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REFINEMENTS ON THE INTERLACING OF EIGENVALUES OF CERTAIN TOTALLY NONNEGATIVE MATRICES

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(*communicated by Chi-Kwong Li*)

Abstract. It has long been known that the eigenvalues of a totally positive matrix interlace the eigenvalues of its maximal leading principal submatrix. Motivated by recent questions arising from studying the roots of certain biorthogonal polynomials, we extend the classical strict interlacing fact to other classes of totally nonnegative matrices.

1. Introduction

An $n \times n$ matrix A is called *totally nonnegative (positive)*, TN (TP), if *all* minors of A are nonnegative (positive) (see also [1, 12, 14]). We write $A \in$ TN (TP). An important subclass of the totally nonnegative matrices, which arises in many motivating applications, is called the *oscillatory matrices*. An $n \times n$ matrix is called oscillatory if it is totally nonnegative and some positive integral power of it is totally positive. The class of oscillatory matrices will be denoted by OSC. Finally, an $n \times n$ matrix $A \in$ TN is called *reducible* if A has a zero-block of size $k \times j$ with $k + j = n$, where the row indices and column indices are contiguous, and the row indices either start with 1 or end in n. If such a matrix is not reducible, then we say it is *irreducible*.

In their pioneering work on the theory of oscillatory matrices and sign-regular matrices, Gantmacher and Krein [11] developed in large part the spectral theory of oscillatory matrices. From the numerous results they established we will only mention a few key facts.

THEOREM 1. [11] *Suppose A is an $n \times n$ oscillatory matrix. Then:*

1. *The eigenvalues of A are real, positive and distinct.*
2. *If B is the $(n-1) \times (n-1)$ leading principal submatrix of A based on the first $n-1$ rows and columns, then B is an oscillatory matrix.*
3. *If B is as in (2) and the eigenvalues of A and B are denoted by $\lambda_1 < \lambda_2 < \cdots < \lambda_n$; $\mu_1 < \mu_2 < \cdots < \mu_{n-1}$, respectively, then*

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \lambda_{n-1} < \mu_{n-1} < \lambda_n \tag{1}$$

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As noted in [11], a classical example of an oscillatory matrix is an entry-wise nonnegative and positive definite tridiagonal matrix – such matrices are referred to as Jacobi matrices and appear naturally in the context of orthogonal polynomials. We remind the reader that any irreducible entry-wise nonnegative tridiagonal matrix is diagonally similar (via a positive diagonal matrix) to a symmetric matrix (see [11, pg. 71]). We will be making use of both of these facts throughout the paper.

Another important fact discovered by Gantmacher and Krein [11] and applied elsewhere (see [8]) is a simple criterion for a totally nonnegative matrix to be oscillatory.

THEOREM 2. *Let $A = [a_{ij}]$ be an $n \times n$ totally nonnegative matrix. Then A is oscillatory if and only if A is invertible and $a_{i,i+1} > 0, a_{i+1,i} > 0$, for each $i = 1, 2, \dots, n-1$.*

In Theorem 1 (3), we refer to the list of inequalities in (1) as *strict interlacing*. If we have instead the list of inequalities $\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \mu_2 \leqslant \cdots \leqslant \lambda_{n-1} \leqslant \mu_{n-1} \leqslant \lambda_n$, then we say the eigenvalues of A and B *interlace*. If A is TN and not OSC, then the eigenvalues of A and B (as defined in Theorem 1 (3)) interlace, but need not strictly interlace (let A be the identity matrix, for example).

From an historical perspective eigenvalue interlacing type results were originally studied in connection with Hermitian matrices (see [13, Thm. 4.3.8]). Of course for Hermitian matrices the location of the principal submatrix is irrelevant, since Hermitian matrices are closed under simultaneous row and column permutations. More precisely, if A is an $n \times n$ Hermitian matrix and $B = A(i)$ is the $(n-1) \times (n-1)$ principal submatrix obtained from A by deleting the ith row and column, then the eigenvalues of A and B interlace. For oscillatory matrices, it is crucial that $i = 1$ or $i = n$. As a matter of completeness we demonstrate the previous claim with an example (see also [15]). Let

$$B = \begin{bmatrix} 1 & 1 & 0.1 \\ 2 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix}. \tag{2}$$

Then B is a TN matrix and the eigenvalues of B are approximately $0, .2111, 3.7889$. Moreover, the eigenvalues of the 2×2 principal submatrix indexed by rows 1 and 3 (which is OSC) are $.5528, 1.4472$. So in this case it is clear that conventional interlacing of the eigenvalues does not hold for all principal submatrices of a TN matrix.

