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ABELIAN SELF–COMMUTATORS IN FINITE FACTORS

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Abstract. An *abelian self-commutator* in a C*-algebra $\mathcal{A}$ is an element of the form $A = X^*X - XX^*$, with $X \in \mathcal{A}$, such that X^*X and XX^* commute. It is shown that, given a finite AW*-factor $\mathcal{A}$, there exists another finite AW*-factor $\mathcal{M}$ of same type as $\mathcal{A}$, that contains $\mathcal{A}$ as an AW*-subfactor, such that any self-adjoint element $X \in \mathcal{M}$ of quasitrace zero is an abelian self-commutator in $\mathcal{M}$.

Introduction

According to the Murray-von Neumann classification, finite von Neumann factors are either of type I_{fin}, or of type II_1. For the non-expert, the easiest way to understand this classification is by accepting the famous result of Murray and von Neumann (see [6]) which states that *every finite von Neumann factor* $\mathcal{M}$ *possesses a unique state-trace* $\tau_{\mathcal{M}}$. Upon accepting this result, the type of $\mathcal{M}$ is decided by so-called *dimension range*: $\mathcal{D}_{\mathcal{M}} = \{\tau_{\mathcal{M}}(P) : P \text{ projection in } \mathcal{M}\}$ as follows. If $\mathcal{D}_{\mathcal{M}}$ is finite, then $\mathcal{M}$ is of type I_{fin} (more explicitly, in this case $\mathcal{D}_{\mathcal{M}} = \{\frac{k}{n} : k = 0, 1, \dots, n\}$ for some $n \in \mathbb{N}$, and $\mathcal{M} \simeq \mathrm{Mat}_n(\mathbb{C})$ – the algebra of $n \times n$ matrices). If $\mathcal{D}_{\mathcal{M}}$ is infinite, then $\mathcal{M}$ is of type II_1, and in fact one has $\mathcal{D}_{\mathcal{M}} = [0, 1]$. From this point of view, the factors of type II_1 are the ones that are interesting, one reason being the fact that, although all factors of type II_1 have the same dimension range, there are uncountably many non-isomorphic ones (by some celebrated results of McDuff of Connes).

In this paper we deal with the problem of characterizing the self-adjoint elements of trace zero, in terms of simpler ones. We wish to carry this investigation in a "Hilbert-space-free" framework, so instead of von Neumann factors, we are going to work within the category of AW*-algebras. Such objects were introduced in the 1950's by Kaplansky ([4]) in an attempt to formalize the theory of von Neumann algebras without any use of pre-duals. Recall that A unital C*-algebra $\mathcal{A}$ is called an *AW*-algebra*, if for every non-empty set $\mathcal{X} \subset \mathcal{A}$, the left annihilator set $\mathbf{L}(\mathcal{X}) = \{A \in \mathcal{A} : AX = 0,\ \forall X \in \mathcal{X}\}$ is the principal right ideal generated by a projection $P \in \mathcal{A}$, that is, $\mathbf{L}(\mathcal{X}) = \mathcal{A}P$.

Much of the theory – based on the geometry of projections – works for AW*-algebras exactly as in the von Neumann case, and one can classify the finite AW*-factors into the types I_{fin} and II_1, exactly as above, but using the following alternative result:

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any finite AW-factor* $\mathcal{A}$ *possesses a unique normalized quasitrace* $q_{\mathcal{A}}$. Recall that a *quasitrace* on a C*-algebra $\mathfrak{A}$ is a map $q : \mathfrak{A} \to \mathbb{C}$ with the following properties:

(i) if $A, B \in \mathfrak{A}$ are self-adjoint, then $q(A + iB) = q(A) + iq(B)$;
(ii) $q(AA^*) = q(A^*A) \geqslant 0$, $\forall A \in \mathfrak{A}$;
(iii) q is linear on all abelian C*-subalgebras of $\mathfrak{A}$,
(iv) there is a map $q_2 : \mathrm{Mat}_2(\mathcal{A}) \to \mathbb{C}$ with properties (i)-(iii), such that

$$q_2\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = q(A), \quad \forall A \in \mathfrak{A}.$$

(The condition that q is *normalized* means that $q(I) = 1$.)

With this terminology, the dimension range of a finite AW*-factor is the set $\mathcal{D}_{\mathcal{A}} = \{q_{\mathcal{A}}(P) \ : \ P \text{ projection in } \mathcal{A}\}$, and the classification into the two types is exactly as above. As in the case of von Neumann factors, one can show that the AW*-factors of type $\mathrm{I}_{\mathrm{fin}}$ are again the matrix algebras $\mathrm{Mat}_n(\mathbb{C})$, $n \in \mathbb{N}$. The type II_1 case however is still mysterious. In fact, a longstanding problem in the theory of AW*-algebras is the following:

KAPLANSKY'S CONJECTURE. *Every AW*-factor of type* II_1 *is a von Neumann factor.*

An equivalent formulation states that: *if* $\mathcal{A}$ *is an AW*-factor of type* II_1, *then the quasitrace* $q_{\mathcal{A}}$ *is linear* (so it is in fact a trace). It is well known (see [3] for example) that Kaplansky's Conjecture implies:

QUASITRACE CONJECTURE. *Quasitraces (on arbitrary C*-algebras) are traces.*

A remarkable result of Haagerup ([3]) states that quasitraces on *exact* C*-algebras are traces, so if $\mathcal{A}$ is an AW*-factor of type II_1, generated (as an AW*-algebra) by an exact C*-algebra, then $\mathcal{A}$ is a von Neumann algebra.

It is straightforward that if q is a quasitrace on some C*-algebra $\mathfrak{A}$, and $A \in \mathfrak{A}$ is some element that can be written as $A = XX^* - X^*X$ for some $X \in \mathfrak{A}$, such that XX^* and X^*X *commute*, then $q(A) = 0$. In this paper we are going to take a closer look at such A's, which will be referred to as *abelian self-commutators*.

Suppose now $\mathcal{A}$ is a finite AW*-factor, which is contained as an AW*-subalgebra in a finite AW*-factor $\mathcal{B}$. Due to the uniqueness of the quasitrace, for $A \in \mathcal{A}$, one has the equivalence $q_{\mathcal{A}}(A) = 0 \Leftrightarrow q_{\mathcal{B}}(A) = 0$, so a sufficient condition for $q_{\mathcal{A}}(A) = 0$ is that A is an abelian self-commutator in $\mathcal{B}$. In this paper we prove the converse, namely: *If* $\mathcal{A}$ *is a finite AW*-factor, and* $A \in \mathcal{A}$ *is a self-adjoint element of quasitrace zero, then there exists a finite AW*-factor* $\mathcal{M}$, *that contains* $\mathcal{A}$ *as an AW*-subfactor, such that* A *is an abelian self-commutator in* $\mathcal{M}$. Moreover, $\mathcal{M}$ can be chosen such that it is of same type as $\mathcal{A}$, and *every self-adjoint element* $X \in \mathcal{M}$ *of quasitrace zero is an abelian self commutator in* $\mathcal{M}$. Specifically, in the type I_n, $\mathcal{M}$ is $\mathcal{A}$ itself, and in the type II_1 case, $\mathcal{M}$ is an ultraproduct.

