ is a pure state if and only if there are orthonormal bases $\{f_i : 1 \leqslant i \leqslant n\}$ and $\{g_i : 1 \leqslant i \leqslant n\}$ of $\mathbb{C}^n$ such that $\phi(A) = \langle A\xi, \xi\rangle$ for all $A \in B \otimes B$, where $\xi = \sum_{i=1}^{n} \frac{f_i \otimes g_i}{\sqrt{n}}$.*

REMARK 1.1. A decomposition of a vector η in $\mathbb{C}^n \otimes \mathbb{C}^n$ into the form $\sum_{i=1}^{n} a_i f_i \otimes g_i$, where $\{f_i : i = 1, \dots, n\}, \{g_j : j = 1, \dots, n\}$ are a pair of orthonormal bases of $\mathbb{C}^n$, is known as a Schmidt decomposition, (see [EK] for the reference to [Sc])). In [EK] it is shown that for any pure state ω on $M_p(\mathbb{C}) \otimes M_q(\mathbb{C})$ there is an orthonormal basis $\{f_i : i = 1, \dots, p\}$ for $\mathbb{C}^p$ and another, $\{g_j : j = 1, \dots, q\}$ for $\mathbb{C}^q$ such that the unit vector η satisfying $\omega = \langle \cdot\eta, \eta\rangle$ admits a Schmidt decomposition $\eta = \sum_{i=1}^{r} a_i f_i \otimes g_i$, where $r \leqslant \min(p, q)$.

In the terminology of quantum computing the vector ξ in the statement of the theorem is said to be a maximally entangled vector which means that r above is maximal (i.e., $r = n$) in the Schmidt decomposition of ξ.

Proof. If $\{f_i\}$ and $\{g_i\}$ are orthonormal bases for $\mathbb{C}^n$ it is straightforward to verify that the pure state $\langle \cdot\xi, \xi\rangle$ is a marginal tracial state on $B \otimes B$, where $\xi = \sum_{i=1}^{n} \frac{f_i \otimes g_i}{\sqrt{n}}$. Conversely, suppose ϕ is a marginal tracial state that is also pure. Then for $1 \leqslant i, j \leqslant n$ there are complex numbers $\lambda_{ij} \in \mathbb{C}$ be such that $\phi(A) = \langle A\xi, \xi\rangle$ where $\xi = \sum_{i,j=1}^{n} \lambda_{ij} f_i \otimes f_j$. For $x \in B$ we have $\phi(x \otimes I) = \langle (x \otimes I)\xi, \xi\rangle = \tau(x \otimes I)$. Let $\{e_{rs} : 1 \leqslant r, s \leqslant n\}$ be matrix units of B corresponding to the orthonormal basis $\{f_i\}$, i.e. $e_{rs} f_i = \delta_{is} f_r$, then clearly

$$\delta_{rs}/n = \tau(e_{rs}) = \phi(e_{rs} \otimes I) = \langle (e_{rs} \otimes I)\xi, \xi\rangle = \sum_{j=1}^{n} \lambda_{sj}\overline{\lambda}_{rj},$$

so that $\Lambda = \left(\sqrt{n}\lambda_{ij}\right)$ is a unitary matrix. For $1 \leqslant i \leqslant n$ set $g_i = \sum_{j=1}^{n} \sqrt{n}\lambda_{ij} f_j$. Since Λ is a unitary matrix the set $\{g_i : 1 \leqslant i \leqslant n\}$ is an orthonormal basis of $\mathbb{C}^n$, and $\phi = \langle \cdot\xi, \xi\rangle$ where $\xi = \sum_{i=1}^{n} \frac{f_i \otimes g_i}{\sqrt{n}}$. □

REMARK 1.2. From the proof above it follows that if ϕ is a pure state on $B \otimes B$ that restricts to the tracial state on $B \otimes I$, i.e. $\phi(x \otimes I) = \tau(x) = \tau_B(x)$ for all $x \in B$, then ϕ is automatically a marginal tracial state on $B \otimes B$. For it follows under these hypotheses that $\phi = \langle \cdot\xi, \xi\rangle$ where $\xi = \sum_{i=1}^{n} \frac{f_i \otimes g_i}{\sqrt{n}}$ and so for any $y \in B$, $\phi(I \otimes y) = \langle (I \otimes y)\xi, \xi\rangle = \left\langle (I \otimes y)(\sum_{i=1}^{n} \frac{f_i \otimes g_i}{\sqrt{n}}), \sum_{i=1}^{n} \frac{f_i \otimes g_i}{\sqrt{n}} \right\rangle = \sum_{i=1}^{n} \langle y g_i, g_i\rangle / n = \tau(y)$.

REMARK 1.3. Using an argument similar to the proof above (see also [EK]) one can show that if $\phi = \langle \cdot\xi, \xi\rangle$ is a pure marginal tracial state then the vector ξ has the following property: let η be any vector in $\mathbb{C}^n \otimes \mathbb{C}^n = \mathcal{H} \otimes \mathcal{H}$, the vector space on which $B \otimes B$ acts; then there is a linear operator $\Phi \in B$ such that $(\Phi \otimes I)\xi = \eta$.