For generally positioned maximal principal submatrices of OSC matrices, the most general eigenvalue interlacing results seems to be contained in the recent work of Pinkus [18].

Our interest in eigenvalue interlacing of totally nonnegative matrices is motivated from studying the roots of certain biorthogonal polynomials (see [20] for background and section 4 for more detailed information).

2. Background and Key Lemmas

For an $n \times n$ matrix A, $\alpha \subseteq \{1, 2, \dots, m\}$, and $\beta \subseteq \{1, 2, \dots, n\}$, the submatrix of A lying in rows indexed by α and the columns indexed by β will be denoted by $A[\alpha|\beta]$. Similarly, $A(\alpha|\beta)$ is the submatrix obtained from A by deleting the rows indexed by α and columns indexed by β. If $\alpha = \beta$, then the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$, and the complementary principal submatrix to $A(\alpha)$. For brevity, if $\alpha = \{i\}$, then $A(\alpha)$ is denoted by $A(i)$.

The $(i,j)^{th}$ standard basis matrix, that is the $n \times n$ matrix whose only nonzero entry is in the $(i,j)^{th}$ position and this entry is a one, will be denoted by E_{ij}. Let us denote the matrix $I + \alpha E_{i,i-1}$ by $L_i(\alpha)$, $i = 2,3,\ldots,n$, where as usual I denotes the identity matrix. Observe that $L_i(\alpha)^{-1} = L_i(-\alpha)$ and that $L_i(\alpha)$ is TN whenever $\alpha \geqslant 0$.

The following lemma, originally proved by Whitney [21] in the 50's, has become a very powerful tool for studying many aspects of totally nonnegative matrices.

LEMMA 3. *Suppose $A = [a_{ij}]$ has the property that $a_{j1} > 0$, and $a_{t1} = 0$ for all $t > j+1$. Then A is TN if and only if $L(-a_{j+1,1}/a_{j1})A$ is TN.*

In essence, Lemma 3 claims that applying row operations on a totally nonnegative matrix to annihilate certain entries preserves total nonnegativity. This result coupled with the fact that total nonnegativity is closed under matrix multiplication was the genesis for the so-called bidiagonal factorization of totally nonnegative matrices (see [7, 10]).

As noted in the introduction if $A = [a_{ij}]$ is an oscillatory matrix, then $a_{i,i+1} > 0, a_{i+1,i} > 0$, for each $i = 1,2,\ldots,n-1$. Hence every oscillatory matrix is irreducible. Furthermore, it is not difficult to verify (by considering 2×2 minors only) that any totally nonnegative matrix $A = [a_{ij}]$ is irreducible if and only if $a_{i,i+1} > 0, a_{i+1,i} > 0$, for each $i = 1,2,\ldots,n-1$. In fact much more is known about the patterns of zero entries in an TN matrix. In the following definition the symbol $*$ in a matrix means the corresponding entry is nonzero.

DEFINITION 4. An $n \times n$ matrix A is said to be *in double echelon form* if

(i) Each row of A has one of the following forms:

1. $(*,*,\cdots,*)$,
2. $(*,\cdots,*,0,\cdots,0)$,
3. $(0,\cdots,0,*,\cdots,*)$, or
4. $(0,\cdots,0,*,\cdots,*,0,\cdots,0)$.

(ii) The first and last nonzero entries in row $i+1$ are not to the left of the first and last nonzero entries in row i, respectively ($i = 1,2,\ldots,n-1$).