The paper is organized as follows. In Section 1 we introduce our notations, and we recall several standard results from the literature, and in Section 2 we prove the main results.

1. Prelimiaries

NOTATIONS. Let $\mathcal{A}$ be a unital C*-algebra.

A. We denote by $\mathcal{A}_{sa}$ the real linear space of self-adjoint elements. We denote by $\mathbf{U}(\mathcal{A})$ the group of unitaries in $\mathcal{A}$. We denote by $\mathbf{P}(\mathcal{A})$ the collection of projections in $\mathcal{A}$, that is, $\mathbf{P}(\mathcal{A}) = \{P \in \mathcal{A}_{sa} : P = P^2\}$.

B. Two elements $A, B \in \mathcal{A}$ are said to be *unitarily equivalent in* $\mathcal{A}$, in which case we write $A \approx B$, if there exists $U \in \mathbf{U}(\mathcal{A})$ such that $B = UAU^*$.

C. Two elements $A, B \in \mathcal{A}$ are said to be *orthogonal*, in which case we write $A \perp B$, if: $AB = BA = AB^* = B^*A = 0$. (Using the Fuglede-Putnam Theorem, in the case when one of the two is normal, the above condition reduces to: $AB = BA = 0$. If both A and B are normal, one only needs $AB = 0$.) A collection $(A_j)_{j\in J} \subset \mathcal{A}$ is said to be orthogonal, if $A_i \perp A_j$, $\forall\, i \neq j$.

Finite AW*-factors have several interesting features, contained in the following well-known result (stated without proof).

PROPOSITION 1.1. *Assume $\mathcal{A}$ is a finite AW*-factor.*

A. For any element $X \in \mathcal{A}$, one has: $XX^* \approx X^*X$.

B. If $X_1, X_2, Y_1, Y_2 \in \mathcal{A}$ are such that $X_1 \approx X_2$, $Y_1 \approx Y_2$, and $X_k \perp Y_k$, $k = 1, 2$, then $X_1 + Y_1 \approx X_2 + Y_2$.

DEFINITION. Let $\mathcal{A}$ be a unital C*-algebra. An element $A \in \mathcal{A}_{sa}$ is called an *abelian self-commutator*, if there exists $X \in \mathcal{A}$, such that

- $(XX^*)(X^*X) = (X^*X)(XX^*)$;
- $A = XX^* - X^*X$.

REMARK 1.1. It is obvious that if $A \in \mathcal{A}_{sa}$ is an abelian self-commutator, then $q(A) = 0$, for any quasitrace q on $\mathcal{A}$.

Abelian self-commutators in finite AW*-factors can be characterized as follows.

PROPOSITION 1.2. *Let $\mathcal{A}$ be a finite AW*-factor. For an element $A \in \mathcal{A}_{sa}$, the following are equivalent:*

(i) A is an abelian self-commutator in $\mathcal{A}$;

(ii) there exists $A_1, A_2 \in \mathcal{A}_{sa}$ with:

- *$A_1A_2 = A_2A_1$;*
- *$A = A_1 - A_2$;*
- *$A_1 \approx A_2$.*

Proof. The implication (i) $\Rightarrow$ (ii) is trivial by Proposition 1.1.

Conversely, assume A_1 and A_2 are as in (ii), and let $U \in \mathbf{U}(\mathcal{A})$ be such that $UA_1U^* = A_2$. Choose a real number $t > 0$, such that $A_1 + tI \geqslant 0$ (for example $t = \|A_1\|$), and define the element $X = (A_1 + tI)^{1/2}U^*$. Notice that $XX^* = A_1 + tI$, and $X^*X = A_2 + tI$, so XX^* and X^*X commute. Now we are done, since $XX^* - X^* = A_1 - A_2 = A$. $\square$

NOTATION. In [3] Haagerup shows that, given a normalized quasitrace q on a unital C*-algebra $\mathcal{A}$, the map $d_q : \mathcal{A} \times \mathcal{A} \to [0, \infty)$, given by

$$d_q(X,Y) = q\big((X-Y)^*(X-Y)\big)^{\frac{1}{3}}, \ \ \forall X,Y \in \mathcal{A},$$

defines a metric. We refer to this metric as the *Haagerup "$\frac{2}{3}$-metric" associated with* q. Using the inequality $|q(X)| \leqslant 2\|X\|$, one also has the inequality

$$d_q(X,Y) \leqslant \sqrt[3]{2}\|X-Y\|^{\frac{2}{3}}, \ \ \forall X,Y \in \mathcal{A}. \tag{1}$$

If $\mathcal{A}$ is a finite AW*-factor, we denote by $q_{\mathcal{A}}$ the (unique) normalized quasitrace on $\mathcal{A}$, and we denote by $d_{\mathcal{A}}$ the Haagerup "$\frac{2}{3}$-metric" associated with $q_{\mathcal{A}}$.

We now concentrate on some issues that deal with the problem of "enlarging" a finite AW*-factor to a "nicer" one. Recall that, given an AW*-algebra $\mathcal{B}$, a subset $\mathcal{A} \subset \mathcal{B}$ is declared an *AW*-subalgebra of* $\mathcal{B}$, if it has the following properties:

(i) $\mathcal{A}$ is a C*-subalgebra of $\mathcal{B}$;
(ii) $\mathbf{s}(A) \in \mathcal{A}$, $\forall A \in \mathcal{A}_{sa}$;
(iii) if $(P_i)_{i\in I} \subset \mathbf{P}(\mathcal{A})$, then $\bigvee_{i\in I} P_i \in \mathcal{A}$.

(In condition (ii) the projection $\mathbf{s}(A)$ is the support of A in $\mathcal{B}$. In (iii) the supremum is computed in $\mathcal{B}$.) In this case it is pretty clear that $\mathcal{A}$ is an AW*-algebra on its own, with unit $I_{\mathcal{A}} = \bigvee_{A\in\mathcal{A}_{sa}} \mathbf{s}(A)$. Below we take a look at the converse statement, namely at the question whether a C*-subalgebra $\mathcal{A}$ of an AW*-algebra $\mathcal{A}$, which is an AW*-algebra on its own, is in fact an AW*-subalgebra of $\mathcal{B}$. We are going to restrict ourselves with the factor case, and for this purpose we introduce the following terminology.

