Operators
and
Matrices
Volume 1, Number 3 (2007), 301–368

INDEFINITE STURM–LIOUVILLE OPERATORS $(\operatorname{sgn} x)\left(-\frac{d^2}{dx^2}+q(x)\right)$ WITH FINITE–ZONE POTENTIALS

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To Edvard Tsekanoskii
on the occasion of his seventieth birthday
with deep appreciation

(*communicated by F. Gesztesy*)

Abstract. The indefinite Sturm-Liouville operator $A = (\operatorname{sgn} x)(-d^2/dx^2+q)$ is studied. It is proved that similarity of A to a selfadjoint operator is equivalent to integral estimates of Cauchy type integrals. Some simple sufficient and necessary conditions for the similarity to a selfadjoint operator in terms of Weyl functions are given. For operators with a finite-zone potential q , the components A_{ess} and A_{disc} of A corresponding to the essential and the discrete spectrums, respectively, are investigated. The main result of the paper is a criterion of similarity of the operator A (resp. A_{ess}) with a finite-zone potential q to a normal (resp. selfadjoint) operator. It is given in terms of the Weyl functions corresponding to the Sturm-Liouville operator $-d^2/dx^2+q$. Jordan structure of the operator A_{disc} is described. An example of a non-definitizable operator A that is similar to a normal operator is presented too.

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Mathematics subject classification (2000): 47E05, 34B24, 34B09, 34L10, 47B50.

Key words and phrases: J-selfadjoint operator, indefinite weight, nonselfadjoint operator, Sturm-Liouville operator, similarity, characteristic functions, algebraic and geometric multiplicities of an eigenvalue, weighted norm inequalities.

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1. Introduction

The main object of the paper is a nonselfadjoint indefinite Sturm-Liouville operator

$$A = (\operatorname{sgn} x)\left(-\frac{d^2}{dx^2} + q(x)\right) =: JL, \qquad \operatorname{dom} A = \operatorname{dom} L, \tag{1.1}$$

where $J : f \to \operatorname{sgn} x \cdot f(\cdot)$ and $L := -\frac{d^2}{dx^2} + q(x)$ is a selfadjoint Sturm-Liouville operator on $L^2(\mathbb{R})$ with a real continuous potential $q(\cdot)$. Note, that both ordinary and partial differential operators with indefinite weights have intensively been investigated during two last decades (see [31, 4, 8, 57, 61, 9, 11, 17, 63, 18, 33, 16, 55, 38]). The operator (1.1) on a finite interval subject to selfadjoint boundary conditions has discrete spectrum. The Riesz basis property of Dirichlet and other boundary value problems for Sturm-Liouville operators with indefinite weights has been investigated in [31, 4, 8, 57, 61, 55, 5].

In general, the operator (1.1) considered in $L^2(\mathbb{R})$ has continuous spectrum. In this case, instead of the Riesz basis property for operators with simple spectrum one considers the property of similarity to a normal operator (to a selfadjoint operator if the spectrum $\sigma(A)$ is real).

Let us recall that two closed operators T_1 and T_2 in a Hilbert space $\mathfrak{H}$ are called similar if there exist a bounded operator V with the bounded inverse V^{-1} in $\mathfrak{H}$ such that $V\operatorname{dom}(T_1) = \operatorname{dom}(T_2)$ and $T_2 = VT_1V^{-1}$.

Using the Krein-Langer technique of definitizable operators on Krein spaces Ćurgus and Langer [8] have obtained the first result in this direction. In particular, their result yields that the J-selfadjoint operator (1.1) is similar to a selfadjoint operator if L is uniformly positive (i.e., $L \geqslant \delta > 0$). Similarity of the operator $(\operatorname{sgn} x)\frac{d^2}{dx^2}$ to a selfadjoint one was proved by Ćurgus and Najman [9]. Later on, one of the authors [32, 33] reproved this result using another approach. More precisely, using the resolvent criterion of similarity to a selfadjoint operator [50, 46] (see also Theorem 3.12 below) he proved in [33] that the operator $A = (\operatorname{sgn} x)\cdot p\left(-i\frac{d}{dx}\right)$ is similar to a selfadjoint operator if and only if the polynomial $p(\cdot)$ is nonnegative. Further, Faddeev and Shterenberg [16] investigated operator (1.1) with decaying potential. They showed, that A is similar to a selfadjoint operator if $L \geqslant 0$ and $\int_{\mathbb{R}}(1+x^2)|q(x)|dx < \infty$.

Analysis of the results mentioned above motivates the following conjecture:

Is J-positivity of the differential operator A in $L^2(\mathbb{R})$ necessary for its similarity to a selfadjoint operator in a Hilbert space.

However, it easily follows from our considerations that this conjecture is false even in the case of a one-zone potential q. Namely, Example 8.2 with $\xi \in [-1/2, -k^2]$ shows that this conjecture is false.

The present paper consists of two parts. In the first part (Sections 3–5 and partially Section 6) we investigate the operator A assuming only that $q(\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R})$ (or it is continuous). We investigate this operator in the framework of extension theory considering it as a (nonselfadjoint) extension of the minimal symmetric operator

$$A_{\min} = A^+_{\min} \oplus A^-_{\min} = L^+_{\min} \oplus (-L^-_{\min}),$$

where $L^+_{\min}$ and $L^-_{\min}$ are minimal Sturm-Liouville operators generated by the differential expression L in $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively. Here $\operatorname{dom} L^{\pm}_{\min} := \{f \in \operatorname{dom} L : \ P_{\pm}f \in \operatorname{dom} L\}$, where $P_{\pm}$ is the orthoprojection in $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}_{\pm})$.

With operators $L^{\pm}_{\min}$ one associates the Weyl functions $m_{\pm}(\lambda)$ corresponding to the extensions $L^{\pm}_N$ of $L^{\pm}_{\min}$, generated by the Neumann problems on $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively, $\operatorname{dom} L^{\pm}_N = \{f \in \operatorname{dom} L : \ f'(\pm 0) = 0\}$.

We obtain necessary and sufficient conditions for the similarity in terms of the Weyl functions $M_+(\lambda) := m_+(\lambda)$ and $M_-(\lambda) = -m_-(-\lambda)$. Note, that $M_{\pm}(\cdot)$ are R-functions (Nevanlinna-Herglotz functions), hence the limit values $M_{\pm}(t) := M_{\pm}(t+i0)$ exist a.e. on $\mathbb{R}$.

It is worth to note that the similarity problem for the operator A gives rise to two weight estimates for the Hilbert transform H in $L^2(\mathbb{R})$. If fact, we show that the following estimates

$$\int\limits_{\mathbb{R}} \frac{\operatorname{Im} M_{\pm}(t) + \operatorname{Im} M_{\mp}(t)}{|M_+(t) - M_-(t)|^2} \left| g^{\pm}(t)\Sigma'_{ac\pm}(t) + i(H(g^{\pm} \cdot d\Sigma_{\pm}))(t) \right|^2 dt \leqslant K_1 \int_{\mathbb{R}} |g^{\pm}(t)|^2 d\Sigma_{\pm}(t), \tag{1.2}$$

(see Theorem 5.2) are necessary for the operator A to be similar to a selfadjoint operator. Here $d\Sigma_{\pm}(\cdot)$ stand for the (spectral) measures from the integral representations of $M_{\pm}(\cdot)$ (see (2.9)).

