SECOND ORDER ARITHMETIC MEANS IN OPERATOR IDEALS

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Abstract. We settle in the negative the question arising from [3] on whether equality of the second order arithmetic means of two principal ideals implies equality of their first order arithmetic means (second order equality cancellation) and we provide fairly broad sufficient conditions on one of the principal ideals for this implication to always hold true. We present also sufficient conditions for second order inclusion cancellations. These conditions are formulated in terms of the growth properties of the ratio of regularity sequence associated to the sequence of s-number of a generator of the principal ideal. These results are then extended to general ideals.

1. Introduction

Operators ideals, the two-sided ideals of the algebra $B(H)$ of bounded linear operators on a separable Hilbert space H, have played an important role in operator theory and operator algebras since they were first studied by J. Calkin [2] in 1941. One of the recurring themes in this subject, from the early years on, was the study of commutator spaces, also called commutator ideals. The introduction of cyclic cohomology in the early 1980's by A. Connes (e.g., see [1]) and the connection with algebraic K-theory by M. Wodzicki in the 1990's provided a powerful motivation for further work on operator ideals and commutator spaces. This work culminated in [3] with the full characterization of commutator spaces in terms of arithmetic mean operations on ideals.

Arithmetic means were first connected, albeit only implicitly, to operator ideals in the study of the commutator space of the trace class in [13] (see also [14], [15]) and then explicitly in [11]. In [3], arithmetic means provided a full characterization of the commutator space of arbitrary ideals and led to the introduction of a number of arithmetic mean ideals derived from a given ideal I, among which I_a (see next section for the formal definitions). For instance, an ideal I coincides with its commutator space (equivalently, it supports no nonzero trace) if and only if it is arithmetically mean stable, that is, $I = I_a$ [3, Theorem 5.6]. The study of the properties of I_a and of other related arithmetic mean ideals and the application of these properties to operator ideals are the focus of a program by the authors of this paper which was announced in [6] and includes [7]–[10] and this paper.

The operation $I \to I_a$ is inclusion preserving and the arithmetic mean cancellation properties for inclusion and equality are deeply linked to the structure of operator ideals. For instance in [9] we characterized when an ideal I has the following first order arithmetic mean inclusion and equality cancellation properties:

(i$'$) $J_a \subset I_a \Rightarrow J \subset I$ if and only if I is am-closed ([9, Lemma 6.1(C)])

(ii$'$) $J_a \supset I_a \Rightarrow J \supset I$ if and only if $I = \widehat{I}$ ([9, Definition 6.10] and preceding remark)

(iii$'$) If I is principal, then $J_a = I_a \Rightarrow J = I$ if and only if I is am-stable ([9, Theorem 6.7(A)])

For the definitions, notations and preliminary properties, see Section 2. Notice first that although the terminology we use here is new and due to [3], the inclusion $J_a \subset I_a$ in the case where both I and J are principal ideals has a time honored history: it reduces to the (infinite) majorization of the s-number sequences of the generators (up to a normalization scalar) $\eta \prec \xi$ in the sense that $\sum_1^n \eta_j \leqslant \sum_1^n \xi_j$ for all n, namely, $\eta_a \leqslant \xi_a$ (see for instance [4], [5], and [12]).

If I is principal, then conditions in (i$'$) (I is am-closed) and in (iii$'$) (I is am-stable) are both equivalent to the regularity of the sequence η of the s-numbers of a generator of I, while the condition in (ii$'$) is strictly stronger and is derived from the construction of a new principal arithmetic mean ideal $\widehat{I}$ (see Section 2). Notice that the first order equality cancellation property (iii$'$) is stated directly for principal ideals. Indeed, am-stability is not a sufficient condition for (iii$'$) even in the case when I is countably generated ([9, Example 5.5]), nor do we know of any natural necessary and sufficient condition for the general ideal case.

Second order arithmetic mean cancellations are considerably more complex even for principal ideals and are the focus of this paper. The questions we address here are: which conditions on an ideal I guarantee that the following second order arithmetic mean inclusion cancellations and equality cancellation hold?

(i) $J_{a^2} \subset I_{a^2} \Rightarrow J_a \subset I_a$

(ii) $J_{a^2} \supset I_{a^2} \Rightarrow J_a \supset I_a$

(iii) $J_{a^2} = I_{a^2} \Rightarrow J_a = I_a$

The first natural "test" question, which was posed by M. Wodzicki, arising from work in [3], is whether equality cancellation (iii) holds automatically for all pairs of principal ideals. Reformulated in terms of the s-number sequences ξ and η of the generators of the two principal ideals, the question asks whether the equivalence $\xi_{a^2} \asymp \eta_{a^2}$ of the sequences of the second order arithmetic means always implies the equivalence $\xi_a \asymp \eta_a$ of the first order means.

The answer to this question is negative and is presented in Example 4.3, one of the main results of this paper. The intuition behind the construction of this example led to the notion (Definition 3.1) of the *ratio of regularity* sequence $r(\xi) := \frac{\xi_a}{\xi}$ for a nonincreasing sequence $\xi \in c_o$ and then, indirectly, to the other results in this paper.

The second natural question is whether, at least when I is principal, the arithmetic stability of I (namely, the regularity of the sequence η of s-numbers of a generator of I) might be the necessary and sufficient condition for (iii), as is the case for first order equality cancellation. We found that while regularity is indeed sufficient, it is "very far"

from being necessary, that is, very far in terms of the ratio of regularity $r(\eta_a) = \frac{\eta_{a^2}}{\eta_a}$ of the sequence η_a.

Indeed, as is easy to show, $1 \leqslant r(\eta_a)_n \leqslant \log n$ for all $n > 1$ and all η (Lemma 3.8). The two "extremal cases" for $r(\eta_a)$ are thus when $r(\eta_a)$ is bounded, which is precisely the case when η is regular (see Corollary 3.10 and preceding discussion) and when $r(\eta_a) \asymp \log$. The latter condition is equivalent to what we call the exponential Δ_2-condition $\sup_m \frac{m^2(\eta_a)_{m^2}}{m(\eta_a)_m} < \infty$ (Proposition 3.11). Surprisingly (for us), it turned out that either of these two extremal cases, η regular or $r(\eta_a) \asymp \log$, is sufficient for I to have the second order equality cancellation property (iii) (Theorem 4.8(i)).

A further investigation of the exponential Δ_2-condition shows that if η satisfies this condition, then so does η_{a^p} for all $p \in \mathbb{N}$ (Corollary 3.12) and hence higher order cancellations also hold (Theorem 4.9).

While we do not know if these two conditions, η regular or $r(\eta_a) \asymp \log$, are necessary for the equality cancellation in (iii) to hold, we know that they are too strong for the inclusion cancellation in (i). Indeed, a weaker sufficient condition for (i) is that $r(\eta_a)$ is equivalent to a monotone sequence (Theorem 4.5(i)).

On the other hand, the two conditions, η regular or $r(\eta_a) \asymp \log$, are too weak for the inclusion cancellation in (ii). Indeed, Example 4.7 shows that the principal ideal generated by the regular sequence $\omega^{1/2}$ (where ω is the harmonic sequence $\langle \frac{1}{n} \rangle$) does not satisfy the inclusion cancellation in (ii).

A sufficient condition for the inclusion cancellation in (ii) is that either $\eta = \widehat{\eta}$ or $r(\eta_a) \asymp \log$ (Theorem 4.6(i), and for the definition of $\widehat{\eta}$ see the end of Section 2).

Sufficient conditions for each of the two cancellation properties (i) and (ii) in the case of a general ideal I are that every sequence in the characteristic set $\Sigma(I)$ of I is pointwise majorized by a sequence satisfying the corresponding condition for the principal ideal case (Theorems 4.5(ii) and 4.6(ii)). For the equality cancellation property (iii) for general ideals however, we have to ask a stricter sufficient condition than for the principal ideal case, as the proof for the "weaker" condition fails (Theorem 4.8(ii)). This corresponds roughly to the fact that first order equality cancellation (iii′) can fail for arithmetically mean stable ideals even for a countably generated ideal [9, Example 5.5].

There is a dual theory to arithmetic mean ideals in the trace class implicit in [3] and developed explicitly in [6]-[9] called arithmetic mean at infinity ideals, and we found strong parallels throughout their development. It seems likely that this parallel would continue with the results of this paper although we have not pursued this line of investigation.

2. Preliminaries and notation

Calkin [2] established a correspondence between the two-sided ideals of $B(H)$ for a complex-valued infinite-dimensional separable Hilbert space H and *characteristic sets*. These are the positive cones of c_o^* (the collection of sequences decreasing to 0)

that are hereditary (i.e., solid) and invariant under *ampliations*

$$c_o^* \ni \xi \to D_m\xi := \langle \xi_1, \ldots, \xi_1, \xi_2, \ldots, \xi_2, \xi_3, \ldots, \xi_3, \ldots \rangle$$

where each entry ξ_i of ξ is repeated m-times. The order-preserving lattice isomorphism $I \to \Sigma(I)$ maps each ideal to its characteristic set $\Sigma(I) := \{s(X) \mid X \in I\}$ where $s(X)$ denotes the sequence of s-numbers of X, i.e., all the eigenvalues of $|X| = (X^*X)^{1/2}$ repeated according to multiplicity, arranged in decreasing order, and completed by adding infinitely many zeroes if X has finite rank. Moreover, for every characteristic set $\Sigma \subset c_o^*$, if I is the ideal generated by $\{\text{diag } \xi \mid \xi \in \Sigma\}$ where diag ξ is the diagonal matrix with entries $\xi_1, \xi_2, \ldots$, then we have $\Sigma = \Sigma(I)$.