DEFINITION. Let $\mathcal{B}$ be an AW*-factor. An AW*-subalgebra $\mathcal{A} \subset \mathcal{B}$ is called an *AW*-subfactor of* $\mathcal{B}$, if $\mathcal{A}$ is a factor, and $\mathcal{A} \ni I$ – the unit in $\mathcal{B}$.

PROPOSITION 1.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be finite AW*-factors. If $\pi : \mathcal{A} \to \mathcal{B}$ be a unital (i.e. $\pi(I) = I$) $*$-homomorphism, then $\pi(\mathcal{A})$ is a AW*-subfactor of $\mathcal{B}$.*

Proof. Denote for simplicity $\pi(\mathcal{A})$ by $\mathcal{M}$. Since $\mathcal{A}$ is simple, π is injective, so $\mathcal{M}$ is $*$-isomorphic to $\mathcal{A}$. Among other things, this shows that $\mathcal{M}$ is a factor, which contains the unit I of $\mathcal{B}$. We now proceed to check the two key conditions (ii) and (iii) that ensure that $\mathcal{M}$ is an AW*-subalgebra in $\mathcal{B}$.

(ii). Start with some element $M \in \mathcal{M}_{sa}$, written as $M = \pi(A)$, for some $A \in \mathcal{A}_{sa}$, and let us show that $\mathbf{s}(M)$ – the support of M in $\mathcal{B}$ – in fact belongs to $\mathcal{M}$. This will be the result of the following:

CLAIM 1. *One has the equality $\mathbf{s}(M) = \pi\big(\mathbf{s}(A)\big)$, where $\mathbf{s}(A)$ denotes the support of A in $\mathcal{A}$.*

Denote the projection $\pi\big(\mathbf{s}(A)\big) \in \mathbf{P}(\mathcal{M})$ by P. First of all, since $\big(I - \mathbf{s}(A)\big)A = 0$ (in $\mathcal{A}$), we have $(I-P)M = 0$ in $\mathcal{B}$, so $(I-P) \perp \mathbf{s}(M)$, i.e. $\mathbf{s}(M) \geqslant P$. Secondly, since $q_{\mathcal{B}} \circ \pi : \mathcal{A} \to \mathbb{C}$ is a quasitrace, we must have the equality

$$q_{\mathcal{B}} \circ \pi = q_{\mathcal{A}}. \tag{2}$$

In particular the projection P has dimension $D_{\mathcal{B}}(P) = D_{\mathcal{A}}\big(\mathbf{s}(A)\big)$. We know however that for a self-adjoint element X in a finite AW*-factor with quasitrace q, one has the equality $q\big(\mathbf{s}(X)\big) = \mu^X\big(\mathbb{R}\setminus\{0\}\big)$, where μ^X is the scalar spectral measure, defined implicitly (using Riesz' Theorem) by

$$\int_{\mathbb{R}} f\, d\mu^X = q\big(f(X)\big), \ \ \forall f \in C_0(\mathbb{R}).$$

So in our case we have the equalities

$$D_{\mathcal{B}}\big(\mathbf{s}(M)\big) \ = \ \mu^M_{\mathcal{B}}\big(\mathbb{R}\setminus\{0\}\big), \tag{3}$$

$$D_{\mathcal{B}}(P) \ = \ D_{\mathcal{A}}\big(\mathbf{s}(A)\big) = \mu^A_{\mathcal{A}}\big(\mathbb{R}\setminus\{0\}\big), \tag{4}$$

where the subscripts indicate the ambient AW*-factor. Since π is a $*$-homomorphism, one has the equality $\pi\big(f(A)\big) = f(M)$, $\forall f \in C_0(\mathbb{R})$, and then by (2) we get

$$\int_{\mathbb{R}} f\, d\mu^M_{\mathcal{B}} = q_{\mathcal{B}}\big(f(M)\big) = (q_{\mathcal{B}}\circ\pi)\big(f(A)\big) = q_{\mathcal{A}}\big(f(A)\big) = \int_{\mathbb{R}} f\, d\mu^A_{\mathcal{A}}, \ \ \forall f \in C_0(\mathbb{R}).$$

In particular we have the equality $\mu^M_{\mathcal{B}} = \mu^A_{\mathcal{A}}$, and then (3) and (4) will force $D_{\mathcal{B}}\big(\mathbf{s}(M)\big) = D_{\mathcal{B}}(P)$. Since $P \geqslant \mathbf{s}(M)$, the equality of dimensions will force $P = \mathbf{s}(M)$.

(iii). Start with a collection of projections $(P_i)_{i\in I} \subset \mathbf{P}(\mathcal{M})$, let $P = \bigvee_{i\in I} P_i$ (in $\mathcal{B}$), and let us prove that $P \in \mathcal{M}$. Write each $P_i = \pi(Q_i)$, with $Q_i \in \mathbf{P}(\mathcal{A})$, and let $Q = \bigvee_{i\in I} Q_i$ (in $\mathcal{A}$). The desired conclusion will result from the following.

CLAIM 2. $P = \pi(Q)$.

Denote by $\mathcal{F}$ the collection of all finite subsets of I, which becomes a directed set with inclusion, and define the nets $P_F = \bigvee_{i\in F} P_i$ (in $\mathcal{B}$) and $Q_F = \bigvee_{i\in F} Q_i$ (in $\mathcal{A}$). On the one hand, if we consider the element $X_F = \sum_{i\in F} Q_i$, then $Q_F = \mathbf{s}(X_F)$ (in $\mathcal{A}$), and $P_F = \mathbf{s}\big(\sum_{i\in I} P_i\big) = \mathbf{s}\big(\pi(X_F)\big)$ (in $\mathcal{B}$), so by Claim 1, we have the equality $P_F = \pi(Q_F)$. On the other hand, we have $Q = \bigvee_{F\in\mathcal{F}} Q_F$ (in $\mathcal{A}$), with the net $(Q_F)_{F\in\mathcal{F}}$ increasing, so we get the equality $D_{\mathcal{A}}(Q) = \lim_{F\in\mathcal{F}} D_{\mathcal{A}}(Q_F)$. Arguing the same way (in $\mathcal{B}$), and using the equalities $P_F = \pi(Q_F)$, we get

$$D_{\mathcal{B}}(P) = \lim_{F\in\mathcal{F}} D_{\mathcal{B}}(P_F) = \lim_{F\in\mathcal{F}} D_{\mathcal{B}}\big(\pi(Q_F)\big) = \lim_{F\in\mathcal{F}} D_{\mathcal{A}}(Q_F) = D_{\mathcal{A}}(Q) = D_{\mathcal{B}}\big(\pi(Q)\big).$$

Finally, since $\big[I - \pi(Q)\big]P_i = \pi\big([I-Q]Q_i\big) = 0$, $\forall i \in I$, we get the inequality $\pi(Q) \geqslant P$, and then the equality $D_{\mathcal{B}}(P) = D_{\mathcal{B}}\big(\pi(Q)\big)$ will force $P = \pi(Q)$. □

COMMENT. In the above proof we employed an argument based on the following property of the dimension function D on a finite AW*-factor $\mathcal{A}$:

(L) *If a net $(P_\lambda)_{\lambda\in\Lambda} \subset \mathbf{P}(\mathcal{A})$ is increasing, then* $D\big(\bigvee_{\lambda\in\Lambda} P_\lambda\big) = \lim_{\lambda\in\Lambda} D(P_\lambda)$.