We show that, in turn, conditions (1.2) yield several (weaker and simpler) necessary conditions for the similarity of A to a selfadjoint operator. One of them reads as follows

$$\left(\frac{1}{|\mathcal{I} \cap E_{\pm}|} \int_{\mathcal{I}} \frac{\operatorname{Im} M_+(t) + \operatorname{Im} M_-(t)}{|M_+(t) - M_-(t)|^2} dt \right) \cdot \left(\frac{1}{|\mathcal{I} \cap E_{\pm}|} \int_{\mathcal{I}} \operatorname{Im} M_{\pm}(t) dt \right) \leqslant C < \infty, \tag{1.3}$$

where $\mathcal{I} (\subset \mathbb{R})$ is any interval, $E_{\pm}$ stand for the topological supports of the functions $\operatorname{Im} M_{\pm}(t) = \operatorname{Im} M_{\pm}(t + i0)$, $t \in \mathbb{R}$, and C does not depend on $\mathcal{I}$.

Besides, condition (1.2) implies the following simpler necessary condition for the similarity

$$\frac{\operatorname{Im} M_+(t) + \operatorname{Im} M_-(t)}{M_+(t) - M_-(t)} \in L^{\infty}(\mathbb{R}). \tag{1.4}$$

Note also, that (1.4) is implied by (1.3) so, the necessary condition (1.3) is stronger. We mention also that (1.2) yields one more necessary condition for the similarity that is formulated in terms of the Poisson integrals and is stronger than (1.3) (see Corollary 5.6).

Moreover, we show (see Lemma 5.7) that inequalities (1.2) are equivalent to the following ones involved only Hilbert transform

$$\int_{\mathbb{R}} \frac{\operatorname{Im} M_{\pm}(t) + \operatorname{Im} M_{\mp}(t)}{|M_{+}(t) - M_{-}(t)|^2} \left|(H(g^{\pm} \cdot d\Sigma_{\pm}))(t)\right|^2 dt \leqslant K_1 \int_{\mathbb{R}} |g^{\pm}(t)|^2 d\Sigma_{\pm}(t). \tag{1.5}$$

So, necessary conditions for the similarity are reduced to two-weight estimates of the Hilbert transform only. In particular, all necessary conditions listed above are implied by (1.5). We conjecture, that if additionally both measures $d\Sigma_{+}$ and $d\Sigma_{-}$ are absolutely continuous, $\Sigma_{\pm} = \Sigma_{ac\pm}$, and $\sigma_{\text{disc}} = \emptyset$, then conditions (1.5) are also sufficient for A to be similar to a selfadjoint operator.

We also show that the (stronger) condition

$$\sup_{\lambda \in \mathbb{C}_{+}} \frac{|M_{+}(\lambda) + M_{-}(\lambda)|}{|M_{+}(\lambda) - M_{-}(\lambda)|} < \infty \tag{1.6}$$

is sufficient for the similarity to a selfadjoint operator though it is not necessary (see Remark 8.1).

The second part of the paper (Sections 6 and 7) is devoted to the spectral analysis of the operator (1.1) with a finite-zone potential $q(\cdot)$.

Recall that a quasi-periodic (in particular, periodic) potential $q(\cdot) = \overline{q(\cdot)}$ is called a finite-zone potential if the spectrum $\sigma(L)$ of the (selfadjoint) operator L has a finite number of bands (equivalently, the resolvent set $\rho(L)$ has a finite number of gaps, that are called also forbidden zones).

We show that the operator $A = (\operatorname{sgn} x)\left(-\frac{d^2}{dx^2} + q\right)$ with a finite-zone potential q has a finite number of eigenvalues, and it has no embedded eigenvalues in the essential spectrum $\sigma_{\text{ess}}(A)$, that is $\sigma_p(A) \cap \sigma_{\text{ess}}(A) = \emptyset$ (equivalently, the essential spectrum of A coincides with purely continuous spectrum). Moreover, we show that the operator A admits the following direct sum decomposition:

$$A = A_{\text{disc}} \dotplus A_{\text{ess}}, \tag{1.7}$$

where A_{ess} is a part of the operator A corresponding to essential spectrum $\sigma_{\text{ess}}(A)$ of A.

We summarize our main results (Theorems 7.1, 7.2 and Corollary 7.4) as follows:

If the potential q is finite-zone, then the part A_{ess} of the operator A is similar to a selfadjoint operator if and only if condition (1.4) is satisfied. Besides, in this case A_{ess} is similar to a selfadjoint operator with absolutely continuous spectrum.

Moreover, A is similar to a normal operator if and only if condition (1.4) is satisfied and all the eigenvalues of A are simple.

In connection with latter statement we mention extremely interesting recent publications [20], [21] where a criterion for a nonselfadjoint Hill operator L (i.e., operator L with periodic $q \neq \bar{q}$) to be similar to a normal operator have been obtained.

The paper is organized as follows. In Section 2 we present some necessary information on finite-zone Sturm-Liouville operators, boundary triplets and the corresponding Weyl functions, and briefly discuss a functional model for a symmetric operator. Some information on Hardy classes and two-weight estimates of the Hilbert transform is presented too.

In Section 3 we establish some new results on similarity of a nonselfadjoint (in particular J-selfadjoint) operator to a selfadjoint one in terms of characteristic functions $\theta_T(\cdot)$ and their $\mathcal{J}$-forms $\omega_\theta(\cdot) := \mathcal{J} - \theta_T(\cdot)\mathcal{J}\theta_T^*(\cdot)$ and $\omega_{\theta^*}(\cdot) := \mathcal{J} - \theta_T^*(\cdot)\mathcal{J}\theta_T(\cdot)$. For example, we show in Proposition 3.4 that if a completely non-selfadjoint operator T without eigenvalues is similar to a selfadjoint operator $T_0 = T_0^*$ and the $\mathcal{J}$-form $\omega_\theta(\cdot)$ is bounded in $\mathbb{C}_+ \cup \mathbb{C}_-$ outside arbitrary small neighborhoods of a finite (for simplicity) number of spectral singularities, then T_0 is purely absolutely continuous. We compute also a characteristic function $\theta_A(\cdot)$ of the operator A and show that the upper diagonal entry of $\theta_A(\cdot)$ is precisely the function $(M_+(\cdot) + M_-(\cdot))(M_+(\cdot) - M_-(\cdot))^{-1}$ appeared in (1.6).

In Section 4 we compute the resolvent and investigate the eigenvalues of the operator A. Moreover, we prove the known fact that the spectrum of A is real if $L \geqslant 0$. Our proof, however, is based on Weyl function technique and differs form the known ones.

In Section 5 we find some necessary and sufficient conditions for A to be similar to a selfadjoint operator. In particular, based on the resolvent criterion of similarity (see [50, 46]) we prove that integral conditions (1.2) are necessary for similarity. Moreover, we show that much simpler conditions (1.3) and (1.4) formulated in terms of the Weyl functions are necessary too and indicate a stronger necessary condition (see Corollary 5.6) formulated in terms of harmonic continuation of two weights appeared in (1.4). Besides, we show that condition (1.6) is sufficient for similarity to a selfadjoint operator.

In Sections 6 and 7 we prove the main result stated above. In particular, we show in Corollary 7.4 that the continuous part A_{ess} of J-nonnegative operator A with a finite-zone potential $q(\cdot)$ is similar to a selfadjoint operator with absolutely continuous spectrum. Moreover, we demonstrate in Example 8.2 that J-nonnegativity of the operator A is not necessary for its similarity to a selfadjoint operator even in the case of one-zone potential q.

