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ON THE OPERATOR EQUATION $AXB + CYD = Z$

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Abstract. Suppose two n-tuples of operators A_k and B_k on a Hilbert space are given, and Φ is the mapping from the set of n-tuples of operators on the Hilbert space into the set of all operators on the space defined by $\Phi(X_1, X_2, \ldots, X_n) = \sum_{k=1}^{n} A_k X_k B_k$. Conditions are given for Φ to be onto.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and n a positive integer, and let $\{A_1, A_2, \cdots, A_n\}$ and $\{B_1, B_2, \cdots, B_n\}$ be two fixed n-tuples of operators on $\mathcal{H}$. We consider the mapping Φ on n-tuples of operators $X = \{X_1, X_2, \cdots, X_n\}$ defined by

$$\Phi(X) = \sum_{k=1}^{n} A_k X_k B_k. \tag{1}$$

The mapping Φ may be considered on various classes of operators. Let $\mathbf{B}(\mathcal{H})$ be the set of all bounded operators on $\mathcal{H}$, let $\mathbf{K}(\mathcal{H})$ be the compact operators, let $\mathbf{T}(\mathcal{H})$ be the trace class, and let $\mathbf{C}_p(\mathcal{H})$ be the Schatten p-class for $p \geqslant 1$. When there is only one Hilbert space under consideration we will write simply $\mathbf{B}$ for $\mathbf{B}(\mathcal{H})$, etc. Thus, for example, $\mathbf{T}(\mathcal{H}) = \mathbf{T} = \mathbf{C}_1$. We write $\mathbf{B}^{(n)}$ for the n-fold direct sum of $\mathbf{B}$ with itself.

We consider the question of when Φ is onto. It is immediate that for this to happen the matrix $\begin{pmatrix} A_1 & \cdots & A_n \end{pmatrix}$ must map the n-fold direct sum of $\mathcal{H}$ with itself onto $\mathcal{H}$. By taking adjoints, one sees that a second necessary condition is that $\begin{pmatrix} B_1^* & \cdots & B_n^* \end{pmatrix}$ must be onto. That more is necessary even in the case $n = 2$ may be seen by taking a nontrivial projection P and putting $A_1 = B_1 = P$ and $A_2 = B_2 = 1 - P$. A complete answer to the question is obtained in the case when $n = 2$, and partial results are obtained in the general case.

Restricting all the X's to be the same operator yields the case of elementary operators, which have been much studied (see [2]). The particular case when $\Phi(X) = AX - XB$ is the familiar Rosenblum operator (see [1, 7]).

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2. The simplest case

The very special case that motivated this study is the following.

THEOREM 2.1. *Let $\mathcal{X}$ be a Banach space, and suppose A and B are bounded operators on $\mathcal{X}$. If the mapping $\Phi(X,Y) = AX + YB$ from $\mathbf{B}(\mathcal{X}) \times \mathbf{B}(\mathcal{X})$ into $\mathbf{B}(\mathcal{X})$ is onto and B is not injective, then A maps $\mathcal{X}$ onto itself.*

Proof. Suppose $Bf = 0$ for some nonzero f in $\mathcal{X}$. Then, for every g in $\mathcal{X}$, there are operators X and Y such that $(AX + YB)f = g$. Since $Bf = 0$, g is in the range of A. $\square$

COROLLARY 2.2. *If $\mathcal{X}$ is finite dimensional, then the map $\Phi(X,Y) = AX + YB$ is onto if and only if at least one of the maps $X \mapsto AX$ or $Y \mapsto YB$ is onto.*

Proof. Sufficiency is immediate. If Φ is onto, then the theorem shows that at least one of A and B is invertible, and hence one of $X \mapsto AX$ or $Y \mapsto YB$ is onto. $\square$

The corollary holds as well for operators on Hilbert space.

