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SPHERICAL POTENTIALS OF COMPLEX ORDER IN WEIGHTED GENERALIZED HÖLDER SPACES WITH RADIAL OSCILLATING WEIGHTS

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Abstract. For the spherical potential and hypersingular operators, in general of complex order including the purely imaginary case, there are proved weighted Zygmund type estimates with radial type weights of the Zygmund-Bary-Stechkin class, which may oscillate between power functions. By means of those estimates there are proved boundedness theorems for these operators in weighted generalized Hölder spaces and it is shown that the potential type operator realizes a certain isomorphism within the frameworks of such spaces.

1. Introduction

We consider the spherical potential operator

$$(K^{\alpha}f)(x) = \frac{1}{\gamma_{n-1}(\alpha)} \int\limits_{\mathbb{S}^{n-1}} \frac{f(\sigma)}{|x-\sigma|^{n-1-\alpha}}\, d\sigma, \quad x \in \mathbb{S}^{n-1}, \quad 0 < \operatorname{Re}\alpha < n-1, \tag{1.1}$$

see for instance [8], p.151, and the related hypersingular integral

$$(D^{\alpha}f)(x) = \frac{1}{\gamma_{n-1}(-\alpha)} \lim_{\varepsilon\to 0} \int\limits_{\substack{\mathbb{S}^{n-1}\\ |x-\sigma|\geqslant\varepsilon}} \frac{f(\sigma)-f(x)}{|x-\sigma|^{n-1+\alpha}}\, d\sigma, \quad x \in \mathbb{S}^{n-1}, \tag{1.2}$$

where $0 \leqslant \operatorname{Re}\alpha < 2, \alpha \neq 0$ and $\gamma_{n-1}(\alpha)$ is the known normalizing constant. We refer to [8], Ch.6, for spherical hypersingular integrals.

Note that the operator (1.1) is well defined (almost everywhere) for all integrable functions $f \in L^1(\mathbb{S}^{n-1})$, while the domain of the operator (1.2) if studied within the frameworks of Lebesgue integrable functions needs a special description and the integral in (1.2) must be interpreted in the principal value sense, see details for the case of Lebesgue spaces in [8], Ch.6. The advantage of using the generalized Hölder spaces (as opposed to say weighted Lebesgue spaces) is that the hypersingular integrals on these spaces converge absolutely everywhere except maybe for the nodes of the weight.

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In papers [10], [11] for the spherical operators (1.1)-(1.2) of real order $\alpha > 0$ and in [12], [13], [14] for complex order with $\mathrm{Re}\,\alpha \in (0,1)$ there were obtained Zygmund type estimates in non-weighted case and weighted case with power weight $\rho(x) = |x-a|^{\mu}, a \in \mathbb{S}^{n-1}$. An extension of such estimates to more general weights was an open problem. This problem is partially solved in this paper by admitting radial type weights $\rho(x) = \prod\limits_{k=1}^{N} \varphi_k(|x-a_k|)$ associated with some points $a_k \in \mathbb{S}^{n-1}$, where the functions φ_k belong to a certain subclass of the Bari-Stechkin class $\Phi^0_{n-1\pm\mathrm{Re}\,\alpha}$ and thereby may oscillate between two power functions.

Using the obtained weighted Zygmund type estimates we prove boundedness theorems for operators (1.1)-(1.2) in weighted generalized Hölder spaces $H^{\omega}(\mathbb{S}^{n-1},\rho)$. The characteristic $\omega(t)$ of the space is assumed to belong to a certain class Φ^{δ}_{β} of Bary-Stechkin type, with the parameters β and δ depending on $\mathrm{Re}\,\alpha$ and n. In its turn, these boundedness theorems are used to prove that the operator K^{α} with $0 < \mathrm{Re}\,\alpha < 1$ realizes the isomorphism

$$K^{\alpha}[H^{\omega}_0(\mathbb{S}^{n-1},\rho)] = H^{\omega_\alpha}_0(\mathbb{S}^{n-1},\rho),$$

and similarly

$$\mathscr{D}^{\pm i\theta}[H^{\omega}_0(\mathbb{S}^{n-1}\rho)] = H^{\omega}_0(\mathbb{S}^{n-1},\rho),$$

where $\omega_\alpha(t) = t^{\mathrm{Re}\,\alpha}\omega(t)$.

The result on the isomorphism is based on the fact that the operator

$$\mathscr{D}^{\alpha} = \frac{1}{b_n} I + D^{\alpha} \qquad \text{with} \qquad b_n = \frac{\Gamma\left(\frac{n-1-\alpha}{2}\right)}{\Gamma\left(\frac{n-1+\alpha}{2}\right)},$$

is left inverse to K^{α}, see [8], Ch.6.

Observe that the corresponding statements in the one-dimensional case, for the Riemann-Liouville fractional integrals and Marchaud fractional derivatives, in a similar setting were obtained in [9], Th. 6, see also [4] and [3], Th. 4.10.

Various constants, the value of which is not important, will be denoted in the sequel by c or C.

The main results are proved for the case $n \geqslant 3$. The case $n = 2$ may be treated similarly but it requires some technical modifications.

2. Preliminaries

We use the following standard notation:

$x = (x_1, x_2, ..., x_n), |x| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2};$

$x \cdot y = x_1y_1 + x_2y_2 + ... + x_ny_n;$

$\mathbb{S}^{n-1} = \{x : x \in \mathbb{R}^n, |x| = 1\}, \quad n \geqslant 2; \quad |\mathbb{S}^{n-1}| = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)};$

$\gamma_{n-1}(\alpha) = 2^{\alpha}\pi^{\frac{n-1}{2}} \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-1-\alpha}{2}\right)}$ (treated as the analytical continuation when $\mathrm{Re}\,\alpha \leqslant 0$),

$\omega(f,h) = \sup\limits_{\substack{|x-\sigma|\leqslant h \\ x,\sigma\in\mathbb{S}^{n-1}}} |f(x) - f(\sigma)|.$

2.1. On some spherical integrals

From the Catalan formula (see for instance, [8], Ch.6)

$$\int\limits_{\mathbb{S}^{n-1}} f(x\cdot\sigma)\,d\sigma = C_n \int\limits_{-1}^{1} f(t)(1-t^2)^{(n-3)/2}\,dt, \quad x\in\mathbb{S}^{n-1},$$

where $C_n = |\mathbb{S}_{n-2}|$, taking into account that $|x-\sigma| = \sqrt{2(1-x\cdot\sigma)}$, we derive that the integral

$$J(a,b) = \int\limits_{a<|x-\sigma|<b} g(|x-\sigma|)\,d\sigma, \quad x\in\mathbb{S}^{n-1}$$

where $0\leqslant a<b\leqslant 2$, may be calculated by the formula

$$J(a,b) = 2^{3-n}C_n \int\limits_a^b g(u)u^{n-2}(4-u^2)^{\frac{n-3}{2}}\,du. \tag{2.1}$$

From (2.1) it follows that

$$|J(a,b)| \leqslant C_n \int\limits_a^b |g(u)|u^{n-2}\,du \tag{2.2}$$

in the case $n\geqslant 3$.

