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SPECTRAL CONTINUITY OF k–TH ROOTS OF HYPONORMAL OPERATORS

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Abstract. Continuity of the set theoretic function spectrum, and some of its distinguished parts, on the class of k-th roots of hyponormal operators is proved.

1. Introduction and notation

A Hilbert space operator T, $T \in B(\mathcal{H})$, is hyponormal if $TT^* \leqslant T^*T$, and an operator $A \in B(\mathcal{H})$ is a k-th root, for some integer $k \geqslant 2$, of a hyponormal operator if A^k is hyponormal. Let $\sqrt[k]{H} = \{A \in B(\mathcal{H}) : A^k \text{ is hyponormal}\}$. Then, this is easily verified, $\sqrt[k]{H}$ is stable under multiplication by scalars, closed in the norm topology and a proper subclass of $B(\mathcal{H})$. Since the restriction of a hyponormal operator to an invariant subspace is again hyponormal, $A \in \sqrt[k]{H}$ implies $A|_M \in \sqrt[k]{H}$ for every invariant subspace M of A. Hyponormal operators satisfy (Bishop's) property (β) [24]. Hence, apply [19, Theorem 3.3.9], operators $A \in \sqrt[k]{H}$ satisfy property (β); in particular, operators $A \in \sqrt[k]{H}$ have SVEP, the single-valued extension property [19, Proposition 1.2.19].

The spectrum $\sigma(T)$ of $T \in B(\mathcal{H})$ is a compact subset of the set $\mathbf{C}$ of complex numbers. The function σ, viewed upon as a function from $B(\mathcal{H})$ into the set of all compact subsets of $\mathbf{C}$ with its Hausdorff metric, is an upper semi-continuous function [15, Problem 103], which fails (in general) to be continuous [15, Problem 102]. Starting with the seminal paper [23] by Newburgh on spectral continuity in a general Banach algebra, characterization of the points of continuity of σ in the algebra $B(\mathcal{H})$ has been carried out by a number of authors, amongst them Conway and Morrel [7, 8] and Apostal et.al. [4, Chapter 14]). It is however a demanding exercise to apply criteria from [7, 8] and [4] to determine the continuity of σ on individual classes of operators; this has led to the development of a number of techniques. It is known that σ is continuous on the class which are either hyponormal operators or p-hyponormal operators or M-hyponormal operators or operators satisfying a growth condition of order 1 or (p,k)-quasihyponormal operators or paranormal operators (see [10], [11], [17], [20] and [14]). The question of whether σ is continuous on the class of k-th roots of operators

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belonging to one of the classes above seems however not to have been considered. This paper considers this problem. It is proved that the points $A \in \sqrt[k]{H}$ are points of continuity of the set theoretic functions the spectrum, the Browder spectrum, the Weyl spectrum, the Weyl surjectivity spectrum, the Weyl approximate point spectrum and the approximate point spectrum. We also point out how our argument extends to cover the class of k-th roots of some of the other classes of operators referred to above.

We introduce now some of our notation and terminology. Any other notation or terminology will be introduced as and when required.

Let $T \in B(\mathcal{H})$, and let $T - \lambda = T - \lambda I$. The ascent of T at λ, $\operatorname{asc}(T-\lambda)$, is the least non-negative integer n such that $(T-\lambda)^{-n}(0) = (T-\lambda)^{-(n+1)}(0)$ and the descent of T at λ, $\operatorname{dsc}(T-\lambda)$, is the least non-negative integer n such that $(T-\lambda)^n\mathcal{H} = (T-\lambda)^{n+1}\mathcal{H}$. We say that T has finite ascent (resp., descent) if $\operatorname{asc}(T-\lambda) < \infty$ (resp., $\operatorname{dsc}(T-\lambda) < \infty$) for all $\lambda \in \mathbf{C}$. The operator T has *the single-valued extension property* at λ_0, SVEP at λ_0 for short, if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \to \mathcal{H}$ which satisfies

$$(T-\lambda)f(\lambda) = 0 \quad \text{for all} \quad \lambda \in \mathcal{D}_{\lambda_0}$$

is the function $f \equiv 0$. Trivially, every operator T has SVEP at points of the resolvent $\rho(T) = \mathbf{C} \setminus \sigma(T)$ and at isolated points of $\sigma(T)$. *We say that A has SVEP if it has SVEP at every $\lambda \in \mathbf{C}$.*