The literature ([1],[4]) often mentions a different feature: the *complete additivity*

(C.A.) *If $(E_i)_{i\in I} \subset \mathbf{P}(\mathcal{A})$ is an orthogonal family, then* $D\big(\bigvee_{i\in I} E_i\big) = \sum_{i\in I} D(E_i)$.

To prove property (L) one can argue as follows. Let $P = \bigvee_{\lambda\in\Lambda} P_\lambda$, so that $D(P) \geqslant D(P_\lambda)$, $\forall \lambda \in \Lambda$. In particular if we take $\ell = \lim_{\lambda\in\Lambda} D(P_\lambda)$ (which exists by monotonicity), we get $D(P) \geqslant \ell$. To prove that in fact we have $D(P) = \ell$, we construct a sequence $\alpha_1 \prec \alpha_2 \prec \dots$ in Λ, such that $\lim_{n\to\infty} D(P_{\alpha_n}) = \ell$, we consider the projection $Q = \bigvee_{n\in\mathbb{N}} P_{\alpha_n} \leqslant P$, and we show first that $D(Q) = \ell$, then $Q = P$. The equality $D(Q) = \ell$ follows from (C.A.) since now we can write $Q = P_{\alpha_1} \vee \bigvee_{n=1}^{\infty} [P_{\alpha_{n+1}} - P_{\alpha_n}]$ with all projections orthogonal, so we get

$$
\begin{aligned}
D(Q) &= D(P_{\alpha_1}) + \sum_{n=1}^{\infty} D(P_{\alpha_{n+1}} - P_{\alpha_n}) \\
&= D(P_{\alpha_1}) + \sum_{n=1}^{\infty} \left[D(P_{\alpha_{n+1}}) - D(P_{\alpha_n})\right] = \lim_{n\to\infty} D(P_{\alpha_n}) = \ell.
\end{aligned}
$$

To prove the equality $Q = P$, we fix for the moment $\lambda \in \Lambda$, and integer $n \geqslant 1$, and some $\mu \in \Lambda$ with $\mu \succ \lambda$ and $\mu \succ \alpha_n$, and we observe that, using the Parallelogram Law, we have:

$$
P_\lambda - P_\lambda \wedge Q \leqslant P_\lambda - P_\lambda \wedge P_{\alpha_n} \approx P_\lambda \vee P_{\alpha_n} - P_{\alpha_n} \leqslant P_\mu - P_{\alpha_n},
$$

so applying the dimension function we get

$$
D(P_\lambda - P_\lambda \wedge Q) \leqslant D(P_\mu - P_{\alpha_n}) = D(P_\mu) - D(P_{\alpha_n}) \leqslant \ell - D(P_{\alpha_n}).
$$

Since the inequality $D(P_\lambda - P_\lambda \wedge Q) \leqslant \ell - D(P_{\alpha_n})$ holds for arbitrary $n \in \mathbb{N}$ and $\lambda \in \Lambda$, taking limit (as $n \to \infty$) yields $D(P_\lambda - P_\lambda \wedge Q) = 0$, which in turn forces $P_\lambda = P_\lambda \wedge Q$, which means that $P_\lambda \leqslant Q$. Since this is true for all $\lambda \in \Lambda$, it will force $Q \geqslant P$, so we must have $Q = P$.

We now recall the ultraproduct construction of finite AW*-factors, discussed for example in [2] and [3].

NOTATIONS. Let $\mathbf{A} = (\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence of finite AW*-factors, and let $q_n : \mathcal{A}_n \to \mathbb{C}$ denote the (unique) normalized quasitrace on $\mathcal{A}_n$. One considers the finite AW*-algebra

$$
\mathbf{A}^\infty = \left\{(X_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathcal{A}_n \,:\, \sup_{n\in\mathbb{N}} \|X_n\| < \infty\right\}.
$$

Given a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, one defines the quasitrace $\tau_{\mathcal{U}} : \mathbf{A}^\infty \to \mathbb{C}$ by

$$
\tau_{\mathcal{U}}(\boldsymbol{x}) = \lim_{\mathcal{U}} q_n(X_n), \quad \forall\, \boldsymbol{x} = (X_n)_{n\in\mathbb{N}} \in \mathbf{A}^\infty.
$$

Next one considers the norm-closed ideal

$$
\mathbf{J}_{\mathcal{U}} = \left\{\boldsymbol{x} \in \mathbf{A}^\infty \,:\, \tau_{\mathcal{U}}(\boldsymbol{x}^*\boldsymbol{x}) = 0\right\}.
$$

It turns out that quotient C*-algebra $\mathbf{A}_{\mathcal{U}} = \mathbf{A}^\infty/\mathbf{J}_{\mathcal{U}}$ becomes a finite AW*-factor. Moreover, its (unique) normalized quasitrace $q_{\mathbf{A}_{\mathcal{U}}}$ is defined implicitly by $q_{\mathbf{A}_{\mathcal{U}}} \circ \Pi_{\mathcal{U}} = \tau_{\mathcal{U}}$, where $\Pi_{\mathcal{U}} : \mathbf{A}^\infty \to \mathbf{A}_{\mathcal{U}}$ denotes the quotient $*$-homomorphism.

The finite AW*-factor $\mathbf{A}_{\mathcal{U}}$ is referred to as the *$\mathcal{U}$-ultraproduct of the sequence* $\mathbf{A} = (\mathcal{A}_n)_{n\in\mathbb{N}}$.

REMARKS 1.2. Let $\mathbf{A} = (\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence of finite factors.

A. With the notations above, if $\boldsymbol{x} = (X_n)_{n\in\mathbb{N}}, \boldsymbol{y} = (Y_n)_{n\in\mathbb{N}} \in \mathbf{A}^\infty$ are elements that satisfy the condition $\lim_{\mathcal{U}} d_{\mathcal{A}_n}(X_n, Y_n) = 0$, then $\Pi_{\mathcal{U}}(\boldsymbol{x}) = \Pi_{\mathcal{U}}(\boldsymbol{y})$ in $\mathbf{A}_{\mathcal{U}}$. This is trivial, since the given condition forces
$$\lim_{\mathcal{U}} q_{\mathcal{A}_n}\big((X_n - Y_n)^*(X_n - Y_n)\big) = 0,$$
i.e. $\boldsymbol{x} - \boldsymbol{y} \in \mathbf{J}_{\mathcal{U}}$.