2.2. The generalized Hölder spaces and classes of characteristics and weights

We recall that a non-negative function ω on $[0,\ell]$ is called almost increasing (or almost decreasing), if there exists a constant $c\geqslant 1$, such that $\omega(x)\leqslant c\omega(y)$ for all $0\leqslant x\leqslant y\leqslant \ell$ (or $\ell\geqslant x\geqslant y\geqslant 0$, respectively), $\ell>0$. In the sequel we take $\ell=2$, when working with radial weights on the unit sphere.

DEFINITION 2.1. We say that $\omega(x)\in W = W([0,2])$, if
1) ω is continuous on $(0,2]$;
2) $\lim\limits_{x\to+0}\omega(x)=0$, $\omega(x)\neq 0$, $0<x\leqslant 2$;
3) $\omega(x)$ is almost increasing.

The following numbers

$$m(w) = \sup_{x>1}\frac{\ln\left[\underline{\lim}_{h\to 0}\frac{w(xh)}{w(h)}\right]}{\ln\,x}, \quad M(w) = \inf_{x>1}\frac{\ln\left[\overline{\lim}_{h\to 0}\frac{w(xh)}{w(h)}\right]}{\ln\,x}, \tag{2.3}$$

introduced in such a form in [5], [6] are well defined for non-negative functions, in particular, for $w\in W$ they are known as the lower and upper indices of the function w. In case $w\in W$ we have $0\leqslant m(w)\leqslant M(w)\leqslant\infty$; we refer also to [3], [7] for properties of functions $w \in W$ related to these indices. Observe that the relations below follow directly from the definitions in (2.3):

$$m\left(\frac{1}{w}\right) = -M(w), \tag{2.4}$$

$$m(u) + m(v) \leqslant m(uv) \leqslant M(uv) \leqslant M(u) + M(v). \tag{2.5}$$

DEFINITION 2.2. Let $\omega \in W([0,2])$. By $H^\omega(\mathbb{S}^{n-1})$ we denote the Banach algebra of functions $f \in C(\mathbb{S}^{n-1})$ such that $\omega(f,t) \leqslant c\omega(t)$, equipped with the norm

$$\|f\|_{H^\omega(\mathbb{S}^{n-1})} = \|f\|_{C(\mathbb{S}^{n-1})} + \sup_{0<t<2} \frac{\omega(f,t)}{\omega(t)}, \tag{2.6}$$

and define

$$H^\omega(\mathbb{S}^{n-1},\rho) = \{f : \rho f \in H^\omega(\mathbb{S}^{n-1})\}$$

with the naturally defined norm. For a given finite set of points

$$\Pi = \{a_1, a_2, \dots, a_N\} \subset \mathbb{S}^{n-1}$$

we introduce also the subspace

$$H_0^\omega(\mathbb{S}^{n-1}) = H_{\Pi,0}^\omega(\mathbb{S}^{n-1}) = \{f \in H^\omega(\mathbb{S}^{n-1}) : f(x) = 0, x \in \Pi\}.$$

Finally, we introduce the weighted space

$$H_0^\omega(\mathbb{S}^{n-1},\rho) = \{f \in H^\omega(\mathbb{S}^{n-1},\rho) : \lim_{x \to a_k \in \Pi} (\rho f)(x) = 0,\ k = 1,...,N\}.$$

The space $H^\omega(\mathbb{S}^{n-1})$ is non-trivial, if $\sup\limits_{t>0} \frac{t}{\omega(t)} < \infty$.

To define the class of admissible characteristics ω and weights $\varphi(|x-a|)$ we need the following definitions (following ideas in [9], [3]).

DEFINITION 2.3. By $W_\mu = W_\mu([0,2])$, $\mu > 0$, we denote the subclass of functions $\varphi \in W$ such that $\frac{\varphi(x)}{x^\mu}$ is almost decreasing on $[0,2]$, and by W^ν, $\nu > 0$, we denote the subclass of functions $\omega \in W$ such that $\frac{\varphi(x)}{x^\nu}$ is almost increasing on $[0,2]$, and

$$W_\mu^\nu = W_\mu \cap W^\nu, \mu > \nu. \tag{2.7}$$

Observe that

$$\varphi \in W_\mu,\ \mu > 0 \quad \Longrightarrow \quad \varphi(x+y) \leqslant c[\varphi(x) + \varphi(y)] \tag{2.8}$$

(with c in general depending on φ).

DEFINITION 2.4. We say that $\varphi \in K^* = K^*([0,2])$ or $\varphi \in K_* = K_*([0,2])$, if φ belongs to W and satisfies the condition

$$|\varphi(x) - \varphi(y)| \leqslant c|x-y|\frac{\varphi(\xi^*)}{\xi^*}, \quad \xi^* = \max(x,y). \tag{2.9}$$

or

$$|\varphi(x) - \varphi(y)| \leqslant c|x-y|\frac{\varphi(\xi_*)}{\xi_*}, \quad \xi_* = \min(x,y), \tag{2.10}$$

respectively.

Observe that functions $\varphi \in K^* \cap W_1$ satisfy the inequalities

$$|\varphi(x) - \varphi(y)| \leqslant c|x-y|\frac{\varphi(x)}{x}, \quad y \geqslant x > 0, \tag{2.11}$$

that is, $K^* \cap W_1 \subset K_* \cap W_1$, and

$$|\varphi(x) - \varphi(y)| \leqslant c\varphi(|x-y|). \tag{2.12}$$

Note also that

$$|\varphi(x) - \varphi(y)| \leqslant c|x-y|\frac{\varphi(x+y)}{x+y} \tag{2.13}$$

for $\varphi \in K^* \cap W^1$, and

$$\left|\frac{\varphi(x) - \varphi(y)}{\varphi(y)}\right| \leqslant c\left[\frac{|x-y|}{x+y} + \frac{\varphi(|x-y|)}{\varphi(y)}\right] \tag{2.14}$$

for $\varphi \in K^* \cap W^1_\nu,\ \nu > 1$. Inequality (2.14) is obtained from (2.13) taking into account that $x+y \leqslant |x-y| + 2|y|$.

DEFINITION 2.5. By $\mathcal{Z}^\delta, \delta \geqslant 0$, we denote the class of functions $\omega \in W$ satisfying the Zygmund condition $\int_0^x \left(\frac{x}{t}\right)^\delta \frac{\omega(t)}{t}\,dt \leqslant c\omega(x)$, and by $\mathcal{Z}_\beta, \beta > 0$, the class of functions $\omega \in W$, satisfying the condition $\int_x^2 \left(\frac{x}{t}\right)^\beta \frac{\omega(t)}{t}\,dt \leqslant c\omega(x)$, and by $\Phi^\delta_\beta = \mathcal{Z}^\delta \cap \mathcal{Z}_\beta, \quad 0 \leqslant \delta < \beta$, we denote the Bari-Stechkin-type class.