T is said to be *left semi-Fredholm* (resp., *right semi-Fredholm*), $T \in \Phi_+(\mathcal{H})$ (resp., $T \in \Phi_-(\mathcal{H})$), if $T\mathcal{H}$ is closed and the deficiency index $\alpha(T) = \dim(T^{-1}(0))$ is finite (resp., the deficiency index $\beta(T) = \dim(\mathcal{H} \setminus T\mathcal{H})$ is finite); T is semi-Fredholm if it is either left semi-Fredholm or right semi-Fredholm, and T is Fredholm, $T \in \Phi(\mathcal{H})$, if it is both left and right semi-Fredholm. The semi-Fredholm index of T, $\operatorname{ind}(T)$, is the (finite or infinite) integer $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. We say that the operator T is *Weyl* (resp., *Browder*) if it is Fredholm of index 0 (resp., Fredholm of finite ascent and descent). The *Weyl spectrum* $\sigma_w(T)$ (resp., *Browder spectrum* $\sigma_b(T)$) of T is the set $\{\lambda : T-\lambda \text{ is not Weyl}\}$ (resp., the set $\{\lambda : T-\lambda \text{ is not Browder}\}$). Let $\operatorname{iso}\sigma(T)$, $\Pi(T) = \{\lambda \in \sigma(T) : \operatorname{asc}(T-\lambda) = \operatorname{dsc}(T-\lambda) < \infty\}$ and $\Pi_0(T) = \{\lambda \in \Pi(T) : T-\lambda \text{ is Fredholm}\}$ denote, respectively, the isolated points of the spectrum, the set of poles (of the resolvent of T) and the set of Riesz points of T.

The (Fredholm) essential spectrum $\sigma_e(T)$ of T is the set $\{\lambda : T-\lambda \text{ is not Fredholm}\}$. Evidently, $\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T)$. Let $\sigma_a(T)$ ($\sigma_s(T)$) denote the approximate point spectrum (resp., the surjectivity spectrum) of T. The Weyl approximate point spectrum (Weyl defect spectrum) of T is the set $\sigma_{wa}(T) == \cap\{\sigma_a(T+K) : K \in \mathcal{K}\}$ (resp., $\sigma_{ws}(T) == \cap\{\sigma_s(T+K) : K \in \mathcal{K}\}$, where $\mathcal{K}$ is the ideal of compact operators in $B(\mathcal{H})$. If we let $\Phi_+^-(\mathcal{H}) = \{T \in \Phi_+(\mathcal{H}) : \operatorname{ind}(T) \leqslant 0\}$ and $\Phi_-^+(\mathcal{H}) = \{T \in \Phi_-(\mathcal{H}) : \operatorname{ind}(T) \geqslant 0\}$, then $\sigma_{wa}(T)$ is the complement in $\mathbf{C}$ of all those λ for which $(T-\lambda) \in \Phi_+^-(\mathcal{H})$ and $\sigma_{ws}(T)$ is the complement in $\mathbf{C}$ of all those λ for which $(T-\lambda) \in \Phi_-^+(\mathcal{H})$ [1, Theorem 3.65]. Evidently, $\sigma_{ws}(T) = \overline{\sigma_{wa}(T^*)}$, where, for a subset S of $\mathbf{C}$, $\overline{S} = \{\overline{\lambda} : \lambda \in S\}$.