2. Main Results

Let $\mathfrak{S}$ be the convex set of all states on $B \otimes B$; then for each $\phi \in \mathfrak{S}$ there is a unique positive element $h_\phi \in T \otimes T$ such that $\phi(a) = \tau(ah_\phi)$, all $a \in B \otimes B$, and $\tau(h_\phi) = \phi(I) = 1$. Conversely, let h be any element of $T \otimes T$ satisfying $\tau(h) = 1 = \|h\|_1$, where $\|\cdot\|_1$ denotes the trace norm on $T \otimes T$; then h must be positive and $\phi^h(a) = \tau(ah), a \in B \otimes B$, is a state on $B \otimes B$, [Sa]. Under this correspondence we shall identify the set of all states on $B \otimes B$ with the set of all positive elements of $B \otimes B$ with trace norm 1.

Let P (resp., Q) be the canonical conditional expectation of $B \otimes B$ onto $B \otimes I$ (resp., $I \otimes B$); then the linear mappings P and Q satisfy the following conditions:

- (i) $P(h) \geqslant 0$ (resp. $Q(h) \geqslant 0$) for $h \geqslant 0$ in $B \otimes B$, and $P(I) = I$ (resp. $Q(I) = I$).
- (ii) $P(axb) = aP(x)b$ for $a, b \in B \otimes I$ and $x \in B \otimes B$, (resp., $Q(cxd) = cQ(x)d$ for $c, d \in I \otimes B$ and $x \in B \otimes B$).
- (iii) $P(xk) = P(kx)$ for $k \in I \otimes B$ and $x \in B \otimes B$ (resp. $Q(x\ell) = Q(\ell x)$ for $\ell \in B \otimes I$ and $x \in B \otimes B$).
- (iv) $P(x^*x) = 0$ (resp. $Q(x^*x) = 0$) if and only if $x = 0$, for $x \in B \otimes B$.
- (v) $\tau(P(x)) = \tau(Q(x)) = \tau(x)$ for $x \in B \otimes B$.
- (vi) $P(x^*) = P(x)^*$ and $Q(x^*) = Q(x)^*$ for $x \in B \otimes B$.
- (vii) $\|P(x)\| \leqslant \|x\|$ and $\|Q(x)\| \leqslant \|x\|$ for all $x \in B \otimes B$.

Since $T \otimes T$ and $B \otimes B$ are identical as sets, the conditional expectation P (resp. Q) may be viewed as a positive linear mapping of $T \otimes T$ onto $T \otimes I$ (resp. of $T \otimes T$ onto $I \otimes T$) and satisfying the same properties above. Concerning the norm we have $\|P(x)\|_1 \leqslant \|x\|_1$ and $\|Q(x)\|_1 \leqslant \|x\|_1$ for $x \in T \otimes T$. In fact,

$$
\begin{aligned}
\|P(x)\|_1 &= \sup_{\|a\| \leqslant 1, a \in B \otimes I} |\tau(aP(x))| \\
&= \sup_{\|a\| \leqslant 1, a \in B \otimes I} |\tau(P(aP(x)))| \\
&= \sup_{\|a\| \leqslant 1, a \in B \otimes I} |\tau(P(a)P(x))| \\
&= \sup_{\|a\| \leqslant 1, a \in B \otimes I} |\tau(P(a)x)| \\
&= \sup_{\|a\| \leqslant 1, a \in B \otimes I} |\tau(ax)| \\
&\leqslant \sup_{\|b\| \leqslant 1, b \in B \otimes B} |\tau(bx)| \\
&\leqslant \|x\|_1.
\end{aligned}
$$

LEMMA 2.1. *A state h on $B \otimes B$ is a marginal tracial state if and only if $P(h) = Q(h) = I$.*

Proof. Suppose h is a marginal tracial state; then $\tau(ah) = \tau(a)$, $a \in B \otimes I$ is the tracial state on $B \otimes I$ and $\tau(ah) = \tau(P(ah)) = \tau(aP(h)) = \tau(a)$, $a \in B \otimes I$. Hence $P(h) = I$, and similarly, $Q(h) = I$. Conversely, if $P(h) = Q(h) = I$ for a positive element h in $T \otimes T$ then $\tau(ah) = \tau(P(ah)) = \tau(aP(h)) = \tau(a)$ for $a \in B \otimes I$ and so h is tracial on $B \otimes I$, and similarly it is tracial on $I \otimes B$, so that h is a marginal tracial state. □

Consider the linear mapping $P + Q$ on $T \otimes T$ and let $V = ker(P + Q)$; then for $v \in V$, $(P + Q)(v) = 0$ and $P(P + Q)(v) = P(v) + PQ(v) = P(v) + \tau(v)I = 0$. Similarly, $Q(v) + \tau(v)I = 0$; hence $(P + Q)(v) + 2\tau(v)I = 0$; therefore $\tau(v) = 0$ and so $P(v) = Q(v) = 0$.

REMARK 2.1. It is straightforward to see that $V = N \otimes N$ where $N \subset B$ is the vector space of all elements $b \in B$ with trace 0.

LEMMA 2.2. *$(I - V) \cap (T \otimes T)^+$ is the set of all marginal tracial states on $B \otimes B$, where $(T \otimes T)^+$ is the set of all positive elements of $T \otimes T$.*

Proof. Let $h \in (I - V) \cap (T \otimes T)^+$ and put $h = I - v$, $v \in V$. Since $h \geqslant 0, \|h\|_1 = \tau(h) = \tau(I - v) = \tau(I) = 1$; hence h is a state on $B \otimes B$. Moreover, $P(I - v) = P(I) - P(v) = I$ and $Q(I - v) = I$ and therefore, by the previous lemma, h is a marginal tracial state.