The class Φ^0_β was introduced and studied in case of monotonic functions ω in [1]. We refer for instance to [3], [7] for properties of functions $\omega \in \Phi^\delta_\beta$, quoting some of them:

$$\Phi^\delta_\beta = \emptyset \quad \text{in the case} \quad \delta \geqslant \beta; \tag{2.15}$$

$$\Phi^{\delta_1}_{\beta_1} \cap \Phi^{\delta_2}_{\beta_2} = \Phi^{\max(\delta_1,\delta_2)}_{\min(\beta_1,\beta_2)} \quad \text{and} \quad \Phi^{\delta_2}_{\beta_2} \subset \Phi^{\delta_1}_{\beta_1},\ \delta_2 \geqslant \delta_1,\ \beta_2 \leqslant \beta_1, \tag{2.16}$$

$$\omega \in \Phi^\delta_\beta \iff \delta < m(w) \leqslant M(w) < \beta, \tag{2.17}$$

$$m(\omega) = \sup\left\{\nu > 0 : \ \frac{\omega(x)}{x^\nu} \ \text{is almost increasing}\right\}, \tag{2.18}$$

$$M(\omega) = \inf\left\{\mu > 0 : \ \frac{\omega(x)}{x^\mu} \ \text{is almost decreasing}\right\}, \tag{2.19}$$

$$\omega \in \Phi^\delta_\beta \iff t^\rho\omega(t) \in \Phi^{\delta+\rho}_{\beta+\rho},\ \rho \geqslant -\delta, \tag{2.20}$$

$$\omega \in \Phi^\delta_\beta \implies c_1 t^{M(\omega)+\varepsilon} \leqslant \omega(t) \leqslant c_2 t^{m(\omega)-\varepsilon}, \qquad 0 \leqslant t \leqslant 2 \tag{2.21}$$

for all $\varepsilon > 0$, $c_i = c_i(\varepsilon), i = 1,2$.

From properties (2.17), (2.18) and (2.19) it is easily derived that for $0 < \nu < \mu < \infty$ the following embeddings hold

$$W^{\nu+\delta}_{\mu-\delta} \subset \Phi^\nu_\mu \subset W^{\nu-\varepsilon}_{\mu+\varepsilon}, \tag{2.22}$$

where $0 < \delta < \frac{\mu-\nu}{2}, \quad 0 < \varepsilon \leqslant \nu$.

From (2.17) and (2.22) the following statement follows

LEMMA 2.6. *For a function $\varphi \in W$ to belong to W_μ^ν, the condition $\nu \leqslant m(\varphi) \leqslant M(\varphi) \leqslant \mu$ is necessary and the condition $\nu < m(\varphi) \leqslant M(\varphi) < \mu$ is sufficient.*

We need also the following lemma (compare with Lemma 8.2 in [6]).

LEMMA 2.7. *Let $\omega \in \Phi := \Phi_1^0$ and let φ be a bounded function on $[0,2]$, satisfying the condition*

$$|\varphi(t) - \varphi(\tau)| \leqslant C\frac{|t-\tau|}{\min(t,\tau)}, \quad t,\tau \in (0,2]. \tag{2.23}$$

Then functions of the form $\rho(x) = \varphi(|x-a|), a \in \mathbb{S}^{n-1}$, are multipliers in $H^\omega_{\Pi,0}(\mathbb{S}^{n-1})$, $\Pi = \{a\}$.

Proof. Let $f \in H_0^\omega(\mathbb{S}^{n-1})$, $x,y \in \mathbb{S}^{n-1}$ and $|x-y| \leqslant h$. We have to show that

$$|\rho(x)f(x) - \rho(y)f(y)| \leqslant C\omega(h). \tag{2.24}$$

Suppose for definiteness that $|x-a| \leqslant |y-a|$. Taking into account that $f(a) = 0$, we have

$$|\rho(x)f(x) - \rho(y)f(y)| \leqslant C\omega(f,h) + \omega(f,|x-a|)|\varphi(|x-a|) - \varphi(|y-a|)|.$$

If $|x-a| \leqslant h$, estimate (2.24) is obvious by the boundedness of φ. In the case $|x-a| > h$, it suffices to make use of (2.23) and the fact that $\frac{\omega(f,t)}{t}$ is almost decreasing (see for instance, [2], p. 50):

$$\frac{\omega(f_0,t_2)}{t_2} \leqslant 2\frac{\omega(f_0,t_1)}{t_1} \quad \text{forall} \quad 0 < t_1 \leqslant t_2. \qquad \Box \tag{2.25}$$

COROLLARY 2.8. *Any function $\varphi(|x-a|), a \in \mathbb{S}^{n-1}$, where $\varphi \in K^* \cap W_\mu([0,2])$ or $\varphi \in K_* \cap W_\mu([0,2])$, $0 < \mu < 2$, is a multiplier in the space $H^\omega_{\Pi,0}(\mathbb{S}^{n-1})$, $\Pi = \{a\}$, $\omega \in \Phi$.*

Indeed, if $0 < \mu \leqslant 1$, then Corollary's statement follows immediately from Lemma 2.7. If $\mu > 1$, then $\varphi(t) = t\varphi_0(t)$, with $\varphi_0(t) = \frac{\varphi(t)}{t}$, where t generates an evident multiplier $t = |x-a|$, while $\varphi_0(t)$ satisfies the condition (2.23), which may be directly checked. Hence $\varphi_0(|x-a|)$ is a multiplier as well.

3. On spherical potentials in non-weighted space $H^\omega(\mathbb{S}^{n-1})$

We will use the following Zygmund type estimates for spherical potentials and spherical hypersingular integrals obtained in [10] for real exponents α and in [12] for complex values of α, including also the purely imaginary case $\alpha = i\theta$ (see also [11], [13] and [14] for some modifications and generalizations).

THEOREM 3.1. *Let $f \in C(\mathbb{S}^{n-1})$ and $0 < Re\,\alpha < 1$. Then*

$$\omega(K^\alpha f, h) \leqslant ch \int\limits_h^2 \frac{\omega(f,t)}{t^{2-\mathrm{Re}\,\alpha}}dt \quad \text{and} \quad \omega(D^\alpha f, h) \leqslant c \int\limits_0^h \frac{\omega(f,t)}{t^{1+\mathrm{Re}\,\alpha}}dt, \qquad 0 < h < 1.$$

For the pure imaginary case $\alpha = i\theta$ the corresponding statement is as follows.

THEOREM 3.2. *Let $f \in C(\mathbb{S}^{n-1})$ and $\theta \in \mathbb{R}^1 \setminus \{0\}$. Then*

$$\omega(D^{i\theta} f, h) \leqslant c \left(\int\limits_0^h \frac{\omega(f,t)}{t}dt + h\int\limits_h^2 \frac{\omega(f,t)}{t^2}dt \right).$$

The following theorems were obtained in [11] and [13] (see also[14]) by means of the above estimates.

THEOREM 3.3. *Let $0 < Re\,\alpha < 1$. If $\omega \in \mathcal{Z}_{1-\mathrm{Re}\,\alpha}$, then the operator K^α is bounded from the space $H^\omega(\mathbb{S}^{n-1})$ to the space $H^{\omega_\alpha}(\mathbb{S}^{n-1})$. If $\omega \in \mathcal{Z}^{\mathrm{Re}\,\alpha}$, then the operator D^α is bounded from the space $H^\omega(\mathbb{S}^{n-1})$ to the space $H^{\omega_{-\alpha}}(\mathbb{S}^{n-1})$.*

THEOREM 3.4. *Let $\omega \in \Phi_1^0$ and $\theta \in \mathbb{R}^1 \setminus \{0\}$. Then the operator $D^{i\theta}$ is bounded in the space $H_0^\omega(\mathbb{S}^{n-1})$.*

In [11] the following lemma was proved, which will be used in the sequel.