2. Results

Let $\{T_n\}$ be a sequence of operators in $B(\mathcal{H})$ such that $\lim_{n\longrightarrow\infty}||T_n - T|| = 0$. The function σ_x, where σ_x is σ or a distinguished part thereof, is said to be continuous at T if σ_x is (both) upper semi-continuous, $\limsup \sigma_x(T_n) \subset \sigma_x(T)$, and lower semi-continuous, $\sigma_x(T) \subset \liminf \sigma_x(T_n)$. It is well known that σ and σ_{ea} are upper semi-continuous at every $T \in B(\mathcal{H})$ ([15, Problem 103] and [9, Theorem 2.1]). For Hilbert space operators, a necessary and sufficient condition for the continuity of σ at T is that σ is continuous at T^*; see Burlando [6, Proposition 3.1], where it is shown that the continuity of σ at T^* is both necessary and sufficient for the continuity of σ at T in a reflexive Banach space. We say that a Banach space operator is polaroid (simply polaroid) if the isolated points of the spectrum of the operator are poles (resp., poles of order one) of the resolvent of the operator. It is well known that the eigen-spaces of a hyponormal operator are reducing; hence, hyponormal operators are simply polaroid. The following theorem is our main result.

THEOREM 2.1. *Operators $A \in \sqrt[k]{H}$ are points of continuity of σ, σ_b, σ_w and σ_{es}.*

Proof. We split the proof into a number of parts, stated below as steps.

Step 1: A has SVEP. This has already been observed, but here is some additional detail. It is well known that hyponormal operators have SVEP; so that A^k has SVEP. Recall from [1, Theorem 2.40] that if $f : \mathcal{C} \longrightarrow \mathbf{C}$ is an analytic function on an open neighbourhood $\mathcal{C}$ of $\sigma(A)$, which is non-constant on each connected component of $\mathcal{C}$, then A has SVEP if and only if $f(A)$ has SVEP. Hence A has SVEP.

Step 2: $\overline{\sigma_{ws}(A)} = \sigma_{wa}(A^*) = \sigma_w(A^*) = \sigma_b(A^*)$. Evidently, $\sigma_{wa}(A^*) \subseteq \sigma_w(A^*)$. Let $\lambda \notin \sigma_{wa}(A^*)$. Then $A^* - \lambda \in \Phi_+^-(\mathcal{H})$, i.e., $A^* - \lambda$ is upper semi-Fredholm and $\operatorname{ind}(A^* - \lambda) \leqslant 0$. Since A has SVEP, the semi-Fredholm property of $A^* - \lambda$ implies that $\operatorname{ind}(A^* - \lambda) \geqslant 0$ [1, Corollary 3.19]. Hence $\operatorname{ind}(A^* - \lambda) = 0$ and $A^* - \lambda$ is Fredholm. But then $\lambda \notin \sigma_w(A^*) \Longrightarrow \sigma_w(A^*) \subset \sigma_{wa}(A^*)$. Hence $\sigma_{wa}(A^*) = \sigma_w(A^*)$. To prove the equality $\sigma_w(A^*) = \sigma_b(A^*)$, we start by observing that $\sigma_w(A^*) \subseteq \sigma_b(A^*)$. Let $\lambda \notin \sigma_w(A^*)$. Then $A^* - \lambda \in \Phi(\mathcal{H})$, which (since A has SVEP) implies that $\operatorname{dsc}(A^* - \lambda) < \infty$ [1, Theorem 3.17(vi)], and this (since $A^* - \lambda \in \Phi(\mathcal{H})$ and $\alpha(A^* - \lambda) = \beta(A^* - \lambda) < \infty$) in turn implies that $\operatorname{asc}(A^* - \lambda) = \operatorname{dsc}(A^* - \lambda) < \infty \Longrightarrow \lambda \notin \sigma_b(A^*)$. Since $\overline{\sigma_{ws}(A)} = \sigma_{wa}(A^*)$ for every operator A, the proof of Step 2 is complete.