B. For $\boldsymbol{x} = (X_n)_{n\in\mathbb{N}} \in \mathbf{A}^\infty$, one has the inequality: $\|\Pi_{\mathcal{U}}(\boldsymbol{x})\| \leqslant \lim_{\mathcal{U}} \|X_n\|$. To prove this inequality we start off by denoting $\lim_{\mathcal{U}} \|X_n\|$ by ℓ, and we observe that given any $\varepsilon > 0$, the set
$$U_\varepsilon = \{n \in \mathbb{N} : \ell - \varepsilon < \|X_n\| < \ell + \varepsilon\}$$
belongs to $\mathcal{U}$, so if we define the sequence $\boldsymbol{x}_\varepsilon = (X_n^\varepsilon)_{n\in\mathbb{N}}$ by
$$X_n^\varepsilon = \begin{cases} X_n & \text{if } n \notin U_\varepsilon \\ 0 & \text{if } n \in U_\varepsilon \end{cases}$$
we clearly have $\lim_{\mathcal{U}} \|X_n^\varepsilon\| = 0$. In particular, by part A, we have $\Pi_{\mathcal{U}}(\boldsymbol{x}) = \Pi_{\mathcal{U}}(\boldsymbol{x} - \boldsymbol{x}_\varepsilon)$. Since $\|X_n - X_n^\varepsilon\| \leqslant \ell + \varepsilon$, $\forall\, n \in \mathbb{N}$, it follows that
$$\|\Pi_{\mathcal{U}}(\boldsymbol{x})\| = \|\Pi_{\mathcal{U}}(\boldsymbol{x} - \boldsymbol{x}_\varepsilon)\| \leqslant \ell + \varepsilon,$$
and since the inequality $\|\Pi_{\mathcal{U}}(\boldsymbol{x})\| \leqslant \ell + \varepsilon$ holds for all $\varepsilon > 0$, it follows that we indeed have $\|\Pi_{\mathcal{U}}(\boldsymbol{x})\| \leqslant \ell$.

EXAMPLE 1.1. Start with a finite AW*-factor $\mathcal{A}$ of type II_1 and a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Let $\mathcal{A}_{\mathcal{U}}$ denote the ultraproduct of the constant sequence $\mathcal{A}_n = \mathcal{A}$. For every $X \in \mathcal{A}$ let $\Gamma(X) = (X_n)_{n\in\mathbb{N}} \in \mathbf{A}^\infty$ be the constant sequence: $X_n = X$. It is obvious that $\Gamma : \mathcal{A} \to \mathbf{A}^\infty$ is a unital $*$-homomorphism, so the composition $\Delta_{\mathcal{U}} = \Pi_{\mathcal{U}} \circ \Gamma : \mathcal{A} \to \mathcal{A}_{\mathcal{U}}$ is again a unital $*$-homomorphism. Using Proposition 2.1 it follows that $\Delta_{\mathcal{U}}(\mathcal{A})$ is an AW*-subfactor in $\mathcal{A}_{\mathcal{U}}$.

Moreover, if $\mathcal{B}$ is some finite AW*-factor, and $\pi : \mathcal{B} \to \mathcal{A}$ is some unital $*$-homomorphism, then the $*$-homomorphism $\pi_{\mathcal{U}} = \Delta_{\mathcal{U}} \circ \pi : \mathcal{B} \to \mathcal{A}_{\mathcal{U}}$ gives rise to an AW*-subfactor $\pi_{\mathcal{U}}(\mathcal{B})$ of $\mathcal{A}_{\mathcal{U}}$.

2. Main Results

We start off with the analysis of the type $\mathrm{I}_{\mathrm{fin}}$ situation, i.e. the algebras of the form $\mathrm{Mat}_n(\mathbb{C})$ – the $n \times n$ complex matrices. To make the exposition a little easier, we are going to use the un-normalized trace $\tau : \mathrm{Mat}_n(\mathbb{C}) \to \mathbb{C}$ with $\tau(I_n) = n$.

The main result in the type $\mathrm{I}_{\mathrm{fin}}$ – stated in a way that will allow an inductive proof – is as follows.

THEOREM 2.1. *Let $n \geqslant 1$ be an integer, and let $X \in \mathrm{Mat}_n(\mathbb{C})_{sa}$ be a matrix with $\tau(X) = 0$.*

A. *For any projection* $P \in \mathrm{Mat}_n(\mathbb{C})$ *with* $PX = XP$ *and* $\tau(P) = 1$, *there exist elements* $A, B \in \mathrm{Mat}_n(\mathbb{C})_{sa}$ *with:*
- $AB = BA$;
- $X = A - B$;
- $A \approx B$;
- $\max\{\|A\|, \|B\|\} \leqslant \|X\|$;
- $A \perp P$.

B. X *is an abelian self-commutator in* $\mathrm{Mat}_n(\mathcal{C})$.

Proof. A. We are going to use induction on n. The case $n = 1$ is trivial, since it forces $X = 0$, so we can take $A = B = 0$. Assume now property A is true for all $n < N$, and let us prove it for $n = N$. Fix some $X \in \mathrm{Mat}_N(\mathbb{C})$ with $\tau(X) = 0$, and a projection $P \in \mathrm{Mat}_N(\mathbb{C})$ with $\tau(P) = 1$ such that $PX = XP$. The case $X = 0$ is trivial, so we are going to assume $X \neq 0$. Let us list the spectrum of X as $\mathrm{Spec}(X) = \{\alpha_1 < \alpha_2 < \cdots < \alpha_m\}$, and let $(E_i)_{i=1}^m$ be the corresponding spectral projections, so that

(i) $\tau(E_i) > 0$, $\forall\, i \in \{1, \ldots, m\}$;
(ii) $E_i \perp E_j$, $\forall\, i \neq j$, and $\sum_{i=1}^m E_i = I_N$;
(iii) $X = \sum_{i=1}^m \alpha_i E_i$, so $\tau(X) = \sum_{i=1}^m \alpha_i \tau(E_i)$.