LEMMA 3.5. *Let $x, y, \sigma \in \mathbb{S}^{n-1}$. If $|x-\sigma| \geqslant 2|x-y|$, then for every $\gamma > 0$ the following inequality holds*

$$\left| |x-\sigma|^{-\gamma} - |y-\sigma|^{-\gamma} \right| \leqslant c \frac{|x-y|}{|x-\sigma|^\gamma (|x-\sigma| + |x-y|)}. \tag{3.1}$$

4. Weighted Zygmund type estimates

We consider the weights of the form

$$\rho(x) = \prod_{k=1}^N \varphi_k(|x-a_k|), \quad a_k \in \mathbb{S}^{n-1}. \tag{4.1}$$

In Section 5. we formulate the boundedness theorems, which is our main goal, for the case of weights (4.1). The proof of these theorems is based on the statements on Zygmund type estimates, obtained in this Section. Since we may separate the singularities of the weight, see Subsection 6.1., it suffices to prove the Zygmund type estimates for the case of a single weight

$$\rho(x) = \varphi(|x-a|), \quad a \in \mathbb{S}^{n-1}. \tag{4.2}$$

4.1. Formulation of the main results

Theorems 4.1-4.3 given below were proved in [10]-[14] in the case of power weights $\varphi(t) = t^\mu$. The proof for our more general weights follows mainly the same ideas, but require an essential modification, related to the usage of the technique of the index numbers of almost monotonic functions. The most modifications is required for the case of purely imaginary order $\alpha = i\theta$. For this reason we will dwell in the sequel only on the proof of Theorem 4.3. The proof of Theorems 4.1-4.2 may be obtained by the same scheme.

4.1.1. The case $0 < \mathbf{Re}\,\alpha < 1$

THEOREM 4.1. *Let $\rho(x)$ be the weight (4.2) with $\varphi \in K^* \cap W_1$ or $\varphi \in K^* \cap W^1_{n-1-\mathrm{Re}\,\alpha}$, let $\rho(x)f(x) \in C(\mathbb{S}^{n-1})$ and $(\rho f)(a) = 0$. Then the potential operator K^α, $0 < Re\,\alpha < 1$, admits the following Zygmund type estimate*

$$\omega(\rho K^\alpha f, h) \leqslant c\psi(h) \int\limits_h^2 \frac{\omega(\rho f, t)}{t^{1-\mathrm{Re}\,\alpha}\psi(t)} dt, \tag{4.3}$$

where $\psi(t) = \begin{cases} \varphi(t), & \text{when } \varphi \in K^* \cap W_1, \\ t, & \text{when } \varphi \in K^* \cap W^1_{n-1-\mathrm{Re}\,\alpha}. \end{cases}$

THEOREM 4.2. *Let $\rho(x)$ be the weight (4.2) with $\varphi \in K^* \cap W_1$ or $\varphi \in K^* \cap W^1_{n-1+\mathrm{Re}\,\alpha}$, let $\rho(x)f(x) \in C(\mathbb{S}^{n-1})$ and $(\rho f)(a) = 0$. Then the hypersingular operator D^α, $0 < Re\,\alpha < 1$, admits the following Zygmund type estimate*

$$\omega(\rho D^\alpha f, h) \leqslant c \left\{ \int\limits_0^h \frac{\omega(\rho f, t)}{t^{1+\mathrm{Re}\,\alpha}} dt + \psi(h) \int\limits_h^2 \frac{\omega(\rho f, t)}{t^{1+\mathrm{Re}\,\alpha}\psi(t)} dt \right\} \tag{4.4}$$

where $\psi(t) = \begin{cases} \varphi(t), & \text{when } \varphi \in K^* \cap W_1, \\ t, & \text{when } \varphi \in K^* \cap W^1_{n-1+\mathrm{Re}\,\alpha}. \end{cases}$

4.1.2. The case $\mathbf{Re}\,\alpha = 0, \alpha \neq 0$

THEOREM 4.3. *Let $\rho(x)$ be the weight (4.2) with $\varphi \in K^* \cap W_1$ or $\varphi \in K^* \cap W^1_{n-1}$, let $\rho(x)f(x) \in C(\mathbb{S}^{n-1})$ and $(\rho f)(a) = 0$. Then*

$$\omega(\rho D^{i\theta} f, h) \leqslant c \left\{ \int\limits_0^h \frac{\omega(\rho f, t)}{t} dt + \psi(h) \int\limits_h^2 \frac{\omega(\rho f, t)}{t\psi(t)} dt \right\}, \tag{4.5}$$

where $\psi(t) = \begin{cases} \varphi(t), & \text{when } \varphi \in K^* \cap W_1, \\ t, & \text{when } \varphi \in K^* \cap W^1_{n-1}. \end{cases}$

REMARK 4.4. The estimates of the type (4.3), (4.4), (4.5) for the case of the weight (4.1) are similar. The proof is the same, requiring only some technical complications. For instance, estimate (4.4) is as follows

$$\omega(\rho D^{\alpha}f,h) \leqslant c\left\{\int_0^h \frac{\omega(\rho f,t)}{t^{1+\mathrm{Re}\,\alpha}}dt + \sum_{k=1}^{N}\psi_k(h)\int_h^2 \frac{\omega(\rho f,t)}{t^{1+\mathrm{Re}\,\alpha}\psi_k(t)}dt\right\} \tag{4.6}$$

with

$$\psi_k(t) = \begin{cases} \varphi_k(t), & \text{when } \varphi_k \in K^* \cap W_1, \\ t, & \text{when } \varphi_k \in K^* \cap W^1_{n-1+\mathrm{Re}\,\alpha}. \end{cases} \tag{4.7}$$

4.2. Proof of Theorem 4.3

We denote $f_0(\sigma) = \rho(\sigma)f(\sigma) = \varphi(|\sigma - a|)f(\sigma)$ for brevity, so that

$$f_0(a) = 0 \quad \text{and} \quad |f_0(\sigma)| \leqslant \omega(f_0, |\sigma - a|) \quad \text{for} \quad f \in H_0^{\omega}(\rho, \mathbb{S}^{n-1}).$$

We have

$$\rho(x)(D^{i\theta}f)(x) = (D^{i\theta}f_0)(x) + g(x), \tag{4.8}$$

where

$$g(x) = \frac{1}{\gamma_{n-1}(-i\theta)}\int_{\mathbb{S}^{n-1}} \frac{\varphi(|x-a|) - \varphi(|\sigma - a|)}{\varphi(|\sigma - a|)|x-\sigma|^{n-1+i\theta}} f_0(\sigma)d\sigma.$$

The estimate of the continuity modulus of the first term in (4.8) is given by Theorem 3.2. For the estimation of $\omega(g,h)$, we denote

$$\Delta(x,\sigma) = \frac{\varphi(|x-a|) - \varphi(|\sigma - a|)}{\varphi(|\sigma - a|)|x-\sigma|^{n-1+i\theta}} \tag{4.9}$$

and observe that

$$|\Delta(x,\sigma)| \leqslant \frac{c}{|\sigma - a||x-\sigma|^{n-2}} \tag{4.10}$$

in view of (2.9) and (2.12). We represent $g(x) - g(y)$ as

$$g(x) - g(y) = \int_{|x-\sigma|<2h} \Delta(x,\sigma)f_0(\sigma)d\sigma - \int_{|x-\sigma|<2h} \Delta(y,\sigma)f_0(\sigma)d\sigma$$

$$+ \int_{|x-\sigma|>2h} \{\Delta(x,\sigma) - \Delta(y,\sigma)\}f_0(\sigma)d\sigma = I_1 + I_2 + I_3. \tag{4.11}$$

We will separately estimate every term. We denote $|x-y| = h$ and suppose that $|x-a| > |y-a|$ for definiteness.