Note in conclusion, that indefinite Sturm-Liouville operators with finite-zone potentials are useful in spectral theory allowing one to construct different counterexamples. Say, neither characteristic function $\theta_A(\cdot)$, no the corresponding $\mathcal{J}$-forms $\omega_{\theta^*}(\cdot)$ and $\omega_\theta(\cdot)$ of any operator A with a finite-zone potential is bounded in $\mathbb{C}_+ \cup \mathbb{C}_-$, while the operator A may be similar to a selfadjoint one. Besides, we show using a one-zone periodic operator that condition (1.6) is not necessary for similarity to a selfadjoint operator (see Remark 8.1). Moreover, we present an example (see Example 8.3) of J-selfadjoint one-zone periodic operator A that is not definitizable (hence is not J-positive), though it is similar to a normal operator and its "continuous part" A_{ess} (see (1.7)) is similar to a selfadjoint operator.

We emphasize that all general results of the paper contained in Sections 3–6 are stated with respect to the operator A, though they are valid (without changes in the proofs) for a special nonselfadjoint extension (a nonselfadjoint coupling of two symmetric operators S_1 and S_2, $n_\pm(S_1) = n_\pm(S_2) = 1$) of the operator $S = S_1 \oplus S_2$ (see Remark 3.3).

The main results of the paper have been announced in our short communication [38] and published as a preprint [39].

NOTATIONS. Throughout the paper $\mathfrak{H}$ and $\mathcal{H}$ denote Hilbert spaces, $\mathcal{C}(\mathfrak{H})$ ($[\mathfrak{H}]$) stands for the set of closed (resp. bounded) linear operators in $\mathfrak{H}$. The domain, kernel and range of an operator $T \in \mathcal{C}(\mathfrak{H})$ is denoted by $\mathrm{dom}(T)$, $\ker(T)$ and $\mathrm{ran}(T)$ respectively; $\sigma(T)$ and $\rho(T)$ denote the spectrum and the resolvent set of $T \in \mathcal{C}(\mathfrak{H})$ respectively; $\mathcal{R}_T(\lambda) := (T - \lambda I)^{-1}$, $\lambda \in \rho(T)$, stands for the resolvent of $T \in \mathcal{C}(\mathfrak{H})$.

As usual, $\sigma_{\mathrm{disc}}(T)$ denotes the discrete spectrum of $T \in \mathcal{C}(\mathfrak{H})$, that is, the set of isolated eigenvalues of finite algebraic multiplicity; the essential spectrum is $\sigma_{\mathrm{ess}}(T) := \sigma(T) \setminus \sigma_{\mathrm{disc}}(T)$; $\sigma_p(T)$ stands for the set of eigenvalues. The continuous spectrum is defined by

$$\sigma_c(T) := \{\lambda \in \mathbb{C} \setminus \sigma_p(T) : \mathrm{ran}(T - \lambda) \neq \overline{\mathrm{ran}(T - \lambda)} = \mathfrak{H}\}.$$

If $T = T^*$ is selfadjoint, then $\sigma_{ac}(T)$ and $\sigma_s(T)$ stand for the absolutely continuous and singular spectra of T respectively. $\mathrm{Lat}\, T$ stands for the set of (closed) invariant subspaces of $T \in \mathcal{C}(\mathfrak{H})$. $\mathrm{span}\{f_1, f_2, \dots\}$ is the closed linear hull of vectors f_1, $f_2, \dots$.

For any interval $\mathcal{I}$ in $\mathbb{R}$ and any Borel measure $d\Sigma$ on $\mathcal{I}$ we denote by $L^2(\mathcal{I}, d\Sigma)$ the Hilbert space of measurable functions f on $\mathcal{I}$ satisfying $\int_{\mathcal{I}} |f|^2 d\Sigma < \infty$. If $\mathcal{I}$ or $d\Sigma$ is fixed, we will write $L^2(d\Sigma)$ or $L^2(\mathcal{I})$. The topological support $\mathrm{supp}\, d\Sigma$ of $d\Sigma$ is the smallest closed set S such that $d\Sigma(\mathbb{R} \setminus S) = 0$. The indicator function of a set S is denoted by $\chi_S(\cdot)$; $\chi_\pm(t) := \chi_{\mathbb{R}_\pm}(t)$.

We say $f \in \mathrm{Hol}(\mathcal{D})$ if $f(\cdot)$ is a holomorphic function on a domain $\mathcal{D}$. As usual $H^2(\mathbb{C}_+)$ and $\mathcal{N}^+(\mathbb{C}_+)$ stand for the Hardy space and the Smirnov class on $\mathbb{C}_+$ respectively (see [19, 42] and Subsection 2.6.). For any interval $\mathcal{I}$ in $\mathbb{R}$ and $\alpha \in (0, 1]$ denote by $\mathrm{Lip}^\alpha(\mathcal{I})$ the Lipschitz class on $\mathcal{I}$ (see, for example, [19]).

We write $f(x) \asymp g(x)$ $(x \to x_0)$, if both $\frac{f}{g}$ and $\frac{g}{f}$ are bounded in a small neighborhood of the point x_0; $f(x) \asymp g(x)$ $(x \in D)$ means that $\frac{f}{g}$ and $\frac{g}{f}$ are bounded on the set D.

2. Preliminaries

2.1. Indefinite Sturm-Liouville operators $(\mathrm{sgn}\, x)\left(-\frac{d^2}{dx^2} + q(x)\right)$

Denote by J the multiplication operator by $\mathrm{sgn}\, x$ in the Hilbert space $L^2(\mathbb{R})$, $J : f(x) \to \mathrm{sgn}\, x f(x)$. Next we consider in $L^2(\mathbb{R})$ the differential expression

$$L = -\frac{d^2}{dx^2} + q(x), \tag{2.1}$$

with a real continuous potential q. Suppose additionally that the minimal operators L^+_{min}, L^-_{min} (see [51], [52]) associated with (2.1) in $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively, have the deficiency indices $(1,1)$. Denote also by L the Sturm-Liouville operator generated in $L^2(\mathbb{R})$ by the differential expression (2.1). It is clear that L is selfadjoint in $L^2(\mathbb{R})$.

The main object of our paper is an indefinite Sturm-Liouville operator

$$A := JL = (\operatorname{sgn} x)\left(-\frac{d^2}{dx^2} + q(x)\right), \qquad \operatorname{dom}(A) := \operatorname{dom}(L), \tag{2.2}$$

in $L^2(\mathbb{R})$. It is easy to see that $A \neq A^*$. Indeed, the operator $A^* = LJ$ is defined by the same differential expression (2.2) on the domain, $\operatorname{dom}(A^*) = J\operatorname{dom} L \neq \operatorname{dom}(A)$, containing functions discontinuous at zero together with the first derivative.

DEFINITION 2.1. Let J be a signature operator on a Hilbert space $\mathfrak{H}$, that is $J = J^* = J^{-1}$. An operator T in $\mathfrak{H}$ is called J-selfadjoint if $JT = (JT)^*$.