If $\xi \in c_o^*$, denote by ξ_a the arithmetic mean sequence of ξ, namely

$$(\xi_a)_n = \frac{1}{n} \sum_1^n \xi_j.$$

If I is an ideal, then the arithmetic mean ideals $_aI$ and I_a, called the *pre-arithmetic mean* and *arithmetic mean* of I, are the ideals with characteristic sets

$$\Sigma(_aI) := \{\xi \in c_o^* \mid \xi_a \in \Sigma(I)\},$$

$$\Sigma(I_a) := \{\xi \in c_o^* \mid \xi = O(\eta_a) \text{ for some } \eta \in \Sigma(I)\}.$$

The ideals $I^o := (_aI)_a$ and $I^- := {_a}(I_a)$ are called the *am-interior* and *am-closure* of I and the following 5-chain of inclusions holds (all of which can be simultaneously proper):

$$_aI \subset I^o \subset I \subset I^- \subset I_a.$$

Simple consequences of the 5-chain are the identities $I_a = (_a(I_a))_a$ and $_aI = {_a}((_aI)_a)$ and the consequent idempotence of the maps $I \to I^-$ and $I \to I^o$. And either derived from these or proven directly are the higher order 5-chains of inclusions:

$$_{a^n}I \subset (_{a^n}I)_{a^n} \subset I \subset {_{a^n}}(I_{a^n}) \subset I_{a^n},$$

the higher order identities $I_{a^n} = (_{a^n}(I_{a^n}))_{a^n}$ and $_{a^n}I = {_{a^n}}((_{a^n}I)_{a^n})$, and the idempotence of the maps $I \to {_{a^n}}(I_{a^n})$ and $I \to (_{a^n}I)_{a^n}$. So $_{a^n}(I_{a^n})$ are called higher order am-closures and form an increasing nest. (Similarly, $(_{a^n}I)_{a^n}$ are called higher order am-interiors and form a decreasing nest.)

Principal ideals are those ideals generated by a single operator X and we denote them by $(X) = (s_n(X))$, i.e., for every $\xi \in c_o^*$, the principal ideal $(\text{diag } \xi)$ is also denoted by (ξ). Then $\Sigma((\xi)) = \{\eta \in c_o^* \mid \eta = O(D_m\xi) \text{ for some } m \in \mathbb{N}\}$ and $(\xi)_a = (\xi_a)$. Since arithmetic mean sequences satisfy the $\Delta_{1/2}$-condition, i.e., $\xi_a \asymp D_2(\xi_a)$ and hence $\xi_a \asymp D_m(\xi_a)$ for all $m \in \mathbb{N}$, one has $\Sigma((\xi)_a) = \{\eta \in c_o^* \mid \eta = O(\xi_a)\}$. As a consequence, $(\eta)_a \subset (\xi)_a$ (resp., $(\eta)_a = (\xi)_a$) if and only if $\eta_a = O(\xi_a)$ (resp., $\eta_a \asymp O(\xi_a)$).

We denote by H the sequence $\langle \sum_1^n \frac{1}{j} \rangle$ and we often use the inequalities

$$\frac{1}{n} + \log n < H_n < 1 + \log n \quad \text{for } n > 1 \tag{1}$$

and

$$\log \frac{n}{m} - \frac{1}{m+1} < \log \frac{n+1}{m+1} < H_n - H_m < \log \frac{n}{m} \quad \text{for } n > m. \tag{2}$$

Given an ideal I and a sequence $\xi \in c_o^*$, the ideal $\widehat{I}$ and the sequence $\widehat{\xi}$ mentioned in the introduction are defined in [9, Definitions 6.10, 6.11] as $\widehat{I} := \cap \{J \mid J_a \supset I_a\}$ and $(\widehat{\xi})_n := (\xi_a)_{\nu(\xi)_n}$ where $\nu(\xi)_n := \min\{k \in \mathbb{N} \mid \sum_{i=1}^{k} \xi_i \geqslant n\xi_1\}$. Then $\overline{(\widehat{\xi})} = (\widehat{\xi})$ for ξ not summable. In particular, we proved there that $\widehat{\omega^p} = \omega^{p'}$ where $\frac{1}{p} - \frac{1}{p'} = 1$ and that if $\xi \asymp \widehat{\xi}$ then ξ is regular (see [ibid., Corollaries 6.15 and 6.16, and Proposition 6.17]).

3. Ratio of regularity

Ideals that coincide with their commutator space (i.e., those which do not support any nonzero trace) were identified in [3] as the arithmetic mean stable ideals, i.e., the ideals I that coincide with their arithmetic mean I_a. Nonzero principal am-stable ideals are precisely the ideals $I = (\xi)$ with a *regular* generator, i.e., $\xi_a \asymp \xi$ (see [5, III.14.3]). Since $\xi \leqslant \xi_a$ holds always, a sequence ξ is regular precisely when $\xi \notin \Sigma(F)$ (i.e., $\xi_n > 0$ for all $n \in \mathbb{N}$) and $\frac{\xi_a}{\xi}$ is bounded. In this paper we will see how properties of the ratio of regularity sequence $\frac{\eta_{a^2}}{\eta_a}$ of η_a relate to second order arithmetic mean cancellations for $I = (\eta)$.

DEFINITION 3.1. Let $\xi \in c_o^* \setminus \Sigma(F)$. The sequence $r(\xi) := \frac{\xi_a}{\xi}$ is called the ratio of regularity of ξ.

Notice that for all $\xi \in c_o^* \setminus \Sigma(F)$, $r(\xi)_1 = 1$, $r(\xi)_n \geqslant 1$ for all n and if m is the first index for which $\xi_m < \xi_1$, then $r(\xi)_n > 1$ for all $n \geqslant m$.

Ratios of regularity sequences have appeared implicitly in the literature and are helpful in analyzing various sequence properties. For instance they are instrumental in deriving the Potter-type inequality characterizing the sequences satisfying the $\Delta_{1/2}$-condition and characterizing regularity, see [3, Theorem 3.10], [7, Proposition 4.14 (proof)-Corollary 4.15]. Their usefulness derives in part from the inversion formula (3) in the next proposition.

PROPOSITION 3.2. *There is a one-to-one correspondence between the sequences* $\xi \in c_o^* \setminus \Sigma(F)$ *with* $\xi_1 = 1$ *and the sequences* $\langle r_n \rangle$ *with* $r_1 = 1$, $(n+1)r_{n+1} \geqslant nr_n + 1$ *for all* n *and* $\log r_n + \sum_2^n \frac{1}{j}(1 - \frac{1}{r_j}) \to \infty$. *This correspondence is given by the map* $\xi \to r(\xi) := \frac{\xi_a}{\xi}$ *with inverse map* $r \to \xi$ *given by*

$$\xi_n = \begin{cases} 1 & \text{for } n = 1 \\ \frac{1}{nr_n} \prod_2^n (1 + \frac{1}{jr_j - 1}) = \frac{1}{nr_n \prod_2^n (1 - \frac{1}{jr_j})} & \text{for } n > 1. \end{cases} \tag{3}$$

Proof. The monotonicity of ξ is equivalent to the inequality for r_n. Indeed, first assume ξ is a monotone nonincreasing sequence with $\xi_1 = 1$ and $\xi_n > 0$ for all n.

Let $r := r(\xi) = \frac{\xi_a}{\xi}$. Then $r_1 = \frac{(\xi_a)_1}{\xi_1} = 1$ and

$$(n+1)r_{n+1}\xi_{n+1} = \sum_1^{n+1} \xi_i = \sum_1^n \xi_i + \xi_{n+1} = nr_n\xi_n + \xi_{n+1},$$

for all n, i.e., the sequences r and ξ satisfy the recurrence relation

$$((n+1)r_{n+1} - 1)\xi_{n+1} = nr_n\xi_n, \quad \text{with } \xi_1 = 1 \text{ and } r_1 = 1. \tag{4}$$

By the monotonicity of ξ,

$$((n+1)r_{n+1} - 1)\xi_{n+1} \geqslant nr_n\xi_{n+1}$$

and hence $(n+1)r_{n+1} \geqslant nr_n + 1$, since $\xi_{n+1} > 0$ for all n.

Conversely, assume that $r = \langle r_n \rangle$ is a sequence with $r_1 = 1$ and $(n+1)r_{n+1} \geqslant nr_n + 1$ for all n and ξ is the sequence given by (3). Then

$$\frac{\xi_{n+1}}{\xi_n} = \frac{nr_n}{(n+1)r_{n+1}} \left(1 + \frac{1}{(n+1)r_{n+1} - 1}\right) = \frac{nr_n}{(n+1)r_{n+1} - 1} \leqslant 1,$$

i.e., ξ is monotone nonincreasing. Moreover, it is easy to verify that ξ and r also satisfy the recurrence relation (4) and hence that $r(\xi) = r$.

It remains to prove that the limit condition on r is necessary and sufficient for $\xi_n \to 0$. Clearly, $\xi_n \to 0$ if and only if

$$\log nr_n - \sum_2^n \log\left(1 + \frac{1}{jr_j - 1}\right) \to \infty. \tag{5}$$

Equivalently,

$$\log r_n + \sum_2^n \left(\frac{1}{j} - \log\left(1 + \frac{1}{jr_j - 1}\right)\right) \to \infty.$$

Since $\frac{1}{x} < \log\left(1 + \frac{1}{x-1}\right) < \frac{1}{x-1}$ for $x > 1$ and hence

$$\sum_2^n \left(\frac{1}{j} - \frac{1}{jr_j}\right) - \sum_2^n \frac{1}{j(j-1)} \leqslant \sum_2^n \left(\frac{1}{j} - \frac{1}{jr_j - 1}\right) < \sum_2^n \left(\frac{1}{j} - \log\left(1 + \frac{1}{jr_j - 1}\right)\right)$$

$$< \sum_2^n \left(\frac{1}{j} - \frac{1}{jr_j}\right),$$

it follows that (5) is equivalent to

$$\log r_n + \sum_2^n \frac{1}{j}\left(1 - \frac{1}{r_j}\right) \to \infty. \qquad \square \tag{6}$$

REMARK 3.3.   Since the sequence in (5) is monotone nondecreasing, sufficient conditions for (6) are $\sup r = \infty$ or $\sum_2^\infty \frac{1}{jr_j} < \infty$. A necessary and sufficient condition for (6) is that $\sup r = \infty$ or $\sum_2^\infty \frac{1}{j}(1 - \frac{1}{r_j}) = \infty$. (These conditions are not mutually exclusive.)