Step 3: $\sigma_a(A^*) \setminus \sigma_{wa}(A^*) = \Pi_{0a}(A^*) = \{\lambda \in \operatorname{iso}\sigma_a(A^*) : 0 < \alpha(T^* - \lambda) < \infty\}$. *In particular, if $\lambda \in \sigma_a(A^*) \setminus \sigma_{wa}(A^*)$, then $\lambda \in \operatorname{iso}\sigma_a(A^*)$.* Since A has SVEP, $\overline{\sigma(A)} = \sigma(A^*) = \sigma_a(A^*)$, $\overline{\sigma_w(A)} = \sigma_w(A^*) = \sigma_{wa}(A^*)$ (Step 2) and $\overline{\Pi_0(A)} = \Pi_0(A^*) = \Pi_0^a(A^*)$, where $\Pi_0^a(A^*) = \{\lambda \in \sigma_a(A^*) : \lambda \in \Pi_0(A^*)\}$. We prove that $\sigma(A) \setminus \sigma_w(A) = \Pi_0(A) = \Pi_{00}(A)$: this, in view of the above, would then imply that $\sigma_a(A^*) \setminus \sigma_{wa}(A^*) = \Pi_{0a}(A^*)$. Since A has SVEP, $\lambda \in \sigma(A) \setminus \sigma_w(A)$ implies that $A - \lambda$ is Fredholm, $\operatorname{ind}(A - \lambda) = 0$ and $\operatorname{asc}(A - \lambda) < \infty$ [1, Theorem 3.16]. Hence $A - \lambda$ is Fredholm and $\operatorname{asc}(A - \lambda) = \operatorname{dsc}(A - \lambda) < \infty$ [1, Theorem 3.4(iv)]. Equivalently, $\lambda \in \Pi_0(A)$, and (since $\Pi_0(A) \subseteq \sigma(A) \setminus \sigma_w(A)$ for every operator A) we conclude that $\sigma(A) \setminus \sigma_w(A) = \Pi_0(A) \subseteq \Pi_{00}(A)$. For the reverse inclusion, let $\lambda \in \Pi_{00}(A)$.

Then $\lambda \in \text{iso}\sigma(A)$: there exists a decomposition $\mathcal{H} = H_0(A-\lambda) \oplus K(A-\lambda)$ of $\mathcal{H}$ such that $A-\lambda|_{H_0(A-\lambda)}$ is quasi-nilpotent (and $K(T-\lambda)$ is semi-regular) [21]. Evidently, $\sigma(A_1) = \sigma(A|_{H_0(A-\lambda)}) = \{\lambda\}$. Since A_1^k is hyponormal, hence polaroid, it follows that $A_1^k = \lambda^k I|_{H_0(A-\lambda)}$. Hence $A_1 = \lambda I|_{H_0(A-\lambda)}$, which implies that $\mathcal{H} = (A-\lambda)^{-1}(0) \oplus K(A-\lambda)$. Since $(A-\lambda)K(A-\lambda) = K(A-\lambda)$, it follows that $\mathcal{H} = (A-\lambda)^{-1}(0) \oplus (A-\lambda)\mathcal{H}$, i.e., $\lambda \in \Pi(A)$. Since $\alpha(A-\lambda) < \infty$, we conclude that $\Pi_{00}(A) = \Pi_0(A)$.

Step 4: If $\{A_n\}$ is a sequence in $\sqrt[k]{H}$ which converges in norm to A, then A^* is a point of continuity of σ_{wa}. The function σ_{wa} being upper semi-continuous [9, Theorem 2.1], we prove that σ_{wa} is lower semi-continuous, i.e., $\sigma_{wa}(A^*) \subset \liminf \sigma_{wa}(A_n^*)$. Assume to the contrary that σ_{wa} is not lower semi-continuous. Then there exists an $\epsilon > 0$, a $\lambda_0 \in \sigma_{wa}(A^*)$ and an ϵ-neighbourhood $(\lambda_0)_\epsilon$ of λ_0 such that $\sigma_{wa}(A_n^*) \cap (\lambda_0)_\epsilon = \emptyset$ for infinitely many values of n. Assume without loss of generality that $\sigma_{wa}(A_n^*) \cap (\lambda_0)_\epsilon = \emptyset$ for all n. Then $A_n^* - \lambda \in \Phi_+^-(\mathcal{H})$ for every $(\lambda_0 \neq)\lambda \in (\lambda_0)_\epsilon$, which implies that $\text{ind}(A_n^* - \lambda) \leqslant 0$, $\alpha(A_n^* - \lambda) < \infty$ and $(A_n^* - \lambda)\mathcal{H}$ is closed. Consequently, $\text{ind}(A_n - \overline{\lambda}) \geqslant 0$ and $\beta(A_n - \overline{\lambda}) < \infty$, which, since A_n has SVEP for all n, implies $\text{ind}(A_n - \overline{\lambda}) \leqslant 0$. Hence $\text{ind}(A_n - \overline{\lambda}) = 0$ and $\alpha(A_n - \overline{\lambda}) = \beta(A_n - \overline{\lambda}) < \infty$ for all n. The continuity of the index implies that $\text{ind}(A - \overline{\lambda}) = \lim_{n \longrightarrow \infty} \text{ind}(A_n - \overline{\lambda}) = 0$, and hence that $A - \overline{\lambda}$ is Fredholm with $\text{ind}(A - \overline{\lambda}) = 0$. But then $A^* - \lambda_0$ is Fredholm with $\text{ind}(A^* - \lambda_0) = 0 \Longrightarrow A^* - \lambda_0 \in \Phi_+^-(\mathcal{H})$, which is a contradiction.