1°. The case $\varphi \in K^* \cap W_1$.

For I_1 we have

$$I_1 = \int_{\substack{|x-\sigma|<2h \\ |\sigma-a|\geqslant|x-\sigma|}} \Delta(x,\sigma)f_0(\sigma)d\sigma + \int_{\substack{|x-\sigma|<2h \\ |\sigma-a|\leqslant|x-\sigma|}} \Delta(x,\sigma)f_0(\sigma)d\sigma =: I_i' + I_1''. \tag{4.12}$$

The term I_i' is estimated as follows

$$|I_i'| \leqslant c \int\limits_{\substack{|x-\sigma|<2h \\ |x-\sigma|<|\sigma-a|}} \frac{\omega(f_0,|\sigma-a|)}{|\sigma-a||x-\sigma|^{n-2}} d\sigma \leqslant c \int\limits_0^h \frac{\omega(f_0,t)}{t} dt$$

by estimate (4.10), property (2.25) and inequality (2.2). For the integral I_1'', by (4.10) and the embedding $\{|\sigma-a|<|\sigma-x|<2h\} \subset \{|\sigma-a|<2h\}$ and inequality (2.2) we obtain

$$|I_1''| \leqslant c \int\limits_{|\sigma-a|<2h} \frac{w(f_0,|\sigma-a|)}{|\sigma-a|^{n-1}} d\sigma \leqslant c \int\limits_0^h \frac{\omega(f_0,t)}{t} dt.$$

Hence

$$|I_1| \leqslant c \int\limits_0^h \frac{\omega(f_0,t)}{t} dt. \tag{4.13}$$

The term I_2 is similarly estimated.

To consider I_3, we first need to estimate the difference $|\Delta(x,\sigma)-\Delta(y,\sigma)|$. We make use of inequalities (2.9), (2.12), (3.1), the fact that φ is almost increasing and $\frac{\varphi(t)}{t}$ is almost decreasing, the assumption $|y-a|<|x-a|$ and the inequality $|\sigma-a| \leqslant |x-\sigma|+|x-a|$, and get

$$|\Delta(x,\sigma)-\Delta(y,\sigma)| \leqslant \frac{ch}{|\sigma-a||x-\sigma|^{n-1}} + \frac{c\varphi(h)}{|\sigma-a||x-\sigma|^{n-2}\varphi(|x-\sigma|)}. \tag{4.14}$$

Since $|x-\sigma|>h$ in I_3 and $\frac{\varphi(t)}{t}$ is almost decreasing, from (4.14) we finally have

$$|\Delta(x,\sigma)-\Delta(y,\sigma)| \leqslant \frac{c\varphi(h)}{|\sigma-a||x-\sigma|^{n-2}\varphi(|x-\sigma|)}. \tag{4.15}$$

We split the integration in I_3:

$$I_3 = \int\limits_{\substack{|x-\sigma|>2h \\ |x-\sigma|<|a-\sigma|}} \cdots d\sigma + \int\limits_{\substack{|x-\sigma|>2h \\ |x-\sigma|>|a-\sigma|>2h}} \cdots d\sigma + \int\limits_{\substack{|x-\sigma|>2h \\ |x-\sigma|>2h>|a-\sigma|}} \cdots d\sigma =: I_3' + I_3'' + I_3'''.$$

By (4.15) and (2.2) we obtain

$$|I_3'| \leqslant c\varphi(h) \int\limits_{|x-\sigma|>2h} \frac{\omega(f_0,|x-\sigma|)}{|x-\sigma|^{n-1}\varphi(|x-\sigma|)} d\sigma \leqslant c\varphi(h) \int\limits_h^2 \frac{\omega(f_0,t)}{t\varphi(t)} dt. \tag{4.16}$$

Similarly

$$|I_3''| \leqslant c\varphi(h) \int\limits_{|x-\sigma|>|a-\sigma|>2h} \frac{\omega(f_0,|\sigma-a|)}{|\sigma-a|^{n-1}\varphi(|\sigma-a|)} d\sigma \leqslant c\varphi(h) \int\limits_h^2 \frac{\omega(f_0,t)}{t\varphi(t)} dt. \tag{4.17}$$

Finally, for I_3''' we observe that the right hand side of (4.15) is dominated by $ch^{2-n}|\sigma - a|^{-1}$, which yields

$$|I_3| \leqslant ch^{2-n} \int\limits_{|a-\sigma|<2h} \frac{\omega(f_0,|\sigma-a|)}{|\sigma-a|} d\sigma \leqslant c\omega(f_0,h)$$

In view of (2.25) we have

$$\omega(f_0,h) \leqslant 2\int\limits_0^h \frac{\omega(f_0,t)\,dt}{t}$$

and then

$$|I_3| \leqslant c\int\limits_0^h \frac{\omega(f_0,t)\,dt}{t}. \tag{4.18}$$

Gathering estimates (4.13), a similar estimate for the term I_2 and estimates (4.16), (4.17) and (4.18), we arrive at (4.5).

2°. The case $\varphi \in K^* \cap W^1_{n-1}$.

We have to estimate the terms I_1, I_2, I_3 in (4.11). For I_1 we make use of splitting (4.12) and instead of (4.10) apply the following estimate

$$|\Delta(x,\sigma)| \leqslant c \begin{cases} \frac{1}{|\sigma-a||x-\sigma|^{n-2}}, & |x-\sigma| \leqslant |\sigma-a| \\ \frac{1}{|\sigma-a|^{n-1}}, & |x-\sigma| > |\sigma-a| \end{cases}, \tag{4.19}$$

which follows directly from (4.9) by assumptions on φ. By property (2.25), the first line in (4.19) and inequality (2.2), we get

$$|I_1'| \leqslant c \int\limits_{\substack{|x-\sigma|<2h \\ |x-\sigma|<|\sigma-a|}} \frac{\omega(f_0,|\sigma-a|)}{|\sigma-a||x-\sigma|^{n-2}} d\sigma \leqslant c\int\limits_0^h \frac{\omega(f_0,t)}{t}dt.$$

For I_1'' taking into account the second line in (4.19), the embedding $\{|\sigma-a| < |\sigma-x| < 2h\} \subset \{|\sigma-a| < 2h\}$ and inequality (2.2), we obtain

$$|I_1''| \leqslant c \int\limits_{|\sigma-a|<2h} \frac{w(f_0,|\sigma-a|)}{|\sigma-a|^{n-1}} d\sigma \leqslant c\int\limits_0^h \frac{\omega(f_0,t)}{t}dt.$$

Therefore,

$$|I_1| \leqslant c\int\limits_0^h \frac{\omega(f_0,t)}{t}dt.$$

The estimate for I_2 is the same.