Lower bound estimates for the rate of decrease of $\frac{\xi_n}{\xi_m}$ for $n \geq m$ are an important tool in the subject. For instance, the Potter-type characterization of a regular sequence states that ξ is regular if and only if $\xi_n \geq \left(\frac{m}{n}\right)^p \xi_m$ for some $0 < p < 1$ and for all $n \geq m$ (see for instance [3, Proposition 4.14]). Since $n(\eta_a)_n$ is monotone increasing for every $\eta \in c_o^*$ (strictly increasing if and only if $\eta_n > 0$ for all n, i.e., if $\eta \notin \Sigma(F)$), it follows that $(\eta_a)_n \geq \left(\frac{m}{n}\right)(\eta_a)_m$, for all $n \geq m$. The next lemma provides both an identity and an upper bound estimate for the rate of decrease of $\frac{(\eta_a)_n}{(\eta_a)_m}$ for $n \geq m$, both of which are needed here.

LEMMA 3.4.   Let $\xi \in c_o^* \setminus \Sigma(F)$ and let $r = r(\xi)$.
(i) For all $n > m$,

$$(\xi_a)_n = \frac{m}{n} \prod_{j=m+1}^n \left(1 + \frac{1}{jr_j - 1}\right)(\xi_a)_m = \frac{\frac{m}{n}}{\prod_{j=m+1}^n \left(1 - \frac{1}{jr_j}\right)}(\xi_a)_m.$$

(ii) If ϕ is a nondecreasing strictly positive sequence with $\phi \leq r$, then

$$(\xi_a)_n \leq \left(\frac{m}{n}\right)^{1 - \frac{2}{\phi_m}}(\xi_a)_m$$

for every $m \in \mathbb{N}$ sufficiently large so that $m\phi_m > 2$, and for every $n \geq m$.

Proof.
(i) By applying (3) to $\frac{1}{\xi_1}\xi$ we have

$$\xi_n = \frac{1}{nr_n} \prod_{j=2}^n \left(1 + \frac{1}{jr_j - 1}\right)\xi_1 = \frac{mr_m}{nr_n} \prod_{j=m+1}^n \left(1 + \frac{1}{jr_j - 1}\right)\xi_m$$

$$= \frac{m}{n} \frac{\xi_n}{(\xi_a)_n} \prod_{j=m+1}^n \left(1 + \frac{1}{jr_j - 1}\right)(\xi_a)_m.$$

(ii) Equality holds trivially for $n = m$, and for $n > m$ one has

$$(\xi_a)_n = \frac{\frac{m}{n}}{\prod_{j=m+1}^n \left(1 - \frac{1}{jr_j}\right)}(\xi_a)_m \qquad \text{by (i)}$$

$$\leq \frac{\frac{m}{n}}{\prod_{j=m+1}^n \left(1 - \frac{1}{j\phi_j}\right)}(\xi_a)_m \qquad \text{since } \phi \leq r$$

$$\leq \frac{\frac{m}{n}}{\prod_{j=m+1}^n \left(1 - \frac{1}{j\phi_m}\right)}(\xi_a)_m \qquad \text{because } \phi \text{ is nondecreasing}$$

$$= \frac{m}{n} e^{-\sum_{j=m+1}^{n} \log\left(1 - \frac{1}{j\phi_m}\right)} (\xi_a)_m$$

$$\leqslant \frac{m}{n} e^{\sum_{j=m+1}^{n} \frac{2}{j\phi_m}} (\xi_a)_m \qquad \text{by the inequality } \log(1 - x) \geqslant -2x \text{ for } 0 \leqslant x < \frac{1}{2},$$

$$= \frac{m}{n} e^{\frac{2}{\phi_m}(H_n - H_m)} (\xi_a)_m$$

$$\leqslant \frac{m}{n} e^{\frac{2}{\phi_m} \log \frac{n}{m}} (\xi_a)_m \qquad \text{by (2)}$$

$$= \left(\frac{m}{n}\right)^{1 - \frac{2}{\phi_m}} (\xi_a)_m. \qquad \square$$

Notice that in general, while the ratio of regularity $\frac{\xi_a}{\xi}$ has downward variations bounded by the inequality $r_{n+1} \geqslant \frac{n}{n+1} r_n + \frac{1}{n+1}$, it can vary abruptly upwards since

$$r(\xi)_{n+1} = \frac{(\xi_a)_{n+1}}{\xi_{n+1}} = \frac{(n+1)(\xi_a)_n}{(n+1)\xi_{n+1}} \geqslant \frac{n(\xi_a)_n}{(n+1)\xi_{n+1}} = \frac{n}{n+1} \frac{\xi_n}{\xi_{n+1}} r(\xi)_n$$

and $\frac{\xi_n}{\xi_{n+1}}$ can be arbitrarily large.

Sequences in c_o^* that are an arithmetic mean of another sequence in c_o^* are however smoother and their ratios of regularity are subject to "slower" upward variations. Indeed, if $\xi = \eta_a$ for some $\eta \in c_o^*$, the upward variation of $\frac{\xi_a}{\xi} = \frac{\eta_{a^2}}{\eta_a}$ is limited by the inequality

$$r(\eta_a)_{n+1} = \frac{(n+1)(\eta_{a^2})_{n+1}}{(n+1)(\eta_a)_{n+1}} \leqslant \frac{(n+1)(\eta_{a^2})_n}{n(\eta_a)_n} = \left(1 + \frac{1}{n}\right) r(\eta_a)_n. \qquad (7)$$

As is easy to verify (cf. [8, Lemma 2.12]), a sequence $\xi \in c_o^*$ is the arithmetic mean of another sequence in c_o^* if and only if $\frac{\xi}{\omega}$ is nondecreasing and concave, i.e.,

$$2n\xi_n \geqslant (n+1)\xi_{n+1} + (n-1)\xi_{n-1} \quad \text{for all } n > 1.$$

Define the *concavity ratio* $c(\xi)$ of a sequence $\xi \in c_o^* \setminus \Sigma(F)$ to be

$$c(\xi)_n := \frac{n\xi_n}{(n+1)\xi_{n+1}}. \qquad (8)$$

Concavity of $\frac{\xi}{\omega}$ is equivalent to the condition: $c(\xi)_n + \frac{1}{c(\xi)_{n+1}} \leqslant 2$. Since we are not going to make use of the concavity ratio beyond a slight improvement in inequality (7) for Corollary 3.7(ii), we will only sketch briefly the proofs of the next proposition (which is an analog of Proposition 3.2) and its Corollaries 3.6–3.7.

PROPOSITION 3.5. *There is a one-to-one correspondence between the sequences* $\xi \in c_o^* \setminus \Sigma(F)$ *with* $\xi_1 = 1$ *and the sequences* $\langle c_n \rangle$ *with* $c_n \geqslant 1 - \frac{1}{n+1}$ *and* $\sum_1^{\infty} \left(\frac{1}{j} + \log c_j\right) = \infty$. *This correspondence is established by the map* $\xi \to c(\xi) := \left\langle \frac{n\xi_n}{(n+1)\xi_{n+1}} \right\rangle$ *with inverse map*

$$\xi_n = \begin{cases} 1 & \text{for } n = 1 \\ \frac{1}{n\prod_1^{n-1} c_j} & \text{for } n > 1. \end{cases} \qquad (9)$$

Moreover,

$$c(\xi)_n = \frac{r(\xi)_{n+1} - \frac{1}{n+1}}{r(\xi)_n} \quad and \quad r(\xi)_n = \begin{cases} 1 & \text{for } n = 1 \\ \frac{1}{n} + \sum_{k=1}^{n-1} \frac{1}{k} \prod_{j=k}^{n-1} c(\xi)_j & \text{for } n > 1. \end{cases}$$

Proof. It is straightforward to verify that (9) provides the inverse of (8) and that the monotonicity of ξ is equivalent to the condition $c_n \geqslant 1 - \frac{1}{n+1}$. Also, $\xi_n \to 0$ if and only if

$$\log \frac{1}{\xi} = \log n + \sum_{1}^{n-1} \log c_j \to \infty$$

which by (1) is equivalent to $\sum_{1}^{\infty} \left(\frac{1}{j} + \log c_j \right) = \infty$. The remaining equations are simple computations. $\square$

COROLLARY 3.6. *There is a one-to-one correspondence between the sequences $\xi = \eta_a$ with $\eta \in c_o^*$, $\eta_1 = 1$ and the sequences $\langle c_n \rangle$ with $c_1 \geqslant \frac{1}{2}$, $c_n > 0$, $c_n + \frac{1}{c_{n+1}} \leqslant 2$ for all $n \in \mathbb{N}$ and $\sum_{1}^{\infty} \left(\frac{1}{j} + \log c_j \right) = \infty$. This correspondence is established by the map $\xi \to c(\xi)$ (Equation (8)). Moreover, $c(\xi)_n \uparrow 1$ and*

$$\eta_n = \begin{cases} 1 & \text{for } n = 1 \\ \frac{1 - c(\xi)_{n-1}}{\prod_{1}^{n-1} c(\xi)_j} & \text{for } n > 1. \end{cases} \quad and \quad r(\eta)_n = \begin{cases} 1 & \text{for } n = 1 \\ \frac{1}{n(1 - c(\xi)_{n-1})} & \text{for } n > 1. \end{cases}$$