THEOREM 2.3. *Suppose $\mathcal{H}$ is a Hilbert space. Then the map $\Phi(X,Y) = AX + YB$ from $\mathbf{B}(\mathcal{H}) \times \mathbf{B}(\mathcal{H})$ into $\mathbf{B}(\mathcal{H})$ is surjective if and only if at least one of the maps $X \mapsto AX$ or $Y \mapsto YB$ is surjective.*

Proof. Again we need only show sufficiency, so suppose Φ is onto. If B has nontrivial nullspace, then Theorem 2.1 implies the conclusion. By taking adjoints, we see that the same is true if A^* has a nontrivial nullspace. To prove the theorem it must be shown that B is bounded below or A is onto. Suppose neither is the case. Then there exists a sequence of unit vectors $\{f_n\}$ such that $\{Bf_n\}$ converges to zero. We may assume that $\{f_n\}$ is orthonormal (see [5] or [6], Lemma 2.5). Similarly there exists an orthonormal sequence $\{g_n\}$ such that $\{A^*g_n\}$ converges to zero. If C is any operator of the form $AX + YB$, then the sequence of inner products $\langle Cf_n, g_n \rangle$ converges to zero. But there exists a partial isometry V such that $Vf_n = g_n$ for each n, and thus $\langle Vf_n, g_n \rangle = 1$ for each n. Hence there are no operators X and Y such that V has the form $AX + YB$. $\square$

Question. Is the statement of the above result false if $\mathcal{H}$ is only assumed to be a Banach space?

3. The case of two summands

The above ideas may be modified to give a complete answer to the original question in the case where $n = 2$. For this we need to recall some ideas concerning duality.

LEMMA 3.1. *When Φ acts as a map from $\mathbf{K}^{(n)}$ into $\mathbf{K}$, then the adjoint $\Phi^\sharp$ maps $\mathbf{T}$ into $\mathbf{T}^{(n)}$ and is given by*

$$\Phi^\sharp(T) = (B_1TA_1, B_2TA_2, \cdots, B_nTA_n).$$

Furthermore, $\Phi^{\sharp\sharp}$ *maps* $\mathbf{B}^{(n)}$ *into* $\mathbf{B}$, *and is given by the same formula as was* Φ *in equation (1).*

Proof. The assertion concerning $\Phi^{\sharp}$ follows from the standard trace relation:

$$\operatorname{tr}\Phi(X)T = \operatorname{tr}\sum_{k=1}^{n} X_k B_k T A_k.$$

For X in $\mathbf{K}^{(n)}, \Phi^{\sharp\sharp}(X) = \Phi(X)$, and $\mathbf{K}^{(n)}$ is weak* dense in $\mathbf{B}^{(n)}$. It follows from weak* continuity of the map $Y \mapsto AYB$ on $\mathbf{B}$ for fixed A and B that the action of Φ on $\mathbf{B}^{(n)}$ is determined by equation (1). $\square$

LEMMA 3.2. *When* Φ *acts as a map from* $\mathbf{C}_p^{(n)}$ *into* $\mathbf{C}_p$ *for* $1 \leqslant p < \infty$, *then* $\Phi^{\sharp}$ *maps* $\mathbf{C}_q$ *into* $\mathbf{C}_q^{(n)}$, *where* q *is the conjugate index and* $\mathbf{C}_\infty = \mathbf{B}$, *and* $\Phi^{\sharp}$ *is given by*

$$\Phi^{\sharp}(T) = (B_1TA_1, B_2TA_2, \cdots, B_nTA_n).$$

The next lemma is elementary.

LEMMA 3.3. *Let* Ψ *be an operator from one Banach space to another.*
1. Ψ *is onto precisely when its adjoint is bounded below.*
2. Ψ *is onto if and only if* $\Psi^{\sharp\sharp}$ *is onto.*

An immediate consequence of Lemma 3.3 is the following.

PROPOSITION 3.4. *The operator* Φ *maps* $\mathbf{K}^{(n)}$ *onto* $\mathbf{K}$ *if and only if it maps* $\mathbf{B}^{(n)}$ *onto* $\mathbf{B}$.