Thus a matrix in double echelon form appears as follows

$$\begin{bmatrix} * & * & 0 & \cdots & 0 \\ * & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * & * \end{bmatrix}.$$

It is not difficult to see that any TN matrix with no zero rows or columns must be in double echelon form (see also [9]). This result is sometimes referred to as shadows' lemma (see [5]), although a more general version is also presented in [5] involving minors of larger sizes.

Rainey and Halbetler [19] were interested in a reduction to tridiagonal form and used Whitney's result and the above criterion for irreducibility to demonstrate a useful factorization result for TN matrices.

THEOREM 5. *If A is an $n \times n$ nonsingular totally nonnegative matrix, then there exists a nonsingular totally nonnegative matrix S and a tridiagonal totally nonnegative matrix T such that*

(i) $TS = SA$, *and*

(ii) *the matrices A and T have the same eigenvalues.*

Cryer [4] extended the above result to the singular case. It should be noted that if A is irreducible and TN, then the TN tridiagonal matrix T produced during this reduction process is reducible, only if it contains a zero line (that is either a zero row or a zero column). For more details consult [9].

We finish this preparatory section with two final remarks on certain spectral properties of irreducible TN matrices.

The first result is concerned with the positive eigenvalues of irreducible TN matrices (see [9]).

THEOREM 6. *Let A be an $n \times n$ irreducible totally nonnegative matrix. Then the positive eigenvalues of A are distinct.*

Finally, we close by mentioning that in a recent paper by Li and Mathias [17] an alternative proof was presented for the eigenvalue interlacing result (Theorem 1) of Gantmacher and Krein. Their key idea was the fact that applying Whitney type row operations to reduce an oscillatory matrix A to tridiagonal form T still yields a similarity between $A[\{2,3,\ldots,n\}]$ and $T[\{2,3,\ldots,n\}]$. Consequently, eigenvalue interlacing of A reduces to the well-known tridiagonal case.

3. Main Results and Proofs

In this section we state and prove our main result, which mildly extends Gantmacher and Krein's original result on the strict interlacing of the eigenvalues of totally positive matrices.

THEOREM 7. *Suppose $A = [a_{ij}]$ is an $n \times n$ totally nonnegative matrix and assume that $A(1)$ (respectively, $A(n)$) is oscillatory. Then strict interlacing holds between the eigenvalues of A and of $A(1)$ (respectively, $A(n)$) if and only if A is irreducible.*

We prove the theorem for the case that $A(1)$ is oscillatory. The other case is proved in a similar way.

Proof. ($\Longrightarrow$) If A is TN, reducible and $A(1)$ is oscillatory, then from the remarks in the previous section it follows that either $a_{12} = 0$ or $a_{21} = 0$, or both are zero. Assume, without loss of generality, that $a_{12} = 0$ (consider transposition otherwise). Since $A(1)$ is OSC, it follows that $A(1)$ does not contain any zero lines. Hence $a_{1k} = 0$ for each $k = 2, 3, \ldots, n$. Thus A has the form,

$$A = \left[\begin{array}{c|c} a_{11} & 0 \\ \hline y & A(1) \end{array}\right].$$

From which it follows that the eigenvalues of $A(1)$ are contained among the eigenvalues of A, and consequently strict interlacing does not hold.

($\Longleftarrow$) Suppose A is TN, irreducible, and that $A(1)$ is OSC. Now if the rank of A is n, then it follows from Theorem 2 that A is OSC. Thus an application of Theorem 1 implies that the eigenvalues of A and of $A(1)$ strictly interlace. Thus assume that A is not invertible. Then, by hypothesis, the rank of A must be $n-1$. Since $A(1)$ is invertible and A is TN, it follows that the coefficient of x (the linear term) in the characteristic polynomial of A is different than zero. Thus the algebraic multiplicity of the eigenvalue zero is equal to one. Therefore A must have $n-1$ positive eigenvalues. Furthermore, by an application of Theorem 6 we know that these $n-1$ positive eigenvalues must be distinct. Hence the eigenvalues of A are distinct real numbers and the eigenvalues of $A(1)$ are also distinct real numbers.