Proof. It is an immediate consequence of the above mentioned concavity of $\frac{\xi}{\omega}$ and of Proposition 3.5 that the sequence $c(\xi)$ satisfies the stated inequalities and the series condition when $\xi = \eta_a$ and $\eta \in c_o^*$. Conversely, it is straightforward to show that if $\langle c_n \rangle$ satisfies these conditions then it is nondecreasing, its limit is 1 and an easy induction shows that $c_n \geqslant 1 - \frac{1}{n+1}$. Thus again by [ibid], $c = c(\xi)$ for the sequence ξ given by (9), which then implies that $\frac{\xi}{\omega}$ is nondecreasing. And by the comment after Equation (8), it is also concave, hence $\xi = \eta_a$ for some $\eta \in c_o^*$. The remaining claims are also easy to verify directly. $\square$

Combining Proposition 3.5 and Corollary 3.6 (wherein $c(\xi)_n \leqslant 1$) we obtain

COROLLARY 3.7. *For $\xi \in c_o^* \setminus \Sigma(F)$ and $\xi_1 = 1$,*
(i) $\xi = \eta_a$ for some $\eta \in c_o^*$ if and only if

$$\frac{r(\xi)_n - \frac{1}{n}}{r(\xi)_{n-1}} + \frac{r(\xi)_n}{r(\xi)_{n+1} - \frac{1}{n+1}} \leqslant 2 \quad (n > 1).$$

(ii) $r(\eta_a)_{n+1} \leqslant r(\eta_a)_n + \frac{1}{n+1} < \left(1 + \frac{1}{n}\right) r(\eta_a)_n \quad (n \geqslant 1)$

From Corollary 3.7(ii) one sees immediately that $r(\eta_a)_n \leqslant H_n$ for all n. Of interest is a direct proof of this fact that avoids ratios of concavity considerations.

LEMMA 3.8. *Let $0 \neq \eta \in c_o^*$.*
(i) $r(\eta_a)_1 = H_1 = 1$ *and if* $\eta_2 > 0$, *then* $r(\eta_a)_n < H_n$ *for all* $n > 1$.
(ii) *If* $\eta_n > 0$ *for all* n, *then* $H_n - r(\eta_a)_n$ *is strictly increasing.*
(iii) *If* η *is not summable, then* $\lim(H_n - r(\eta_a)_n) = \infty$.

Proof.
(i) $H_1 = (\frac{\eta_{a^2}}{\eta_a})_1 = 1$. For $n > 1$,

$$
\begin{aligned}
n(\eta_{a^2})_n &= \sum_{j=1}^{n} \frac{1}{j} \sum_{i=1}^{j} \eta_i = \sum_{i=1}^{n} \sum_{j=i}^{n} \frac{\eta_i}{j} = \eta_1 H_n + \sum_{i=2}^{n} \eta_i(H_n - H_{i-1}) \\
&= H_n \sum_{i=1}^{n} \eta_i - \sum_{i=2}^{n} \eta_i H_{i-1} = H_n n(\eta_a)_n - \sum_{i=2}^{n} \eta_i H_{i-1}.
\end{aligned}
$$

Thus

$$
H_n - r(\eta_a)_n = \frac{\sum_{i=2}^{n} \eta_i H_{i-1}}{\sum_{i=1}^{n} \eta_i} > 0 \tag{10}
$$

by the assumption that $\eta_2 > 0$.

(ii) Using (10),

$$
\begin{aligned}
H_{n+1} - \left(\frac{\eta_{a^2}}{\eta_a}\right)_{n+1} - \left(H_n - \left(\frac{\eta_{a^2}}{\eta_a}\right)_n\right) &= \frac{\sum_{i=2}^{n+1} \eta_i H_{i-1}}{\sum_{i=1}^{n+1} \eta_i} - \frac{\sum_{i=2}^{n} \eta_i H_{i-1}}{\sum_{i=1}^{n} \eta_i} \\
&= \eta_{n+1} \frac{H_n(\sum_{i=1}^{n} \eta_i) - \sum_{i=2}^{n} \eta_i H_{i-1}}{(\sum_{i=1}^{n+1} \eta_i)(\sum_{i=1}^{n} \eta_i)} \\
&= \eta_{n+1} \frac{n(\eta_{a^2})_n}{(\sum_{i=1}^{n+1} \eta_i)(\sum_{i=1}^{n} \eta_i)} \\
&= \frac{(\frac{\eta_{a^2}}{\eta_a})_n}{(n+1)(\frac{\eta_a}{\eta})_{n+1}} > 0,
\end{aligned}
$$

where the third equality was obtained in (i).

(iii) Elementary from (10) since $H_n \to \infty$. □

This lemma tells us that there are two extreme cases for the ratio of regularity for η_a: when $r(\eta_a) \asymp 1$ (i.e., η_a is regular) and when $r(\eta_a) \asymp \log$ (by which we mean more precisely: $\alpha \log n \leqslant r(\eta_a)_n \leqslant \beta \log n$ for some $\alpha, \beta > 0$ and all $n \geqslant 2$). As we will see in the next section, both cases play a special role for second order arithmetic mean cancellation.

First we obtain some elementary comparisons between the ratios of regularity for η and for η_a evaluated at pairs of indices.

LEMMA 3.9. *Let* $\eta \in c_o^* \setminus \Sigma(F)$.
(i) $\frac{m}{n}((\eta_a)_m - \eta_n) + \eta_n \leqslant (\eta_a)_n \leqslant \frac{m}{n}((\eta_a)_m - \eta_m) + \eta_m \ \forall n \geqslant m$.
(ii) $\frac{m}{n}((\eta_{a^2})_m + (H_n - H_m)(\eta_a)_m) \leqslant (\eta_{a^2})_n \leqslant \frac{m}{n}(\eta_{a^2})_m + (H_n - H_m)(\eta_a)_n \ \forall n \geqslant m$.
(iii) *Let* $n = [m(\frac{\eta_a}{\eta})_m]$ *(integer part). Then* $r(\eta_a)_n > \frac{1}{2} \log r(\eta)_m$.

Proof.

(i) The inequalities are identities for $n = m$. For $n > m$,

$$n(\eta_a)_n = \sum_1^n \eta_j = \sum_1^m \eta_j + \sum_{m+1}^n \eta_j \leqslant m(\eta_a)_m + (n - m)\eta_m,$$

and also

$$n(\eta_a)_n \geqslant m(\eta_a)_m + (n - m)\eta_n.$$

(ii) The inequalities are identities for $n = m$. For $n > m$,

$$
\begin{aligned}
n(\eta_{a^2})_n &= m(\eta_{a^2})_m + \sum_{m+1}^n \frac{1}{j} j(\eta_a)_j \\
&\leqslant m(\eta_{a^2})_m + \sum_{m+1}^n \frac{1}{j} n(\eta_a)_n \\
&= m(\eta_{a^2})_m + (H_n - H_m)n(\eta_a)_n,
\end{aligned}
$$

and also

$$
\begin{aligned}
n(\eta_{a^2})_n &\geqslant m(\eta_{a^2})_m + \sum_{m+1}^n \frac{1}{j} m(\eta_a)_m \\
&= m(\eta_{a^2})_m + (H_n - H_m)m(\eta_a)_m.
\end{aligned}
$$

(iii) By the definition of n, $\frac{n+1}{m} > (\frac{\eta_a}{\eta})_m \geqslant \frac{n}{m}$. Hence by (i), (ii), (2), and the trivial inequality $(\eta_{a^2})_m \geqslant (\eta_a)_m$,

$$
\begin{aligned}
\left(\frac{\eta_{a^2}}{\eta_a}\right)_n &> \frac{m\left((\eta_{a^2})_m + (\eta_a)_m \log \frac{n+1}{m+1}\right)}{m(\eta_a)_m + (n - m)\eta_m} \\
&> \frac{(\eta_a)_m \left(1 + \log \frac{n+1}{m+1}\right)}{(\eta_a)_m + \frac{n}{m}\eta_m} \\
&> \frac{\log \frac{n+1}{m}}{2} \\
&> \frac{1}{2}\log\left(\frac{\eta_a}{\eta}\right)_m. \qquad \square
\end{aligned}
$$

This proof evolved from work of K. Davidson with the second named author. The original proof was obtained by the authors using ratios of concavity.

Lemma 3.9 provides a direct and quantitative proof for the following result (proven implicitly in [16, Theorem IRR] and explicitly in [3, Theorem 3.10]; see also [ibid, Remark 3.11]).

COROLLARY 3.10. *For $\eta \in c_o^*$, η is regular if and only if η_a is regular.*

Now we consider the second "extreme" case, namely when $r(\eta_a) \asymp \log$, meaning in the sense that, except for $n = 1$, $\alpha \log n \leqslant r(\eta_a)_n \leqslant \beta \log n$ for some $\alpha, \beta > 0$.

PROPOSITION 3.11.   *Let* $0 \neq \eta \in c_0^*$. *Then* $r(\eta_a) \asymp \log$ *if and only if* $\sup\limits_m \frac{m^2(\eta_a)_{m^2}}{m(\eta_a)_m} < \infty$.