THEOREM 3.5. *Suppose* Φ *acts on* $\mathbf{K}^{(2)}$ *and both* $(A_1 \quad A_2)$ *and* $(B_1^* \quad B_2^*)$ *map* $\mathcal{H}\bigoplus\mathcal{H}$ *onto* $\mathcal{H}$. *Then* Φ *is onto precisely when both operators in at least one of the pairs* $\{A_1, B_1^*\}, \{A_2, B_2^*\}, \{A_1, A_2\}$ *or* $\{B_1^*, B_2^*\}$ *are onto.*

Proof. Suppose A_1 and B_1^* are onto. Then A_1 has a right inverse, say C_1, and B_1 has a left inverse, say D_1. Then it is easy to see that the map $X \mapsto A_1XB_1$ is onto, for the image of C_1XD_1 is then X. Consequently Φ is onto. The case where A_2 and B_2^* are onto is essentially the same. Suppose A_1 and A_2 are onto. Then each of them has a right inverse, say C_1 and C_2. Furthermore, since $(B_1^* \quad B_2^*)$ is onto, there exist D_1 and D_2 such that $D_1B_1 + D_2B_2 = 1$. Hence if K is a given compact operator, then we can put $X_1 = C_1KD_1$ and $X_2 = C_2KD_2$ to produce a compact pair $X = \{X_1, X_2\}$ such that $\Phi(X) = K$. Thus Φ is onto in this case also, and the final case may be handled similarly.

For the converse, suppose Φ is onto. Then $\Phi^{\sharp}$ is bounded below; i.e., there exists $\varepsilon > 0$ such that $\|\Phi^{\sharp}(T)\|_1 \geqslant \varepsilon\|T\|_1$ for every T in $\mathbf{C}_1$. Apply $\Phi^{\sharp}$ to $T = g \otimes f$ where f and g are unit vectors, and observe that the last inequality becomes

$$\max\{\|A_1^*f\| \ \|B_1g\|, \|A_2^*f\| \ \|B_2g\|\} \geqslant \varepsilon. \tag{2}$$

We consider the two cases where A_1^* is and is not bounded below. If A_1^* is bounded below and B_1 is also, then both A_1 and B_1^* are onto. Suppose then that A_1^* is bounded below, but B_1 isn't. It then follows from inequality (2) that A_2^* is also bounded below, and so each of A_1 and A_2 is onto. Thus we need only consider the second case where A_1^* is not bounded below. In this case (2) implies that B_2 is bounded below. Either A_2^* is also bounded below, in which case we are done, or A_2^* is not. In the latter case (2) implies that B_1 is bounded below, so the result follows in this case also. □

COROLLARY 3.6. *Suppose Φ acts on $\mathbf{B}^{(2)}$ and both $(A_1 \quad A_2)$ and $(B_1^* \quad B_2^*)$ are onto. Then Φ is onto precisely when both operators in at least one of the pairs $\{A_1, B_1^*\}, \{A_2, B_2^*\}, \{A_1, A_2\}$ or $\{B_1^*, B_2^*\}$ are onto.*

Proof. This follows from Lemma 3.1, Lemma 3.3, and Theorem 3.5. □

Theorem 2.3 is an immediate corollary of the above Corollary.

COROLLARY 3.7. *Suppose Φ acts on $\mathbf{C}_p^{(2)}$ for $1 < p < \infty$ and both $(A_1 \quad A_2)$ and $(B_1^* \quad B_2^*)$ are onto. Then Φ is onto precisely when both operators in at least one of the pairs $\{A_1, B_1^*\}, \{A_2, B_2^*\}, \{A_1, A_2\}$ or $\{B_1^*, B_2^*\}$ are onto.*

Proof. The proof of sufficiency is the same as that of the theorem. The proof of necessity differs only in that when Φ acts on $\mathbf{C}_p$ and is onto, then $\|\Phi^\sharp(T)\|_q \geqslant \varepsilon \|T\|_q$ for some $\varepsilon > 0$. The rest of the proof is exactly the same. □

In [3] it is shown that the operator defined by $\phi(X) = AX - XB$ is onto precisely when the right spectrum of A is disjoint from the left spectrum of B. Theorem 2.3 may be interpreted as saying that the operator defined by $\phi(X_1, X_2) = AX_1 - X_2 B$ is onto precisely when 0 is not in the intersection of the right spectrum of A and the left spectrum of B.