It is clear that A is a J-selfadjoint operator. We will investigate the operator A in the framework of extension theory of symmetric operators. For this purpose we recall the following

DEFINITION 2.2. ([1]) Let S be a closed symmetric operator with equal finite deficiency indices (n,n), $n < \infty$. A closed operator $\widetilde{S}$ is called a quasi-selfadjoint extension of S if

$$S \subset \widetilde{S} \subset S^* \qquad \text{and} \qquad \dim\big(\operatorname{dom}(\tilde{S})/\operatorname{dom}(S)\big) = n.$$

Let

$$A_{\min} := A \cap A^* \qquad \text{and} \qquad A_{\min}^{\pm} := \pm L_{min}^{\pm}. \tag{2.3}$$

It is easily seen that

$$A_{\min} = A_{\min}^{-} \oplus A_{\min}^{+}, \qquad \operatorname{dom}(A_{\min}) := \{y \in \operatorname{dom}(L) : \ y(0) = y'(0) = 0\}. \tag{2.4}$$

It is clear that $A_{\min}$ is a simple symmetric operator with deficiency indices $(2,2)$ and A is its quasi-selfadjoint extension. Indeed,

$$\begin{aligned} \operatorname{dom}(A) \ := \ & \big\{y \in \operatorname{dom}\big((A_{\min}^{+})^*\big) \oplus \big((A_{\min}^{-})^*\big) : \\ & y(+0) = y(-0),\ y'(+0) = y'(-0)\big\}, \end{aligned} \tag{2.5}$$

and $\dim(\operatorname{dom}(A)/\operatorname{dom}(A_{\min})) = 2$.

Note in conclusion that if q is bounded, then $\operatorname{dom}(A) := \operatorname{dom}(L) = W_2^2(\mathbb{R})$, the Sobolev space, and $\operatorname{dom}(A_{\min}) = W_2^{2,0}(\mathbb{R}) := \{y \in W_2^2(\mathbb{R}) : \ y(0) = y'(0) = 0\}$.

2.2. Weyl functions

Recall definition of the Weyl functions of the Sturm-Liouville operator (2.1) assuming as before, the limit point cases at $\pm\infty$. Denote by $s(x,\lambda)$ and $c(x,\lambda)$ the solutions of

$$-y''(x) + q(x)y(x) = \lambda y(x)$$

satisfying the following initial conditions

$$s(0,\lambda) = \frac{d}{dx}c(0,\lambda) = 0, \qquad \frac{d}{dx}s(0,\lambda) = c(0,\lambda) = 1.$$

According to Weyl theory (see [45]) there exists the function $m_{\pm}(\lambda)$ on $\mathbb{C}_+ \cup \mathbb{C}_-$ such that

$$s(\cdot,\lambda) \mp m_{\pm}(\lambda)c(\cdot,\lambda) \in L^2(\mathbb{R}_{\pm}). \tag{2.6}$$

The functions $m_{\pm}$ are called *the Weyl function of* $L^{\pm}_{min}$ corresponding to the initial condition $y'(0) = 0$. The functions

$$M_{\pm}(\lambda) := \pm m_{\pm}(\pm\lambda) \tag{2.7}$$

are said to be *the Weyl function of* $A^{\pm}_{\min}$ (corresponding to the initial condition $y'(0) = 0$).

Define

$$\psi_{\pm}(\cdot,\lambda) := \begin{cases} -\left(s_{\pm}(\cdot,\pm\lambda) - M_{\pm}(\lambda)c(\cdot,\pm\lambda)\right), & x \in \mathbb{R}_{\pm}, \\ 0 & x \in \mathbb{R}_{\mp}. \end{cases} \tag{2.8}$$

It is easily seen that $\psi_{\pm}(\cdot,\lambda) \in L^2(\mathbb{R}_{\pm})$ for $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ and $(A^{\pm}_{\min})^* \psi_{\pm}(x,\lambda) = \lambda\, \psi_{\pm}(x,\lambda)$.

Recall that a function $m(\lambda)$ is called *an* R*-function (Herglotz or Nevanlinna function)* [1, 29] if it is holomorphic in $\mathbb{C}_+ \cup \mathbb{C}_-$,

$$\operatorname{Im}\lambda \cdot \operatorname{Im} m(\lambda) > 0 \quad \text{for } \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \qquad \text{and} \qquad m(\bar{\lambda}) = \overline{m(\lambda)}.$$

The set of all *R-functions* is denoted by (R) (see [29]).

The functions $m_{\pm}(\cdot)$, as well as $M_{\pm}(\cdot)$ are R-functions (see [45]). Moreover, it follows from (2.7) and the known integral representation of $m_{\pm}(\cdot)$ (see [44, 52]) that $M_{\pm}(\cdot)$ admit the following integral representations

$$M_{\pm}(\lambda) = \int_{\mathbb{R}} \frac{d\Sigma_{\pm}(t)}{t-\lambda} \qquad \text{and} \qquad \int_{\mathbb{R}} \frac{d\Sigma_{\pm}(t)}{1+|t|} < \infty. \tag{2.9}$$

with (nonunique) nondecreasing scalar functions $\Sigma_{\pm}(\cdot)$. Note that $\Sigma_{\pm}(\cdot)$ in (2.9) are uniquely determined by the following normalized conditions:

$$2\Sigma_{\pm}(t) = \Sigma_{\pm}(t+0) + \Sigma_{\pm}(t-0), \quad \Sigma_{\pm}(0) = 0.$$

Note also that (2.9) gives a holomorphic continuation of $M_{\pm}(\cdot)$ to $\mathbb{C} \setminus \operatorname{supp} d\Sigma_{\pm}$.

Moreover, the known asymptotic relations for $m_{\pm}(\cdot)$ (see [44]) yield

$$M_{\pm}(\lambda) \;=\; \pm\frac{i}{\sqrt{\pm\lambda}} + O\left(\frac{1}{\lambda}\right), \ \ (\lambda \to \infty,\ 0 < \delta < \arg\lambda < \pi - \delta) \tag{2.10}$$

$$\Sigma_{\pm}(t) \;=\; \pm\frac{2}{\pi}\sqrt{\pm t} \, \pm \Sigma_{\pm}(\pm\infty) \, + o(1), \quad t \to \pm\infty. \tag{2.11}$$

Here and below $\sqrt{z}$ is the branch of the multifunction on the complex plane $\mathbb{C}$ with the cut along $\mathbb{R}_+$, singled out by the condition $\sqrt{-1} = i$. We assume that $\sqrt{\lambda} \geqslant 0$ for $\lambda \in [0,+\infty)$.

Consider the operator

$$A^{\pm}_0 := (A^{\pm}_{\min})^* \upharpoonright \operatorname{dom}(A^{\pm}_0), \quad \operatorname{dom}(A^{\pm}_0) = \{y \in \operatorname{dom}\left((A^{\pm}_{\min})^*\right) : \ y'(\pm 0) = 0\}. \tag{2.12}$$

Clearly, $A_0^{\pm} = (A_0^{\pm})^*$. The functions $\Sigma_{\pm}$ are called *the spectral functions* of the operators $A_0^{\pm}$ [45, 52]. It is known that the generalized Fourier transforms $\mathcal{F}_{\pm}$, defined by

$$(\mathcal{F}_{\pm}f)(t) := \underset{x_1 \to \pm\infty}{\mathrm{l.i.m.}} \pm \int_0^{x_1} f(x)c(x, \pm t)dx, \tag{2.13}$$

are isometric operators from $L^2(\mathbb{R}_{\pm})$ onto $L^2(\mathbb{R}, d\Sigma_{\pm})$. Here l.i.m. denotes the strong limit in $L^2(\mathbb{R}, d\Sigma_{\pm})$.

The operator $\widehat{A_0^{\pm}} := \mathcal{F}_{\pm}A_0^{\pm}\mathcal{F}_{\pm}^{-1}$ is the operator of multiplication by t in $L^2(\mathbb{R}, d\Sigma_{\pm}(t))$, $\widehat{A_0^{\pm}} : g(t) \to tg(t)$ (see [45, 52]). Note that $\sigma(A_0^{\pm}) = \operatorname{supp} d\Sigma_{\pm}$.