Proof. Assume $\gamma := \sup\limits_m \frac{m^2(\eta_a)_{m^2}}{m(\eta_a)_m} < \infty$. For all $n \geqslant m$, it follows from Lemma 3.9(ii) that

$$(\eta_{a^2})_n > \frac{m}{n}(H_n - H_m)(\eta_a)_m$$

and hence by (2)

$$\left(\frac{\eta_{a^2}}{\eta_a}\right)_n > \frac{m}{n}\frac{(\eta_a)_m}{(\eta_a)_n}(H_n - H_m) > \frac{m}{n}\frac{(\eta_a)_m}{(\eta_a)_n}\log\frac{n+1}{m+1}$$

Set $m = [\sqrt{n}] + 1$. Then

$$
\begin{aligned}
\left(\frac{\eta_{a^2}}{\eta_a}\right)_n &> \frac{m(\eta_a)_m}{m^2(\eta_a)_{m^2}}\frac{m^2(\eta_a)_{m^2}}{n(\eta_a)_n}\log\frac{n+1}{\sqrt{n}+1} \\
&\geqslant \frac{1}{\gamma}\frac{m^2(\eta_a)_{m^2}}{n(\eta_a)_n}\log\frac{n+1}{\sqrt{n}+1} \qquad \text{by the definition of } \gamma \\
&\geqslant \frac{1}{\gamma}\log\frac{n+1}{\sqrt{n}+1} \qquad \text{by the monotonicity of } k(\eta_a)_k \text{ since } m^2 > n \\
&\geqslant \frac{1}{3\gamma}\log n \qquad \text{for } n \text{ sufficiently large.}
\end{aligned}
$$

On the other hand, by Lemma 3.8(i) and (2), $r(\eta_a)_n < H_n < \log n + 1$, so $r(\eta_a) \asymp \log$.

Conversely, assume $r(\eta_a) \asymp \log$, i.e., $\alpha \log n \leqslant r(\eta_a)_n \leqslant \beta \log n$ for some $\alpha, \beta > 0$ and all $n \geqslant 2$. Then

$$\frac{m^2(\eta_a)_{m^2}}{m(\eta_a)_m} = \frac{\left(\frac{\eta_{a^2}}{\eta_a}\right)_m}{\left(\frac{\eta_{a^2}}{\eta_a}\right)_{m^2}}\frac{m^2(\eta_{a^2})_{m^2}}{m(\eta_{a^2})_m} \leqslant \frac{\beta}{\alpha}\frac{\log m}{\log m^2}\frac{m^2(\eta_{a^2})_{m^2}}{m(\eta_{a^2})_m} = \frac{\beta}{2\alpha}\frac{m^2(\eta_{a^2})_{m^2}}{m(\eta_{a^2})_m}$$

and similarly,

$$\frac{m^2(\eta_a)_{m^2}}{m(\eta_a)_m} \geqslant \frac{\alpha}{2\beta}\frac{m^2(\eta_{a^2})_{m^2}}{m(\eta_{a^2})_m},$$

i.e.,

$$\frac{m^2(\eta_a)_{m^2}}{m(\eta_a)_m} \asymp \frac{m^2(\eta_{a^2})_{m^2}}{m(\eta_{a^2})_m}. \tag{11}$$

Now, from Lemma 3.4(ii) applied to $\xi = \eta_a$, $\phi = \alpha \log$, and $n = m^2$, we have

$$\frac{m^2(\eta_{a^2})_{m^2}}{m(\eta_{a^2})_m} \leqslant m\left(\frac{m}{m^2}\right)^{1 - \frac{2}{\alpha\log m}} = e^{\frac{2}{\alpha}}. \qquad \square$$

By analogy to the Δ_2-condition for monotone sequences ($\sup_m \frac{\phi(2m)}{\phi(m)} < \infty$), we say that η satisfies the *exponential* Δ_2*-condition* if $\sup_m \frac{m^2(\eta_a)_{m^2}}{m(\eta_a)_m} < \infty$, i.e., $\sum_{i=1}^{m^2} \eta_i \leqslant c \sum_{i=1}^m \eta_i$, for some $c > 0$ and all m.

An obvious consequence of (11) is:

COROLLARY 3.12. *Let $0 \neq \eta \in c_o^*$. If η satisfies the exponential Δ_2-condition then η_{a^p} satisfies the exponential Δ_2-condition for every $p \in \mathbb{N}$.*

EXAMPLE 3.13. Among the c_o^*-sequences that satisfy the exponential Δ_2-condition are all summable sequences, all sequences $\langle \frac{\log^p n}{n} \rangle$, $\langle \frac{\log^p n(\log \log n)^q}{n} \rangle$, etc. (starting from wherever they become monotone decreasing). Among sequences that do not satisfy the exponential Δ_2-condition are all regular sequences, and the sequences in Examples 4.3 and 4.7. Likewise for the sequence (starting from wherever it becomes monotone decreasing)

$$\eta := \left\langle \frac{e^{\int_{e^e}^n \frac{1}{t \log \log t}}}{n \log^2 \log n} \right\rangle \quad \text{for which} \quad r(\eta_a) \asymp \log \log n.$$

We skip the verification of the stated properties for these sequences.

This last example shows that the condition that $r(\eta_a)$ be equivalent to a monotone sequence (see Theorem 4.5) is more general than that $r(\eta_a)$ be equivalent to $\log$ or 1 (see Theorems 4.6 and 4.8).

Both regular sequences and sequences that satisfy the exponential Δ_2-condition are special cases of sequences η for which $r(\eta_a)$ is equivalent to a monotone sequence. In that case, since $r(\eta_a) \geqslant 1$, $r(\eta_a)$ is either equivalent to a sequence increasing to infinity or to a constant sequence (when η is regular); in either case, it is equivalent to a nondecreasing sequence so that Lemma 3.4(ii) applies.

LEMMA 3.14. *If $0 \neq \eta \in c_o^*$ and $r(\eta_a)$ is equivalent to a monotone sequence, then there are constants $K, M > 0$ for which, if $m \geqslant M$, then $(\eta_{a^2})_n \leqslant K \frac{m}{n} \log \frac{n}{m} (\eta_a)_m$ for some $n > m$.*

Proof. Assume without loss of generality that $r(\eta_a)$ is equivalent to a monotone nondecreasing sequence ϕ and assume for simplicity's sake that $\phi \leqslant r(\eta_a) \leqslant \beta \phi$ for some $\beta > 0$. Let $M \in \mathbb{N}$ be an integer for which $M \phi_M > 2$. Then for all $n \geqslant m \geqslant M$, by Lemma 3.4(ii) applied to η_a,

$$(\eta_{a^2})_n \leqslant \left(\frac{m}{n}\right)^{1-\frac{2}{\phi_m}} (\eta_{a^2})_m \leqslant \beta \phi_m \left(\frac{m}{n}\right)^{1-\frac{2}{\phi_m}} (\eta_a)_m = \beta \phi_m \frac{m}{n} e^{\frac{2 \log \frac{n}{m}}{\phi_m}} (\eta_a)_m.$$

Choosing $n := [me^{\phi_m}]$ one has $\frac{n}{m} \leqslant e^{\phi_m} < \frac{n+1}{m}$ so $\phi_m < \log \frac{n+1}{m} < 2 \log \frac{n}{m} \leqslant 2\phi_m$. Thus

$$(\eta_{a^2})_n \leqslant 2\beta e^2 \frac{m}{n} \log \frac{n}{m} (\eta_a)_m. \qquad \square$$

4.   Arithmetic mean cancellations of second order

First order lower arithmetic mean cancellation characterizes am-closed ideals, i.e., ideals I for which $I = I^- := {}_a(I_a)$. Indeed for a fixed ideal I,

$$J_a \subset I_a \Rightarrow J \subset I \quad \text{if and only if} \quad I = {}_a(I_a) \quad [9, \text{Lemma } 6.1(\text{C})].$$

The second order analog of this property involves second order am-closure, ${}_{a^2}(I_{a^2})$.

PROPOSITION 4.1. *For a fixed ideal I,*

$$J_{a^2} \subset I_{a^2} \Rightarrow J_a \subset I_a \quad \text{if and only if} \quad I^- = {}_{a^2}(I_{a^2}).$$

Proof. Necessity follows from the general identity $I_{a^2} = ({}_{a^2}(I_{a^2}))_{a^2}$ (see Section 2) which implies from the hypothesis that $({}_{a^2}(I_{a^2}))_a \subset I_a$ and hence

$$I^- = {}_a(I_a) \supset {}_a(({}_{a^2}(I_{a^2}))_a) = {}_a(({}_a[{}_a(I_{a^2})])_a) = {}_a[{}_a(I_{a^2})] = {}_{a^2}(I_{a^2}) \supset I^-$$

where the third equality holds from the general identity ${}_a(({}_aL)_a) = {}_aL$ and the last inclusion also holds for any ideal. Conversely, if $I^- = {}_{a^2}(I_{a^2})$ and $J_{a^2} \subset I_{a^2}$, then $J \subset {}_{a^2}(J_{a^2}) \subset {}_{a^2}(I_{a^2})$ and hence

$$J_a \subset ({}_{a^2}(I_{a^2}))_a = (I^-)_a = ({}_a(I_a))_a = I_a. \qquad \square$$

REMARK 4.2.

(i) The last step in the above proof shows that if $I^- = {}_{a^2}(I_{a^2})$, then $I_a = (I_a)^{-o}$. It is easy to see that the converse also holds, i.e., that $I^- = {}_{a^2}(I_{a^2})$ if and only if $I_a = (I_a)^{-o}$.

(ii) Since $I \subset I^- \subset {}_{a^2}(I_{a^2})$ for arbitrary ideals I, if $I = {}_{a^2}(I_{a^2})$ then $I^- = {}_{a^2}(I_{a^2})$. The converse implication fails in general. For instance, if L is the countably generated ideal provided by [9, Example 5.5] for which $L \subsetneq L_a = L_{a^2}$, then ${}_{a^2}(L_{a^2}) = L_a$ and hence $L \neq {}_{a^2}(L_{a^2})$ but $L^- = L_a = {}_{a^2}(L_{a^2})$. Furthermore, the converse implication can fail even when I is principal. Indeed, if $I = (\eta)$ is not am-stable, then by [8, Theorem 2.11], it is not am-closed, hence $I \neq {}_{a^2}(I_{a^2})$ (recall that ${}_aJ$ is am-closed for every ideal J). However by Theorem 4.5(i) below, if $r(\eta_a)$ is equivalent to a monotone sequence (e.g., $\eta = \omega$ hence $r(\eta_a) \asymp \log$), then $I^- = {}_{a^2}(I_{a^2})$ by Proposition 4.1.