4. Three or more summands

We now consider the case of three summands:

$$\Phi(X_1, X_2, X_3) = A_1 X_1 B_1 + A_2 X_2 B_2 + A_3 X_3 B_3, \tag{3}$$

where we assume $(A_1 \quad A_2 \quad A_3)$ and $(B_1^* \quad B_2^* \quad B_3^*)$ are both onto. Here are five conditions each of which implies Φ is onto.

1. Each of A_1, A_2 and A_3 is onto.
2. Each of A_1, A_2 and $(B_1^* \quad B_2^*)$ is onto.
3. Each of A_1 and B_1^* is onto.
4. Each of $(A_1 \quad A_2)$ and B_1^* and B_2^* is onto.
5. Each of B_1^*, B_2^* and B_3^* is onto.

There are others which differ from these in a trivial way by a permutation of the indices. There are also others that differ from these in a nontrivial way.

We will give two examples not covered by the above conditions. In the first Φ is onto; in the second it is not.

EXAMPLE 4.1. *Fix three nonzero subspaces that are pairwise orthogonal and sum to $\mathcal{H}$, and let P_k be the projection onto the k^{th} of these subspaces. Put $A_k = B_k = 1 - P_k$. It is immediate that none of the five conditions above is satisfied. Also for each pair of distinct indices j,k it is clear that $\begin{pmatrix} A_j & A_k \end{pmatrix}$ and $\begin{pmatrix} B_j^* & B_k^* \end{pmatrix}$ are onto. To see that Φ is onto in this case write each operator on $\mathcal{H}$ as a 3×3 matrix relative to the decomposition*

$$\mathcal{H} = P_1\mathcal{H} + P_2\mathcal{H} + P_3\mathcal{H}.$$

Then the P_k are diagonal matrices with two identities and one zero on the main diagonal. It is easy to see that $\Phi(X_1,X_2,X_3)$ is a 3×3 matrix in which each of the nine entries can be freely specified.

EXAMPLE 4.2. *Decompose $\mathcal{H}$ into a direct sum of two infinite dimensional subspaces and let $A_1 = B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, et $A_2 = B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and let $A_3 = B_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. It may be shown that this example has the same properties as the last in that none of the five conditions above is satisfied and for each pair of distinct indices j,k each of $\begin{pmatrix} A_j & A_k \end{pmatrix}$ and $\begin{pmatrix} B_j^* & B_k^* \end{pmatrix}$ is onto. However a calculation shows for each operator X on $\mathcal{H}$ that A_3XB_3 has all four entries the same, and therefore it follows that Φ cannot be onto.*

We show that if Φ is onto then a related operator in which the A's and B's are all projections is also onto. We first show that the A's and B's can be replaced by positive operators without changing the range. For an operator A, let $\operatorname{Ran} A$ be its range.

THEOREM 4.1. *Suppose $A_1,\cdots,A_n,B_1,\cdots,B_n$ are operators on $\mathcal{H}$ and*

$$\Phi(X_1,X_2,\cdots,X_n) = \sum_{k=1}^{n} A_kX_kB_k.$$

Suppose in addition $A_1',\cdots,A_n',B_1',\cdots,B_n'$ are operators on $\mathcal{H}$ and Φ' is defined analogously to Φ using these operators. If $\operatorname{Ran} A_i' \supset \operatorname{Ran} A_i$ and $\operatorname{Ran}(B_i')^* \supset \operatorname{Ran}(B_i)^*$ for $1 \leqslant i \leqslant n$, then $\operatorname{Ran}\Phi' \supset \operatorname{Ran}\Phi$.

