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ON THE RELATION BETWEEN $XX^{[*]}$ AND $X^{[*]}X$ IN AN INDEFINITE INNER PRODUCT SPACE

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Abstract. The relations between the canonical forms for pairs $(XX^{[*]},H)$ and $(X^{[*]}X,H)$, with nilpotent matrices $XX^{[*]}$ and $X^{[*]}X$, are studied. We show that for some specific cases which are obtained by imposing restrictions on $\operatorname{rank} X$ or $\operatorname{rank} X^{[*]}X$ or both, the relations between these canonical forms can be found.

1. Introduction

Let $H \in \mathbb{C}^{n\times n}$ be a hermitian, invertible matrix. Then this matrix induces the following *indefinite inner product*

$$[\mathbf{x},\mathbf{y}] \overset{def}{=} \langle H\mathbf{x},\mathbf{y}\rangle$$

for $\mathbf{x},\mathbf{y} \in \mathbb{C}^n$, where $\langle .,.\rangle$ denotes the usual inner product in $\mathbb{C}^n$. For a matrix $X \in \mathbb{C}^{n\times n}$ we define the H-adjoint of X by

$$X^{[*]} \overset{def}{=} H^{-1}X^*H.$$

It is easily shown that $X^{[*]}$ is the unique matrix which satisfies the equation

$$[X\mathbf{x},\mathbf{y}] = [\mathbf{x},X^{[*]}\mathbf{y}]$$

for all $\mathbf{x},\mathbf{y} \in \mathbb{C}^n$. We call a matrix $A \in \mathbb{C}^{n\times n}$ H-*selfadjoint* if we have $A^{[*]} = A$. For this category of matrices we can define a *canonical form* (J,P) for the matrix pair (A,H), in which the matrix J is the Jordan canonical form for A and where the matrix P, which replaces H, is a matrix with a simple structure. See the next section for more details, see also [3, 4].

We will consider the matrices $X^{[*]}X$ and $XX^{[*]}$. One easily verifies that both of these matrices are H-selfadjoint, so both pairs $(X^{[*]}X,H)$ and $(XX^{[*]},H)$ can be written in canonical form. We will search for the connections between these two canonical forms. It turns out that the main case of interest is the case where $X^{[*]}X$ and $XX^{[*]}$ are nilpotent, and we shall focus on this case. The problem is connected to the

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problem treated in [1], although in the present paper the problem is discussed in the finite dimensional case only.

In Section 2 we state the problem, and discuss preliminary material. In particular we discuss the part of the canonical form connected to the nonzero eigenvalues of $X^{[*]}X$ in this section. From Section 3 onwards we assume that $X^{[*]}X$, and hence also $XX^{[*]}$, is nilpotent. In Section 3 we discuss the case where $X^{[*]}X$ is nilpotent and $\operatorname{rank} X^{[*]}X = \operatorname{rank} X$. Several special cases that are, at least to a large extent, corollaries of the result obtained in Section 3 are discussed in Section 4. Section 5 is concerned with the case where $X^{[*]}X$ is nilpotent and $\operatorname{rank} X = n-1$. Finally, in Section 6 we treat the case $X^{[*]}X = 0$. Sections 3 and 6 are opposite extremes: Section 3 treats the case where $\operatorname{Ker} X^{[*]}X$ is as small as possible given X, while Section 6 treats the case where $\operatorname{Ker} X^{[*]}X$ is as large as possible.

2. Problem Formulation and Preliminaries

Let us first define the following matrix blocks. We define the $n \times n$ *Jordan block*, associated with the real eigenvalue λ as

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}.$$

For a nonreal eigenvalue λ from the upper half of the complex plane we define $K_{2n}(\lambda) = J_n(\lambda) \oplus J_n(\bar{\lambda})$, a 'double' Jordan block associated with both eigenvalues λ and $\bar{\lambda}$. Note that the Jordan canonical form of an H-selfadjoint matrix has such couples of Jordan blocks (see e.g. [3]). Further we define the *standard involutary permutation matrix block* (sip matrix block) of an appropriate order by

$$P = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \iddots & 0 \\ 0 & 1 & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

We also define the following property of two matrix pairs (A_1, H_1) and (A_2, H_2). We say that these two pairs are *unitarily similar*, denoted by $(A_1, H_1) \approx (A_2, H_2)$, if there exists an invertible matrix T for which holds

$$A_2 = T^{-1}A_1T \qquad \text{and} \qquad H_2 = T^*H_1T. \tag{1}$$

With these definitions in place, we can now present the following theorem (see [3], p.33).

THEOREM 1. *Let A be an H-selfadjoint matrix and let*

$$J = diag[J(\lambda_1), ..., J(\lambda_\alpha), K(\lambda_{\alpha+1}), ..., K(\lambda_\beta)]$$

be a Jordan canonical form for A consisting of Jordan blocks $J(\lambda_i)$ associated with eigenvalues λ_i of A, where $\lambda_1,...,\lambda_\alpha$ are real and $\lambda_{\alpha+1},...,\lambda_\beta$ are non-real and from the upper half of the complex plane. Define further, for some ordered set of signs $\varepsilon = \{\varepsilon_1, \varepsilon_2, ..., \varepsilon_\alpha\}$ with $\varepsilon_i = \pm 1$ for $1 \leqslant i \leqslant \alpha$:

$$P_{\varepsilon,J} = diag[\varepsilon_1 P_1, \varepsilon_2 P_2, ..., \varepsilon_\alpha P_\alpha, P_{\alpha+1}, ..., P_\beta]$$

where P_i is a sip matrix of the same size as $J(\lambda_i)$ for $1 \leqslant i \leqslant \alpha$ and of the same size as $K(\lambda_i)$ for $\alpha+1 \leqslant i \leqslant \beta$. Then

$$(A,H) \approx (J, P_{\varepsilon,J})$$

for some set ε that is uniquely determined by (A,H), up to permutations of signs corresponding to equal Jordan blocks. Conversely, if for some set of signs ε, the pairs (A,H) and $(J,P_{\varepsilon,J})$ are unitarily similar, then A is H-selfadjoint.

We call the pair $(J,P_{\varepsilon,J})$ the canonical form for (A,H).

In order to find connections between the canonical forms for $(X^{[*]}X,H)$ and $(XX^{[*]},H)$, we need to know the connections between the Jordan canonical forms for $X^{[*]}X$ and $XX^{[*]}$. We will present a general theorem concerning the connections between the Jordan canonical forms of the products AB and BA, where $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times n}$ (see [5]).

THEOREM 2. *The elementary divisors of AB and BA which do not have zero as a root coincide with those of BA.*
If $\lambda^{n_1}, \lambda^{n_2}, ...$ is the sequence of elementary divisors of AB with zero as a root and $n_1 \geqslant n_2 \geqslant ...$ is the sequence of the (nonincreasing) exponents of these elementary divisors, made infinitely with adjunction of zeros, and $m_1 \geqslant m_2 \geqslant ...$ is the corresponding sequence of exponents of the elementary divisors of BA with zero as a root, then $|n_j - m_j| \leqslant 1$.
Conversely, if $C \in \mathbb{C}^{n\times n}$ and $D \in \mathbb{C}^{m\times m}$ satisfy these conditions, there exist matrices $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times n}$ such that $C = AB$ and $D = BA$.

The implication of this theorem for the Jordan canonical forms for the matrices AB and BA is that for nonzero eigenvalues λ, all blocks $J(\lambda)$ appearing in the Jordan canonical form for AB also appear in the Jordan canonical form for BA. For eigenvalue $\lambda = 0$, the blocks differ at most 1 in order. The signs attached to the Jordan blocks associated with *nonreal* eigenvalues in the canonical form for (AB,H) and (BA,H) are both $+1$ according to Theorem 1. For the signs attached to the Jordan blocks associated with *real nonzero* eigenvalues we have the following proposition.