Step 5: If $\{A_n\}$ is a sequence in $\sqrt[k]{H}$ which converges in norm to A, then σ is continuous at A^*. Since A has SVEP, $\sigma(A^*) = \sigma_a(A^*)$. In view of the upper semi-continuity of the function σ, it would therefore suffice to prove that $\sigma_a(A^*) \subset \liminf \sigma_a(A_n^*)$ for every sequence $\{A_n\} \in \sqrt[k]{H}$ such that A_n converges in norm to A. Let $\lambda \in \sigma_a(A^*)$. Then we have two possibilities: either $\lambda \in \sigma_{wa}(A^*)$ or $\lambda \in \sigma_a(T^*) \setminus \sigma_{wa}(A^*)$. Since

$$\sigma_{wa}(A^*) \subset \liminf \ \sigma_{wa}(A_n^*) \subset \liminf \ \sigma_a(A_n^*)$$

(see Step 4), the proof for the case in which $\lambda \in \sigma_{wa}(A^*)$ follows. Now let $\lambda \in \sigma_a(A^*) \setminus \sigma_{wa}(A^*)$. Then, see Step 3, $\lambda \in \text{iso}\sigma_a(A^*)$. Hence $\lambda \in \liminf \ \sigma_a(A_n^*) = \liminf \ \sigma(A_n^*)$) for all n [18, Theorem IV.3.16].

Step 6: Completing the proof. It is immediate from Step 4 that σ_{ws} is continuous at $A \in \sqrt[k]{H}$ and, since σ is continuous at A if and only if σ is continuous at A^*, the continuity of σ at $A \in \sqrt[k]{H}$ follows from Step 5. Finally, since $\overline{\sigma_w(T^*)} = \sigma_w(T)$ and $\overline{\sigma_b(T^*)} = \sigma_b(T)$, the continuity of σ_w and σ_b at $T \in \sqrt[k]{H}$ follows from Steps 2 and 4. □

Continuity of σ_e and σ_a.

Let $\mathcal{U} = B(\mathcal{H})/\mathcal{K}$ denote the Calkin algebra, and let $\pi : B(\mathcal{H}) \longrightarrow \mathcal{U}$ denote the quotient map. (Recall that $\mathcal{K} \subset B(\mathcal{H})$ is the two sided ideal of compact operators.) We say that an operator $A \in B(\mathcal{H})$ is in $\text{ess}\sqrt[k]{H}$ if $\pi(A)^k = \pi(A^k)$ is hyponormal.

THEOREM 2.2. *σ_e is continuous on $\text{ess}\sqrt[k]{H}$.*

Proof. Since $\mathcal{U}$ is a C^*-algebra, there is a Hilbert space $\mathcal{H}_0$ and an isometric $*$-isomorphism $\nu : \mathcal{U} \longrightarrow B(\mathcal{H}_0)$ such that the essential spectrum $\sigma_e(A)$ $(= \sigma(\pi(A)))$ is the spectrum of the operator $\nu o\pi(A) \in B(\mathcal{H}_0)$. Since the operator $(\nu o\pi(A))^k = \nu o\pi(A^k) \in B(\mathcal{H}_0)$ is hyponormal, Theorem 2.1 applies. $\square$

Although σ_a is generally not a continuous function, it is upper semi-continuous [8]. The following lemma, which proves the continuity of σ_{wa} at points $A \in \sqrt[k]{H}$, will be required in our proof of the continuity of σ_a on $\sqrt[k]{H}$.