Since $\tau(P) = 1$ and P commutes with X, there exists a unique index $i_0 \in \{1, \ldots, m\}$ such that $P \leqslant E_{i_0}$. Since none of the inclusions $\mathrm{Spec}(X) \subset (0, \infty)$ or $\mathrm{Spec}(X) \subset (-\infty, 0)$ is possible, there exists $i_1 \in \{1, \ldots, m\}$, $i_1 \neq i_0$, such that one of the following inequalities holds

$$\alpha_{i_1} < 0 \leqslant \alpha_{i_0}, \tag{5}$$
$$\alpha_{i_1} > 0 \geqslant \alpha_{i_0}. \tag{6}$$

Choose then a projection $Q \leqslant E_{i_1}$ with $\tau(Q) = 1$, and let us define the elements $S = \alpha_{i_0}(P - Q)$ and $Y = X - S$. Notice that

$$Y = \sum_{i \neq i_0, i_0} \alpha_i E_i + \alpha_{i_1}(E_{i_1} - Q) + \alpha_{i_0}(E_{i_0} - P) + (\alpha_{i_1} + \alpha_{i_0})Q,$$

so in particular we have $Y \perp P$. Notice also that either one of (5) or (6) yields

$$|\alpha_{i_1} + \alpha_{i_0}| \leqslant \max\{|\alpha_{i_0}|, |\alpha_{i_1}|\} \leqslant \|X\|,$$

so we have $\|Y\| \leqslant \|X\|$. Finally, since both Y an Q belong to the subalgebra

$$\mathcal{A} = (I_N - P)\mathrm{Mat}_N(\mathbb{C})(I_N - P),$$

which is $*$-isomorphic to $\mathrm{Mat}_{N-1}(\mathbb{C})$, using the inductive hypothesis, with Y and Q (which obviously commute), there exist $A_0, B_0 \in \mathcal{A}$, with
- $A_0 B_0 = B_0 A_0$;
- $Y = A_0 - B_0$;
- $A_0 \approx B_0$;
- $\max\{\|A_0\|, \|B_0\|\} \leqslant \|Y\| \leqslant \|X\|$;
- $A_0 \perp Q$.

It is now obvious that the elements $A = A_0 - \alpha_{i_0} Q$ and $B = B - \alpha_{i_0} P$ will satisfy the desired hypothesis (at one point, Proposition 1.1.B is invoked).

B. This statement is obvious from part A, since one can always start with an arbitrary projection $P \leqslant E_m$, with $\tau(P) = 1$, and such a projection obviously commutes with X. $\square$

In preparation for the type II_1 case, we have the following approximation result.

LEMMA 2.1. *Let $\mathcal{A}$ be an AW*-factor of type II_1, and let $\varepsilon > 0$ be a real number. For any element $X \in \mathcal{A}_{sa}$, there exists an AW*-subfactor $\mathcal{B} \subset \mathcal{A}$, of type $\mathrm{I}_{\mathrm{fin}}$, and an element $B \in \mathcal{B}_{sa}$ with*

(i) $d_{\mathcal{A}}(X, B) < \varepsilon$;
(ii) $q_{\mathcal{A}}(X) = q_{\mathcal{A}}(B)$;
(iii) $\|B\| \leqslant \|X\| + \varepsilon$.

Proof. We begin with the following

PARTICULAR CASE. *Assume X has finite spectrum.*

Let $\mathrm{Spec}(X) = \{\alpha_1 < \alpha_2 < \cdots < \alpha_m\}$, and let $E_1, \dots, E_m$ be the corresponding spectral projections, so that

- $D(E_i) > 0$, $\forall\, i \in \{1, \dots, m\}$;
- $E_i \perp E_j$, $\forall\, i \neq j$, and $\sum_{i=1}^m E_i = I$;
- $X = \sum_{i=1}^m \alpha_i E_i$, so $q_{\mathcal{A}}(X) = \sum_{i=1}^m \alpha_i D(E_i)$.

For any integer $n \geqslant 2$ define the set $Z_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right\}$, and let $\theta_n : \{1, \dots, m\} \to Z_n$ be the map defined by

$$\theta_n(i) = \max\left\{\zeta \in Z_n \,:\, \zeta \leqslant D(E_i)\right\}.$$

For every $i \in \{1, \dots, m\}$, and every integer $n \geqslant 2$, chose $P_{ni} \in \mathbf{P}(\mathcal{A})$ be an arbitrary projection with $P_{ni} \leqslant E_i$, and $D(P_{ni}) = \theta_n(i)$. Notice that, for a fixed $n \geqslant 2$, the projections $P_{n1}, \dots, P_{nm}$ are pairwise orthogonal, and have dimensions in the set Z_n, hence there exists a subfactor $\mathcal{B}_n$ of type I_n, that contains them. Define then the element $H_n = \sum_{i=1}^m \alpha_i P_{ni} \in (\mathcal{B}_n)_{sa}$. Note that $\|H_n\| \leqslant \|X\|$. We wish to prove that

(A) $\lim_{n\to\infty} d_{\mathcal{A}}(X, H_n) = 0$;
(B) $\lim_{n\to\infty} q_{\mathcal{A}}(H_n) = q_{\mathcal{A}}(X)$.

To prove these assertions, we first observe that, for each $n \geqslant 2$, the elements X and H_n commute, and we have

$$X - H_n = \sum_{i=1}^m \alpha_i (E_i - P_{ni}).$$

In particular, one has

$$|q_{\mathcal{A}}(X) - q_{\mathcal{A}}(H_n)| \leqslant \sum_{i=1}^m |\alpha_i| \cdot D(E_i - P_{ni}) \leqslant m\|X\| \cdot \max\left\{D(E_i - P_{ni})\right\}_{i=1}^m. \quad (7)$$

Likewise, since

$$(X - H_n)^*(X - H_n) = \sum_{i=1}^{m} \alpha_i^2 (E_i - P_{ni}),$$

we have

$$\begin{aligned} q_{\mathcal{A}}\big((X - H_n)^*(X - H_n)\big) &= \sum_{i=1}^{m} \alpha_i^2 \cdot D(E_i - P_{ni}) \\ &\leqslant m\|X\|^2 \cdot \max\big\{D(E_i - P_{ni})\big\}_{i=1}^{m}. \end{aligned} \tag{8}$$

By construction however we have $D(E_i - P_{ni}) < \frac{1}{n}$, so the estimates (7) and (8) give

$$|q_{\mathcal{A}}(X) - q_{\mathcal{A}}(H_n)| \leqslant \frac{m\|X\|}{n}$$
$$d_{\mathcal{A}}(X, H_n) \leqslant \sqrt[3]{\frac{m\|X\|^2}{n}},$$

which clearly give the desired assertions (A) and (B).

Using the conditions (A) and (B), we immediately see that, if we define the numbers $\beta = q_{\mathcal{A}}(X)$, $\beta_n = q_{\mathcal{A}}(H_n)$, and the elements $B_n = H_n + (\beta - \beta_n)I \in \mathcal{B}_n$, then the sequence $(B_n)_{n\geqslant 2}$ will still satisfy $\lim_{n\to\infty} d_{\mathcal{A}}(X, B_n) = 0$, but also $q_{\mathcal{A}}(B_n) = q_{\mathcal{A}}(B_n)$, and

$$\|B_n\| \leqslant \|H_n\| + |\beta - \beta_n\| \leqslant \|X\| + |\beta - \beta_n|,$$

which concludes the proof of the Particular Case.