PROPOSITION 3. *If $\lambda_1 \neq 0$ is a real eigenvalue of $X^{[*]}X \in \mathbb{C}^{n\times n}$ with the corresponding sign ε_1 attached to the Jordan block $J(\lambda_1)$ in the canonical form for $(X^{[*]}X,H)$, then the corresponding sign attached to this Jordan block in the canonical form for $(XX^{[*]},H)$ is* $\operatorname{sign}(\lambda_1)\varepsilon_1$. *This proposition also holds for $X^{[*]}X$ and $XX^{[*]}$ interchanged.*

Proof. Suppose that $\mathbf{x}_k, \mathbf{x}_{k-1}, \dots, \mathbf{x}_0$, with $\mathbf{x}_j \in \mathbb{C}^n$, is a Jordan chain of $X^{[*]}X$, corresponding to the nonzero real eigenvalue λ, so we have:

$$\begin{aligned} X^{[*]}X\mathbf{x}_j &= \lambda\mathbf{x}_j + \mathbf{x}_{j-1} \qquad \text{for } j = 1, \dots, k \\ X^{[*]}X\mathbf{x}_0 &= \lambda\mathbf{x}_0. \end{aligned}$$

Now put $\mathbf{y}_j = X\mathbf{x}_j$ for $j = 0, \dots, k$. Then $\mathbf{y}_k, \mathbf{y}_{k-1}, \dots, \mathbf{y}_0$ is a Jordan chain of $XX^{[*]}$. To see this, we can write

$$XX^{[*]}\mathbf{y}_j = XX^{[*]}X\mathbf{x}_j = X(\lambda\mathbf{x}_j + \mathbf{x}_{j-1}) = \lambda\mathbf{y}_j + \mathbf{y}_{j-1} \qquad \text{for } j = 1, \dots, k,$$

and

$$XX^{[*]}\mathbf{y}_0 = XX^{[*]}X\mathbf{x}_0 = \lambda X\mathbf{x}_0 = \lambda\mathbf{y}_0.$$

We can use these Jordan chains to determine the signs attached to the Jordan blocks corresponding to λ in both canonical forms. Note that Jordan chains are used for an (appropriate) Jordan basis T in the basis transformation described in (1). Combining this with Theorem 1 we have $P_{\varepsilon,J} = T^*HT$. Writing out this product, it is easily verified that, for example, the sign attached to the Jordan block corresponding to λ in the canonical form for $(X^{[*]}X, H)$ is equal to the sign of $[\mathbf{x}_0, \mathbf{x}_k] = \langle H\mathbf{x}_0, \mathbf{x}_k \rangle$. To determine the sign attached to the corresponding block in the canonical form for $(XX^{[*]}, H)$ we calculate

$$\langle H\mathbf{y}_0, \mathbf{y}_k \rangle = \langle HX\mathbf{x}_0, X\mathbf{x}_k \rangle = \langle X^*HX\mathbf{x}_0, \mathbf{x}_k \rangle = \left\langle HX^{[*]}X\mathbf{x}_0, \mathbf{x}_k \right\rangle = \lambda \langle H\mathbf{x}_0, \mathbf{x}_k \rangle .$$

For $X^{[*]}X$ and $XX^{[*]}$ interchanged, similar equations hold. This proves the proposition. $\square$

So the only remaining cases of interest are the Jordan blocks associated with eigenvalue $\lambda = 0$. Therefore, from now on we will only consider matrices X for which $X^{[*]}X$ is nilpotent.

A motivation for the search for connections between the canonical forms for the pairs $(XX^{[*]})$ and $(X^{[*]}X, H)$ can be found in the results of the articles [2] and [6]. The first article contains a corollary (Corollary 6) which states that a matrix $X \in \mathbb{C}^{n\times n}$ admits an H-*polar decomposition*: $X = UA$ (with an H-unitary U and an H-selfadjoint A) if and only if $(XX^{[*]}, H)$ and $(X^{[*]}X, H)$ have the *same* canonical form (see also [7] and [8]). The matrices $XX^{[*]}$ and $X^{[*]}X$ need not be nilpotent in this corollary. The second article contains a theorem (Theorem 4.4) which states that a matrix $X \in \mathbb{C}^{n\times n}$ admits an H-polar decomposition if and only if the pair $(X^{[*]}X, H)$ satisfies three conditions of which the second one concerns the part of the canonical form for $(X^{[*]}X, H)$ corresponding to the *zero eigenvalue*.

3. The case $\operatorname{rank} X^{[*]}X = \operatorname{rank} X$

As already stated before, from now on we will only consider matrices X for which $X^{[*]}X$ is nilpotent. For simplicity from now on we shall denote $J_n(0)$ by J_n.

We make the following general observation.

PROPOSITION 4. *Let X be such that $X^{[*]}X$ is a nilpotent matrix and such that* $\operatorname{rank} X^{[*]}X = \operatorname{rank} X$. *Then $(X^{[*]}X, H)$ and $(XX^{[*]}, H)$ cannot have the same canonical form unless $X = 0$.*

Proof. Assume that $(X^{[*]}X, H)$ and $(XX^{[*]}, H)$ have the same canonical form. According to [2, Corollary 6], X admits an H-polar decomposition: $X = UA$, with an H-unitary U and an H-selfadjoint A. Clearly $\operatorname{Ker} X = \operatorname{Ker} A$ and $A^2 = X^{[*]}X$. Then, because $\operatorname{rank}(X^{[*]}X) = \operatorname{rank} X$, we have $\operatorname{Ker} X = \operatorname{Ker} X^{[*]}X$, and so:

$$\operatorname{Ker} A = \operatorname{Ker} X = \operatorname{Ker} X^{[*]}X = \operatorname{Ker} A^2.$$

The matrix A is also nilpotent, and considering the Jordan canonical form of A we see that it follows from this that $A = 0$, and hence $X = 0$. $\square$

COROLLARY 5. *If* $\operatorname{rank} X^{[*]}X = \operatorname{rank} X$ *and $X \neq 0$ then X does not allow an H-polar decomposition.*

Let us consider the case where $\operatorname{Ker} X^{[*]}X = \operatorname{Ker} X$, or, in other words, $\operatorname{rank} X^{[*]}X = \operatorname{rank} X$. In that case the following theorem holds:

THEOREM 6. *Assume that X is a matrix for which* $\operatorname{Ker} X^{[*]}X = \operatorname{Ker} X$. *Let the canonical form of $(X^{[*]}X, H)$ be given by*

$$\oplus_{j=1}^{k} J_{n_j} \oplus \oplus_{j=k+1}^{l} J_1, \qquad \oplus_{j=1}^{k} \varepsilon_j P_{n_j} \oplus \oplus_{j=k+1}^{l} \varepsilon_j, \tag{2}$$

where we assume that $n_j > 1$ for $j = 1, \ldots, k$. Then the canonical form of $(XX^{[*]}, H)$ is given by

$$\oplus_{j=1}^{k} J_{n_j-1} \oplus \oplus_{j=1}^{k} J_1 \oplus \oplus_{j=k+1}^{l} J_1, \qquad \oplus_{j=1}^{k} \varepsilon_j P_{n_j-1} \oplus \oplus_{j=1}^{k} \delta_j \oplus \oplus_{j=k+1}^{l} \varepsilon_j, \tag{3}$$

and the numbers $\delta_j = \pm 1$ satisfy the equation one obtains from comparing the signature of H in both canonical forms:

$$\sum_{n_j \text{ is odd}} \varepsilon_j = \sum_{n_j \text{ is even}} \varepsilon_j + \sum_{j=1}^{k} \delta_j.$$

In other words,

$$\sum_{j=1}^{k} \delta_j = \sum_{n_j \text{ is odd}} \varepsilon_j - \sum_{n_j \text{ is even}} \varepsilon_j. \tag{4}$$

Before proving the theorem, let us note that equation (4) determines the number of $+1$'s and -1's among the δ_j's, that is, the δ_j's can be computed from (4) up to a reordering of the one-by-one blocks in the canonical form (3).