The idea now is to use the elementary matrices L_i to eliminate the entries in the first column (and row) of A up to $a_{21} > 0$ (and $a_{12} > 0$). Suppose A is of the form

$$A = \left[\begin{array}{c|c} a_{11} & x^t \\ \hline y & A(1) \end{array}\right],$$

and assume that an invertible TN elementary matrix L_i ($i > 2$) is partitioned conformally with A and denoted by

$$L_i = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & L' \end{array}\right].$$

Then applying the similarity $L_iA(L_i)^{-1}$ yields the matrix

$$B = L_iA(L_i)^{-1} = \left[\begin{array}{c|c} a_{11} & x^t(L')^{-1} \\ \hline L'y & L'A(1)(L')^{-1} \end{array}\right]. \tag{3}$$

There are two things to note about the matrix B in (3). Firstly, it follows that $B(1)$ is similar to $A(1)$, and since $A(1)$ is invertible, so is $B(1)$. Secondly, if B is formed by applying Whitney's theorem (Lemma 3), then B will remain TN. Now suppose $A = [a_{ij}]$ satisfies $a_{j+1,1} > 0$, for some j with $2 \leqslant j+1 \leqslant n$ and $a_{t1} = 0$ for all $t > j+1$ (if $j+1 = n$, then ignore t). Then there exists an irreducible totally nonnegative matrix $B = [b_{ij}]$ as in (3) such that $b_{j+1,1} = 0$. Define i_1 to be the smallest column index such that $a_{j+2,i_1} > 0$. (Note that since A is irreducible $i_1 \leqslant j+1$.) If $j+1 = 2$, then proceed to column i_1. Since A is irreducible, $a_{st} > 0$ for all s,t with $|s-t| \leqslant 1$, hence it follows that $i_1 = 2$, in the case when $j+1 = 2$. Otherwise $j+1 > 2$. Use row $j \geqslant 2$ to eliminate $a_{j+1,1}$ (note that since A is irreducible $a_{j1} > 0$), via the elementary nonsingular matrix $L_j(\alpha)$. Consider the matrix $B = L_j(\alpha)A(L_j(\alpha))^{-1}$. By Lemma 3,

$L_j(\alpha)A$ is totally nonnegative, and since $(L_j(\alpha))^{-1}$ is totally nonnegative, we have that B is totally nonnegative. Clearly the $(j+1,1)^{st}$ entry of B is zero. Observe that the only possible zero row that could arise in B is necessarily the $(j+1)^{st}$ row. However, this is impossible since $B(1)$ is similar to the OSC matrix $A(1)$ and $j+1 \geqslant 2$. Then we must show that B is irreducible. There are two cases to consider: (1): $i_1 < j+1$; or (2): $i_1 = j+1$. Suppose that the first nonzero entry in row $j+1$ of B is in column t, where $2 \leqslant t \leqslant i_1$. If $i_1 < j+1$, then we have $t \leqslant i_1 < j+1$ and in this case $b_{kl} > 0$ whenever $|k-l| \leqslant 1$, so in particular, $b_{j+1,j} > 0$, $b_{j+1,j+1} > 0$, and $b_{j+1,j+2} > 0$. Otherwise, suppose $i_1 = j+1$. Now if $t < i_1$, then the same reasoning as above applies. Therefore assume that $t = i_1 = j+1$. After eliminating $a_{j+1,1}$, the worst possible case for row $j+1$ of $L_j(\alpha)A$ is that it has the form $[0,0,\ldots,0,*,\ldots]$, where the first nonzero occurs in column $j+1$. However, upon premultiplying $L_j(\alpha)A$ by $L_j(\alpha)^{-1}$ we add a positive multiple of column $j+1$ to column j. By the definition of t, it follows that the $(j+1,j+1)^{st}$ entry of $L_j(\alpha)A$ is nonzero, and hence $b_{j+1,j} > 0$. Consider the entry $b_{j+1,j+2}$. To show that this entry must be positive, suppose otherwise, i.e., assume $b_{j+1,j+2} = 0$. But in this case $a_{j,j+2}$ must be positive since it was used to eliminate this entry. Hence it is not possible that $b_{j+1,j+2} = 0$; otherwise the 2×2 minor based on rows $\{j, j+1\}$ and columns $\{j+1, j+2\}$ is negative, which is a contradiction. Therefore B is irreducible since $b_{kl} > 0$ for all k,l with $|k-l| \leqslant 1$. Thus since B is irreducible it follows that $B(1)$ must be OSC. Continue this process until we have eliminated all of the entries in column 1, below row 2. According to the above analysis A will then be similar to an irreducible TN matrix C whose principal submatrix is similar to $A(1)$. To continue the proof, we proceed on to column 2 and eliminate the entries in column 2 below row 3. This process is continued until all the entries below the subdiagonal of A have been eliminated. We then consider transposition, and apply the above process to the transpose of the resulting matrix. Once all of the entries below the subdiagonal of the transposed matrix have been eliminated, we arrive at the final matrix D. It is crucial to observe that D is an irreducible, TN, tridiagonal matrix whose eigenvalues are identical with the eigenvalues of A. Furthermore, the eigenvalues of $D(1)$ are equal to the eigenvalues of $A(1)$. This follows since both similarity and transposition preserves eigenvalues. Thus we have reduced interlacing to the classical tridiagonal case and since D is irreducible and has rank $n-1$ strict interlacing follows (see [11, pg. 68, item 2°]). This completes the proof. $\square$