Having proven the Particular Case, we now proceed with the general case. Start with an arbitrary element $X \in \mathcal{A}_{sa}$, and pick a sequence $(T_n)_{n\in\mathbb{N}} \subset \mathcal{A}_{sa}$ of elements with finite spectrum, such that $\lim_{n\to\infty} \|T_n - X\| = 0$. (This can be done using Borel functional calculus.) Using the norm-continuity of the quasitrace, we have $\lim_{n\to\infty} q_{\mathcal{A}}(T_n) = q_{\mathcal{A}}(X)$, so if we define $X_n = T_n + \big(q_{\mathcal{A}}(X) - q_{\mathcal{A}}(T_n)\big)I$, we will still have $\lim_{n\to\infty} \|X_n - X\| = 0$, but also $q_{\mathcal{A}}(X_n) = q_{\mathcal{A}}(X)$. In particular, there exists some $k \geqslant 1$, such that

- $d_{\mathcal{A}}(X_k, X) < \varepsilon/2$;
- $\|X_k\| < \|X\| + \varepsilon/2$.

Finally, applying the Particular Case, we can also find an AW*-subfactor $\mathcal{B} \subset \mathcal{A}$, of type $\mathrm{I}_{\mathrm{fin}}$, and an element $B \in \mathcal{B}_{sa}$ with

- $d_{\mathcal{A}}(X_k, B) < \varepsilon/2$;
- $\|B\| < \|X_k\| + \varepsilon/2$;
- $q_{\mathcal{A}}(X_k) = q_{\mathcal{A}}(B)$.

It is then trivial that B satisfies conditions (i)-(iii). $\square$

We are now in position to prove the main result in the type II_1 case.

THEOREM 2.2. *Let $\mathbf{A} = (\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence of AW*-factors of type II_1, let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and let $X \in (\mathbf{A}_{\mathcal{U}})_{sa}$ be an element with $q_{\mathbf{A}_{\mathcal{U}}}(X) = 0$. Then there exist elements $A, B \in (\mathbf{A}_{\mathcal{U}})_{sa}$ with*

- *$AB = BA$;*

- $X = A - B$;
- $A \approx B$;
- $\max\{\|A\|, \|B\|\} \leqslant \|X\|$.

In particular, X is an abelian self-commutator in $\mathbf{A}_{\mathcal{U}}$.

Proof. Write $X = \Pi_{\mathcal{U}}(\boldsymbol{x})$, where $\boldsymbol{x} = (X_n)_{n\in\mathbb{N}} \in \mathbf{A}^{\infty}$. Without loss of generality, we can assume that all X_n's are self-adjoint, and have norm $\leqslant \|X\|$.

Consider the elements $\tilde{X}_n = X_n - q_{\mathcal{A}_n}(X_n)I \in \mathcal{A}_n$. Remark that, since X_n is self-adjoint, we have $|q_{\mathcal{A}_n}(X_n) \leqslant \|X_n\| \leqslant \|X\|$, so we have $\|\tilde{X}_n\| \leqslant 2\|X\|$, $\forall n \in \mathbb{N}$, hence the sequence $\tilde{\boldsymbol{x}} = (\tilde{X}_n)_{n\in\mathbb{N}}$ defines an element in $\mathbf{A}^{\infty}$. By construction, we have $\lim_{\mathcal{U}} q_{\mathcal{A}_n}(X_n) = 0$, and $d_{\mathcal{A}_n}(\tilde{X}_n, X_n) = \left|q_{\mathcal{A}_n}(X_n)\right|^{\frac{2}{3}}$, so by Remark 1.2.A it follows that $X = \Pi_{\mathcal{U}}(\boldsymbol{x}) = \Pi_{\mathcal{U}}(\tilde{\boldsymbol{x}})$.

Use Lemma 2.1 to find, for each $n \in \mathbb{N}$, an AW*-subfactor $\mathcal{B}_n$ of $\mathcal{A}_n$ of type $\mathrm{I}_{\mathrm{fin}}$, and and elements $Y_n \in (\mathcal{B}_n)_{sa}$ with

(i) $d_{\mathcal{A}_n}(Y_n, \tilde{X}_n) < \frac{1}{n}$;
(ii) $q_{\mathcal{A}_n}(Y_n) = 0$, $\forall n \in \mathbb{N}$;
(iii) $\|Y_n\| \leqslant \|\tilde{X}_n\| + \frac{1}{n}$.

Furthermore, using Theorem 2.1, for each $n \in \mathbb{N}$, combined with the fact that $q_{\mathcal{B}_n} = q_{\mathcal{A}_n}\big|_{\mathcal{B}_n}$ (which implies the equality $q_{\mathcal{B}_n}(Y_n) = 0$), there exist elements $A_n, B_n \in (\mathcal{B}_n)_{sa}$ such that

(A) $A_nB_n = B_nA_n$;
(B) $Y_n = A_n - B_n$;
(C) $A_n \approx B_n$;
(D) $\max\{\|A_n\|, \|B_n\|\} \leqslant \|Y_n\|$.

Choose $U_n \in \mathbf{U}(\mathcal{B}_n)$, such that $B_n = U_nA_nU_n^*$.

Let us view the sequences $\boldsymbol{a} = (A_n)_{n\in\mathbb{N}}$, $\boldsymbol{b} = (B_n)_{n\in\mathbb{N}}$, $\boldsymbol{u} = (U_n)_{n\in\mathbb{N}}$ as elements in the AW*-algebra $\mathbf{A}^{\infty}$, and let us define the elements $A = \Pi_{\mathcal{U}}(\boldsymbol{a})$, $B = \Pi_{\mathcal{U}}(\boldsymbol{b})$, and $U = \Pi_{\mathcal{U}}(\boldsymbol{u})$ in $\mathbf{A}_{\mathcal{U}}$. Obviously A and B are self-adjoint. Since $\boldsymbol{u}$ is unitary in $\mathbf{A}^{\infty}$, it follows that U is unitary in $\mathbf{A}_{\mathcal{U}}$. Moreover, since by construction we have $\boldsymbol{uau}^* = \boldsymbol{b}$, we also have the equality $UAU^* = B$, so $A \approx B$ in $\mathbf{A}_{\mathcal{U}}$. Finally, since by construction we also have $\boldsymbol{ab} = \boldsymbol{ba}$, we also get the equality $AB = BA$. Since by condition (D) we have

$$\max\{\|A_n\|, \|B_n\|\} \leqslant \|Y_n\| \leqslant \|\tilde{X}_n\| + \tfrac{1}{n} \leqslant \|X\| + |q_{\mathcal{A}_n}(X_n)| + \tfrac{1}{n},$$