Proof. Let us first define $n_j = 1$ for $k + 1 \leqslant j \leqslant l$ and denote $N_m = \sum_{j=1}^{m} n_j$ for $1 \leqslant m \leqslant l$, finally define $N_0 = 0$. Let us further assume that $(X^{[*]}X, H)$ is already in canonical form, then we can write

$$\operatorname{Ker} X = \operatorname{Ker} X^{[*]}X = \operatorname{span}\left\{\mathbf{e}_1, \mathbf{e}_{N_1+1}, \mathbf{e}_{N_2+1}, \ldots, \mathbf{e}_{N_k+1}, \mathbf{e}_{N_k+2}, \ldots, \mathbf{e}_n\right\}.$$

Denoting the i-th column of X by $\mathbf{x}_i$, this leads to the following notation for the matrix X:

$$X = \Bigg(\underbrace{\mathbf{0}\ \mathbf{x}_2\ \cdots\ \mathbf{x}_{N_1}}_{n_1 \text{ columns}}\ \underbrace{\mathbf{0}\ \mathbf{x}_{N_1+2}\ \cdots\ \mathbf{x}_{N_2}}_{n_2 \text{ columns}}\ \cdots\ \underbrace{\mathbf{0}\ \mathbf{x}_{N_{k-1}+2}\ \cdots\ \mathbf{x}_{N_k}}_{n_k \text{ columns}}\ \underbrace{\mathbf{0}\ \mathbf{0}\ \cdots\ \mathbf{0}}_{l-k \text{ columns}} \Bigg).$$

From this, it follows that we have (note that $H^{-1} = H$)

$$X^{[*]} = HX^*H = \begin{pmatrix} \varepsilon_1 \mathbf{x}_{N_1}^* \\ \vdots \\ \varepsilon_1 \mathbf{x}_2^* \\ \mathbf{0} \\ \hline \varepsilon_2 \mathbf{x}_{N_2}^* \\ \vdots \\ \varepsilon_2 \mathbf{x}_{N_1+2}^* \\ \mathbf{0} \\ \hline \vdots \\ \hline \varepsilon_k \mathbf{x}_{N_k}^* \\ \vdots \\ \varepsilon_k \mathbf{x}_{N_{k-1}+2}^* \\ \mathbf{0} \\ \hline \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} H. \tag{5}$$

We now want to compute the product $X^{[*]}X$, using the notations above. In order to simplify the notation of this product, let us denote (for $1 \leqslant i, j \leqslant l$)

$$A_{i,j} = \begin{pmatrix} 0 & \varepsilon_i[\mathbf{x}_{N_{(j-1)}+2}, \mathbf{x}_{N_i}] & \varepsilon_i[\mathbf{x}_{N_{(j-1)}+3}, \mathbf{x}_{N_i}] & \cdots & \varepsilon_i[\mathbf{x}_{N_j}, \mathbf{x}_{N_i}] \\ 0 & \varepsilon_i[\mathbf{x}_{N_{(j-1)}+2}, \mathbf{x}_{N_i-1}] & \varepsilon_i[\mathbf{x}_{N_{(j-1)}+3}, \mathbf{x}_{N_i-1}] & \cdots & \varepsilon_i[\mathbf{x}_{N_j}, \mathbf{x}_{N_i-1}] \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \varepsilon_i[\mathbf{x}_{N_{(j-1)}+2}, \mathbf{x}_{N_{(i-1)}+2}] & \varepsilon_i[\mathbf{x}_{N_{(j-1)}+3}, \mathbf{x}_{N_{(i-1)}+2}] & \cdots & \varepsilon_i[\mathbf{x}_{N_j}, \mathbf{x}_{N_{(i-1)}+2}] \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

an $n_i \times n_j$ matrix block. Writing out the product $X^{[*]}X$, using the notations of X, $X^{[*]}$ and $A_{i,j}$ from above, we find

$$X^{[*]}X = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,l} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,l} \\ \vdots & \vdots & \ddots & \vdots \\ A_{l,1} & A_{l,2} & \cdots & A_{l,l} \end{pmatrix}.$$

When we equate this expression to the Jordan canonical form in (2), we find

$$A_{i,j} = \begin{cases} J_{n_i} & \text{for} \quad i = j \\ \mathbf{0} & \text{for} \quad i \neq j \end{cases}$$

This gives us important information concerning the indefinite inner products of all pairs of columns of X:

$$[\mathbf{x}_i, \mathbf{x}_j] = \begin{cases} \varepsilon_p & \text{for} \quad i + j = N_p + N_{(p-1)} + 2 \ \text{ and } \ N_{(p-1)} + 2 \leqslant i,j \leqslant N_p \\ 0 & \text{else} \end{cases} \tag{6}$$

In particular, using the notation of $X^{[*]}$ in (5), we see that

$$X^{[*]}\mathbf{x}_{N_i+2} = \mathbf{e}_{N_i+1} \in \operatorname{Ker} X$$

for $0 \leqslant i \leqslant k-1$. So we have

$$\operatorname{Ker} XX^{[*]} \supseteq \operatorname{span} \left\{ \mathbf{x}_2, \mathbf{x}_{N_1+2}, \ldots, \mathbf{x}_{N_{(k-1)}+2} \right\}. \tag{7}$$

Moreover, for $0 \leqslant i \leqslant k-1$ and $N_i + 1 \leqslant j \leqslant N_{(i+1)}$ we have that

$$X^{[*]}\mathbf{x}_j = \mathbf{e}_{j-1}$$

and so

$$XX^{[*]}\mathbf{x}_j = \mathbf{x}_{j-1}.$$

Combining these results, it appears that we can extend all k eigenvectors spanning the subspace of $\operatorname{Ker} XX^{[*]}$ mentioned in (7) to Jordan chains of length $n_1 - 1$, $n_2 - 1$, $\ldots$, $n_k - 1$ respectively:

$$\begin{array}{c} \mathbf{x}_{N_1} \overset{XX^{[*]}}{\longrightarrow} \mathbf{x}_{N_1-1} \overset{XX^{[*]}}{\longrightarrow} \ldots \overset{XX^{[*]}}{\longrightarrow} \mathbf{x}_2 \overset{XX^{[*]}}{\longrightarrow} \mathbf{0} \\ \mathbf{x}_{N_2} \overset{XX^{[*]}}{\longrightarrow} \mathbf{x}_{N_2-1} \overset{XX^{[*]}}{\longrightarrow} \ldots \overset{XX^{[*]}}{\longrightarrow} \mathbf{x}_{N_1+2} \overset{XX^{[*]}}{\longrightarrow} \mathbf{0} \\ \vdots \\ \mathbf{x}_{N_k} \overset{XX^{[*]}}{\longrightarrow} \mathbf{x}_{N_k-1} \overset{XX^{[*]}}{\longrightarrow} \ldots \overset{XX^{[*]}}{\longrightarrow} \mathbf{x}_{N_{(k-1)}+2} \overset{XX^{[*]}}{\longrightarrow} \mathbf{0} \end{array} \tag{8}$$

This means that we can expect k Jordan blocks of order $n_j - 1$ for $1 \leqslant j \leqslant k$ in the Jordan canonical form for $XX^{[*]}$, unless we can extend one or more of the Jordan chains mentioned in (8). We will show, however, that such an extention is impossible. Let us therefore assume that we can extend one or more chains, this means that there exists a vector $\mathbf{y} \in \mathbb{C}^n$ such that we have

$$XX^{[*]}\mathbf{y} = \alpha_1 \mathbf{x}_{N_1} + \alpha_2 \mathbf{x}_{N_2} + \ldots \alpha_k \mathbf{x}_{N_k} \overset{def}{=} \mathbf{v},$$

for some (complex) numbers α_i, not all equal to zero. Then we have $\mathbf{v} \in \operatorname{Im} XX^{[*]} = (\operatorname{Ker} XX^{[*]})^{[\perp]}$. This means in particular that $\mathbf{v}$ is H-orthogonal to all vectors spanning the subspace of $\operatorname{Ker} XX^{[*]}$ mentioned in (7). When we calculate these indefinite inner products, using (6), we find

$$\begin{aligned} [\mathbf{v}, \mathbf{x}_2] &= \alpha_1 \varepsilon_1 \\ [\mathbf{v}, \mathbf{x}_{N_1+2}] &= \alpha_2 \varepsilon_2 \\ &\vdots \\ [\mathbf{v}, \mathbf{x}_{N_{(k-1)}+2}] &= \alpha_k \varepsilon_k, \end{aligned}$$

so we must conclude that $\alpha_i = 0$ for $1 \leqslant i \leqslant k$. Here we have a contradiction.