We note that simple eigenvalues played a role in the above analysis, but it seems that assuming $A(1)$ is OSC is critical. For example, if we had just assumed that A is TN, $A(1)$ irreducible, and that the eigenvalues of both A and $A(1)$ are distinct, then strict interlacing need not hold. Consider the matrix

$$A = \begin{bmatrix} 10 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then A and $A(1)$ satisfy the hypothesis above but both are singular, so strict interlacing does not hold. Of course, if $A(1)$ has distinct positive eigenvalues only, then $A(1)$ is OSC and we are back to Theorem 7.

In addition, given the example in (2) it is clear that the principal submatrix of interest must be (at least) based upon contiguous index sets (observe that B in (2) satisfies the hypotheses in Theorem 7). Thus there is no generalization of Theorem 7 that consists of replacing $A(1)$ by $A(i)$, where $i \in \{2,3,\ldots,n-1\}$.

Also, it is not clear that you can consider smaller oscillatory submatrices, that is, assume that $A(\{1,2\})$ is OSC and hope to ensure some alternate version of strict interlacing. For example, consider the irreducible TN matrix:

$$A = \begin{bmatrix} 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then $A(\{1,2\})$ is OSC, but both A and $A(1)$ are singular, and A has zero as an eigenvalue of mutliplicity two.

However, we can offer a mild generalization of Theorem 7 to include smaller sized contiguous principal submatrices. A simple consequence of (1) (consult [13] for the Hermitian case) is that if A is an $n \times n$ TN matrix and B is an $(n-r)\times(n-r)$ principal submatrix of A based on a contiguous index set, and if the eigenvalues of A and B are denoted by $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$; $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_{n-r}$, respectively, then

$$\lambda_k \leqslant \mu_k \leqslant \lambda_{k+r}, \tag{4}$$

where $k = 1,2,\ldots,n-r$. When A is OSC the interlacing is strict, and we have

$$\lambda_k < \mu_k < \lambda_{k+r}, \tag{5}$$

where $k = 1,2,\ldots,n-r$.

We are now in a position to extend Theorem 7 to include certain smaller sized principal submatrices.