LEMMA 2.3. *If $\{A_n\}$ is a sequence of operators in $\sqrt[k]{H}$ such that A_n converges in norm to A, then $\sigma_{wa}(A) \subset \liminf \sigma_{wa}(A_n)$.*

Proof. If σ_{wa} is not lower semi-continuous, then there exists an $\epsilon > 0$, an integer n_0, a $\lambda \in \sigma_{wa}(A)$ and an ϵ-neighbourhood $(\lambda)_\epsilon$ of λ such that $\sigma_{wa}(A_n) \cap (\lambda)_\epsilon = \emptyset$ for all $n > n_0$. Choose an n sufficiently large so that $||A_n - A|| = ||(A_n - \lambda) - (A - \lambda)|| < \epsilon$. Then the punctured neighbourhood theorem for semi-Fredholm operators implies that $A - \lambda \in \Phi_+^-(\mathcal{H})$. This is a contradiction. $\square$

THEOREM 2.4. *σ_a is continuous on $\sqrt[k]{H}$.*

Proof. To prove the continuity of σ_a at A, we have only to prove that σ_a is lower semi-continuous at A. Let $\lambda \in \sigma_a(A)$. Then either $\lambda \in \sigma_a(A) \setminus \sigma_{wa}(A)$ or $\lambda \in \sigma_{wa}(A)$. Since

$$\sigma_{wa}(A) \subset \liminf \sigma_{wa}(A_n) \subset \liminf \sigma_a(A_n),$$

the proof for the case in which $\lambda \in \sigma_{wa}(A)$ follows. Now let $\lambda \in \sigma_a(A) \setminus \sigma_{wa}(A)$. Then, since A has SVEP, $\lambda \in \mathrm{iso}\sigma_a(A)$ [1, Theorem 3.23]; hence, [18, Theorem IV.3.16], $\lambda \in \liminf \sigma_a(A_n)$. $\square$

Extension to some other classes of Hilbert space operators.

An operator $T \in B(\mathcal{H})$ is
p-hyponormal, $0 < p \leqslant 1$, if $|T^*|^{2p} \leqslant |T|^{2p}$;
w-hyponormal if $|\tilde{T}^*| \leqslant |T| \leqslant |\tilde{T}|$, where, for the polar decomposition $T = U|T|$ of T, $\tilde{T}$ is the Aluthge transform $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ of T;
M-hyponormal if there exists a number $M \geqslant 1$ such that $|T^* - \overline{\lambda}|^2 \leqslant M|T - \lambda|^2$ for all complex λ;
q-quasihyponormal for some positive integer q if $T^{*q}(|T|^2 - |T^*|^2)T^q \geqslant 0$;
(q,p)-quasihyponormal for some positive integer q and $0 < p \leqslant 1$ if $T^{*q}(|T|^{2p} - |T^*|^{2p})T^q \geqslant 0$;
of class $\mathcal{A}$ if $|T|^2 \leqslant |T^2|$;
totally $*$-paranormal if $||(T - \lambda)^* x||^2 \leqslant ||(T - \lambda)^2 x||$ for all $\lambda \in \mathbf{C}$ and every unit vector $x \in \mathcal{H}$,
and T is *paranormal* if $||Tx||^2 \leqslant ||T^2 x||$ for every unit vector $x \in \mathcal{H}$.

The following inclusions are known to be proper: hyponormal $\subset$ p-hyponormal $\subset$ w-hyponormal $\subset$ paranormal and $\mathcal{A} \subset$ paranormal (see [12, P 144] for an appropriate reference). Observe that a $(q,1)$-quasihyponormal operator is q-quasihyponormal.

Let $\mathcal{P}$ denote any one of the classes of operators defined above. The argument of the proof of Theorem 2.1 extends to classes $\mathcal{P}$.

THEOREM 2.5. *If $A \in B(\mathcal{H})$ is in $\sqrt[k]{\mathcal{P}}$, then σ, σ_a, σ_{ws}, σ_b and σ_w are continuous at A. Furthermore, σ_e is continuous on* $\mathrm{ess}\sqrt[k]{\mathcal{P}}$.