Proof. For operators S and T, $\operatorname{Ran} S \subset \operatorname{Ran} T$ is equivalent to the existence of an operator U such that $S = TU$ (see [4])), and consequently $\operatorname{Ran} S^* \subset \operatorname{Ran} T^*$ is equivalent to $S = V^*T$ for some operator V. Thus the hypothesis on the A's and B's imply the existence of $U_1,\dots,U_n,V_1,\dots,V_n$ such that $A_i'U_i = A_i$ and $V_i^*B_i' = B_i$ for $1 \leqslant i \leqslant n$. Hence

$$\Phi(X_1,\dots,X_n) = \sum_{i=1}^{n} A_i'U_iX_iV_i^*B_i',$$

which shows that $\operatorname{Ran}\Phi \subset \operatorname{Ran}\Phi'$. $\square$

COROLLARY 4.2. *If $\operatorname{Ran} A_i = \operatorname{Ran} A_i'$ and $\operatorname{Ran} B_i^* = \operatorname{Ran}(B_i')^*$ for $1 \leqslant i \leqslant n$, then $\operatorname{Ran}\Phi = \operatorname{Ran}\Phi'$.*

Proof. Two applications of the Theorem imply the Corollary. □

COROLLARY 4.3. *If*

$$\Phi(X_1, X_2, \cdots, X_n) = \sum_{k=1}^{n} A_k X_k B_k$$

for given operators $A_1, \cdots, A_n, B_1, \cdots, B_n$ on $\mathcal{H}$, then there are positive operators $P_1, \cdots, P_n, Q_1, \ldots, Q_n$ on $\mathcal{H}$ such that for each i the operators A_i and P_i have the same range, B_i^* and Q_i have the same range, and the operator Ψ defined by

$$\Psi(X_1, X_2, \cdots, X_n) = \sum_{k=1}^{n} P_k X_k Q_k$$

has the same range as Φ.

Proof. Let A_i and B_i^* have polar decompositions $A_i = P_i U_i$ and $B_i^* = Q_i V_i$ respectively for $1 \leqslant i \leqslant n$. Then $\operatorname{Ran} A_i = \operatorname{Ran} P_i$ and $\operatorname{Ran} B_i^* = \operatorname{Ran} Q_i$ for $1 \leqslant i \leqslant n$, so the result follows from the preceding Corollary. □

THEOREM 4.4. *Suppose $A_1, \cdots, A_n, B_1, \cdots, B_n$ are all operators on $\mathcal{H}$ and Φ, defined by*

$$\Phi(X_1, X_2, \cdots, X_n) = \sum_{k=1}^{n} A_k X_k B_k,$$

is onto. Then there are projections $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ such that the range of each P_k is included in that of the corresponding A_k, the range of each Q_k is included in that of the corresponding B_k^*, and the operator Ψ defined by

$$\Psi(X_1, X_2, \cdots, X_n) = \sum_{k=1}^{n} P_k X_k Q_k$$

is onto.

Proof. By Corollary 4.3, we may assume that the A_k and B_k are all positive operators. Suppose Φ is onto. The set of surjective operators on a Banach space is open in the norm topology, since an operator is surjective if and only if its adjoint is bounded below, and the set of operators that are bounded below is easily seen to be open. Therefore if each A_k and B_k is given a small perturbation, then the operator that results from Φ is also onto. Subtract from each A_k an operator of the form $A_k E_k$, where E_k is a spectral projection of A_k such that $\| A_k E_k \| < \varepsilon$ for a suitably small positive ε. Perturb the B_k similarly. The ranges of the perturbed operators are then included in the ranges of the original ones. Thus we may assume that each A_k and each B_k has closed range.