Conversely suppose h is a marginal tracial state on $B \otimes B$; then $P(I - h) = I - I = 0$ and $Q(I - h) = 0$; hence $I - h \in V$ and $h = I - (I - h)$. □

A consequence of the following result is that for $n \geqslant 2$ the unique tracial state on $B \otimes B$ is never an extremal marginal tracial state.

THEOREM 2.3. *Let h_0 be a marginal tracial state on $B \otimes B$. Then the following conditions are equivalent:*

- (*i*) *h_0 is extremal among the marginal tracial states on $B \otimes B$,*
- (*ii*) $(R(h_0)T \otimes TR(h_0)) \cap V = \{0\}$,
- (*iii*) $(R(h_0)T \otimes TR(h_0)) \cap \{\lambda I + V : \lambda \in \mathbb{C}\} = \{\lambda h_0 : \lambda \in \mathbb{C}\}$,

where $R(h_0)$ is the range projection of h_0.

Proof. (*i*) $\Longrightarrow$ (*ii*): Suppose there is a nonzero element v in $(R(h_0)T \otimes TR(h_0)) \cap V$. Since $R(h_0)T \otimes TR(h_0)$ and V are selfadjoint, $v^* \in (R(h_0)T \otimes TR(h_0)) \cap V$ and so $v + v^*, iv - iv^* \in (R(h_0)T \otimes TR(h_0)) \cap V$; hence without loss of generality we may assume that v is self-adjoint. Then there is a positive real number λ such that $-\lambda R(h_0) \leqslant v \leqslant \lambda R(h_0)$. Therefore there exists a positive number μ such that $-\mu h_0 \leqslant -\lambda R(h_0) \leqslant v \leqslant \lambda R(h_0) \leqslant \mu h_0$. Hence $\mu h_0 \pm v \geqslant 0$ and so $h_0 \pm \frac{v}{\mu} \geqslant 0$. Then $\|h_0 \pm \frac{v}{\mu}\|_1 = \tau(h_0 \pm \frac{v}{\mu}) = \tau(h_0) = 1$. Moreover, $P(h_0 \pm \frac{v}{\mu}) = P(h_0) = I$ (resp. $Q(h_0 \pm \frac{v}{\mu}) = Q(h_0) = I$) and so $h_0 \pm \frac{v}{\mu}$ are marginal tracial states and $h_0 = \frac{(h_0 + \frac{v}{\mu}) + (h_0 - \frac{v}{\mu})}{2}$, a contradiction.

$(ii) \implies (iii)$: Clearly $h_0 \in R(h_0)T \otimes TR(h_0)$ and $h_0 = I + (h_0 - I)$ with $h_0 - I \in V$; hence $\{\lambda h_0 : \lambda \in \mathbb{C}\} \subset (R(h_0)T \otimes TR(h_0)) \cap \{\lambda I + V : \lambda \in \mathbb{C}\}$. If there is an element k in $(R(h_0)T \otimes TR(h_0)) \cap \{\lambda I + V : \lambda \in \mathbb{C}\}$ that is not a scalar multiple of h_0 then $k = \lambda_1 I + v$ for some $\lambda_1 \in \mathbb{R}$ and $v \in V$. If $\lambda_1 = 0$ then $k = v \neq 0$ and $v \in R(h_0)T \otimes TR(h_0)$, which contradicts (ii). If $\lambda_1 \neq 0$ then $\frac{k}{\lambda_1} = I + \frac{v}{\lambda_1} \notin \{\lambda h_0 : \lambda \in \mathbb{C}\}$ and $\frac{k}{\lambda_1} \in (R(h_0)T \otimes TR(h_0)) \cap \{\lambda I + V : \lambda \in \mathbb{C}\}$. We have $\frac{k}{\lambda_1} - h_0 = I + \frac{v}{\lambda_1} - (I - (I - h_0)) = \frac{v}{\lambda_1} + (I - h_0) \in V$ and $\frac{v}{\lambda_1} + (I - h_0) \neq 0$; moreover $\frac{v}{\lambda_1} + (I - h_0) \in R(h_0)T \otimes TR(h_0) \cap V$. This contradicts (ii).

$(iii) \implies (i)$: If h_0 is not extremal then there are two marginal tracial states h_1 and h_2 on $B \otimes B$ such that $h_0 \neq h_i$ for $i = 1,2$ and $h_0 = \frac{h_1 + h_2}{2}$. Therefore $0 \leqslant h_i \leqslant 2h_0$ for $i = 1,2$, and so by the preceding lemma it follows that $h_i \in (R(h_0)T \otimes TR(h_0)) \cap \{\lambda I + V : \lambda \in \mathbb{C}\}$ for $i = 1,2$. If $h_1 = \lambda h_2$ then $\tau(h_1) = \lambda \tau(h_2) = \lambda$, hence $\lambda = 1$ and so $\dim((R(h_0)T \otimes TR(h_0)) \cap \{\lambda I + V : \lambda \in \mathbb{C}\}) \geqslant 2$. □

REMARK 2.2. In [P1] K. R. Parthasarathy provides necessary and sufficient conditions for a state, with restrictions ρ_1 on $B \otimes I$ and ρ_2 on $I \otimes B$, to be extremal among all states with the same restrictions. In the appendix to this paper we show that our condition (ii) above is equivalent to Parthasarathy's condition for marginal tracial states to be extremal.