Proof. An examination of the proof of Theorem 2.1 shows that the properties of the class of hyponormal operators that play a role in the proof of the Theorem 2.1 are: (i) Hyponormal operators have SVEP; (ii) the class of hyponormal operators is closed in the uniform topology; (iii) if $\lambda \in \mathrm{iso}\sigma(T)$, then $H_0(T-\lambda) = (T-\lambda)^{-1}(0)$. Here the integer 1 in $(T-\lambda)^{-1}(0)$ is of little consequence; $(T-\lambda)^{-n}(0)$ for some positive integer n would do the job just as well. We show in the following that operators in $\mathcal{P}$ share these properties with hyponormal operators. Evidently it would suffice to consider classes of operators which are either M-hyponormal or (q,p)-quasihyponormal or totally $*$-paranormal or paranormal. Let $\mathcal{P}_0$ denote the class of operators which are either M-hyponormal or (q,p)-quasihyponormal or totally $*$-paranormal or paranormal.

If $T \in \mathcal{P}_0$, then T has SVEP; see [12, pp 144–146] for M-hyponormal, (q,p)-quasihyponormal and totally $*$-paranormal operators and [2] for paranormal operators. It is well known that the restriction of an operator $T \in \mathcal{P}_0$ to an invariant subspace is again in the same class. Since operators $T \in \mathcal{P}_0$ are polaroid (see [13, p 276] and [25, Theorem 6]), it follows that operators $T \in \mathcal{P}_0$ satisfy property (iii) above. We are thus left to prove that if $\{T_n\}$ is a sequence of operators in a class $\mathcal{P}_0$ such that $\lim_{n\longrightarrow\infty} ||T_n - T|| = 0$, then $T \in \mathcal{P}_0$. The proof here is almost the same for each of the constituent classes of $\mathcal{P}_0$: we give a brief outline for paranormal and (q,p)-quasihyponormal operators. Choose n large enough so that $||T - T_n|| < \epsilon$ for some (arbitrary) $\epsilon > 0$. If T_n is paranormal, then for every unit vector $x \in \mathcal{H}$

$$\begin{aligned} ||Tx||^2 &\leqslant ||(T-T_n)x||^2 + 2||(T-T_n)x||\,||T_nx|| + ||T_nx||^2 \\ &\leqslant 3\epsilon^2 + 2\epsilon||T|| + ||T_nx||^2 \leqslant 3\epsilon^2 + 2\epsilon||T|| + ||T_n^2x|| \\ &\leqslant 4\epsilon^2 + 4\epsilon||T|| + ||T^2x||. \end{aligned}$$

If T_n is (q,p)-quasihyponormal, then

$$|||T^*|^pT^qx|| \leqslant ||(|T^*|^p - |T_n^*|^p)T^qx|| + |||T_n^*|^p(T^q - T_n^q)x|| + |||T_n^*|^pT_n^qx||$$

and

$$|||T_n^*|^pT_n^qx|| \leqslant |||T_n|^pT_n^qx|| \leqslant ||(|T|^p - |T_n|^p)T_n^qx|| + |||T|^p(T^q - T_n^q)x|| + |||T|^pT^qx||$$

for all $x \in \mathcal{H}$. Since $||A^r - B^r|| \leqslant |||A-B|^r||$ for positive operators $A, B \in B(\mathcal{H})$ and $0 \leqslant r \leqslant 1$ [5, p 293, X.7], and since $(|A-B|^{2r}x,x) \leqslant ||x||^{2(1-r)}(|A-B|^2x,x)^r$ for all $A, B \in B(\mathcal{H})$, $x \in \mathcal{H}$ and $0 \leqslant r \leqslant 1$ by the Hölder-McCarthy inequality [22], it follows that

$$|||T^*|^pT^qx|| \leqslant \epsilon^{2p}f_1 + \epsilon f_2 + |||T|^pT^qx||,$$

where f_1 and f_2 are (finite valued) functions of ϵ and (powers of) $||T||$ and $||x||$. $\square$

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