To estimate I_3, we make use of the inequality

$$|\Delta(x,\sigma)-\Delta(y,\sigma)| \leqslant \frac{ch}{|\sigma-a||x-\sigma|^{n-1}}+\frac{ch\varphi(|x-\sigma|)}{\varphi(|\sigma-a|)|x-\sigma|^n} \tag{4.20}$$

which is obtained by direct estimations with the help of (2.9), (2.14), (3.1), the inequalities $|x-a| \leqslant |x-\sigma|+|\sigma-a|$ and $|y-\sigma| \leqslant h+|x-\sigma|$, and the assumption $|y-a|<|x-a|$. We find it convenient to rewrite (4.20) in the form

$$|\Delta(x,\sigma)-\Delta(y,\sigma)| \leqslant ch \begin{cases} \frac{1}{|\sigma-a||x-\sigma|^{n-1}}, & |x-\sigma| \leqslant |\sigma-a|; \\ \frac{\varphi(|x-\sigma|)}{\varphi(|\sigma-a|)|x-\sigma|^n}, & |x-\sigma|>|\sigma-a|. \end{cases} \tag{4.21}$$

We split I_3:

$$I_3 = \int\limits_{\substack{|x-\sigma|>2h \\ |x-\sigma|<|\sigma-a|}} \cdots d\sigma + \int\limits_{\substack{|x-\sigma|>2h \\ |x-\sigma|>|\sigma-a|}} \cdots d\sigma =: I_3' + I_3''$$

For I_3' by the first line in (4.21), property (2.25) and inequality (2.2), we get

$$|I_3'| \leqslant ch \int\limits_{|x-\sigma|>2h} \frac{\omega(f_0,|x-\sigma|)}{|x-\sigma|^n} d\sigma \leqslant ch \int\limits_h^2 \frac{\omega(f_0,u)}{u^2} du.$$

For I_3'' we use the second line in (4.21) and the fact that $\frac{\varphi(t)}{t^{n-1}}$ is almost decreasing and obtain

$$|I_3''| \leqslant ch \int\limits_{|a-\sigma|>2h} \frac{\omega(f_0,|\sigma-a|)}{|\sigma-a|^n} d\sigma.$$

Hence, by (2.2), we arrive at the same estimate as for I_3'.

Collecting all the estimates for I_1, I_2 and I_3, we get at (4.5).

5. The main results

In the theorems of this section we denote

$$w_\alpha(t) = t^{\operatorname{Re}\alpha}\omega(t), \quad w_{-\alpha}(t) = t^{-\operatorname{Re}\alpha}\omega(t).$$

5.1. Boundedness statements

In the theorems below, $\rho(x)$ is the weight (4.1) and ψ_k is the notation introduced in (4.7).

5.1.1. The case $0 < \mathbf{Re}\,\alpha < 1$

THEOREM 5.1. *Let* $\varphi_k \in K^* \cap W_1$ or $\varphi_k \in K^* \cap W^1_{n-1-\mathrm{Re}\,\alpha}$, *and let*

$$\omega \in W \quad \text{and} \quad \frac{t\omega(t)}{\psi_k(t)} \in \mathcal{Z}_{1-\mathrm{Re}\,\alpha}. \tag{5.1}$$

Then the operator K^α *is bounded from* $H_0^\omega(\mathbb{S}^{n-1},\rho)$ *to* $H_0^{\omega\alpha}(\mathbb{S}^{n-1},\rho)$.

Theorem 5.1 may be reformulated in terms of the index numbers as follows.

THEOREM 5.2. *Let* $\omega \in W, \varphi_k \in W \cap K^*$, *and let the index numbers of the weight functions* φ_k *satisfy conditions*

$$0 < m(\varphi_k) \leqslant M(\varphi_k) < 1 \quad \text{or} \quad 1 < m(\varphi_k) \leqslant M(\varphi_k) < n-1-Re\,\alpha. \tag{5.2}$$

If

$$0 \leqslant m(\omega) \leqslant M(\omega) < 1 - Re\,\alpha \qquad \text{and} \qquad m\left(\frac{\varphi_k}{\omega}\right) > \mathrm{Re}\,\alpha, \quad k = 1,...,N, \tag{5.3}$$

then the operator K^α *is bounded from* $H_0^\omega(\mathbb{S}^{n-1},\rho)$ *to* $H_0^{\omega\alpha}(\mathbb{S}^{n-1},\rho)$.

REMARK 5.3. The condition $m(\varphi_k) > \mathrm{Re}\,\alpha + M(\omega)$ is sufficient for the second inequality in (5.3) to hold.
Indeed, from formulas (2.4)-(2.5) it follows that

$$m(\varphi_k) > \mathrm{Re}\,\alpha + M(\omega) \quad \Longleftrightarrow \quad m(\varphi_k) + m\left(\frac{1}{w}\right) > \alpha \quad \Longrightarrow \quad m\left(\frac{\varphi_k}{\omega}\right) > \mathrm{Re}\,\alpha.$$

The corresponding statements for the fractional differentiation operators D^α look as follows.

THEOREM 5.4. *Let* $\varphi_k \in K^* \cap W_1$ or $\varphi_k \in K^* \cap W^1_{n-1+\mathrm{Re}\,\alpha}$, *let* $\omega \in \mathcal{Z}^{\mathrm{Re}\,\alpha}$ *and let additionally*

$$\frac{\omega(t)}{\varphi_k(t)} \in \mathcal{Z}_{\mathrm{Re}\,\alpha} \tag{5.4}$$

in the case of "small" weights $\varphi_k \in K^* \cap W_1$. *Then the operator* D^α *is bounded from* $H_0^\omega(\mathbb{S}^{n-1},\rho)$ *to* $H_0^{\omega-\alpha}(\mathbb{S}^{n-1},\rho)$.

THEOREM 5.5. *Let* $\omega \in W, \varphi_k \in W \cap K^*$ *and*

$$0 < m(\varphi_k) \leqslant M(\varphi_k) < 1 \quad \text{or} \quad 1 < m(\varphi_k) \leqslant M(\varphi_k) < n-1+Re\,\alpha.$$

Let also

$$m(\omega) > Re\,\alpha \quad \text{and} \quad M\left(\frac{\omega}{\varphi_k}\right) < \mathrm{Re}\,\alpha, \quad k = 1,...,N. \tag{5.5}$$

Then the operator D^α *is bounded from* $H_0^\omega(\mathbb{S}^{n-1},\rho)$ *to* $H_0^{\omega-\alpha}(\mathbb{S}^{n-1},\rho)$.