To complete our knowledge of the Jordan canonical form for $XX^{[*]}$, we will consider $\operatorname{Ker} XX^{[*]}$ again. Note that

$$\dim \operatorname{Ker} X^{[*]} = \dim \operatorname{Ker} X = l$$

and

$$\operatorname{Ker} X^{[*]} \cap \operatorname{span} \left\{ \mathbf{x}_2, \mathbf{x}_{N_1+2}, \ldots, \mathbf{x}_{N_{(k-1)}+2} \right\} = \{\mathbf{0}\}.$$

So we may, using (7), simply add the k vectors mentioned in the span above to the l vectors of $\operatorname{Ker} X^{[*]}$ when we construct $\operatorname{Ker} XX^{[*]}$ and we see that $\dim \operatorname{Ker} XX^{[*]} \leqslant l + k$. But for our Jordan basis for $XX^{[*]}$, we have already the $N_k - k$ vectors from the Jordan chains in (8), associated with k eigenvectors of $XX^{[*]}$. To complete the Jordan basis, we need another $k + (l - k) = l$ vectors in Jordan chains associated with all remaining eigenvectors respectively. Since there are still l eigenvectors remaining, we may conclude that there are l Jordan chains of length 1 and $\dim \operatorname{Ker} XX^{[*]} = l + k$. So the Jordan canonical form for $XX^{[*]}$ is given by

$$J \stackrel{def}{=} \oplus_{j=1}^{k} J_{n_j-1} \oplus \oplus_{j=1}^{k} J_1 \oplus \oplus_{j=k+1}^{l} J_1,$$

as in (3).

We will now consider the signs that are attached to the Jordan blocks of J. As a Jordan basis for $XX^{[*]}$ we can use the Jordan chains of (8) completed with an H-orthonormal basis for $\operatorname{Ker} X^{[*]}$. The existence of such a basis is guaranteed by the fact that $\operatorname{Im} X$ is H-nondegenerate. To show this, let $\mathbf{x} \in \operatorname{Im} X \cap (\operatorname{Im} X)^{[\perp]} = \operatorname{Im} X \cap \operatorname{Ker} X^{[*]}$. Let $\mathbf{x} = X\mathbf{y}$. Then $\mathbf{y} \in \operatorname{Ker} X^{[*]}X$. From $\operatorname{rank} X^{[*]}X = \operatorname{rank} X$ it follows that $\operatorname{Ker} X^{[*]}X = \operatorname{Ker} X$, so $\mathbf{y} \in \operatorname{Ker} X$. But then $\mathbf{x} = \mathbf{0}$. It follows that $\operatorname{Im} X$ is H-nondegenerate and hence $\operatorname{Ker} X^{[*]} = (\operatorname{Im} X)^{[\perp]}$ is H-nondegenerate and admits an H-orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_l\}$. So for a Jordan basis for $XX^{[*]}$ we can write

$$\mathscr{S} = \left\{ \underbrace{\mathbf{x}_2, \ldots, \mathbf{x}_{N_1}}_{n_1-1 \text{ columns}}, \ldots, \underbrace{\mathbf{x}_{N_{k-1}+2}, \ldots, \mathbf{x}_{N_k}}_{n_k-1 \text{ columns}}, \underbrace{\mathbf{v}_1, \ldots, \mathbf{v}_l}_{l \text{ columns}} \right\}.$$

Putting the vectors of $\mathscr{S}$ as columns in a matrix S, we compute $P \stackrel{def}{=} S^*HS$. Let us therefore first introduce some notation. We define, for $1 \leqslant i,j \leqslant k$

$$P_{i,j} = \begin{pmatrix} \left[\mathbf{x}_{N_{(i-1)}+2}, \mathbf{x}_{N_{(j-1)}+2}\right] & \cdots & \left[\mathbf{x}_{N_{(i-1)}+2}, \mathbf{x}_{N_j}\right] \\ \vdots & \iddots & \vdots \\ \left[\mathbf{x}_{N_i}, \mathbf{x}_{N_{(j-1)}+2}\right] & \cdots & \left[\mathbf{x}_{N_i}, \mathbf{x}_{N_j}\right] \end{pmatrix},$$

an $(n_i - 1) \times (n_j - 1)$ matrix block. Then we have

$$P = \left(\begin{array}{cccc|ccc} P_{1,1} & P_{1,2} & \cdots & P_{1,k} & & & \\ P_{2,1} & P_{2,2} & & \vdots & & & \\ \vdots & & \ddots & & & & \\ P_{k,1} & \cdots & & P_{k,k} & & & \\ \hline & & & & [\mathbf{v}_1, \mathbf{v}_1] & & \\ & & & & & \ddots & \\ & & & & & & [\mathbf{v}_l, \mathbf{v}_l] \end{array}\right),$$

where all empty places are indefinite inner products $[\mathbf{x}_i, \mathbf{v}_j]$, which are equal to zero because $\mathbf{x}_i \in \operatorname{Im} X$ and $\mathbf{v}_j \in \operatorname{Ker} X^{[*]} = (\operatorname{Im} X)^{\perp}$. Further, we have $[\mathbf{v}_i, \mathbf{v}_i] \overset{def}{=} \delta_i = \pm 1$ for $1 \leqslant i \leqslant l$. When we use (6), we find that $P_{i,j} = \mathbf{0}$ for $i \neq j$ and for $i = j$ we have

$$P_{i,i} = \begin{pmatrix} 0 & \cdots & \varepsilon_i \\ \vdots & \iddots & \vdots \\ \varepsilon_i & \cdots & 0 \end{pmatrix},$$

so the basis $\mathscr{S}$ is appropriate, the canonical form for $(XX^{[*]}, H)$ is (J, P), where J is as in (3.) and P as above. We also see that the signs of the blocks in $(X^{[*]}X, H)$ of order 2 and higher are preserved, as in the theorem. In order to determine the signs δ_i, attached to the remaining blocks of order 1 in (J, P), for $1 \leqslant i \leqslant l$, we notice that this can uniquely be done (up to permutations of the order of the blocks) since the signature of H is preserved under the congruence transformation S^*HS, so we have $\operatorname{sig} H = \operatorname{sig} P$. Writing out this equation leads to (4). $\square$

4. Several Corollaries

4.1. The case where $\operatorname{rank} X = 1$

Note that in this case we also have $\operatorname{rank} X^{[*]} = 1$. We shall write $X = \alpha\beta^*$, with $\alpha, \beta \in \mathbb{C}^n$ both nonzero vectors. Note further that in this case there are only three possibilities:

i) $X^{[*]}X = XX^{[*]} = 0$. This is a trivial case.
ii) $\operatorname{rank} X^{[*]}X = 1$ and $XX^{[*]} = 0$. Here we can apply Theorem 6 directly.
iii) $\operatorname{rank} XX^{[*]} = 1$ and $X^{[*]}X = 0$. Here we can apply Theorem 6 for X and $X^{[*]}$ interchanged.