COROLLARY 8. *Suppose A is an $n\times n$ totally nonnegative matrix and assume that $B = A[\{i,i+1,\ldots,i+n-r-1\}]$ is an $(n-r)\times(n-r)$ contiguous principal submatrix of A that is oscillatory. Then strict interlacing holds (in the sense of (5)) between the eigenvalues of A and of B if and only if at least one of $A[\{i-1,i,i+1,\ldots,i+n-r-1\}]$ or $A[\{i,i+1,\ldots,i+n-r-1,i+n-r\}]$ is irreducible.*

Proof. Suppose that $C = A[\{i-1,i,i+1,\ldots,i+n-r-1\}]$ is irreducible (in the event that $i=1$ it is evident that the only submatrix of interest is $A[\{i,i+1,\ldots,i+n-r-1,i+n-r\}]$, to which a similar argument will apply). Let the eigenvalues of C be $\nu_1 < \nu_2 < \cdots < \nu_{n-r+1}$ (from the proof of Theorem 7 we know that the eigenvalues of C are distinct). Applying Theorem 7 to the submatrix C, we may conclude that strict interlacing holds between the eigenvalues of C and B; that is, $\nu_k < \mu_k < \nu_{k+1}$, for $k = 1,2,\ldots,n-r$. These strict inequalities combined with the usual interlacing inequalities that hold between the eigenvalues of A and C (see (5)) imply

$$\lambda_k \leqslant \nu_k < \mu_k < \nu_{k+1} \leqslant \lambda_{k+r}.$$

Hence strict interlacing holds between the eigenvalues of A and B, as desired.

The converse follows easily from the proof of Theorem 7 in the sense that if both $A[\{i-1,i,i+1,\dots,i+n-r-1\}]$ or $A[\{i,i+1,\dots,i+n-r-1,i+n-r\}]$ are reducible, then A will be block triangular with B as one of its main diagonal blocks. $\square$

4. Applications to Biorthogonal Polynomials

In the theory of coupled random matrices it is of interest to study the biorthogonal polynomials associated with the measure $d\mu(x,y) = e^{-V(x)-W(y)+2\tau xy}$, where V and W are polynomials of even degree and $\tau \in \mathbb{R}\setminus\{0\}$ (see, e.g., [2], [6], [16]). The *biorthogonal polynomials* related to this measure are defined as two families of polynomials $\{p_j\}$ and $\{q_k\}$, so that $\deg p_j = j$ and $\deg q_k = k$ and

$$\int_{\mathbb{R}}\int_{\mathbb{R}} p_j(x)q_k(y)d\mu(x,y) = 0,\ j \neq k. \tag{6}$$

When we normalize the polynomials p_j and q_k to be monic, they are uniquely defined by (6). While traditional orthogonal polynomials have the interlacing roots property (see, e.g., [3]) it is still an open question whether biorthogonal polynomials exhibit this property (see [6]). In [20] it was recently shown that the biorthogonal polynomials related to the measure $e^{-x^4-y^2+2\tau xy}dxdy$ have the interlacing property. To be precise, the following was the main result in [20].

THEOREM 9. [20] *Let $d\mu(x,y) = e^{-x^4-y^2+2\tau xy}dxdy$. The roots of the polynomials q_k and q_{k-1} interlace. When k is even the interlacing is strict.*

For the particular measure above the polynomials p_j are traditional orthogonal polynomials with respect to the measure $e^{-x^4+\tau^2x^2}dx$ on $\mathbb{R}$, and therefore they have the interlacing roots property automatically. As a corollary of Theorem 7 we now have the following sharpening of Theorem 9.

COROLLARY 10. *Let $d\mu(x,y) = e^{-x^4-y^2+2\tau xy}dxdy$. The roots of the polynomials q_k and q_{k-1} also interlace strictly when k is odd.*

In order to prove Corollary 10, we need to recall some of the arguments used in [20]. As the polynomials $p_j(x)$ are classical orthogonal polynomials they satisfy a three term recurrence relation which in this case has the form

$$xp_j(x) = p_{j+1}(x) + a_j^2 p_{j-1}(x),\ j = 0,1,\dots, \tag{7}$$

where $p_{-1} \equiv 0$, $a_0 = 0$ and $a_j > 0$, $j \geqslant 1$. The polynomials q_k satisfy a five term recurrence relation of the form

$$yq_k(y) = q_{k+1}(y) + b_k q_{k-1}(y) + c_k q_{k-3}(y) \tag{8}$$

(see [2, Section 2.1]). In fact, the polynomial $q_n(y)$ is the characteristic polynomials of the banded Hessenberg matrix

$$A_n = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ b_1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & b_2 & 0 & 1 & 0 & \cdots \\ c_3 & 0 & b_3 & 0 & 1 & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \\ & & c_{n-1} & 0 & b_{n-1} & 0 \end{pmatrix}.$$