Let P_k and Q_k be the projections on the ranges of A_k and B_k respectively. The assertion then follows from Corollary 4.2. □

The following proposition establishes a necessary condition for the operator Φ of equation (1) to be onto. We need the following notation: given operators $A_1, \cdots, A_n$, for each set of indices $J \subset \{1, \cdots, n\}$ let A_J be the operator determined by the row matrix having entries A_j for all $j \in J$. That is, if $J = \{j_1, j_2, \cdots, j_m\}$, then

$$A_J = \begin{pmatrix} A_{j_1} & A_{j_2} & \cdots & A_{j_m} \end{pmatrix}.$$

Also, let

$$A_J^{\circledast} = \begin{pmatrix} A_{j_1}^* & A_{j_2}^* & \cdots & A_{j_m}^* \end{pmatrix}.$$

If $J = \emptyset$, then let A_J be the zero operator.

PROPOSITION 4.5. *Suppose Φ is defined as in equation (1), and Φ is onto. Let $J = \{j_1, j_2, \cdots, j_m\}$ be any set of indices, and let $K = \{k_1, k_2, \cdots, k_p\}$ be the complementary set of indices, so $J \cap K = \emptyset$ and $J \cup K = \{1, 2, \cdots, n\}$. Then A_J or $B_K^{\circledast}$ is onto.*

Proof. According to Lemma 3.3, since Φ is onto, $\Phi^\sharp$ is bounded below. Applying Lemma 3.2 to $T = f \otimes g$ with f and g unit vectors, we see that there is an $\varepsilon > 0$ such that

$$\max\{\|A_1^* g\| \|B_1 f\|, \|A_2^* g\| \|B_2 f\|, \cdots, \|A_n^* g\| \|B_n f\|\} \geqslant \varepsilon. \tag{4}$$

If A_J is not onto, then A_J^* is not bounded below, and a unit vector g can be chosen to make $A_J^* g$ arbitrarily small. But then (4) implies that $\max\{\|B_k f\| : k \in K\} > \delta$ for some $\delta > 0$ and all unit vectors f. This implies $B_K^{\circledast}$ is onto. $\square$

We prove sufficiency of the above condition only in the case where the A_k's and B_k's are commuting sets of projections. The projection hypothesis can certainly be weakened, but Example 4.2 shows that the commutativity hypothesis cannot be removed.

THEOREM 4.6. *Suppose $\{P_1, P_2, \ldots, P_n\}$ and $\{Q_1, Q_2, \ldots, Q_n\}$ are two sets of commuting projections and*

$$\Phi(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} P_i X_i Q_i.$$

Then Φ is onto precisely when for each subset J of $\{1, 2, \ldots, n\}$, if K is the complementary set of indices, then P_J or Q_K is onto.

Proof. Only sufficiency remains to be established. To facilitate an induction argument we prove a slight generalization of the theorem where the P_k's are projections on a Hilbert space $\mathcal{H}$ and the Q_k's are projections on a Hilbert space $\mathcal{K}$ and the X_k are operators from $\mathcal{K}$ to $\mathcal{H}$.

In the case where $n = 1$ the hypothesis implies both P_1 and Q_1 are the identity operator, and the assertion is trivial in this case.

Suppose inductively that the proposition is true when there are $n-1$ projections in each family and $\{P_1, \ldots, Q_n\}$ and $\{Q_1, \ldots, Q_n\}$ satisfying the hypothesis are given.