The first possibility occurs when $\mathrm{Im}X$ is H-neutral and $\mathrm{Im}X^*$ is H^{-1}-neutral, the second case when $\mathrm{Im}X$ is not H-neutral and $\mathrm{Im}X^*$ is H^{-1}-neutral, and the third case occurs when $\mathrm{Im}X$ is H-neutral and $\mathrm{Im}X^*$ is not H^{-1}-neutral. Let us assume that the third possibility occurs. As a sample result we state the conclusion of applying Theorem 6. Let the signs in the sign characteristic of $(X^{[*]}X, H)$ be n_+ plus ones, and n_- minus ones. Then the Jordan canonical form of $XX^{[*]}$ consists of one block of size two, and $n-2$ blocks of size one, while the sign corresponding to the block of order two in the sign characteristic of $(XX^{[*]}, H)$ is given by $\mathrm{sign}(\beta^*H^{-1}\beta)$, and there are $n_+ - 1$ plus ones, and $n_- - 1$ minus ones corresponding to the $n-2$ blocks of order one.

4.2. The case where $\operatorname{rank} X^{[*]}X = n-1$

Note that in this case we have

$$\dim\ \operatorname{Ker} X^{[*]}X = 1.$$

Further, we have

$$\operatorname{Ker} X^{[*]}X \supset \operatorname{Ker} X \neq \{\mathbf{0}\}.$$

It now follows that

$$\dim\ \operatorname{Ker} X = 1,$$

so we can apply Theorem 6 directly. The result is as follows: let us denote the (single) sign in the sign characteristic of $(X^{[*]}X, H)$ by ε. Then the Jordan canonical form of $XX^{[*]}$ consists of one block of size $n-1$ and one block of size one. The signs in the sign characteristic of $(XX^{[*]}, H)$ are as follows. The signs are both ε when n is odd, while in case n is even, the sign corresponding to the block of size $n-1$ is ε, and the sign corresponding to the block of size 1 is $-\varepsilon$.

5. The case where $\operatorname{rank} X = n-1$

Recall that we still assume that $X^{[*]}X$ is nilpotent. In this section we discuss the case where $\operatorname{rank} X = n-1$. In that case we have $\dim \operatorname{Ker} X^{[*]}X \leqslant 2$ and $\dim \operatorname{Ker} XX^{[*]} \leqslant 2$. If one of the two (or both) has a one-dimensional kernel, then we are in the case where either $\operatorname{rank} X^{[*]}X = n-1$, or $\operatorname{rank} XX^{[*]} = n-1$, a case that was treated before in Section 4.2. So, we may assume that

$$\dim\ \operatorname{Ker} X^{[*]}X = \dim\ \operatorname{Ker} XX^{[*]} = 2.$$

Let us further assume that

$$X^{[*]}X = J_k \oplus J_{n-k}, \quad H = \varepsilon_1 P_k \oplus \varepsilon_2 P_{n-k}, \quad k \leqslant n-k. \tag{9}$$

We denote X by $X = (\,\mathbf{x}_1 \quad \cdots \quad \mathbf{x}_n\,)$, that is, the vector $\mathbf{x}_j$ denotes the j-th column of X. Since $\operatorname{Ker} X$ is a one-dimensional subspace of $\operatorname{Ker} X^{[*]}X$ we have that there are complex numbers α and β, not both zero, such that $\operatorname{Ker} X = \operatorname{span}\{\alpha\mathbf{e}_1 + \beta\mathbf{e}_{k+1}\}$. (We can normalize either α or β to 1 if we wish.) In other words,

$$\alpha\mathbf{x}_1 + \beta\mathbf{x}_{k+1} = 0.$$

With these assumptions and notations in place, we can state the following theorem.

THEOREM 7. *Let X be of rank $n-1$, with both $X^{[*]}X$ and $XX^{[*]}$ of rank $n-2$. Let the pair $(X^{[*]}X, H)$ be in canonical form (9), and let α and β, not both zero, be so that $\alpha\mathbf{x}_1 + \beta\mathbf{x}_{k+1} = 0$. Then the following hold:*

a. *if $\mathbf{x}_{k+1} = 0$ (and so $\alpha = 0$), then $XX^{[*]} \approx J_{k+1} \oplus J_{n-k-1}$ with corresponding signs ε_1 and ε_2.*

b. *if $\mathbf{x}_{k+1} \neq 0$ then there are the following possibilities*
   - *i) $2k \neq n$. In this case $XX^{[*]} \approx J_{k-1} \oplus J_{n-k+1}$ with corresponding signs ε_1 and ε_2.*
   - *ii) $2k = n, \varepsilon_1 = \varepsilon_2$. In this case $XX^{[*]} \approx J_{k-1} \oplus J_{k+1}$ with corresponding signs both equal to ε_1.*
   - *iii) $2k = n, \varepsilon_1 = -\varepsilon_2$ and $|\alpha| \neq |\beta|$. Then $XX^{[*]} \approx J_{k-1} \oplus J_{k+1}$ with corresponding signs $sign(|\alpha|^2 - |\beta|^2)\varepsilon_1$ and $sign(|\alpha|^2 - |\beta|^2)\varepsilon_2$.*
   - *iv) $2k = n, \varepsilon_1 = -\varepsilon_2$ and $|\alpha| = |\beta|$. Then $XX^{[*]} \approx J_k \oplus J_k$ with corresponding signs $+1$ and -1.*

Proof. We start by making several general observations that are independent of the special case at hand. Since

$$X = \begin{pmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_k & \mathbf{x}_{k+1} & \cdots & \mathbf{x}_n \end{pmatrix}$$

we have

$$X^{[*]} = H^{-1}X^*H = \begin{pmatrix} \varepsilon_1\mathbf{x}_k^* \\ \vdots \\ \varepsilon_1\mathbf{x}_1^* \\ \varepsilon_2\mathbf{x}_n^* \\ \vdots \\ \varepsilon_2\mathbf{x}_{k+1}^* \end{pmatrix} H,$$

and so

$$XX^{[*]} = \begin{pmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_k & \mathbf{x}_{k+1} & \cdots & \mathbf{x}_n \end{pmatrix} \begin{pmatrix} \varepsilon_1\mathbf{x}_k^* \\ \vdots \\ \varepsilon_1\mathbf{x}_1^* \\ \varepsilon_2\mathbf{x}_n^* \\ \vdots \\ \varepsilon_2\mathbf{x}_{k+1}^* \end{pmatrix} H.$$

The fact that $X^{[*]}X = J_k \oplus J_{n-k}$ gives identities for $\mathbf{x}_i^*H\mathbf{x}_j$. These identities will be used frequently in the proof.

In particular, we have $X^{[*]}\mathbf{x}_1 = 0, X^{[*]}\mathbf{x}_{k+1} = 0$, and considering the second and the $k+2$-th column in $X^{[*]}X$ we see that $X^{[*]}\mathbf{x}_2 = \mathbf{e}_1, X^{[*]}\mathbf{x}_{k+2} = \mathbf{e}_{k+1}$, and hence it follows that

$$XX^{[*]}\mathbf{x}_2 = \mathbf{x}_1, \qquad XX^{[*]}\mathbf{x}_{k+2} = \mathbf{x}_{k+1}.$$

In a similar way it follows that, more generally

$$XX^{[*]}\mathbf{x}_j = \mathbf{x}_{j-1}, j = 2, \cdots, k,\ j = k+2, \cdots, n. \tag{10}$$

Also, we shall use that $\mathbf{x}_1^* H \mathbf{x}_k = 0$ and $\mathbf{x}_{k+2}^* H \mathbf{x}_k = 0$.