Using integration by parts one can show (see [20]) that the coefficients a_k, b_k and c_k are related in the following way:

$$b_k = 2(a_{k+1}^2 + a_k^2 + a_{k-1}^2)a_k^2 = \frac{k}{2} + \tau^2 a_k^2,\ k \geqslant 1, \tag{9}$$

$$c_{k+2} = 2\tau^2 a_k^2 a_{k+1}^2 a_{k+2}^2,\ k \geqslant 1. \tag{10}$$

A crucial result in [20] was that (9) and (10) imply that the tridiagonal matrices

$$G_m = \begin{pmatrix} b_1 & 1 & 0 & \cdots \\ c_3 & b_3 & 1 & \ddots \\ & \ddots & \ddots & \ddots \\ \cdots & 0 & c_{2m-1} & b_{2m-1} \end{pmatrix}, L_m := \begin{pmatrix} b_2 & 1 & 0 & \cdots \\ c_4 & b_4 & 1 & \ddots \\ & \ddots & \ddots & \ddots \\ \cdots & 0 & c_{2m} & b_{2m} \end{pmatrix}$$

are oscillatory. We now have what we need to prove the Corollary 10. We let e_j denote the jth standard vector containing all zeros except in the jth position. The size of the vector e_j should be clear from the context.

Proof of Corollary 10. Reordering the rows and columns of A_n in such a way that the odd numbered rows and columns come first and then the even rows and numbers, we easily see that $xI_{2m} - A_{2m}$ is permutation similar to

$$\begin{pmatrix} xI_m & -\begin{pmatrix} e_1^T & 0 \\ L_{m-1} & e_{m-1} \end{pmatrix} \\ -G_m & xI_m \end{pmatrix} \tag{11}$$

and that $xI_{2m+1} - A_{2m+1}$ is permutation similar to

$$\begin{pmatrix} xI_{m+1} & -\begin{pmatrix} e_1^T \\ L_m \end{pmatrix} \\ -\begin{pmatrix} G_m & e_m \end{pmatrix} & xI_m \end{pmatrix}. \tag{12}$$

In general we have the observation that for $x \neq 0$

$$\det \begin{pmatrix} xI_k & P \\ Q & xI_l \end{pmatrix} = x^l \det(xI_k - \frac{1}{x}PQ) = x^{l-k}\det(x^2 I_k - PQ) = x^{k-l}\det(x^2 I_l - QP).$$

If we apply this observation to (11) and (12) we get that

- the eigenvalues of A_{2m} are the positive and negative square roots of the eigenvalues of
$$\begin{pmatrix} e_1^T & 0 \\ L_{m-1} & e_{m-1} \end{pmatrix} G_m, \tag{13}$$
- the eigenvalues of A_{2m+1} are the positive and negative square roots of the eigenvalues of
$$\begin{pmatrix} e_1^T \\ L_m \end{pmatrix} \begin{pmatrix} G_m & e_m \end{pmatrix}, \tag{14}$$
where the zero eigenvalue should be counted only once.

Notice that the matrices in (13) and (14) are easily seen to be totally nonnegative (as they are products of totally nonnegative matrices) and irreducible. In addition, the matrix in (13) is nonsingular and has positive sub- and superdiagonal entries, and is therefore oscillatory (Theorem 2). As

$$\begin{pmatrix} e_1^T \\ L_m \end{pmatrix} \begin{pmatrix} G_m & e_m \end{pmatrix} = \begin{pmatrix} e_1^T & 0 \\ L_{m-1} & e_{m-1} \\ c_{2m}e_{m-1}^T & b_{2m} \end{pmatrix} \begin{pmatrix} G_m & e_m \end{pmatrix},$$

we have that the matrix in (13) is obtained from the matrix in (14) by removing the last row and column. We now obtain from Theorem 7 that the roots of q_{2m} and q_{2m+1} strictly interlace. $\square$

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