Suppose $f \in L^2(\mathbb{R})$. Let $f_{\pm} := P_{\pm}f \in L^2(\mathbb{R}_{\pm})$ where $P_{\pm}$ is the orthoprojection in $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}_{\pm})$. The following two representations of the resolvent $\mathcal{R}_{A_0^{\pm}}$ are known (see [45, 52]):

$$(\mathcal{R}_{A_0^{\pm}}(\lambda)f_{\pm})(x) = \int_{\mathbb{R}} \frac{c(x, \pm t)\,(\mathcal{F}_{\pm}f_{\pm})(t)\,d\Sigma_{\pm}(t)}{t - \lambda}, \tag{2.14}$$

$$\begin{aligned}(\mathcal{R}_{A_0^{\pm}}(\lambda)f_{\pm})(x) = \mp\psi_{\pm}(x, \lambda) \int_0^{\pm x} c(s, \pm\lambda)f(s)ds \mp c(x, \pm\lambda) \cdot \\ \cdot \int_{\pm x}^{\pm\infty} \psi_{\pm}(s, \lambda)f(s)ds.\end{aligned} \tag{2.15}$$

2.3. Definitizable operators

The spectral theory of linear operators in Kreĭn spaces can be found in [3], [43]. Here we give some basic definitions.

Consider a Hilbert space $\mathfrak{H}$ with a scalar product $(\cdot, \cdot)$. Let J be a fundamental symmetry in $\mathfrak{H}$, that is $J = J^{-1} = J^*$. We put $[\cdot, \cdot] := (J\cdot, \cdot)$. Then the pair $\mathcal{K} = (\mathfrak{H}, [\cdot, \cdot])$ is a Kreĭn space (see the literature cited above). If $J \neq I$, then the sesqulinear form $[\cdot, \cdot]$ is indefinite.

Let T be a densely defined operator in $\mathfrak{H}$. Then J-adjoint operator $T^{[*]}$ is defined by

$$[Tf, g] = [f, T^{[*]}g], \qquad f \in \operatorname{dom}(T), \;\; g \in \operatorname{dom}(T^{[*]}).$$

Clearly, $T^{[*]} = JT^*J$, where T^* is the adjoint operator with respect to the scalar product $(\cdot, \cdot)$. An operator T is called *J-selfadjoint* if $T = T^{[*]}$. Evidently, this definition is equivalent to Definition 2.1 and $T = T^{[*]} \iff T = JT^*J$.

DEFINITION 2.3. ([43]) A J-selfadjoint operator T is called definitizable if $\rho(T) \neq \emptyset$ and there exist a real polynomial p such that

$$[p(T)f, f] \geqslant 0 \quad \text{for} \quad f \in \operatorname{dom}(p(T)).$$

Definitizable operators have spectral functions with critical points. Thus theirs spectral properties are close to spectral properties of selfadjoint operators in some sense (see [43]).

Operators of the form (2.2) are J-selfadjoint. In this case, $\mathfrak{H} = L^2(\mathbb{R})$ and J is a multiplication operator by $\operatorname{sgn} x$. Such operators can be nondefinitizable. The following theorem gives a criterion of definitizability.

THEOREM 2.1. ([36, 37]) *Let $A = (\operatorname{sgn} x)(-d^2/dx^2 + q(x))$ be an operator of the form (2.2). Then A is definitizable if and only if the sets $\operatorname{supp} d\Sigma_+$ and $\operatorname{supp} d\Sigma_-$ (see Subsection 2.2. for definitions) are separated by a finite number of points, i.e., there exists a finite ordered set*

$$\{\alpha_j\}_{j=1}^{2n-1}, \quad -\infty = \alpha_0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_{2n-1} < \alpha_{2n} = +\infty,$$

such that

$$\operatorname{supp} d\Sigma_- \subset \bigcup_{k=0}^{n-1} [\alpha_{2k}, \alpha_{2k+1}], \qquad \operatorname{supp} d\Sigma_+ \subset \bigcup_{k=0}^{n-1} [\alpha_{2k+1}, \alpha_{2k+2}].$$

Several conditions of definitizability in abstract terms were given in [27] and [28].

Spectral properties of some classes of definitizable differential operators were studied in [8, 17, 11]; see also references in [11].

DEFINITION 2.4. An operator T is called J-nonnegative if

$$[Tf, f] \geqslant 0 \quad \text{for} \quad f \in \operatorname{dom}(T).$$

Denote the *root subspace* (the algebraic eigensubspace) of T for λ by $\mathfrak{L}_\lambda(T)$, that is

$$\mathfrak{L}_\lambda(T) := \operatorname{span}\{\ker(T-\lambda)^k : \ k \in \mathbb{Z}_+\}.$$

PROPOSITION 2.2. ([56], see also [3, 64]) *Let T be a J-nonnegative operator. Then*

(i) $\sigma_p(A) \cap (\mathbb{C}_+ \cup \mathbb{C}_-) = \emptyset$.

(ii) If $\lambda \in \sigma_p(T)$ and $\lambda \neq 0$, then the eigenvalue λ is semisimple, i.e., $\mathfrak{L}_\lambda = \ker(T-\lambda)$.

(iii) If $0 \in \sigma_p(T)$, then $\mathfrak{L}_0 = \ker T^2$ (in general, $\mathfrak{L}_0 \neq \ker T$).

2.4. Finite-zone potentials

Following [44] we recall a definition of Sturm-Liouville operator with a finite-zone potential. Let $N \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Consider sets of real numbers $\{\overset{l}{\mu}_j\}_{j=0}^{N+1}$, $\{\overset{r}{\mu}_j\}_0^N$, $\{\xi_j\}_1^N$ such that

$$-\infty = \overset{l}{\mu}_0 < \overset{r}{\mu}_0 < \overset{l}{\mu}_1 < \overset{r}{\mu}_1 < \cdots < \overset{l}{\mu}_N < \overset{r}{\mu}_N < \overset{l}{\mu}_{N+1} = +\infty,$$

$\xi_j \in [\overset{l}{\mu}_j, \overset{r}{\mu}_j]$, $j = 1, \ldots, N$. Define polynomials $R(\lambda)$, $P(\lambda)$ by

$$P(\lambda) = \prod_{j=1}^{N} (\lambda - \xi_j), \qquad R(\lambda) = (\lambda - \overset{r}{\mu}_0) \prod_{j=1}^{N} (\lambda - \overset{l}{\mu}_j)(\lambda - \overset{r}{\mu}_j). \tag{2.16}$$