COROLLARY 2.4. *(cf. [P1]) If h_0 is an extremal marginal tracial state then $R(h_0) < I$, i.e., h_0 is not invertible.*

Proof. If $R(h_0) = I$ then $(R(h_0)T \otimes TR(h_0)) \cap V = V = \{0\}$. On the other hand, $V = N \otimes N$ where $N = \{a \in B : \tau_B(a) = 0\}$, where τ_B is the tracial state on B. Hence $V \neq \{0\}$. □

Let $\mathcal{H}$ be the set of all Hilbert-Schmidt class matrices of B; then $\mathcal{H} = B$ as sets and the inner product of $\mathcal{H}$ is given by $\langle a, b \rangle = \tau_B(b^* a)$, for $a, b \in B$, where τ_B is the tracial state on B. The norm on $\mathcal{H}$ is given by $\|a\|_2 = \tau_B(a^* a)^{\frac{1}{2}}, a \in \mathcal{H}$. $\mathcal{H} \otimes \mathcal{H}$ coincides with $B \otimes B$ as a set and $\mathcal{H} \otimes \mathcal{H}$ is a Hilbert space with inner product given by $\langle a \otimes b, c \otimes d \rangle = \tau((c \otimes d)^*(a \otimes b))$ for $a, b, c, d \in B$ and extended to all of $\mathcal{H} \otimes \mathcal{H}$ by linearity. Moreover, since $B = \mathbb{C}I + N$ it follows that $\mathcal{H} \otimes \mathcal{H} = B \otimes B = \mathbb{C}I \oplus (N \otimes I + I \otimes N) \oplus (N \otimes N)$ where we abuse notation slightly by using I to denote the identity on B and also on $B \otimes B$.

LEMMA 2.5. *$\{\lambda I : \lambda \in \mathbb{C}\} \oplus (N \otimes I + I \otimes N) \oplus (N \otimes N)$ is an orthogonal decomposition of $\mathcal{H} \otimes \mathcal{H}$.*

Proof. For $\lambda \in \mathbb{C}$ we have $\tau((\lambda I)^*(N \otimes I + I \otimes N)) = \overline{\lambda} \tau(N \otimes I + I \otimes N) = 0$ and $\tau((N \otimes I + I \otimes N)^*(N \otimes N)) = \tau(N^* N \otimes N + N \otimes N^* N) = \tau(N^* N \otimes I)\tau(I \otimes N) + \tau(N \otimes I)\tau(I \otimes N^* N) = 0$. □

Henceforth we shall often use the notation B rather than T or $\mathcal{H}$ because they are the same as sets.

Let E be a linear subspace of $B \otimes B$. We shall use the notation E^o to denote the orthogonal complement of E with respect to the scalar product defined by the tracial state τ on $B \otimes B$. We shall also simplify notation by using R to denote the range projection $R(h_0)$ of a fixed extremal marginal tracial state h_0. Then we have the following lemma.

LEMMA 2.6. $B \otimes B = \mathbb{C}I + (N \otimes I + I \otimes N) + (RB \otimes BR)^o$.

Proof. By Theorem 2.3, $(RB \otimes BR) \cap V = \{0\}$; hence $B \otimes B = ((RB \otimes BR) \cap V)^o = (RB \otimes BR)^o + V^o = (RB \otimes BR)^o + (B \otimes I + I \otimes B)$. □

The following lemma is an immediate consequence of the preceding lemma.

LEMMA 2.7. $R(B \otimes I + I \otimes B)R = \mathbb{C}R + R(N \otimes I + I \otimes N)R = RB \otimes BR$.

REMARK 2.3. Note that the conclusion of the lemma shows that $R \neq I$ and so we recapture the result of Corollary 2.4.

The identification made in the preceding lemma between $RB \otimes BR$ and $\mathbb{C}R + R(N \otimes I + I \otimes N)R$ is trivial when $\dim(R) = 1$ and seems to be rather puzzling otherwise, since it does not seem intuitive to us that $\mathbb{C}R + R(N \otimes I + I \otimes N)R$ is isomorphic to a full matrix algebra for $\dim(R) > 1$. For this reason the lemma suggests that any extremal marginal tracial state must in fact be pure.

In what follows we shall make use of the following observation (see Corollary 2.10). Since $h_0 \in RB \otimes BR$ and $\tau(h_0 R(N \otimes I + I \otimes N)R) = \tau((N \otimes I + I \otimes N)h_0) = 0$, we have the orthogonal decomposition:

$$RB \otimes BR = \mathbb{C}h_0 \oplus R(N \otimes I + I \otimes N)R.$$

Moreover, if $W = \{a \in RB \otimes BR : \tau(a) = 0\}$ then by the preceding equation $W = h_0^{\frac{1}{2}} R(N \otimes I + I \otimes N) R h_0^{\frac{1}{2}} = h_0^{\frac{1}{2}} (N \otimes I + I \otimes N) h_0^{\frac{1}{2}}$.

Our next main goal, in Theorem 2.9, is to obtain necessary and sufficient conditions for a extremal marginal tracial state to be a pure state. To prove this result it will be helpful to study two norms on the linear subspace $\mathbb{C}I + V$ of $B \otimes B$. First note that since $B \otimes B$ has the orthogonal decomposition $B \otimes B = (\mathbb{C}I + V) \oplus (N \otimes I + I \otimes N)$ we may identify $\mathbb{C}I + V$ with the quotient space $B \otimes B/(N \otimes I + I \otimes N)$ of $B \otimes B$.