The condition $I^- = {}_{a^2}(I_{a^2})$ is not very "transparent," not even for principal ideals. A natural question is whether this condition might be automatically satisfied for all ideals. As is easy to see (cf. proof of Theorem 4.5 below), if second order inclusion cancellation were to hold for all pairs of principal ideals, it would hold also for all pairs of general ideals. Furthermore, if equality cancellation were to hold for all pairs of principal ideals, inclusion cancellation would then also hold for all pairs of principal ideals. Indeed, given two principal ideals I and J and setting $L := I + J$, we see that $J_{a^2} \subset I_{a^2}$ is equivalent to $L_{a^2} = I_{a^2}$ which would then imply $L_a = I_a$ and hence $I_a \supset J_a$.

It is trivial to see that first order equality cancellation can fail to hold for pairs of principal ideals, e.g., because all nonzero principal ideals contained in the trace class have the same arithmetic mean ideal (ω).

Whether second order equality cancellation holds for all pairs of principal ideals or not is indeed a reformulation of a question asked by M. Wodzicki. The following example answers this question in the negative.

We need first the following identities for "step sequences."

Let $\zeta \in c_o^*$ be a step sequence based on a strictly increasing sequence of indices m_k starting with $m_0 = 0$, i.e., $\zeta_j = \epsilon_k$ for $m_k < j \leqslant m_{k+1}$ for some strictly decreasing sequence $\epsilon_k \to 0$. We will need the following formulas for the sequences ζ_a and ζ_{a^2}:

$$j(\zeta_a)_j = \sum_{1}^{j} \zeta_j = \begin{cases} j\epsilon_o & \text{for } 0 < j \leqslant m_1 \\ m_k(\zeta_a)_{m_k} + (j - m_k)\epsilon_k & \text{for } m_k < j \leqslant m_{k+1} \text{ and } k \geqslant 1 \end{cases}$$

hence

$$(\zeta_a)_j = \begin{cases} \epsilon_o & \text{for } 0 < j \leqslant m_1 \\ \frac{m_k}{j}((\zeta_a)_{m_k} - \epsilon_k) + \epsilon_k & \text{for } m_k < j \leqslant m_{k+1} \text{ and } k \geqslant 1. \end{cases} \tag{12}$$

Consequently, for $k > 0$ and $m_k < j \leqslant m_{k+1}$,

$$\begin{aligned} j(\zeta_{a^2})_j &= m_k(\zeta_{a^2})_{m_k} + \sum_{i=m_k+1}^{j} \left(\frac{m_k}{i}((\zeta_a)_{m_k} - \epsilon_k) + \epsilon_k \right) \\ &= m_k(\zeta_{a^2})_{m_k} + m_k\left((\zeta_a)_{m_k} - \epsilon_k \right)(H_j - H_{m_k}) + (j - m_k)\epsilon_k, \end{aligned}$$

and for $k = 0$ and $0 < j \leqslant m_1$, $j(\zeta_{a^2})_j = j\epsilon_0$. Therefore

$$(\zeta_{a^2})_j = \begin{cases} \epsilon_o & \text{for } 0 < j \leqslant m_1 \\ \frac{m_k}{j}\left((\zeta_{a^2})_{m_k} - \epsilon_k + ((\zeta_a)_{m_k} - \epsilon_k)(H_j - H_{m_k}) \right) + \epsilon_k & \text{for } m_k < j \leqslant m_{k+1}. \end{cases} \tag{13}$$

EXAMPLE 4.3. Principal ideals $J \subset I$ for which $J_{a^2} = I_{a^2}$ but $J_a \neq I_a$.

Construction. To define the principal ideals it suffices to provide sequences ξ and η in c_o^* to generate them with the properties that $\xi \leqslant \eta$ and $\xi_{a^2} \asymp \eta_{a^2}$ but for which $\eta_a \neq O(\xi_a)$ (see Section 2 on principal ideals). Construct inductively an increasing sequence of positive integers m_k with $m_1 = 1$, then define the sequence of indices $n_k := [e^{k^2} m_k]$ and the sequence δ_k defined recursively from $\delta_1 = 1$ and $\delta_{k+1} = e^{-k^2}\delta_k$, i.e., $\delta_k = e^{-\sum_{p=1}^{k-1} p^2}$ for $k > 1$. Now, using the sequences m_k, n_k, and δ_k, define the two monotone sequences ξ and η setting $\xi_1 = \eta_1 = 1$ and for every $k \geqslant 1$,

$$\xi_j := e^{-k^2}\delta_k \text{ for } m_k < j \leqslant m_{k+1}$$

and

$$\eta_j := \begin{cases} ke^{-k^2}\delta_k & \text{for } m_k < j \leqslant n_k \\ e^{-k^2}\delta_k & \text{for } n_k < j \leqslant m_{k+1}. \end{cases}$$

Notice that $\eta_{m_k} = \xi_{m_k} = \delta_k$ for every k. Clearly, $\xi \leqslant \eta$ and both sequences are in c_o^*. Assume the construction of the sequence m_k up to $k \geqslant 1$ and choose $m_{k+1} > n_k$ and sufficiently large to insure that $(\eta_{a^2})_{m_{k+1}} \leqslant (1 + \frac{1}{k})e^{-k^2}\delta_k$, which can be achieved using Equation (13). As a consequence,

$$\delta_k = \eta_{m_k} = \xi_{m_k} \leqslant (\xi_a)_{m_k} \leqslant (\eta_a)_{m_k} \leqslant (\eta_{a^2})_{m_k} \leqslant (1 + \frac{1}{k})\delta_k \ \text{ and} \qquad (14)$$

$$\delta_k \leqslant (\xi_a)_{m_k} \leqslant (\xi_{a^2})_{m_k} \leqslant (1 + \frac{1}{k})\delta_k$$

and hence

$$(\xi_a)_{m_k} \sim (\eta_a)_{m_k} \sim (\xi_{a^2})_{m_k} \sim (\eta_{a^2})_{m_k} \sim \delta_k \qquad (15)$$

where $\sim$ denotes asymptotic. From (12), (15) and

$$\frac{m_k}{n_k} = e^{-k^2}\frac{e^{k^2}m_k}{[e^{k^2}m_k]} \sim e^{-k^2} \qquad (16)$$

we have

$$\frac{(\xi_a)_{n_k}}{\delta_k} = \frac{m_k}{n_k}\left(\frac{(\xi_a)_{m_k}}{\delta_k} - e^{-k^2}\right) + e^{-k^2} \sim 2e^{-k^2} \qquad (17)$$

and

$$\frac{(\eta_a)_{n_k}}{\delta_k} = \frac{m_k}{n_k}\left(\frac{(\xi_a)_{m_k}}{\delta_k} - ke^{-k^2}\right) + ke^{-k^2} \sim ke^{-k^2}. \qquad (18)$$

As a consequence, $\eta_a \neq O(\xi_a)$, and hence $(\xi)_a \neq (\eta)_a$.

From (13), (15) and (16) we have

$$\frac{(\xi_{a^2})_{n_k}}{\delta_k} = \frac{m_k}{n_k}\left\{\frac{(\xi_{a^2})_{m_k}}{\delta_k} - e^{-k^2} + \left(\frac{(\xi_a)_{m_k}}{\delta_k} - e^{-k^2}\right)(H_{n_k} - H_{m_k})\right\} + e^{-k^2}$$
$$\sim e^{-k^2}\left(1 + \log\frac{n_k}{m_k}\right) + e^{-k^2} \sim k^2 e^{-k^2} \qquad (19)$$

and

$$\frac{(\eta_{a^2})_{n_k}}{\delta_k} = \frac{m_k}{n_k}\left\{\frac{(\eta_{a^2})_{m_k}}{\delta_k} - ke^{-k^2} + \left(\frac{(\eta_a)_{m_k}}{\delta_k} - ke^{-k^2}\right)(H_{n_k} - H_{m_k})\right\} + ke^{-k^2}$$
$$\sim e^{-k^2}\left(1 + \log\frac{n_k}{m_k}\right) + ke^{-k^2} \sim k^2 e^{-k^2} \qquad (20)$$

As an aside relating to the sufficient condition in Theorem 4.5 below we note the equalities $\limsup\frac{\xi_{a^2}}{\xi_a} = \limsup\frac{\eta_{a^2}}{\eta_a} = \infty$, while from (15), $\liminf\frac{\xi_{a^2}}{\xi_a} = \liminf\frac{\eta_{a^2}}{\eta_a} = 1$, and hence neither $\frac{\xi_{a^2}}{\xi_a}$ nor $\frac{\eta_{a^2}}{\eta_a}$ are equivalent to a monotone sequence. The conclusions of this example together with Theorem 4.5 of course also implies this.