by Remark 1.2.B (combined with $\lim_{\mathcal{U}} q_{\mathcal{A}_n}(X_n) = 0$), we get the inequality

$$\max\{\|A\|, \|B\|\} \leqslant \|X\|.$$

The proof of the Theorem will then be finished, once we prove the equality $X = A - B$. For this purpose, we consider the sequences $\tilde{\boldsymbol{x}} = (\tilde{X}_n)_{n\in\mathbb{N}}$ and $\boldsymbol{y} = (Y_n)_{n\in\mathbb{N}}$, both viewed as elements in $\mathbf{A}^{\infty}$. On the one hand, since by construction we have $\boldsymbol{y} = \boldsymbol{a} - \boldsymbol{b}$, we get the equality $\Pi_{\mathcal{U}}(\boldsymbol{y}) = A - B$. On the other hand, since $\lim_{n\to\infty} d_{\mathcal{A}}(Y_n, \tilde{X}_n) = 0$, we also have $\lim_{\mathcal{U}} d_{\mathcal{A}}(Y_n, \tilde{X}_n) = 0$, so by Remark 1.2.A we get the equalities $X = \Pi_{\mathcal{U}}(\boldsymbol{x}) = \Pi_{\mathcal{U}}(\tilde{\boldsymbol{x}}) = \Pi_{\mathcal{U}}(\boldsymbol{y})$, i.e. $X = A - B$. $\square$

COMMENT. Assume $\mathcal{A}$ an AW*-factor of type II_1, and let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Following Example 1.1, $\mathcal{A}$ is identified with the AW*-subfactor $\Delta_{\mathcal{U}}(\mathcal{A})$ of $\mathcal{A}_{\mathcal{U}}$.

Under this identification, by Theorem 2.2, every element $A \in \mathcal{A}_{sa}$ of quasitrace zero is an abelian self-commutator in $\mathcal{A}_{\mathcal{U}}$.

In connection with this observation, it is legitimate to ask whether A is in fact an abelian self-commutator in $\mathcal{A}$ itself. The discussion below aims at answering this question in a somewhat different spirit, based on the results from [5].

DEFINITION. Let $\mathcal{A}$ be an AW*-factor of type II_1. An element $A \in \mathcal{A}_{sa}$ is called an *abelian approximate self-commutator in* $\mathcal{A}$, if there exist commuting elements $A_1, A_2 \in \mathcal{A}_{sa}$ with $A = A_1 - A_2$, and such that A_1 and A_2 are approximately unitary equivalent, i.e. there exists a sequence $(U_n)_{n\in\mathbb{N}}$ of unitaries in $\mathcal{A}$ such that $\lim_{n\to\infty} \|U_n A_1 U_n^* - A_2\| = 0$. By [5, Theorem 2.1] the condition that A_1 and A_2 are approximately unitary equivalent – denoted by $A_1 \sim A_2$ – is equivalent to the condition $q_{\mathcal{A}}(A_1^k) = q_{\mathcal{A}}(A_2^k)$, $\forall k \in \mathbb{N}$. In particular, it is obvious that abelian approximate self-commutators have quasitrace zero.

With this terminology, one has the following result.

THEOREM 2.3. *Let $\mathcal{A}$ be an AW*-factor of type II_1, and let $X \in \mathcal{A}_{sa}$ be an element with $q_{\mathcal{A}}(X) = 0$. If $D\big(\mathbf{s}(X)\big) < 1$, then X can be written as a sum $X = X_1 + X_2$, where X_1, X_2 are two commuting abelian approximate self-commutators in $\mathcal{A}$.*

Proof. Let $P = I - \mathbf{s}(X)$. Using the proof of Theorem 5.2 from [5], there exist elements $A_1, A_2, B_1, B_2, Y_1, Y_2, S_1, S_2 \in \mathcal{A}_{sa}$, with the following properties:

(i) $A_1, A_2, B_1, B_2, Y_1, Y_2, S_1, S_2$ all commute;

(ii) $A_1 \sim B_1$, $A_2 \sim B_2$, $Y_1 \sim S_1$, $Y_2 \sim S_2$, and $S_1 + S_2$ is spectrally symmetric, i.e. $(S_1 + S_2) \sim -(S_1 + S_2)$;

(iii) $A_1 \perp A_2, B_1, P$, $A_2 \perp B_2, P$, $B_1 \perp B_2, P$, and $B_2 P = P B_2 = Y_1 + Y_2$;

(iv) $Y_1, Y_2 \perp S_1, S_2$;

(v) $Y_1, Y_2, S_1, S_2 \in P\mathcal{A}P$;

(vi) $X = A_1 - B_1 + A_2 - B_2 + Y_1 + Y_2$.

Consider then the elements

$$V_1 = A_1 + Y_1 - S_2; \qquad V_2 = A_2 + \tfrac{1}{2}(S_1 + S_2);$$
$$W_1 = B_1 + S_1 - Y_2; \qquad W_2 = B_2 - \tfrac{1}{2}(S_1 + S_2).$$

Using the orthogonal additivity of approximate unitary equivalence (Corollary 2.1 from [5]), and the above conditions, it follows that $V_1 \sim W_1$ and $V_2 \sim W_2$. Since V_1, V_2, W_1, W_2 all commute, it follows that the elements $X_1 = V_1 - W_1$ and $X_2 = V_2 - W_2$ are abelian approximate self-commutators, and they commute. Finally, one has $X_1 + X_2 = A_1 - B_1 + A_2 - B_2 + Y_1 + Y_2 = X$. $\square$

COROLLARY 2.1. *Let $\mathcal{A}$ be an AW*-factor of type II_1, and let $X \in \mathcal{A}_{sa}$ be an element with $q_{\mathcal{A}}(X) = 0$. There exist two commuting abelian approximate self-commutators $X_1, X_2 \in \text{Mat}_2(\mathcal{A})$ – the 2×2 matrix algebra – such that*

$$X_1 + X_2 = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}. \tag{9}$$

(According to Berberian's Theorem (see [1]), the matrix algebra $\mathrm{Mat}_2(\mathcal{A})$ is an AW*-factor of type II_1.)

Proof. Denote the matrix algebra $\mathrm{Mat}_2(\mathcal{A})$ by $\mathcal{A}_2$, and let $\tilde{X} \in \mathcal{A}_2$ denote the matrix in the right hand side of (9). It is obvious that, if we consider the projection

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

then $\mathbf{s}(\tilde{X}) \leqslant E$. Since $D_{\mathcal{A}_2}(E) = \frac{1}{2} < 1$, and $q_{\mathcal{A}_2}(\tilde{X}) = \frac{1}{2} q_{\mathcal{A}}(X) = 0$, the desired conclusion follows immediately from Theorem 2.3. □

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