The first norm we impose on $\mathbb{C}I + V$ is the C^*-norm $\|\cdot\|$ which the space inherits as a subspace of $B \otimes B$. The other is the quotient norm $|||\cdot|||$ given by $|||a||| = \inf_{y \in N \otimes I + I \otimes N} \|a + y\|, a \in \mathbb{C}I + V$. Clearly $|||a||| \leqslant \|a\|$, all $a \in \mathbb{C}I + V$.

Let $\|\cdot\|_1$ be the trace norm on $T \otimes T$. Then $\|a\|_1 = \tau((a^* a)^{\frac{1}{2}}), a \in T \otimes T$, and is the dual norm of the C^*-norm $\|\cdot\|$ on $B \otimes B$. Let $\|\cdot\|^*$ be the dual norm on $\mathbb{C}I + V (\subset T \otimes T)$ with respect to the norm $\|\cdot\|$ on $\mathbb{C}I + V (\subset B \otimes B)$. By the general theory of Banach spaces

$$(B \otimes B/(N \otimes I + I \otimes N))^* = (N \otimes I + I \otimes N)^o = \mathbb{C}I + V \subset T \otimes T$$

with respect to the norm $\|\cdot\|_1$; hence we have, for $f \in \mathbb{C}I + V$,

$$\|f\|_1 = \sup_{|||x||| \leqslant 1, x \in \mathbb{C}I + V} |f(x)| = \sup_{\|x\| \leqslant 1, x \in B \otimes B} |f(x)|, \qquad \text{and}$$

$$\|f\|^* = \sup_{\|x\| \leqslant 1, x \in \mathbb{C}I + V} |f(x)|,$$

and so $\|f\|^* \leqslant \|f\|_1$.

Suppose $f \in \mathbb{C}I+V$ is a state on $B\otimes B$; then $1=\|f\|_1=f(I)$. By restriction f may be viewed as a linear functional on $\mathbb{C}I+V$, and we have $\|f\|_1 = \sup_{\||x|\|\leqslant 1, x\in\mathbb{C}I+V} |f(x)|$. Let Δ be the set of all such linear functionals. Then Δ coincides with the set of all positive elements h in $\mathbb{C}I+V$ such that $\tau(h)=1$. Since $f(I)\leqslant \|f\|^* \leqslant \|f\|_1$, $\|f\|^* = \|f\|_1$ for $f\in\Delta$.

Let h_0 be an extreme point of Γ and let $I(h_0)$ be the set of all states k on $B\otimes B$ such that $\tau(ah_0)=\tau(ak)$ for $a\in\mathbb{C}I+V$; then $k-h_0\in(\mathbb{C}I+V)^o = N\otimes I+I\otimes N$ and therefore there is a selfadjoint element a in $N\otimes I+I\otimes N$ such that $k=h_0+a$. Conversely we have the following:

LEMMA 2.8. *Let h be an arbitrary state on $B\otimes B$ and consider its restriction to $\mathbb{C}I+V$; then there is a unique selfadjoint element ℓ in $\mathbb{C}I+V$ such that $\tau(xh)=\tau(x\ell)$ for $x\in\mathbb{C}I+V$. In fact $\ell=(I-(P-Q)^2)(h)$.*

Proof. A straightforward calculation shows that $(P-Q)^2(P-Q)^2=(P-Q)^2$, and $((P-Q)^2)^*=(P-Q)^2$ so that $(P-Q)^2$ is the orthogonal projection of $B\otimes B$ onto $N\otimes I+I\otimes N$ in the Hilbert space $B\otimes B$. Hence $I-(P-Q)^2$ is the orthogonal projection of $B\otimes B$ onto $\mathbb{C}I+V$. Hence $\ell=(I-(P-Q)^2)(h)\in\mathbb{C}I+V$.

Next we show that $\tau(xh)=\tau(x\ell)$ for all $x\in\mathbb{C}I+V$. To see this note that for any $a,b\in B\otimes B$, $\tau(P(a)b)=\tau(P(a)P(b))=\tau(aP(b))$. Similarly $\tau(Q(a)b)=\tau(aQ(b))$. Therefore $\tau((P-Q)^2(a)b)=\tau((P-Q)(a)(P-Q)(b))=\tau(a(P-Q)^2(b))$, and so $\tau((I-(P-Q)^2)(a)b)=\tau(a(I-(P-Q)^2)(b))$. Now replace a with $x\in\mathbb{C}I+V$ and b with h in this equation. From the preceding paragraph $x=(I-(P-Q)^2)(x)$, so that $\tau(xh)=\tau(x\ell)$ for all $x\in\mathbb{C}I+V$. The uniqueness of ℓ is straightforward. □

The following gives a characterization of the extreme points of Γ that are in fact pure states. As pointed out in [P1] all extreme points are pure in the case $n=2$, i.e. when the algebra B is isomorphic to the matrix algebra $M_2(\mathbb{C})$.