Hence we have that

$$XX^{[*]}(\alpha \mathbf{x}_2 + \beta \mathbf{x}_{k+2}) = \alpha \mathbf{x}_1 + \beta \mathbf{x}_{k+1} = 0.$$

We can now describe $\operatorname{Ker} XX^{[*]}$ completely:

$$\operatorname{Ker} XX^{[*]} = \operatorname{span}\{\alpha \mathbf{x}_2 + \beta \mathbf{x}_{k+2}, -\bar{\beta}\mathbf{x}_1 + \bar{\alpha}\mathbf{x}_{k+1}\}. \tag{11}$$

Note that the latter vector is indeed non-zero, for if we assume that $-\bar{\beta}\mathbf{x}_1 + \bar{\alpha}\mathbf{x}_{k+1} = 0$ then it follows that there is a number γ such that $\alpha\gamma = -\bar{\beta}, \beta\gamma = \bar{\alpha}$. Then $\alpha|\gamma|^2 = -\bar{\beta}\bar{\gamma} = -\alpha$ and $\beta|\gamma|^2 = -\beta$. Now since at least one of α and β is non-zero this is not possible.

A final general remark we make is that $\operatorname{Im} XX^{[*]} = (\operatorname{Ker} XX^{[*]})^{[\perp]}$, as is well-known (and easy to check).

Proof of case a. Consider the case where $\mathbf{x}_{k+1} = 0$. Then $\alpha = 0$, and we may and will take $\beta = 1$. Then $\operatorname{Ker} XX^{[*]} = \operatorname{span}\{\mathbf{x}_{k+2}, \mathbf{x}_1\}$. From (10) we see that there are two Jordan chains, namely

$$\begin{aligned} &\mathbf{x}_k \quad , \dots , \mathbf{x}_1 \\ &\mathbf{x}_n \quad , \dots , \mathbf{x}_{k+2}, \end{aligned}$$

which we order in such a way that $XX^{[*]}$ maps a vector in this chain to the next vector, ending up finally with a vector in the kernel of $XX^{[*]}$. Note that these Jordan chains are of length k and $n-k-1$, respectively. Further, $\mathbf{x}_k$ is H-orthogonal to both $\mathbf{x}_1$ and $\mathbf{x}_{k+2}$. This is seen from the fact that $\mathbf{x}_k^* H \mathbf{x}_1$ is up to a factor ε_1 the $1,1$-entry in $X^{[*]}X$, which is zero, and that $\mathbf{x}_k^* H \mathbf{x}_{k+2}$ is up to a factor ε_1 the $1, k+2$-entry in $X^{[*]}X$, which is zero as well. (Here it is assumed that $k > 1$, but the case $k = 1$ can be done analogously.) Hence it follows that $\mathbf{x}_k$ is in $\operatorname{Im} XX^{[*]}$. So there is a vector $\mathbf{y}$ such that $XX^{[*]}\mathbf{y} = \mathbf{x}_k$. It follows that $X^{[*]}\mathbf{y} = \mathbf{e}_k + \delta \mathbf{e}_{k+1}$ for some number δ. Now we have a Jordan chain of length $k+1$:

$$\mathbf{y}, \mathbf{x}_k, \dots, \mathbf{x}_1,$$

and the corresponding sign in the sign characteristic of $(XX^{[*]}, H)$ is the sign of the number $\langle H\mathbf{y}, \mathbf{x}_1 \rangle$. Thus we compute

$$\langle H\mathbf{y}, \mathbf{x}_1 \rangle = \mathbf{x}_1^* H \mathbf{y} = \frac{1}{\varepsilon_1}(X^{[*]}\mathbf{y})_k = \varepsilon_1.$$

Next, we compute the sign corresponding to the Jordan chain of length $n-k-1$ given by $\mathbf{x}_n, \dots, \mathbf{x}_{k+2}$. That sign is determined by the sign of the number

$$\langle H\mathbf{x}_n, \mathbf{x}_{k+2} \rangle = \mathbf{x}_{k+2} H \mathbf{x}_n = \frac{1}{\varepsilon_2}(X^{[*]}X)_{n-1,n} = \varepsilon_2.$$

Proof of case b. In all cases presented in case b we have $\mathbf{x}_{k+1} \neq 0$, and hence $\alpha \neq 0$. Then $\operatorname{Ker} XX^{[*]} = \operatorname{span}\{\alpha \mathbf{x}_2 + \beta \mathbf{x}_{k+2}, \mathbf{x}_{k+1}\}$. As a first observation note that the following are certainly Jordan chains:

$$\begin{aligned} &\mathbf{x}_n, \dots, \mathbf{x}_{k+1}, \\ &\alpha \mathbf{x}_k + \beta \mathbf{x}_{2k}, \dots, \alpha \mathbf{x}_2 + \beta \mathbf{x}_{k+2}. \end{aligned} \tag{12}$$

The first chain has length $n-k$, the second has length $k-1$. The question is whether and which one of these chains can be extended with an extra vector. In order to decide this we make use of a general fact:

Fact. If A is H selfadjoint, and we have a Jordan chain corresponding to the eigenvalue 0: $A\mathbf{y}_j = \mathbf{y}_{j-1}, j = 1,\dots,k$, $A\mathbf{y}_0 = 0$, then $\langle H\mathbf{y}_0, \mathbf{y}_j\rangle = 0$ is $j < k$.

Note that $\langle H\mathbf{x}_{k+1}, \mathbf{x}_n\rangle = \mathbf{x}_n^* H\mathbf{x}_{k+1} = 0$ because the $k+1, k+1$-entry in $X^{[*]}X$ is zero. For the other Jordan chain we have (again using knowledge of the entries of $X^{[*]}X$):

$$\begin{aligned}
&\langle H(\alpha\mathbf{x}_2 + \beta\mathbf{x}_{k+2}), \alpha\mathbf{x}_k + \beta\mathbf{x}_{2k}\rangle \\
&\quad = |\alpha|^2\mathbf{x}_k^* H\mathbf{x}_2 + \bar{\alpha}\beta\mathbf{x}_k^* H\mathbf{x}_{k+2} + \alpha\bar{\beta}\mathbf{x}_{2k}^* H\mathbf{x}_2 + |\beta|^2\mathbf{x}_{2k}^* H\mathbf{x}_{k+2} \\
&\quad = |\alpha|^2\varepsilon_1 + \bar{\alpha}\beta\cdot 0 + \alpha\bar{\beta}\cdot 0 + |\beta|^2\delta\varepsilon_2,
\end{aligned} \tag{13}$$

where

$$\delta = \begin{cases} 0 & \text{if} \quad 2k \neq n, \\ 1 & \text{if} \quad 2k = n. \end{cases}$$

We see that the number in (13) is nonzero in the following cases: $2k \neq n$ or $\varepsilon_1 = \varepsilon_2$ or $2k = n$, $\varepsilon_1 = -\varepsilon_2$, $|\alpha|^2 \neq |\beta|^2$. So it follows that in each of these cases the lengths of the Jordan chains of $XX^{[*]}$ are $k-1$ and $n-k+1$. The sign corresponding to the block of size $k-1$ is determined by the sign of the number (13). Now we need to find a Jordan chain that can be extended with a vector. It is now that we split the argument into several cases.