REMARK 5.6. The condition $m(\varphi_k) > M(\omega) - \mathrm{Re}\,\alpha$ is sufficient for the second inequality in (5.5) to hold.

5.1.2. The case $\operatorname{Re}\alpha = 0$

In the case of purely imaginary order we have the following statement.

THEOREM 5.7. *Let*

$$\varphi_k \in K^* \cap W_1 \quad \text{or} \quad \varphi_k \in K^* \cap W^1_{n-1} \qquad \text{and} \qquad \frac{t\omega(t)}{\psi_k(t)} \in \Phi^0_1, \tag{5.6}$$

which certainly holds when

$$0 \leqslant m(\varphi_k) \leqslant M(\varphi_k) < 1 \quad \text{or} \quad 1 < m(\varphi_k) \leqslant M(\varphi_k) < n-1. \tag{5.7}$$

and

$$0 < m(\omega) \leqslant M(\omega) < 1 \qquad \text{and} \qquad 0 < m\left(\frac{\varphi_k}{\omega}\right) \leqslant M\left(\frac{\varphi_k}{\omega}\right) < 1, \quad k = 1, ..., N. \tag{5.8}$$

Then the operator $D^{i\theta}$ is bounded in the space $H^\omega_0(\mathbb{S}^{n-1}, \rho)$.

REMARK 5.8. The conditions

$$\min\{m(\varphi_k), 1\} > M(\omega), \quad \max\{M(\varphi_k), 1\} < m(\omega) + 1, \quad k = 1, ..., N,$$

are sufficient for (5.8) to hold.

5.2. Theorems on the isomorphism

THEOREM 5.9. *I. Let $\varphi_k \in K^* \cap W_1$ or $\varphi_k \in K^* \cap W^1_{n-1-\operatorname{Re}\alpha}$, and*

$$\omega \in \mathcal{Z}^0 \quad \text{and} \quad \frac{t\omega(t)}{\psi_k(t)} \in \mathcal{Z}_{1-\operatorname{Re}\alpha}. \tag{5.9}$$

Then the operator K^α maps the space $H^\omega_0(\mathbb{S}^{n-1}, \rho)$ isomorphically onto $H^{\omega\alpha}_0(\mathbb{S}^{n-1}, \rho)$.

II. Let $\theta \in \mathbb{R}^1$, $\theta \neq 0$. If $\varphi_k \in K^* \cap W_1$ or $\varphi_k \in K^* \cap W^1_{n-1}$, and

$$\omega \in \mathcal{Z}^0 \quad \text{and} \quad \frac{t\omega(t)}{\psi_k(t)} \in \mathcal{Z}_1, \tag{5.10}$$

then the operator $D^{i\theta}$ maps the space $H^\omega_0(\mathbb{S}^{n-1}, \rho)$ isomorphically onto itself.

Theorem 5.9 has the following equivalent reformulation in terms of the index numbers.

THEOREM 5.10. *Let $\omega \in W, \varphi_k \in W \cap K^*$.*

I. Under conditions (5.2) and (5.3), the operator K^α maps the space $H^\omega_0(\mathbb{S}^{n-1}, \rho)$ isomorphically onto $H^{\omega^\alpha}_0(\mathbb{S}^{n-1}, \rho)$.

II. Let $\theta \in \mathbb{R}^1$, $\theta \neq 0$. Under conditions (5.7) and (5.8), the operator $D^{i\theta}$ maps the space $H^\omega_0(\mathbb{S}^{n-1}, \rho)$ isomorphically onto itself.

6. Proofs

6.1. Reduction to the case of a single weight

The theorems on boundedness and isomorphism are formulated below for the case of the product of weights, but the proof may be given for the case of a single weight $\rho(x) = \varphi(|x-a|), a \in \mathbb{S}^{n-1}$. To show that this is possible, it suffices to separate the singularities by introducing the standard partition of unity. Indeed, let $1 = \sum_{k=1}^{N} g_k(x)$ with $g_k(x) \in C^\infty(\mathbb{S}^{n-1})$ and $g_k \equiv 1$ in some neighborhood of the point a_k and $g_k \equiv 0$ in a neighborhood of all other points a_j with $j \neq k$. We have

$$\frac{\rho(x)}{\rho(y)} = \prod_{k=1}^{N} \frac{\varphi_k(|x-a_k|)}{\varphi_k(|y-a_k|)} = \sum_{j=1}^{N} A_j(x)B_j(y)\frac{\varphi_j(|x-a_j|)}{\varphi_j(|y-a_j|)} \tag{6.1}$$

where

$$A_j(x) = \prod_{\substack{k=1\\k\neq j}}^{N} \varphi_k(|x-a_k|),\ B_j(y) = g_j(y)\left(\prod_{\substack{k=1\\k\neq j}}^{N} \varphi_j(|y-a_k|)\right)^{-1}$$

are multipliers in $H_0^\omega(\mathbb{S}^{n-1})$. For $A_j(x)$ this is valid by Corollary 2.8. Since $\mu_j(y) \equiv 0$ in some neighbourhood of a_k, $k \neq j$, we may represent $B_j(y)$ as

$$B_j(y) = g_j(y)\prod_{\substack{k=1\\k\neq j}}^{N} (\widetilde{\varphi}_j(|y-a_k|))^{-1}, \textit{ where } \widetilde{\varphi}_j(x) = \begin{cases} \varphi_j(|x-a_j|), & \varepsilon \leqslant |x-a_j| \leqslant 2\\ \varphi_j(\varepsilon), & 0 \leqslant |x-a_j| \leqslant \varepsilon, \end{cases}$$

ε being sufficiently small and this is a product of multipliers which is verified by means of the same Lemma 2.7.

For the operator $D^{i\theta}$, for example, according to (6.1) we have

$$\rho D^{i\theta}\frac{1}{\rho}f = \sum_{j=1}^{N} A_j\rho_j D^{i\theta}\frac{1}{\rho_j}B_j f.$$

Since A_j and B_j are multipliers in $H_0^\omega(\mathbb{S}^{n-1})$ (and A_j, B_j are equal to zero at the points a_k with $k \neq j$), it suffices to prove boundedness of $\rho_j S\frac{1}{\rho_j}D^{i\theta}$ in $H_0^\omega(\mathbb{S}^{n-1})$, with the zero index related to the set $\Pi = \{a_j\}$ consisting only of the point a_j.

Similarly, the boundedness for the operator K^α is considered with the only difference that one has to check that A_j are multipliers in $H_0^{\omega\alpha}(\mathbb{S}^{n-1})$.

6.2. On the vanishing property $\rho K^\alpha f\,|_{x=a} = 0$

We recall that we prove the boundedness of the operators to weighted generalized Hölder spaces with the vanishing property $\lim\limits_{x\to a} \rho(x)f(x) = 0$. So we have to check that this property holds for functions in the range of our operators. We consider the case of purely imaginary order $\alpha = i\theta$, the proof for other cases being similar.

According to (4.8) we have $\rho(x)(D^{i\theta}f)(x)\Big|_{x=a} = (D^{i\theta}f_0)(a)+g(a)$ and it remains to observe that $(D^{i\theta}f_0)(a) = -g(a)$.