Then there exist (see [44, Lemma8.1.1]) real polynomials $S(\lambda)$ and $Q(\lambda)$ of degrees $\deg S = N+1$ and $\deg Q = N-1$ respectively and such that

$$S(\lambda) = \prod_{j=0}^{N}(\lambda - \tau_j), \quad \tau_0 \in (-\infty, \overset{r}{\mu}_0], \quad \tau_j \in [\overset{l}{\mu}_j, \overset{r}{\mu}_j], \quad j \in \{1,\dots,N\}, \tag{2.17}$$

and such that the following identity holds

$$P(\lambda)S(\lambda) - Q^2(\lambda) = R(\lambda). \tag{2.18}$$

According to [44, formulas (8.1.9) and (8.1.10)] (see also [45, formulas (5.1.8)]) the functions

$$m_\pm(\lambda) := \pm\frac{P(\lambda)}{Q(\lambda) \mp i\sqrt{R(\lambda)}} \tag{2.19}$$

are the Weyl functions corresponding to the Neumann boundary value problems on $\mathbb{R}_\pm$ for some Sturm-Liouville operator $L = -d^2/dx^2 + q(x)$ with a bounded quasi-periodic potential $q = \bar{q}$. Here the multifunction $\sqrt{R(\cdot)}$ is considered on $\mathbb{C}$ with cuts along the union of segments $[\overset{r}{\mu}_0, \overset{l}{\mu}_1] \cup [\overset{r}{\mu}_1, \overset{l}{\mu}_2] \cup \dots \cup [\overset{r}{\mu}_{N-1}, \overset{l}{\mu}_N]$ and the semi-axes $[\overset{r}{\mu}_N, +\infty)$. The branch $\sqrt{R(\cdot)}$ of the multifunction is chosen in such a way that $\sqrt{R(x_0+i0)} > 0$ for some $x_0 \in (\overset{r}{\mu}_N, +\infty)$. In this case $\operatorname{Im} m_+(x+i0) > 0$ for $x \in (\overset{r}{\mu}_0, \overset{l}{\mu}_1) \cup (\overset{r}{\mu}_1, \overset{l}{\mu}_2) \cup \dots \cup (\overset{r}{\mu}_N, +\infty)$ and both $m_\pm(\cdot)$ are R-functions.

DEFINITION 2.5. A (quasi-periodic) potential $q = \bar{q}$ is called a finite-zone potential if the Weyl functions $m_\pm(\,\cdot\,)$ of $L_\pm$ defined by (2.6) admit representations (2.19).

Assume q to be a finite-zone potential. Then q is an analytic function, and the nth derivative $\frac{d^n}{dx^n}q$ is bounded on $\mathbb{R}$ for any $n \in \mathbb{N}$. Moreover, the spectrum of $L = -d^2/dx^2 + q(x)$ is absolutely continuous, and

$$\sigma(L) = \sigma_{ac}(L) = [\overset{r}{\mu}_0, \overset{l}{\mu}_1] \cup [\overset{r}{\mu}_1, \overset{l}{\mu}_2] \cup \dots \cup [\overset{r}{\mu}_N, +\infty).$$

Combining (2.19) with (2.7), we get

$$M_\pm(\lambda) = \frac{P(\pm\lambda)}{Q(\pm\lambda) \mp i\sqrt{R(\pm\lambda)}}. \tag{2.20}$$

Using (2.18), we rewrite (2.20) as

$$M_\pm(\lambda) = \frac{Q(\pm\lambda) \pm i\sqrt{R(\pm\lambda)}}{S(\pm\lambda)}. \tag{2.21}$$

2.5. Boundary triplets and abstract Weyl functions

2.5.1. Weyl functions and spectra of proper extensions.

Let $\mathfrak{H}$ and $\mathcal{H}$ be separable Hilbert spaces.

DEFINITION 2.6. A closed linear relation Θ in $\mathcal{H}$ is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$.

EXAMPLE 2.1. For any closed operator B in $\mathcal{H}$ its graph $G(B)$ is a closed relation in $\mathcal{H}$.

Let S be a closed densely defined symmetric operator in $\mathfrak{H}$ with equal deficiency indices $n_+(S) = n_-(S)$, where $n_\pm(S) := \dim \mathfrak{N}_{\pm i}$ and $\mathfrak{N}_\lambda := \ker(S^* - \lambda)$.

DEFINITION 2.7. ([1]) A closed extension $\tilde{S}$ of S is called a proper extension if $S \subset \tilde{S} \subset S^*$. The set of all proper extensions is denoted by Ext_S.

Recall the definition of a boundary triplet.

DEFINITION 2.8. ([23]) A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ consisting of an auxiliary Hilbert space $\mathcal{H}$ and linear mappings $\Gamma_j : \operatorname{dom}(S^*) \longrightarrow \mathcal{H}, \quad j \in \{0,1\}$, is called a boundary triplet for the operator S^* if the following conditions are satisfied:

(i) The second Green formula

$$(S^* f, g) - (f, S^* g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \qquad f, g \in \operatorname{dom}(S^*), \quad (2.22)$$

holds;

(ii) The mapping $\Gamma : \operatorname{dom}(S^*) \longrightarrow \mathcal{H} \oplus \mathcal{H}$, $\Gamma f := \{\Gamma_0 f, \Gamma_1 f\}$ is surjective.

Definition 2.8 allows one to describe the set Ext_S in the following way (see [12, 13]).

PROPOSITION 2.3. ([12, 13]) *Let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for* S^*. *Then the mapping* Γ *establishes a bijective correspondence* $\tilde{S} \to \Theta := \Gamma(\operatorname{dom}(\tilde{S}))$ *between the set* Ext_S *and the set of closed linear relations in* $\mathcal{H}$.

By Proposition 2.3 the following definition is natural.

DEFINITION 2.9. Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for the operator S^*.

(i) Denote $S_\Theta = \tilde{S}$, if $\Theta = \Gamma(\operatorname{dom}(\tilde{S}))$ that is

$$S_\Theta := S^* | D_\Theta, \text{ where } \operatorname{dom}(S_\Theta) = D_\Theta := \{f \in \operatorname{dom}(S^*) : \{\Gamma_0 f, \Gamma_1 f\} \in \Theta\}. \quad (2.23)$$

(ii) If $\Theta = G(B)$ is the graph of $B \in \mathcal{C}(\mathcal{H})$ then $\operatorname{dom}(S_\Theta)$ determined by the equation $\operatorname{dom}(S_B) = D_B := D_\Theta = \ker(\Gamma_1 - B\Gamma_0)$. We set $S_B := S_\Theta$.

Let us make the following remarks.

REMARK 2.1. 1) The deficiency indices $n_\pm(S)$ are equal to the dimension of $\mathcal{H}$, i.e., $\dim(\mathcal{H}) = n_\pm(S)$.

2) There exist two self-adjoint extensions $S_j := S^* | \ker(\Gamma_j)$ which are naturally associated to a boundary triplet. According to Definition 2.9 $S_j = S_{\Theta_j}, j \in \{0,1\}$, where $\Theta_0 = \{0\} \times \mathcal{H}$, $\Theta_1 = \mathcal{H} \times \{0\}$. Conversely, if S_0 is a self-adjoint extension of A, then there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ such that $S_0 = S^* | \ker(\Gamma_0)$.

3) Θ is the graph of a closed operator B if and only if $\tilde{S}$ and S_0 are disjoint, i.e., $\operatorname{dom}(\tilde{S}) \cap \operatorname{dom}(S_0) = \operatorname{dom}(S)$.

4) $\Theta = G(B)$ with $B \in [\mathcal{H}]$ if and only if $\tilde{S}$ and S_0 are transversal, i.e., $\tilde{S}$ and S_0 are disjoint and $\operatorname{dom}(\tilde{S}) + \operatorname{dom}(S_0) = \operatorname{dom}(S^*)$.

DEFINITION 2.10. ([14]) A proper extension $\tilde{S} \in \mathrm{Ext}_S$ is called an almost solvable if there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and an operator $B \in [\mathcal{H}]$ such that

$$\mathrm{dom}\,(\tilde{S}) = \mathrm{dom}\,(S_B) := \ker(\Gamma_1 - B\Gamma_0). \tag{2.24}$$

The set of almost solvable extensions is denoted by $\mathcal{A}s_S$.

Note that the class $\mathcal{A}s_S$ is sufficiently wide. A proper extension T having two regular points $\lambda_\pm \in \mathbb{C}_\pm$ belongs to $\mathcal{A}s_S$, $T \in \mathcal{A}s_S$. All quasiselfadjoint extensions are in $\mathcal{A}s_S$.