Now we give the crux of the proof, that $(\eta_{a^2})_j \leqslant 2(\xi_{a^2})_j$ for j sufficiently large, implying that $(\xi)_{a^2} = (\eta)_{a^2}$. If $m_k < j \leqslant n_k$, from (13)–(14) we have for all $k \geqslant 2$,

$$
\begin{aligned}
\frac{2(\xi_{a^2})_j - (\eta_{a^2})_j}{\delta_k} &= \frac{m_k}{j}\left\{ \frac{2(\xi_{a^2})_{m_k} - (\eta_{a^2})_{m_k}}{\delta_k} + (k-2)e^{-k^2} \right. \\
&\quad + \left. \left(\frac{2(\xi_a)_{m_k} - (\eta_a)_{m_k}}{\delta_k} + (k-2)e^{-k^2}\right)(H_j - H_{m_k})\right\} + (2-k)e^{-k^2} \\
&\geqslant \frac{m_k}{j}\left\{\frac{2(\xi_{a^2})_{m_k}-(\eta_{a^2})_{m_k}}{\delta_k} + \frac{2(\xi_a)_{m_k}-(\eta_a)_{m_k}}{\delta_k}(H_j-H_{m_k}) - ke^{-k^2}\frac{j}{m_k}\right\} \\
&\geqslant \frac{m_k}{j}\left\{1 - \frac{1}{k} + \left(1 - \frac{1}{k}\right)\left(\log\frac{j}{m_k} - \frac{1}{m_k+1}\right) - ke^{-k^2}\frac{j}{m_k}\right\} \\
&\geqslant \frac{m_k}{j}\left\{\frac{1}{2}\left(\frac{1}{2} + \log\frac{j}{m_k}\right) - ke^{-k^2}\frac{j}{m_k}\right\},
\end{aligned}
$$

where the second last inequality follows from (14) and (2). Notice $1 < \frac{j}{m_k} \leqslant \frac{n_k}{m_k} \leqslant e^{k^2}$ and then elementary calculus shows that the function $\phi(x) := \frac{1}{4} + \frac{1}{2}\log x - ke^{-k^2}x$ attains its absolute minimum on the interval $[1, e^{k^2}]$ for $x = 1$ and $\phi(1) = \frac{1}{4} - ke^{-k^2} > 0$. Thus $2(\xi_{a^2})_j > (\eta_{a^2})_j$ for all $m_k < j \leqslant n_k$. For $n_k < j \leqslant m_{k+1}$, again from (13) we have

$$
\begin{aligned}
&\frac{2(\xi_{a^2})_j - (\eta_{a^2})_j}{\delta_k} \\
&= \frac{n_k}{j}\left(\frac{2(\xi_{a^2})_{n_k} - (\eta_{a^2})_{n_k}}{\delta_k} - e^{-k^2} + \left(\frac{2(\xi_a)_{n_k} - (\eta_a)_{n_k}}{\delta_k} - e^{-k^2}\right)(H_j - H_{n_k})\right) + e^{-k^2}.
\end{aligned}
\tag{21}
$$

Since $\frac{2(\xi_{a^2})_{n_k}-(\eta_{a^2})_{n_k}}{\delta_k} \sim k^2 e^{-k^2}$ by (19)-(20) and $\frac{2(\xi_a)_{n_k}-(\eta_a)_{n_k}}{\delta_k} \sim -ke^{-k^2}$ by (17)-(18), for k sufficiently large we have

$$
\frac{2(\xi_{a^2})_{n_k} - (\eta_{a^2})_{n_k}}{\delta_k} \geqslant \frac{1}{2}k^2 e^{-k^2} \quad \text{and} \quad \frac{2(\xi_a)_{n_k} - (\eta_a)_{n_k}}{\delta_k} \geqslant -2ke^{-k^2}.
$$

Thus from (21) and (2), for k sufficiently large,

$$
\begin{aligned}
\frac{2(\xi_{a^2})_j - (\eta_{a^2})_j}{e^{-k^2}\delta_k} &\geqslant \frac{n_k}{j}\left\{\frac{1}{2}k^2 - 1 + (-2k-1)(H_j - H_{n_k}) + \frac{j}{n_k}\right\} \\
&\geqslant \frac{n_k}{j}\left\{\frac{1}{3}k^2 - 3k\log\frac{j}{n_k} + \frac{j}{n_k}\right\}.
\end{aligned}
$$

Again, elementary Calculus shows that the function $\psi(x) := \frac{1}{3}k^2 - 3k\log x + x$ attains its absolute minimum on the interval $[1, \infty)$ for $x = 3k$ and $\psi(3k) > 0$ when k is sufficiently large. Thus for large k, $2(\xi_{a^2})_j \geqslant (\eta_{a^2})_j$ also for all $n_k < j \leqslant m_{k+1}$, which completes the proof. $\square$

REMARK 4.4. The construction in Example 4.3 illustrates some features of the behavior of these "step sequences," their first order and second order arithmetic means. On a long interval of constancy of a sequence ζ, both the first and second order means ζ_a and ζ_{a^2} approach the value of the sequence ζ, with ζ_a approaching it faster than ζ_{a^2}. This "resetting the clock" greatly simplifies the computations.

Following a step down, ζ_{a^2} decreases much slower than ζ_a. Thus in the construction of the example, by having different step sizes for η and ξ it would have been relatively straightforward to achieve large ratios for $\frac{\eta_a}{\xi_a}$. The delicate point was to do so while simultaneously keeping the difference $2\xi_{a^2} - \eta_{a^2}$ positive not only on the first part of the interval but on the second one as well, where the two sequences are equal (which was not automatic due to the delay in the decrease of η_{a^2}).

A consequence of Example 4.3 is that not all ideals, and not even all principal ideals, satisfy the necessary and sufficient condition $I^- = {}_{a^2}(I_{a^2})$ for the cancellation $J_{a^2} \subset I_{a^2} \Rightarrow J_a \subset I_a$ to hold (Proposition 4.1). However, we are still left wanting a more usable condition. Clearly, am-stability is trivially sufficient even for general ideals, since in this case $I = I_a = {}_aI$ and hence $I = I^- = {}_{a^2}(I_{a^2})$. It is, however, far from necessary. Indeed a much more general sufficient condition is provided by the following theorem.

THEOREM 4.5.

(i) *Let $I = (\eta)$ be principal ideal, let $r(\eta_a)$ be equivalent to a monotone sequence and let J be an arbitrary ideal. Then $J_{a^2} \subset I_{a^2} \Rightarrow J_a \subset I_a$.*

(ii) *A sufficient condition on a general ideal I for the second order am-inclusion cancellation implication in (i) to hold for arbitrary ideals J is that every $\mu \in \Sigma(I)$ is dominated by some $\eta \in \Sigma(I)$ (i.e., $\mu \leqslant \eta$) for which $r(\eta_a)$ is equivalent to a monotone sequence.*

Proof.

(i) It suffices to prove the cancellation property for the case that J itself is principal. Indeed, if J is a general ideal with $J_{a^2} \subset I_{a^2}$ and if $\rho \in \Sigma(J_a)$, i.e., $\rho \leqslant \xi_a$ for some $\xi \in \Sigma(J)$, then $(\xi)_{a^2} \subset J_{a^2} \subset I_{a^2}$. We claim that $(\xi_a) \subset I_a$, whence $(\rho) \subset I_a$ and hence $J_a \subset I_a$ by the arbitrariness of ρ.

Since $I = (\eta)$ is principal, so are I_a and I_{a^2}, indeed, $I_a = (\eta_a)$ and $I_{a^2} = (\eta_{a^2})$. Thus the inclusions $(\xi_a) \subset I_a$ (resp., $(\xi)_{a^2} \subset I_{a^2}$) are equivalent to the conditions $\xi_a = O(\eta_a)$ (resp., $\xi_{a^2} = O(\eta_{a^2})$). Thus, to prove the claim, it suffices to prove that if $\frac{\xi_a}{\eta_a}$ is unbounded, then so is $\frac{\xi_{a^2}}{\eta_{a^2}}$. By Lemma 3.14, there are constants $K, M > 0$ for which, if $m \geqslant M$, then

$$(\eta_{a^2})_{n_m} \leqslant K \frac{m}{n_m} \log \frac{n_m}{m} (\eta_a)_m \quad \text{for some } n_m > m.$$

By Lemma 3.9(ii) and (2),

$$(\xi_{a^2})_{n_m} \geqslant \frac{m}{n_m}(H_{n_m} - H_m)(\xi_a)_m > \frac{m}{n_m} \log \frac{n_m + 1}{m + 1}(\xi_a)_m \geqslant \frac{1}{2}\frac{m}{n_m} \log \frac{n_m}{m}(\xi_a)_m.$$

Thus

$$\left(\frac{\xi_{a^2}}{\eta_{a^2}}\right)_{nm} \geqslant \frac{1}{2K}\left(\frac{\xi_a}{\eta_a}\right)_m.$$

Hence the unboundedness of $\frac{\xi_a}{\eta_a}$ implies the unboundedness of $\frac{\xi_{a^2}}{\eta_{a^2}}$.

(ii) Assume that $J_{a^2} \subset I_{a^2}$ and that $\rho \in \Sigma(J_a)$, i.e., $\rho \leqslant \xi_a$ for some $\xi \in \Sigma(J)$. Then $\xi_{a^2} \in \Sigma(I_{a^2})$, i.e., $\xi_{a^2} = O(\mu_{a^2})$ for some $\mu \in \Sigma(I)$, hence $\xi_{a^2} = O(\eta_{a^2})$ for some $\eta \in \Sigma(I)$ for which $r(\eta_a)$ is equivalent to a monotone sequence. By (i), the inclusion $(\xi_{a^2}) \subset (\eta_{a^2})$ implies the inclusions $(\rho) \subset (\xi_a) \subset (\eta_a) \subset I_a$. By the arbitrariness of ρ, we conclude that $J_a \subset I_a$. $\quad\square$

We do not know if the condition in part (i), that $r(\eta_a)$ is equivalent to a monotone sequence, is necessary for the inclusion cancellation in Theorem 4.5. However, we see from Example 4.7 below that it is not sufficient for the second order reverse inclusion cancellation. Sufficient conditions for that cancellation to hold are given by the following theorem.

THEOREM 4.6.

(i) *Let $I = (\eta)$ be a principal ideal, let $r(\eta_a) \asymp \log$ or $\eta \asymp \widehat{\eta}$ (see Section 2 end), and let J be an arbitrary ideal. Then $J_{a^2} \supset I_{a^2} \Rightarrow J_a \supset I_a$.*

(i) *A sufficient condition on a general ideal I for the second order inclusion am-cancellation implication in (i) to hold for arbitrary ideals J is that every $\mu \in \Sigma(I)$ is dominated by some $\eta \in \Sigma(I)$ for which $r(\eta_a) \asymp \log$ or $\eta \asymp \widehat{\eta}$.*

Proof.