The case where both P_1 and Q_1 are the identity is again trivial, so consider the case where $P_1 = 1$ and Q_1 is a proper projection. Then for an arbitrary operator Y from $\mathcal{K}$ to $\mathcal{H}$,

$$Y = P_1 Y Q_1 + Y(1 - Q_1).$$

The hypothesis implies both $(P_2 \quad \dots \quad P_n)$ and $(Q_1 \quad \dots \quad Q_n)$ are onto. Let $\mathcal{K}' = (1 - Q_1)\mathcal{K}$ and $Q_k' = (1 - Q_1)Q_k$ for $2 \leqslant k \leqslant n$. Then the induction hypothesis implies that Φ' defined by

$$\Phi'(Z_2, \dots, Z_n) = \sum_{k=2}^{n} P_k Z_k Q_k'$$

is onto. Thus there are operators Z_k from $\mathcal{K}'$ to $\mathcal{H}$ such that $\Phi'(Z_2, \dots, Z_n) = Y(1 - Q_1)$. Taking $X_1 = Y$ and $X_k = Z_k(1 - Q_1)$ for $2 \leqslant k \leqslant n$ shows that Y is in the range of Φ, so Φ is onto in this case. The case of P_1 proper and $Q_1 = 1$ is similar.

Finally suppose both P_1 and Q_1 are proper projections, so that both $(P_2 \quad \dots \quad P_n)$ and $(Q_2 \quad \dots \quad Q_n)$ are onto. For an arbitrary operator Y from $\mathcal{K}$ into $\mathcal{H}$ observe

$$Y = P_1 Y Q_1 + (1 - P_1)Y + P_1 Y(1 - Q_1). \tag{5}$$

It must be shown that the last two terms have the form $\sum_{k=2}^{n} P_k X_k Q_k$. We separately consider each of the terms.

For $(1 - P_1)Y$ consider the projections $P_k' = P_k(1 - P_1)$ for $2 \leqslant k \leqslant n$ on $\mathcal{H}' = (1 - P_1)\mathcal{H}$ and $Q_2, \dots, Q_n$ on $\mathcal{K}$. For operators $Z_2, \dots, Z_n$ from $\mathcal{K}$ to $\mathcal{H}'$ let

$$\Phi_1(Z_2, \dots, Z_n) = \sum_{k=2}^{n} P_k' Z_k Q_k, \tag{6}$$

so $\Phi_1(Z_2, \dots, Z_n) = \Phi(0, (1 - P_1)Z_2, \dots, (1 - P_1)Z_n)$. Suppose $J \subset \{2, \dots, n\}$ and P_J' is not onto $\mathcal{H}'$. Put $J' = \{1\} \cup J$, and observe that $P_{J'}$ cannot be onto $\mathcal{H}$. Thus Q_K is onto where

$$K = \{1, \dots, n\} \setminus J' = \{2, \dots, n\} \setminus J.$$

Thus the condition of the theorem is satisfied by the projections P_k' and Q_k for $2 \leqslant k \leqslant n$, and, by induction, Φ_1 is onto. Hence $(1 - P_1)Y$ is in the range of Φ.

For $P_1 Y(1 - Q_1)$ consider $\mathcal{K}' = (1 - Q_1)\mathcal{K}$ and $Q_k' = (1 - Q_1)Q_k$ for $2 \leqslant k \leqslant n$. For operators $W_2, \dots, W_n$ from $\mathcal{K}'$ into $\mathcal{H}$ define

$$\Phi_2(W_2, \dots, W_n) = \sum_{k=2}^{n} P_k W_k Q_k'.$$

Suppose $J \subset \{2, \dots, n\}$ and P_J is not onto $\mathcal{H}$. Put $K = \{2, \dots, n\} \setminus J$ and $K' = \{1\} \cup K$. Then $Q_{K'}$ is onto $\mathcal{K}$, and it follows that Q_K' is onto $\mathcal{K}'$. By induction, Φ_2 is onto, and therefore $P_1 Y(1 - Q_1) = \Phi_2(W_2, \dots, W_n)$ for some choice of $W_2, \dots, W_n$. Thus

$$P_1 Y(1 - Q_1) = \Phi(0, W_2(1 - Q_1), \dots, W_n(1 - Q_1)),$$

and, assembling all the pieces, we see that

$$Y = \Phi(Y, (1-P_1)Z_2 + W_2(1-Q_1), \ldots, (1-P_1)Z_n + W_n(1-Q_1)).$$

Hence Φ is onto. $\square$

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