First assume that $2k < n$. We are looking for a vector of the form $\mathbf{y}_{n-k} = \gamma\mathbf{x}_n + \nu(\alpha\mathbf{x}_k + \beta\mathbf{x}_{2k})$ in $\operatorname{Im} XX^{[*]} = (\operatorname{Ker} XX^{[*]})^{[\perp]}$. Indeed, if we find such a vector, then there is a vector $\mathbf{y}_{n-k+1}$ such that the vectors $(XX^{[*]})^j\mathbf{y}_{n-k+1}$, with $j = 0, 1, \dots, n-k+1$ form a Jordan chain of length $n-k+1$. Now note that since the $k+1$-th column of $X^{[*]}X$ is zero one has

$$\langle H(\gamma\mathbf{x}_n + \nu(\alpha\mathbf{x}_k + \beta\mathbf{x}_{2k}), \mathbf{x}_{k+1}\rangle = 0.$$

So, we are just looking for γ and ν to satisfy

$$\begin{aligned}
0 &= \langle H(\gamma\mathbf{x}_n + \nu(\alpha\mathbf{x}_k + \beta\mathbf{x}_{2k}), \alpha\mathbf{x}_2 + \beta\mathbf{x}_{k+2}\rangle \\
&= \bar{\alpha}(\gamma\mathbf{x}_2^* H\mathbf{x}_n + \nu\alpha\mathbf{x}_2^* H\mathbf{x}_k + \nu\beta\mathbf{x}_2^* H\mathbf{x}_{2k}) \\
&+ \bar{\beta}(\gamma\mathbf{x}_{k+2}^* H\mathbf{x}_n + \nu\alpha\mathbf{x}_{k+2}^* H\mathbf{x}_k + \nu\beta\mathbf{x}_{k+2}^* H\mathbf{x}_{2k}) \\
&= |\alpha|^2\nu\varepsilon_1 + \bar{\beta}\gamma\varepsilon_2.
\end{aligned}$$

So, we can take $\gamma = 1$ and solve for ν: $\nu = -\frac{\bar{\beta}\gamma\varepsilon_2}{|\alpha|^2\varepsilon_1}$. So, for this case we have the following Jordan chain:

$$\mathbf{y}_{n-k+1}, \mathbf{x}_n + \nu(\alpha\mathbf{x}_k + \beta\mathbf{x}_{2k}), \dots, \mathbf{x}_{k+1}.$$

We compute the corresponding sign by computing $\langle H\mathbf{y}_{n-k+1}, \mathbf{x}_{k+1}\rangle$. To do so, we observe that

$$X^{[*]}\mathbf{y}_{n-k+1} = \nu\alpha\mathbf{e}_k + \nu\alpha\mathbf{e}_{2k} + \mathbf{e}_n + \mathbf{z},$$

where $\mathbf{z} \in \operatorname{Ker} X$. So

$$\langle H\mathbf{y}_{n-k+1}, \mathbf{x}_{k+1}\rangle = \mathbf{x}_{k+1}^* H\mathbf{y}_{n-k+1} = \frac{1}{\varepsilon_2}\mathbf{e}_n^* X^{[*]}\mathbf{y}_{n-k+1} = \frac{1}{\varepsilon_2} = \varepsilon_2.$$

This finishes the case b.i.

We assume from now on that $2k = n$. As was argued above, in the cases under consideration in b.ii and b.iii we know that there are two Jordan chains of lengths $k-1$ and $k+1$, respectively, and the sign corresponding to the chain of length $k-1$ is the sign of $|\alpha|^2\varepsilon_1 + |\beta|^2\varepsilon_2$. Like in the previous case it remains to determine a Jordan chain of length $k+1$, and find the corresponding sign. We argue as in the case b.i, but now look for a vector $\mathbf{y}_{k+1}$ such that $XX^{[*]}\mathbf{y}_{k+1} = \gamma\mathbf{x}_{2k} + \nu(\alpha\mathbf{x}_k + \beta\mathbf{x}_{2k})$ for some γ and ν. We determine γ and ν from the condition that this vector must be in the image of $XX^{[*]}$, which is the H-orthogonal of the kernel of $XX^{[*]}$, and the relevant equation now becomes:

$$\begin{aligned} 0 &= \langle H\gamma\mathbf{x}_{2k} + \nu(\alpha\mathbf{x}_k + \beta\mathbf{x}_{2k}), \alpha\mathbf{x}_2 + \beta\mathbf{x}_{k+2}\rangle \\ &= \overline{\beta}\gamma\varepsilon_2 + \nu(|\alpha|^2\varepsilon_1 + |\beta|^2\varepsilon_2). \end{aligned}$$

Since $|\alpha|^2\varepsilon_1 + |\beta|^2\varepsilon_2 \neq 0$ we can solve for ν and take $\gamma = 1$:

$$\nu = -\frac{\overline{\beta}\varepsilon_2}{|\alpha|^2\varepsilon_1 + |\beta|^2\varepsilon_2}.$$

Then we get the vector $\mathbf{y}_{k+1}$ as desired, and we form a Jordan chain by taking the vectors $(XX^{[*]})^j\mathbf{y}_{k+1}$ for $j = 0, \dots, k$. Then it is easily seen that $\mathbf{y}_2 = \mathbf{x}_{k+2} + \nu(\alpha\mathbf{x}_2 + \beta\mathbf{x}_{k+2})$, and hence $\mathbf{y}_1 = \mathbf{x}_{k+1}$. So to get the sign corresponding to this Jordan chain we compute $\langle H\mathbf{y}_{k+1}, \mathbf{x}_{k+1}\rangle$. In order to do this we use the fact that $X^{[*]}\mathbf{y}_{k+1} = \nu\alpha\mathbf{e}_k + (1+\nu\beta)\mathbf{e}_{2k} + \mathbf{z}$ for some $\mathbf{z}$ in the kernel of X. So:

$$\begin{aligned} \langle H\mathbf{y}_{k+1}, \mathbf{x}_{k+1}\rangle &= \mathbf{x}_{k+1}^* H\mathbf{y}_{k+1} = \frac{1}{\varepsilon_2}\mathbf{e}_{2k}^* X^{[*]}\mathbf{y}_{k+1} = \frac{1}{\varepsilon_2}(1+\nu\beta) \\ &= \varepsilon_2 - \frac{|\beta|^2}{|\alpha|^2\varepsilon_1 + |\beta|^2\varepsilon_2} = \frac{|\alpha|^2\varepsilon_1\varepsilon_2}{|\alpha|^2\varepsilon_1 + |\beta|^2\varepsilon_2}. \end{aligned}$$

This proves the cases b.ii and b.iii.

Finally, we consider the case b.iv. In this case we have that $\alpha\mathbf{x}_k + \beta\mathbf{x}_{2k}$ is H-orthogonal to $\operatorname{Ker} XX^{[*]}$. Indeed $\langle H\mathbf{x}_{k+1}, \alpha\mathbf{x}_k + \beta\mathbf{x}_{2k}\rangle = 0$ is straightforward, while $\langle H(\alpha\mathbf{x}_2 + \beta\mathbf{x}_{k+2}), \alpha\mathbf{x}_k + \beta\mathbf{x}_{2k}\rangle = 0$ is just what we have computed in (13). Hence $\alpha\mathbf{x}_k + \beta\mathbf{x}_{2k}$ is in $\operatorname{Im} XX^{[*]}$. So, there is a vector $\mathbf{y}$ so that $XX^{[*]}\mathbf{y} = \alpha\mathbf{x}_k + \beta\mathbf{x}_{2k}$. Then we have two Jordan chains of length k:

$$\begin{aligned} &\mathbf{x}_{2k}, \dots, \mathbf{x}_{k+1}, \\ &\mathbf{y}, \alpha\mathbf{x}_k + \beta\mathbf{x}_{2k}, \dots, \alpha\mathbf{x}_2 + \beta\mathbf{x}_{k+2}. \end{aligned}$$

Now $\langle H\mathbf{x}_{k+1}, \mathbf{x}_n\rangle = 0$. We now use the following general fact:

Fact. Suppose that $A = J_k \oplus J_k$, and $H = P_k \oplus P_k$. Then for any Jordan chain $\mathbf{y}_k, \dots, \mathbf{y}_1$ of length k we have $\langle H\mathbf{y}_k, \mathbf{y}_1\rangle \neq 0$.