THEOREM 2.9. *Let h_0 be an extreme point of Γ, the set of marginal tracial states on $B\otimes B$. Then h_0 is a pure state on $B\otimes B$ if and only if $(I-(P-Q)^2)(RB\otimes BR)\subset RB\otimes BR$.*

Proof. Suppose h_0 is a pure state; then $RB\otimes BR=\mathbb{C}R$ and so $h_0=n^2R$. By Lemma 2.1, $(P-Q)(h_0)=0$, so

$$(I-(P-Q)^2)(R)=\frac{1}{n^2}(I-(P-Q)^2)(h_0))=\frac{1}{n^2}h_0=R.$$

Hence $(I-(P-Q)^2)\mathbb{C}R=\mathbb{C}R$.

For the converse suppose $(I-(P-Q)^2)(RB\otimes BR)\subset RB\otimes BR$. By the proof of Lemma 2.8, for any $h\in RB\otimes BR$, $(I-(P-Q)^2)(h)\in\mathbb{C}I+V$. Therefore $(I-(P-Q)^2)(h)\in(\mathbb{C}I+V)\cap(RB\otimes BR)$. Combining this with the results of Theorem 2.3 shows $(I-(P-Q)^2)(h)\in\{\lambda h_0:\lambda\in\mathbb{C}\}$, and so $(I-(P-Q)^2)(a)\in\{\lambda h_0:\lambda\in\mathbb{C}\}$ for $a\in RB\otimes BR$; hence

$$(I-(P-Q)^2)(a)=\tau(a)h_0,\qquad a\in RB\otimes BR. \tag{2.1}$$

By Lemma 2.1, $P(h_0) = I = Q(h_0)$. Therefore $(P-Q)^2(h_0) = 0$ so that we may rewrite the equation above as

$$a - \tau(a)h_0 = (P-Q)^2(a - \tau(a)h_0). \tag{2.2}$$

Hence $a - \tau(a)h_0 \in N \otimes I + I \otimes N$ for $a \in RB \otimes BR$.

Now assume (to obtain a contradiction) that h_0 is not a pure state. Then if $r = \operatorname{rank}(R)$, $r > 1$, $RB \otimes BR$ is isomorphic to the algebra of $r \times r$ matrices, and therefore the linear space $W = \{a \in RB \otimes BR : \tau(a) = 0\}$ has dimension $r^2 - 1$.

From the preceding paragraph W is also a subset of $N \otimes I + I \otimes N$. Let $a \neq 0$ be a selfadjoint element of W satisfying $\|a\|_2 = 1$, then there are selfadjoint elements ℓ, m of N such that $a = \ell \otimes I + I \otimes m$.

We show $2\ell \otimes m = h_0 - I$. To see this apply Eq. 2.2 to a^2 to see that

$$a^2 - h_0 = (P-Q)^2(a^2 - h_0) \tag{2.3.}$$

But

$$\begin{aligned}(P-Q)^2(a^2 - h_0) &= (P-Q)^2(a^2)\\ &= (P-Q)^2(\ell^2 \otimes I + I \otimes m^2 + 2\ell \otimes m)\\ &= (P-Q)^2(\ell^2 \otimes I + I \otimes m^2)\\ &= (P-Q)^2(\ell^2 \otimes I + I \otimes m^2 - I)\\ &= \ell^2 \otimes I + I \otimes m^2 - I,\end{aligned}$$

where the last equality follows from the observation that $\ell^2 \otimes I + I \otimes m^2 - I \in N \otimes I + I \otimes N$, since

(*i*) $\tau(\ell^2 \otimes I + I \otimes m^2 - I) = \tau(a^2) - 1 = 0$, and
(*ii*) $\ell^2 \otimes I + I \otimes m^2 - I$ is orthogonal to $N \otimes N$.

From the preceding calculation and Equation 2.3 we have $2\ell \otimes m = h_0 - I$ for all self-adjoint a in W with $\|a\|_2 = 1$.

By Corollary 2.4, $h_0 \neq I$ hence $\ell \otimes m \neq 0$, hence $\ell \neq 0$ and $m \neq 0$. Recall [T] that for any functional ρ on B the map $x \otimes y \to \rho(y)x$, $x, y \in B$ extends by linearity to a well-defined (right) slice map from $B \otimes B$ to B. If we define ρ by $\rho(y) = \tau(m^* y), y \in B$ then the corresponding right slice map sends $\ell \otimes m$ to $\tau(m^* m)\ell \neq 0$. Similarly there is a left slice map that sends $\ell \otimes m$ to $\tau(\ell^* \ell)m \neq 0$. But if $b = \ell_1 \otimes I + I \otimes m_1$ is any other selfadjoint element of W with $\|b\|_2 = 1$ then $2\ell_1 \otimes m_1 = h_0 - I = 2\ell \otimes m$, so applying the left and right slice maps above to $\ell_1 \otimes m_1$ shows that ℓ_1 (resp., m_1) is a scalar multiple of ℓ (resp. of m). It follows immediately that if $c = \ell_2 \otimes I + I \otimes m_2$ is *any* selfadjoint element of W, then independent of $\|c\|_2$, ℓ_2 and m_2 are scalar multiples of ℓ and m respectively. Since $W = W^*$ it then follows that for any element d of W, d lies in the subspace of $N \otimes I + I \otimes N$ spanned by $\ell \otimes I, I \otimes m$. Hence W is at most two-dimensional. Since W has dimension $r^2 - 1$, the only possibility is $r = 1$. Hence h_0 must be a pure state on $B \otimes B$. $\square$

PROBLEM. Let h_0 be an extreme point of Γ and let h be an arbitrary state on $B \otimes B$ with $h \in RB \otimes BR$. Can we conclude that $\|(I - (P-Q)^2)(h)\|_1 \leqslant 1$?.