6.3. Proof of the statements on boundedness

The statements of Theorems 5.1, 5.4, 5.7, which are given in terms of Zygmund type conditions, are in fact direct consequences of the corresponding Zygmund estimates of Theorems 4.1, 4.2 and 4.3, respectively. We demonstrate this on example of Theorem 5.1.

We recall that we may deal with the weight

$$\rho(x) = \varphi(|x-a|)$$

and we use the notation $\psi(t) = \begin{cases} \varphi(t), & \text{when } \varphi \in K^* \cap W_1, \\ t, & \text{when } \varphi \in K^* \cap W^1_{n-1-\mathrm{Re}\,\alpha}. \end{cases}$

According to (2.6), we have to show that

$$\frac{\omega(\rho K^\alpha f, h)}{h^{\mathrm{Re}\,\alpha}\omega(h)} \leqslant c < \infty \quad \text{for} \quad f \in H_0^\omega(\mathbb{S}^{n-1},\rho).$$

Under conditions of Theorem 5.1, Zygmund estimate (4.3) is applicable. Then by (4.3) with condition (5.1) in mind we get

$$\frac{\omega(\rho K^\alpha f, h)}{h^{\mathrm{Re}\,\alpha}\omega(h)} \leqslant \frac{c\|\rho f\|_{H_0^\omega(\mathbb{S}^{n-1})}}{\omega(h)} \int\limits_h^2 \frac{\psi(h)}{\psi(t)} \left(\frac{h}{t}\right)^{-\mathrm{Re}\,\alpha} \frac{\omega(t)}{t}\,dt \leqslant c\|f\|_{H_0^\omega(\mathbb{S}^{n-1},\rho)}.$$

Taking also into account the statement of Subsection 6.2., we obtain $\|K^\alpha f\|_{H_0^{\omega_\alpha}(\mathbb{S}^{n-1},\rho)} \leqslant c\|f\|_{H_0^\omega(\mathbb{S}^{n-1},\rho)}$.

As regards the reformulation of Theorem 5.1 made in Theorem 5.2, it suffices to observe that conditions (5.2) are sufficient for the conditions $\varphi_k \in K^* \cap W_1$ or $\varphi_k \in K^* \cap W^1_{n-1-\mathrm{Re}\,\alpha}$ to hold according to Lemma 2.6, while conditions (5.3) are equivalent to (5.1) by property (2.17).

Similarly one can prove Theorems 5.4, 5.5, 5.7.

6.4. Proofs of the theorems on isomorphism

We dwell on the proof of Part I of Theorems 5.9 and 5.10. The proof of part II of these theorems for the pure imaginary order $\alpha = i\theta$ follows the same lines with slight modifications.

By Theorems 5.1 and 5.4, we have

$$K^\alpha \ : \ H_0^\omega(\mathbb{S}^{n-1},\rho) \to H_0^{\omega_\alpha}(\mathbb{S}^{n-1},\rho) \tag{6.2}$$

and

$$\mathscr{D}^\alpha = \frac{1}{b_n}I + D^\alpha : H_0^{\omega\alpha}(\mathbb{S}^{n-1},\rho) \to H_0^{\omega}(\mathbb{S}^{n-1},\rho). \tag{6.3}$$

To state that the results in (6.2) and (6.3) already guarantee the existence of an isomorphism between the spaces $H_0^{\omega}(\mathbb{S}^{n-1},\rho)$ and $H_0^{\omega\alpha}(\mathbb{S}^{n-1},\rho)$, it remains to prove that the range of the operator K^α coincides with the space $H_0^{\omega\alpha}(\mathbb{S}^{n-1},\rho)$:

$$K^\alpha(H_0^{\omega}(\mathbb{S}^{n-1},\rho)) = H_0^{\omega\alpha}(\mathbb{S}^{n-1},\rho). \tag{6.4}$$

We do not have an independent characterization of the range $K^\alpha(H_0^{\omega}(\mathbb{S}^{n-1},\rho))$, but in the case of the Lebesgue spaces $L_p(\mathbb{S}^{n-1})$, a characterization of the range $K^\alpha(L_p(\mathbb{S}^{n-1}))$ is known and runs as follows, see [8], Th. 7.70.

THEOREM 6.1. *A function f belongs to $K^\alpha(L_p(\mathbb{S}^{n-1}))$, $1<p<\infty$, if and only if $f \in L_p(\mathbb{S}^{n-1})$ and $D^\alpha f \in L_p(\mathbb{S}^{n-1})$.*

REMARK 6.2. Observe that in the case of Lebesgue integrable functions, the hypersingular integral $D^\alpha f$ is treated as the limit of the corresponding truncated integrals over $\{\sigma \in \mathbb{S}^{n-1} : |x-\sigma| > \varepsilon\}$ with respect to the norm of the space $L_p(\mathbb{S}^{n-1})$. In our case, the hypersingular integral on functions in the range $K^\alpha(L_p(\mathbb{S}^{n-1}))$ is absolutely convergent for all $x \in \mathbb{S}^{n-1})$ except for probably the points $a_k,\ k=1,...N$.

Therefore, to state that a function $f \in H_0^{\omega\alpha}(\mathbb{S}^{n-1},\rho)$ belongs to the range $K^\alpha(H_0^{\omega}(\rho))$, it suffices to prove that there exists $p>1$ such that conditions $f \in L_p(\mathbb{S}^{n-1})$ and $D^\alpha f \in L_p(\mathbb{S}^{n-1})$ of Theorem 6.1 are satisfied for $f \in H_0^{\omega\alpha}(\rho)$. This will yield

$$H_0^{\omega\alpha}(\mathbb{S}^{n-1},\rho) \subset K^\alpha(L_p)$$

and then Theorem 6.1 and mapping (6.3) will guarantee that coincidence (6.4) holds.

Let us consider for definiteness the case $\varphi \in K^* \cap W^1_{n-1-Re\alpha}$. Since

$$|f(x)| \leqslant C\frac{|x-a|^\alpha \omega(|x-a|)}{\varphi(|x-a|)},$$

we easily obtain from the assumptions of Theorems 5.1 and 5.4 that

$$|f(x)| \leqslant \frac{C}{|x-a|^{n-1-\alpha}} \in L^p(\mathbb{S}^{n-1}),\ \ 1<p<\frac{n-1}{n-1-\operatorname{Re}\alpha}.$$

It remains to prove that $D^\alpha f \in L_p(\mathbb{S}^{n-1})$ for some $p>1$. Taking into account (2.2), we have

$$\int\limits_{S_{n-1}} |\rho D^\alpha f)(\sigma)|^p\, d\sigma \leqslant c \int\limits_{S_{n-1}} \left|\frac{\omega(\rho D^\alpha f, |a-\sigma|)}{\rho(\sigma)}\right|^p d\sigma \leqslant c\int\limits_0^2 \left|\frac{\omega(t)}{\varphi(t)}\right|^p t^{n-2}dt \leqslant C\int\limits_0^2 \frac{dt}{t^\lambda},$$

where $\lambda = pm\left(\frac{\omega}{\varphi}\right) + n-2-\varepsilon$, so that $\lambda < 1$ under the choice of ε sufficiently small and p sufficiently close to 1.

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