In [12, 13] the concept of Weyl function was generalized to an arbitrary symmetric operator T with infinite deficiency indices $n_+(A) = n_-(A)$. Recall some basic facts about Weyl functions.

DEFINITION 2.11. ([12, 13]) Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*. The Weyl function of T corresponding to the boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a unique mapping

$$M(\cdot): \ \rho(T_0) \longrightarrow [\mathcal{H}] \tag{2.25}$$

satisfying

$$\Gamma_1 f_\lambda = M(\lambda)\Gamma_0 f_\lambda, \quad f_\lambda \in \mathfrak{N}_\lambda = \ker(S^* - \lambda I), \quad \lambda \in \rho(S_0). \tag{2.26}$$

It is well known (see [12, 13]) that the above implicit definition of the Weyl function is correct and $M(\cdot)$ is an operator-valued R-function satisfying $0 \in \rho(\mathrm{Im}\,(M(i)))$ (see [15]). The Weyl function immediately provides some information about the "spectral properties" of proper extensions. We confine ourselves to the case of almost solvable extensions of the symmetric operator S.

PROPOSITION 2.4. ([13, 14]) *Suppose that* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is a boundary triplet for* S^*, $M(\cdot)$ *is the corresponding Weyl function,* $\lambda \in \rho(S_0)$ *and* $B \in [\mathcal{H}]$. *Then:*

1) $\lambda \in \rho(S_B)$ *if and only if* $0 \in \rho(B - M(\lambda))$;

2) $\lambda \in \sigma_i(S_B)$ *if and only if* $0 \in \sigma_i(B - M(\lambda))$, $i \in \{p, r, c\}$.

We demonstrate applicability of Proposition 2.4 by describing a discrete spectrum of the operator A.

PROPOSITION 2.5. *Let* $S := A_{\min}$ *be a (minimal) symmetric operator defined by (2.4) and let* $M_\pm(\cdot)$ *be defined by (2.7). Then*

(i) $\Pi = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ *defined by*

$$\Gamma_0,\ \Gamma_1: \ \mathrm{dom}\,(A^*_{\min}) \to \mathcal{H} = \mathbb{C}^2, \tag{2.27}$$
$$\Gamma_0 f = \begin{pmatrix} f(+0), \\ f'(-0) \end{pmatrix}, \qquad \Gamma_1 f = \begin{pmatrix} f'(+0) \\ -f(-0) \end{pmatrix},$$

forms a boundary triplet for the operator $S^* = A^*_{\min}$;

(ii) *The corresponding Weyl function is*

$$M(\lambda) := M_\Pi(\lambda) = \mathrm{diag}\left(-M_+^{-1}(\lambda), M_-(\lambda)\right); \tag{2.28}$$

(iii) *The operator* $A = JL$ *defined by (2.2) is a quasi-selfadjoint extension of* S *and it is determined by*

$$A = S^*|\operatorname{dom} A, \quad \operatorname{dom} A = \ker(\Gamma_1 - B\Gamma_0), \quad \textit{where} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.29}$$

that is $A = S_B$*;*

(iv) $\rho(A) \neq \emptyset$ *and* $\lambda_0 \in \rho(A) \cap \mathbb{C}_\pm$ *if and only if* $M_+(\lambda_0) \neq M_-(\lambda_0)$*. Moreover,* $\rho(A) \cap \mathbb{R} = \cup_j(\alpha_j, \beta_j)$ *where* (α_j, β_j) *is such an interval that both* M_+ *and* M_- *admit holomorphic continuation trough* (α_j, β_j) *and* $M_+(x+i0) \neq M_-(x+i0)$*,* $x \in (\alpha_j, \beta_j)$*.*

(v) *The sets* $\sigma_p(A) \cap \mathbb{C}_\pm$ *are at most countable with possible limit points belonging to* $\mathbb{R} \cup \{\infty\}$*. Moreover,* $\lambda_0 \in \sigma_p(A) \cap \mathbb{C}_\pm$ *if and only if* $M_+(\lambda_0) = M_-(\lambda_0)$*. In the latter case* $\dim \mathfrak{L}_{\lambda_0}(A) = \mathfrak{m}(\lambda_0)$*, where* $\mathfrak{m}(\lambda_0)$ *is the multiplicity of* λ_0 *as a zero of the analytic function* $M_+(\lambda) - M_-(\lambda)$*;*

(vi) *The spectrum* $\sigma(A)$ *is symmetric with respect to the real line, that is* $\lambda_0 \in \sigma_p(A) \Longleftrightarrow \overline{\lambda}_0 \in \sigma_p(A)$ *and* $\dim \mathfrak{L}_{\lambda_0}(A) = \dim \mathfrak{L}_{\overline{\lambda}_0}(A)$ *(equivalently* $\lambda_0 \in \sigma(A) \Longleftrightarrow \lambda_0 \in \sigma(A^*)$ *and* $\dim \mathfrak{L}_{\lambda_0}(A) = \dim \mathfrak{L}_{\lambda_0}(A^*)$*.*

Proof. (i)–(iii) These statements are obvious.

(iv) By Proposition 2.4 $\lambda_0 \in \rho(A)$ if and only if $0 \in \rho(B - M(\lambda_0)$, that is

$$\begin{aligned} \det\big(B - M(\lambda)\big) &= \det \begin{pmatrix} M_+^{-1}(\lambda) & 1 \\ -1 & -M_-(\lambda) \end{pmatrix} \\ &= M_+^{-1}(\lambda) \cdot [M_+(\lambda) - M_-(\lambda)] \neq 0. \end{aligned} \tag{2.30}$$

Note that due to (2.10) $M_+(\cdot)$ and $M_-(\cdot)$ have different asymptotic behavior along any semi-axes $t \cdot e^{i\varphi}$, $t > 0$ with $\varphi \in (0, \pi/2)$. Hence $M_+(\cdot) - M_-(\cdot) \not\equiv 0$, that is the determinant $\det\big(B - M(\cdot)\big)$ does not vanish identically and $\rho(A) \neq \emptyset$.

The last statement follows from Proposition 2.4 and the identity

$$(B - M(\lambda))^{-1} = \frac{1}{M_+(\lambda) - M_-(\lambda)} \begin{pmatrix} -M_+(\lambda)M_-(\lambda) & -M_+(\lambda) \\ M_+(\lambda) & 1 \end{pmatrix}.$$

(v) By Proposition 2.4 $\sigma(S_B) \cap \mathbb{C}_\pm$ coincides with the set of zeros of the determinant $\det\big(B - M(\cdot)\big)$ in $\mathbb{C}_\pm$. Due to (2.30) $\sigma(S_B) \cap \mathbb{C}_\pm$ coincides with the set of zeros of $M_+(\cdot) - M_-(\cdot)$ in $\mathbb{C}_\pm$ since $M_+(\cdot)$ has no zeros in $\mathbb{C}_\pm$. The analytic function $M_+(\cdot) - M_-(\cdot)$ does not vanish identically, hence it has at most countable set of zeros in both $\mathbb{C}_+$ and $\mathbb{C}_-$. The remaining statements follow from analyticity of $M_+(\cdot) - M_-(\cdot)$ and Proposition 2.4.