If so then h_0 is a pure state. In fact, $\tau((I-(P-Q)^2)(h)) = \tau(h) = 1$ and so $(I-(P-Q)^2)(h)$ is a state. Since $(I-(P-Q)^2)(h) \in \mathbb{C}I + V$, it is a marginal tracial state. Now put $h_0 = \sum_{j=1}^m \lambda_j n^2 e_j$, where $\lambda_j > 0$ for $j = 1, 2, \ldots, m$ with $\sum_{j=1}^m \lambda_j = 1$ and $\{e_j : j = 1, 2, \ldots, m\}$ is a family of mutually orthogonal one-dimensional projections in $B \otimes B$ such that $\sum_{j=1}^m e_j = R$. Then $(I-(P-Q)^2)(h_0) = h_0 = \sum_{j=1}^m \lambda_j n^2 (I-(P-Q)^2)(e_j)$. Since h_0 is extreme in Γ, $n^2(I-(P-Q)^2)(e_j) = h_0$ for all j; hence $(I-(P-Q)^2)(R) = (I-(P-Q)^2)\left(\sum_{j=1}^m e_j\right) = \frac{m}{n^2} h_0$ and so $h_0 = \frac{n^2}{m} R$. Hence for an arbitrary projection p of $RB \otimes BR$, $(I-(P-Q)^2)(p) = \tau(p) h_0$, and so by Theorem 2.9, h_0 is pure.

COROLLARY 2.10. *Let h_0 be an extremal marginal tracial state on $B \otimes B$. Then the following conditions are equivalent.*

- (i) *h_0 is a pure state on $B \otimes B$.*
- (ii) *for any $k \in RB \otimes BR$ the restriction of k to $\mathbb{C}I + V$ is $\tau(k) h_{0\restriction \mathbb{C}I+V}$.*
- (iii) *for $\ell \in h_0^{1/2}(N \otimes I + I \otimes N) h_0^{1/2}$, the restriction of ℓ to $\mathbb{C}I + V$ is 0.*
- (iv) $h_0^{1/2}(N \otimes I + I \otimes N) h_0^{1/2} \subset (N \otimes I + I \otimes N)$.
- (v) $h_0^{1/2}(\mathbb{C}I + V) h_0^{1/2} \subset \mathbb{C}I + V$.

Proof. $(i) \implies (ii)$: $R = R(h_0)$ is a rank one projection so $h_0 = n^2 R$ and $k = \lambda R$. Then $\tau(k) = \lambda \tau(R) = \lambda \frac{1}{n^2}$, so $k = \frac{\lambda}{n^2} h_0$.

$(ii) \implies (iii)$: By the remark following Lemma 2.7, since $h_0^{1/2}(N \otimes I + I \otimes N) h_0^{1/2} = W = \{a \in RB \otimes BR : \tau(a) = 0\}$, so by (ii) the restriction of ℓ to $\mathbb{C}I + V$ must be 0.

$(iii) \implies (iv)$: For $a \in \mathbb{C}I + V$, by Lemma 2.8 we have $\tau(a(I-(P-Q)^2)(h_0^{1/2}(N \otimes I + I \otimes N) h_0^{1/2}) = 0$. On the other hand, $\tau(a(I-(P-Q)^2)(h_0^{1/2}(N \otimes I + I \otimes N) h_0^{1/2}) = \tau((I-(P-Q)^2)(a)(h_0^{1/2}(N \otimes I + I \otimes N) h_0^{1/2}) = \tau(a h_0^{1/2}(N \otimes I + I \otimes N) h_0^{1/2}) = 0$ for $a \in \mathbb{C}I + V$; hence $h_0^{1/2}(N \otimes I + I \otimes N) h_0^{1/2} \subset (\mathbb{C}I + V)^o = N \otimes I + I \otimes N$.

$(iv) \implies (v)$: By the above equality $0 = \tau((\mathbb{C}I + V) h_0^{1/2}(N \otimes I + I \otimes N) h_0^{1/2}) = \tau(h_0^{1/2}(\mathbb{C}I + V) h_0^{1/2}(N \otimes I + I \otimes N))$; hence $h_0^{1/2}(\mathbb{C}I + V) h_0^{1/2} \subset (N \otimes I + I \otimes N)^o = (\mathbb{C}I + V)$.

$(v) \implies (iv)$: clear.

$(iv) \implies (iii)$: $W = h_0^{1/2}(N \otimes I + I \otimes N) h_0^{1/2} \subset (N \otimes I + I \otimes N)$; hence $(I-(P-Q)^2)(W) \subset (I-(P-Q)^2)(N \otimes I + I \otimes N) = 0$.

$(iii) \implies (i)$: $RB \otimes BR = \mathbb{C}h_0 + W$ and $(I-(P-Q)^2)(\mathbb{C}h_0 + W) = \mathbb{C}h_0 \subset RB \otimes BR$; hence by Theorem 2.9, h_0 is a pure state. This completes the proof. □

3. Appendix

Let ρ_1 and ρ_2 be states on B and let $\mathcal{E}(\rho_1, \rho_2)$ be the convex set consisting of all states ρ on $B \otimes B$ which restrict on $B \otimes I$ to ρ_1 and on $I \otimes B$ to ρ_2, i.e., $\rho(x \otimes I) = \rho_1(x)$ and $\rho(I \otimes y) = \rho_2(y)$, for $x, y \in B$. In [P1] Parthasarathy gives a necessary and sufficient condition for ρ to be extremal in $\mathcal{E}(\rho_1, \rho_2)$. We show how Parthasarathy's criterion is related to ours in the case $\rho_1 = \rho_2 = \tau_B$ by showing that it is equivalent to condition (ii) of Theorem 2.3.