(i) As in the proof of Theorem 4.5(i), we reduce the proof to the case where J also is principal. Indeed, $I = (\eta)$ and $I_{a^2} \subset J_{a^2}$ implies that $\eta_{a^2} \leqslant \xi_{a^2}$ for some $\xi \in \Sigma(J)$. In the case when $r(\eta_a) \asymp \log$, by Lemma 3.8(i) and (2),

$$(\eta_a)_n \leqslant K\frac{(\eta_{a^2})_n}{\log n} \leqslant K\frac{(\xi_{a^2})_n}{\log n} = K\frac{r(\xi_a)}{\log n}(\xi_a)_n \leqslant K\frac{H_n}{\log n}(\xi_a)_n \leqslant 2K(\xi_a)_n$$

for some $K > 0$ and all $n > 2$. In the case when $\eta \asymp \widehat{\eta}$, by [9, Proposition 6.17], η is regular. Then by [ibid, Definition 6.10 and Corollary 6.15], $(\eta_{a^2}) \subset (\xi_{a^2})$ implies $\widehat{(\eta_a)} \subset (\xi_a)$ and so $(\eta_a) = (\eta) = (\widehat{\eta}) = \widehat{(\eta)} = \widehat{(\eta_a)} \subset (\xi_a)$. In either case, this shows that $I_a = (\eta_a) \subset (\xi_a) \subset J_a$.

(ii) For every $\mu \in \Sigma(I_a)$, there is a $\xi \in \Sigma(I)$ for which $\mu \leqslant \xi_a$. By the hypothesis, there is also an $\eta \in \Sigma(I)$ with $\xi \leqslant \eta_a$ for which $r(\eta_a) \asymp \log$ or $\eta \asymp \widehat{\eta}$. Since $(\eta)_{a^2} \subset I_{a^2} \subset J_{a^2}$, by (i) we obtain $(\mu) \subset (\xi_a) \subset (\eta)_a \subset J_a$. Since μ is arbitrary, we conclude that $I_a \subset J_a$. $\quad\square$

As mentioned earlier, one sees from the last example listed in Example 4.3 that the condition that $r(\eta_a)$ be equivalent to a monotone sequence (as in Theorem 4.5) is more general than that $r(\eta_a)$ be equivalent to *log* or 1 (as in Theorems 4.6 and 4.8).

Notice that neither the condition that $r(\eta_a)$ is equivalent to a monotone sequence, nor the more restrictive condition that $r(\eta_a)$ is bounded, i.e., that η is regular, are sufficient for the cancellation in part (i) to hold. Recall that (ω^p) is am-stable for all

$0 < p < 1$ but $(\omega^p) \neq \widehat{(\omega^p)} = (\omega^{p'})$ for $\frac{1}{p} - \frac{1}{p'} = 1$ by [9, Corollary 6.16]. Thus we see that

$$(\omega^{1/2}) = (\omega^{1/2})_a \subset (\xi)_a \not\Rightarrow (\omega^{1/2}) \subset (\xi).$$

The analogous cancellation for second order ideals is a priori different, although the same conclusion holds.

EXAMPLE 4.7.   A principal ideal (ξ) for which $(\omega^{1/2})_{a^2} \subset (\xi)_{a^2}$ but for which $(\omega^{1/2})_a \not\subset (\xi)_a$.

Construction. We construct inductively an increasing sequence of positive integers m_k starting with $m_1 = 0$ and define $\xi_1 := 1$ and $\xi_j = \epsilon_k := \frac{1}{m_k}$ for $m_k < j \leq m_{k+1}$ for all $k > 1$. Assume the construction up to m_k. Choose $m_{k+1} > e^{2m_k}$ and so that $(\xi_{a^2})_{m_{k+1}} \leq \frac{1+1/k}{m_k}$. This can be achieved using Equation (13) by choosing m_{k+1} sufficiently large. Then

$$\frac{1}{m_k} = \xi_{m_{k+1}} \leq (\xi_a)_{m_{k+1}} \leq (\xi_{a^2})_{m_{k+1}} \leq \frac{1+1/k}{m_k}.$$

For $m_k < j \leq m_{k+1}$ it follows from (12) that

$$\frac{m_k}{j}\left(\frac{1}{m_{k-1}} - \frac{1}{m_k}\right) + \frac{1}{m_k} \leq (\xi_a)_j \leq \frac{m_k(1 + \frac{1}{k-1})}{jm_{k-1}} + \frac{1}{m_k},$$

from which follows, as j increases, the asymptotic

$$(\xi_a)_j \sim \frac{m_k}{jm_{k-1}} + \frac{1}{m_k}.$$

In particular, if $j = [\frac{m_k^2}{m_{k-1}}]$, then $(\xi_a)_j \sim \frac{2}{m_k}$, while $(\omega^{1/2})_j \sim \frac{m_{k-1}^{1/2}}{m_k}$. But then $\omega^{1/2} \neq O(\xi_a)$ and hence $(\omega^{1/2})_a = (\omega^{1/2}) \not\subset (\xi)_a$. We now show that $\xi_{a^2} \geq \omega^{1/2}$ so $(\omega^{1/2})_{a^2} = (\omega^{1/2}) \subset (\xi)_{a^2}$. When $m_k < j \leq m_{k+1}$ and $k \geq 2$, by (13) and (2),

$$\begin{aligned}
j(\xi_{a^2})_j - j\omega_j^{1/2} &= m_k\left((\xi_{a^2})_{m_k} - \frac{1}{m_k} + \left((\xi_a)_{m_k} - \frac{1}{m_k}\right)(H_j - H_{m_k})\right) + \frac{j}{m_k} - j^{1/2} \\
&\geq m_k\left(\frac{1}{m_{k-1}} - \frac{1}{m_k}\right)\left(1 + \log\frac{j+1}{m_k+1}\right) + \frac{j}{m_k} - j^{1/2} \\
&\geq \frac{m_k}{2m_{k-1}}\left(1 + \frac{1}{2}\log\frac{j}{m_k}\right) + \frac{j}{m_k} - j^{1/2}.
\end{aligned}$$

Define the function $\phi(x) := \frac{m_k}{2m_{k-1}}\left(1 + \frac{1}{2}\log\frac{x}{m_k}\right) + \frac{x}{m_k} - x^{1/2}$ for $x \geq m_k$. Elementary calculus and the quadratic form in $\frac{1}{\sqrt{x}}$ of ϕ' with two real roots shows that the function ϕ has an absolute minimum on the interval $[m_k, \infty)$ at $x_{m_k} := \left(\frac{m_k}{4}\left(1 + \sqrt{1 - \frac{4}{m_{k-1}}}\right)\right)^2$ and a direct computation shows that $\phi(x_{m_k}) > 0$ for k sufficiently large, because of the assumption that $m_k \geq e^{2m_{k-1}}$. This proves that $\omega_j^{1/2} \leq (\xi_{a^2})_j$ for all j and hence $(\omega^{1/2}) \subset (\xi)_{a^2}$. $\square$

For equality cancellation we can slightly relax the sufficient conditions for the principal ideal case from those of the general case.

THEOREM 4.8.

(i) *Let $I = (\eta)$ be principal ideal, let $r(\eta_a) \asymp \log$ or $\eta \asymp \eta_a$, and let J be an arbitrary ideal. Then $J_{a^2} = I_{a^2} \Rightarrow J_a = I_a$.*

(ii) *A sufficient condition on a general ideal I for the second order am-equality cancellation implication in (i) to hold for arbitrary ideals J is that every $\xi \in \Sigma(I)$ is dominated by some $\eta \in \Sigma(I)$ for which $r(\eta_a) \asymp \log$ or $\eta \asymp \widehat{\eta}$.*

Proof.

(i) Whether $r(\eta_a) \asymp \log$ or $\eta \asymp \eta_a$, $r(\eta_a)$ is equivalent to a monotone sequence, hence by Theorem 4.5(i), $J_a \subset I_a$. If $r(\eta_a) \asymp \log$, then by Theorem 4.6(i) we can also conclude that $J_a \supset I_a$. If $\eta \asymp \eta_a$, i.e., η is regular, then $\eta \leqslant \xi_{a^2}$ for some $\xi \in \Sigma(J)$. But $\xi_{a^2} \in \Sigma((\eta)_{a^2}) = \Sigma((\eta))$, hence $\xi_{a^2} = O(\eta)$. Then $\xi_{a^2} \asymp \eta$ and thus ξ_{a^2} is regular. It follows that ξ is regular (cf. Corollary 3.10) and hence $\eta = O(\xi)$. Thus $I \subset J$ and in particular, $I_a \subset J_a$.

(ii) Follows combining Theorem 4.5(i) and Theorem 4.6(ii) and recalling that $\eta \asymp \widehat{\eta}$ implies that η is regular and hence that $r(\eta_a)$ is equivalent to a monotone sequence. $\square$

We do not know if in (ii) the last condition $\eta \asymp \widehat{\eta}$ can be replaced by the more general condition of regularity, $\eta \asymp \eta_a$.

Because of Corollary 3.12 we can extend Theorems 4.5, 4.6, and 4.8 to higher order arithmetic means.

THEOREM 4.9.   *Let $p \in \mathbb{N}$. If every c_0^*-sequence in the characteristic set $\Sigma(I)$ of an ideal I is dominated by some η in its characteristic set that satisfies the pth order exponential Δ_2-condition $\sup\limits_{m} \frac{m^2 (\eta_{a^p})_{m^2}}{m(\eta_{a^p})_m} < \infty$, then $J_{a^{p+1}} \supset I_{a^{p+1}}$ implies $J_{a^p} \supset I_{a^p}$, $J_{a^{p+1}} \subset I_{a^{p+1}}$ implies $J_{a^p} \subset I_{a^p}$, and $J_{a^{p+1}} = I_{a^{p+1}}$ implies $J_{a^p} = I_{a^p}$.*

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