Proof. Any Jordan chain of length k is of the form

$$\mathbf{y}_1 = a_1\mathbf{e}_1 + b_1\mathbf{e}_{k+1}, \mathbf{y}_2 = a_1\mathbf{e}_2 + b_1\mathbf{e}_{k+2} + a_2\mathbf{e}_1 + b_2\mathbf{e}_{k+1}, \ldots, \mathbf{y}_k = a_1\mathbf{e}_k + b_1\mathbf{e}_{2k} + \mathbf{x},$$

where $\mathbf{x} \in \operatorname{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_{k-1}, \mathbf{e}_{k+1}, \ldots, \mathbf{e}_{2k-1}\}$, and a_1 and b_1 not both zero. Then

$$\begin{aligned}\langle H\mathbf{y}_k, \mathbf{y}_1\rangle &= \langle H(a_1\mathbf{e}_1 + b_1\mathbf{e}_{k+1}), a_1\mathbf{e}_k + b_1\mathbf{e}_{2k} + \mathbf{x}\rangle \\ &= \langle H(a_1\mathbf{e}_1 + b_1\mathbf{e}_{k+1}), a_1\mathbf{e}_k + b_1\mathbf{e}_{2k}\rangle = |a_1|^2 + |b_1|^2 > 0.\end{aligned}$$

Now applying this fact we conclude that in case b.iv the two blocks of size k must have opposite signs in the sign characteristic. □

6. The case $X^{[*]}X = 0$.

In case $X^{[*]}X = 0$ we have that $\operatorname{Im}X$ is H-neutral. Indeed, $X^{[*]}X = 0$ is equivalent to $\operatorname{Im}X \subset \operatorname{Ker}X^{[*]} = (\operatorname{Im}X)^{[\perp]}$. Let $\mathcal{N}_0$ be a subspace that is skewly linked to $\operatorname{Im}X$, that is, $\mathcal{N}_0$ is an H-neutral subspace such that for any vector $\mathbf{x}$ in $\operatorname{Im}X$ there is a vector $\mathbf{y}$ in $\mathcal{N}_0$ with $[\mathbf{x}, \mathbf{y}] \neq 0$. Observe that $\operatorname{Im}X \dot{+} \mathcal{N}_0$ is a nondegenerate subspace. Let $\mathcal{N}_1$ be the subspace $(\operatorname{Im}X \dot{+} \mathcal{N}_0)^{[\perp]}$. Then $\mathcal{N}_1$ is H-nondegenerate as well. With respect to the decomposition $\mathbb{C}^n = \operatorname{Im}X \dot{+} \mathcal{N}_0 \dot{+} \mathcal{N}_1$ we have, after an appropriate choice of basis,

$$X = \begin{pmatrix} X_1 & X_2 & X_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & H_3 \end{pmatrix}. \tag{14}$$

Note that X_1 is nilpotent, and if we denote $j = \dim \operatorname{Im}X$ then

$$\operatorname{rank}\begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix} = j.$$

In these terms we compute $XX^{[*]}$. First,

$$X^{[*]} = \begin{pmatrix} 0 & X_2^* & 0 \\ 0 & X_1^* & 0 \\ 0 & H_3^{-1}X_3^* & 0 \end{pmatrix}.$$

Then

$$XX^{[*]} = \begin{pmatrix} 0 & Z & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$Z = X_1X_2^* + X_2X_1^* + X_3H_3^{-1}X_3^*. \tag{15}$$

Note that Z is a hermitian matrix. Clearly, the Jordan canonical form of $XX^{[*]}$ depends on the rank of Z. For instance, if Z happens to be full rank then there are j Jordan chains of length two, while if $Z = 0$ then $XX^{[*]} = 0$. Let us denote by k the rank of Z.

Observe that $Z\mathbf{x} = 0$ if and only if

$$\langle HX^{[*]}\tilde{\mathbf{x}}, X^{[*]}\tilde{\mathbf{x}}\rangle = 0,$$

where $\tilde{\mathbf{x}} = \begin{pmatrix} 0 & \mathbf{x}^T & 0 \end{pmatrix}^T$. Moreover, since $\operatorname{rank} \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix} = j$, the second block column of $X^{[*]}$ is a one-to-one map between the kernel of Z and the isotropic subspace of $\mathrm{Im}X^{[*]}$, i.e., $\mathrm{Im}X^{[*]} \cap (\mathrm{Im}X^{[*]})^{[\perp]}$. Therefore, the dimension of the kernel of Z is equal to the dimension of

$$\mathrm{Im}X^{[*]} \cap (\mathrm{Im}X^{[*]})^{[\perp]} = (\operatorname{Ker} X)^{[\perp]} \cap \operatorname{Ker} X.$$

So, the dimension of $\operatorname{Ker} XX^{[*]}$ is $j + (j-k) + (n-2j) = n-k$, where $j-k$ is the dimension of $(\operatorname{Ker} X)^{[\perp]} \cap \operatorname{Ker} X$. Jordan chains of length one are obtained from vectors of the form

$$\begin{pmatrix} 0 \\ 0 \\ \mathbf{x} \end{pmatrix},$$

with corresponding signs in the sign characteristic equal to the signs in the sign characteristic of the pair $(X^{[*]}X, H)$ corresponding to these vectors (and determined by H_3), as well as from vectors of the form

$$\begin{pmatrix} 0 \\ \mathbf{x} \\ 0 \end{pmatrix}, \qquad \mathbf{x} \in \operatorname{Ker} Z,$$

and of the form

$$\begin{pmatrix} \mathbf{x} \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{x} \in (\mathrm{Im}Z)^{\perp}$$

and the signs corresponding to the span of the latter two groups of eigenvectors are an equal number of $+1$'s and -1's, that is, $j-k$ plus 1's and $j-k$ minus 1's. Indeed, the latter observation is based on considering the signature of H. The contribution to this from blocks of order two is an equal number of $+1$'s and -1's, and so the contribution of the span of the last two types of eigenvectors that correspond to blocks of order one must also be an equal number of $+1$'s and -1's.

Next, observe that if $Z\mathbf{x} \neq 0$, then

$$\begin{pmatrix} 0 \\ \mathbf{x} \\ 0 \end{pmatrix}, \begin{pmatrix} Z\mathbf{x} \\ 0 \\ 0 \end{pmatrix}$$

is a Jordan chain of length two for $XX^{[*]}$. Furthermore,

$$\left\langle H \begin{pmatrix} Z\mathbf{x} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathbf{x} \\ 0 \end{pmatrix} \right\rangle = \langle Z\mathbf{x}, \mathbf{x} \rangle .$$

In conclusion, the signs in the sign characteristic of the pair $(XX^{[*]}, H)$ corresponding to blocks of size two are completely determined by Z. Hence, we have the following theorem.

THEOREM 8. *Assume that* $X^{[*]}X = 0$, *let* X *and* H *be in the form* (14), *and let* Z *be given by* (15). *Let* k *be the rank of* Z, *and let* j *denote the dimension of* ImX. *Then* k *is determined by*

$$k = \dim \operatorname{Im} X - \dim \left(\operatorname{Ker} X \cap (\operatorname{Ker} X)^{[\perp]} \right).$$

Let κ_+ *be the number of positive eigenvalues of* H_3, *let* κ_- *be the number of negative eigenvalues of* H_3, *let* ν_+ *be the number of positive eigenvalues of* Z, *and finally, let* ν_- *be the number of negative eigenvalues of* Z.

Then the Jordan canonical form of $XX^{[*]}$ *has* $n-2k$ *Jordan blocks of size one, and* k *Jordan blocks of size two. The signs in the sign characteristic of* $(XX^{[*]}, H)$ *corresponding to the Jordan blocks of size one are as follows: the number of* $+1$ *'s is* $\kappa_+ + (j-k)$ *and the number of* -1 *'s is* $\kappa_- + (j-k)$. *The signs in the sign characteristic of* $(XX^{[*]}, H)$ *corresponding to the Jordan blocks of size two are as follows: the number of* $+1$ *'s is* ν_+, *while the number of* -1 *'s is* ν_-.

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