The following lemma can be found in [P2] (see also [P1]).

LEMMA A.1. *Let h be a state on $B \otimes B$ with rank $r < m$, where $m = n^2$. Then there exists a positive invertible $r \times r$ matrix K, a permutation matrix σ of $B \otimes B$, and an $r \times (m-r)$ matrix A such that*

$$\sigma h \sigma^{-1} = \begin{bmatrix} K & KA \\ A^*K & A^*KA \end{bmatrix}$$

THEOREM A.2. *(cf. [P1]) Let h be a marginal tracial state on $B \otimes B$ with rank $r < m$. Then h is not an extremal marginal tracial state if and only if there is a selfadjoint $r \times r$ matrix L such that*

$$\sigma^{-1} \begin{bmatrix} L & LA \\ A^*L & A^*LA \end{bmatrix} \sigma$$

is in V.

Proof. By condition (ii) of Theorem 2.3 the state h on $B \otimes B$ is not extremal if and only if $(RT \otimes TR) \cap V$ is nontrivial, where $R = R(h)$ and $V = N \otimes N$. Let σ be a permutation matrix in $B \otimes B$ such that $h^\sigma = \sigma h \sigma^{-1}$ has the form of the matrix in the lemma, and let $R^\sigma = \sigma R(h) \sigma^{-1}$. It follows from spectral theory that $R^\sigma = \mathit{lim}_{j \to \infty} h_j^\sigma$ where $h_1^\sigma = h^\sigma, h_{j+1}^\sigma = (h_j^\sigma)^{1/2}$. Moreover, for any positive matrix C in $B \otimes B$ (or in fact, for any positive operator C on a Hilbert space) we may obtain $C^{1/2}$ as a limit $C^{1/2} = \lim_{j \to \infty} C_j$ where $C_0 = 0$ and $C_{j+1} = C_j + \frac{1}{2}(C - C_j^2)$ [Sz]. Therefore $C^{1/2}$ is a limit of linear combinations of powers of C. It follows that the successive square roots h_j^σ of the lemma all have the form

$$\begin{bmatrix} Z_j & Z_jA \\ A^*Z_j & A^*Z_jA \end{bmatrix}$$

and therefore there is an $r \times r$ matrix Q such that the projection R^σ has the form

$$\begin{bmatrix} Q & QA \\ A^*Q & A^*QA \end{bmatrix}.$$

If $(RT \otimes TR) \cap V$ is nontrivial then there is a nonzero selfadjoint element D in $(RT \otimes TR) \cap V$ and $D^\sigma = R^\sigma D^\sigma R^\sigma$. From this equation and the form of R^σ it follows that D^σ has the form $\begin{bmatrix} L & LA \\ A^*L & A^*LA \end{bmatrix}$, as in the statement of the lemma, with $L = L^*$, $L \neq 0$, and with $D = \sigma^{-1} D^\sigma \sigma$ in V.

To prove the converse suppose there is a selfadjoint $r \times r$ matrix L such that the matrix $D = \sigma^{-1} \begin{bmatrix} L & LA \\ A^*L & A^*LA \end{bmatrix} \sigma$ is in V. Since K is invertible it is not difficult to show that $R^\sigma D^\sigma R^\sigma = D^\sigma$ (because the range of the matrix D^σ is contained in the range of the matrix h^σ), so that $D = RDR$ and therefore $D \in (RT \otimes TR) \cap V$. $\square$

REFERENCES

[EK] A. EKERT AND P. L. KNIGHT, *Entangled quantum systems and the Schmidt decomposition*, Am. J. Phys. **63** (1995), 415–423.

[P1] K. R. PARTHASARATHY, *Extremal quantum states in coupled systems*, Ann. Inst. H. Poincaré **41** (2005), 257–268.

[P2] K. R. PARTHASARATHY, *On extremal correlations*, J. Stat. Planning and Inf. **103** (2002), 73–80.

[Sa] S. SAKAI, *C^* -algebras and W^* -algebras*, Springer-Verlag, 1971.

[Sc] E. SCHMIDT, *Zur theorie der linearen und nichtlinearen integralgleichungen*, Math. Annalen **63** (1906), 433–476.

[Sz] B. SZ.-NAGY, *Spektraldarstellung linearer transformationen des Hilbertschen raumes*, Springer-Verlag, 1942.

[T] J. TOMIYAMA, *Applications of Fubini type theorem to the tensor products of C^* -algebras*, Tôhoku Math. J. (2) **19** (1967), 340–344.

[VW] K. G. H. VOLLBRECHT AND R. F. WERNER, *Why two qubits are special*, J. Math. Phys. **41** (2000), 6772–6782.

(Received November 16, 2006)

Geoffrey L. Price
Department of Mathematics 9E
United States Naval Academy
Annapolis, MD 21402, USA
e-mail: glp@usna.edu

Shôichirô Sakai
5-1-6-205, Odawara
Aoba-Ku
Sendai, Japan 980-0003
e-mail: JZU00243@nifty.com