(vi) Note that $M_+(\lambda_0) - M_-(\lambda_0) = 0$ yields

$$M_+(\overline{\lambda}_0) - M_-(\overline{\lambda}_0) = \overline{M_+(\lambda_0) - M_-(\lambda_0)} = 0.$$

A similar implication is valid for jth derivative. This completes the proof. $\square$

2.5.2. A functional model of a symmetric operator.

Next we recall construction of a functional model of a symmetric operator following [15], [48] (see also [22]). We need only the case of the deficiency indices $(1,1)$.

Let $\Sigma(t)$ be a nondecreasing scalar function satisfying the conditions

$$\int_{\mathbb{R}} \frac{1}{1+t^2} d\Sigma(t) < \infty, \quad \int_{\mathbb{R}} d\Sigma(t) = \infty ,$$
$$\Sigma(t) = \frac{1}{2}(\Sigma(t-0)+\Sigma(t+0)), \quad \Sigma(0) = 0. \tag{2.31}$$

The operator of multiplication $Q_\Sigma : f(t) \to tf(t)$ is selfadjoint in $L^2(\mathbb{R}, d\Sigma)$. Consider its restriction

$$\widehat{T}_\Sigma = Q_\Sigma \upharpoonright \mathrm{dom}\,(\widehat{T}_\Sigma), \qquad \mathrm{dom}\,(\widehat{T}_\Sigma) = \left\{ f \in \mathrm{dom}\, Q_\Sigma : \int_{\mathbb{R}} f(t) d\Sigma(t) = 0 \right\}.$$

Then $\widehat{T}_\Sigma$ is a simple densely defined symmetric operator in $L^2(\mathbb{R}, d\Sigma)$ with deficiency indices $(1,1)$. The adjoint operator $\widehat{T}_\Sigma^*$ has the form

$$\mathrm{dom}\,(\widehat{T}_\Sigma^*) = \{ f = f_Q + t(t^2+1)^{-1}h \ : f_Q \in \mathrm{dom}\,(Q_\Sigma),\ h \in \mathbb{C} \},$$
$$\widehat{T}_\Sigma^* f = tf_Q - (t^2+1)^{-1}h.$$

Let $C \in \mathbb{R}$. Define linear mappings Γ_0^Σ, $\Gamma_1^{\Sigma,C}$: $\mathrm{dom}\,(\widehat{T}_\Sigma^*) \to \mathbb{C}$ by

$$\Gamma_0^\Sigma f = h, \qquad \Gamma_1^{\Sigma,C} f = Ch + \int_{\mathbb{R}} f_Q(t) d\Sigma(t), \tag{2.32}$$

where

$$f = f_Q + t(t^2+1)^{-1}h \in \mathrm{dom}\,(\widehat{T}_\Sigma^*), \qquad f_Q \in \mathrm{dom}\,(Q_\Sigma), \qquad h \in \mathbb{C}.$$

Then $\{\mathbb{C}, \Gamma_0^\Sigma, \Gamma_1^{\Sigma,C}\}$ is a boundary triplet for $\widehat{T}_\Sigma^*$. The function

$$M_{\Sigma,C}(\lambda) := C + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d\Sigma(t), \quad \lambda \in \mathbb{C} \setminus \mathrm{supp}\, d\Sigma, \tag{2.33}$$

is the corresponding Weyl function of $\widehat{T}_\Sigma$.

2.6. Some facts from Hardy spaces theory

2.6.1. The Hilbert transform in weighted spaces

Let us recall some facts of Hardy spaces theory following [19] and [42].

Let μ be a Borel measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} (1+t^2)^{-1} d\mu(t) < \infty$. As usual we denote by $u(\lambda) = \mathcal{P}_\lambda(\mu)$ its harmonic extension (the Poisson integral) at the point $\lambda = x+iy \in \mathbb{C}_+$,

$$u(x+iy) := \mathcal{P}_\lambda(\mu) := (P_y * \mu)(x) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2} d\mu(t). \tag{2.34}$$

For any function $\varphi \in L^1(dt/1+t^2)$ we put $\mathcal{P}_\lambda(\varphi) := \mathcal{P}_\lambda(\mu)$ where $\mu = \varphi dx$.

Moreover, assuming that $\int_{\mathbb{R}}(1+|t|)^{-1}d\mu(t) < \infty$ one introduces the harmonic conjugate $\widetilde{u}(\cdot)$ of $u(\cdot)$ by setting

$$\widetilde{u}(x+iy) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{x-t}{(x-t)^2+y^2}d\mu(t). \tag{2.35}$$

Here we require the normalization $\lim_{y\to+\infty}\widetilde{u}(x+iy) = 0$. By Fatou theorem for a.e. $x \in \mathbb{R}$ the limit $\lim_{y\to 0} u(x,y) =: u(x+i0)$ exists and $u(x+i0) = \mu'(x)$. Moreover, the limit $\lim_{y\to 0}\widetilde{u}(x+iy) =: \widetilde{u}(x+i0)$ exists a.e. and coincides with the Hilbert transform of μ, that is

$$\widetilde{u}(x+i0) = (H\mu)(x) := \frac{1}{\pi}\lim_{\delta\to 0}\int\limits_{|x-t|>\delta}\frac{1}{x-t}d\mu(t). \tag{2.36}$$

If $f \in L^p(\mathbb{R})$ with $p \in [1,\infty)$, then by definition $(Hf)(x) := (H\mu)(x)$ with $\mu = f\,dx$. The operator H is a unitary operator on $L^2(\mathbb{R})$.

Recall the Helson-Szegö theorem [25] (see also [19]).

THEOREM 2.6. *(Helson, Szegö) Let $d\mu$ be a positive Borel measure on $\mathbb{R}$ which is finite on compact sets. There is a constant K such that*

$$\int_{\mathbb{R}} |Hf(x)|^2 d\mu(x) \leqslant K\int_{\mathbb{R}} |f(x)|^2 d\mu(x)$$

for all $f \in L^2(\mathbb{R}) \cap L^2(\mathbb{R}, d\mu)$ if and only if μ is absolutely continuous, $d\mu(x) = w(x)dx$, and

$$\log w(x) = u + Hv, \quad u \in L^\infty(\mathbb{R}), \quad \|v\|_{L^\infty(\mathbb{R})} < \pi/2. \tag{2.37}$$

Theorem 2.6, the Helson-Szegö theorem, provides a necessary and sufficient condition for the Hilbert transform to be bounded on $L^2(d\mu)$.

Another solution to this problem has been obtained by Muckenhoupt [49] and Hunt, Muckenhoupt and Wheeden [26].

THEOREM 2.7. *(Hunt, Muckenhoupt, Wheeden) Let $d\mu$ be a positive Borel measure on $\mathbb{R}$ which is finite on compact sets. Then the inequality*

$$\int_{\mathbb{R}} |Hf(x)|^2 d\mu(x) \leqslant K_2\int_{\mathbb{R}} |f(x)|^2 d\mu(x)$$

with K_2 independent of $f \in L^2(\mathbb{R}) \cap L^2(\mathbb{R}, d\mu)$ holds if and only if $d\mu(x) = w(x)dx$ and the density $w(x)$ satisfies the following (A_2)-condition:

$$\sup_{\mathcal{I}}\left(\frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} w(t)dt\right)\left(\frac{1}{|\mathcal{I}|}\int_{\mathcal{I}}\left(\frac{1}{w(t)}\right)dt\right) < \infty. \tag{2.38}$$

In (2.38) sup is taken over the set of all (closed) intervals $\mathcal{I} \subset \mathbb{R}$.

We will write $w \in (A_2)$ if (2.38) is satisfied.

It is well known that the necessary part of the condition (2.38) remains valid (with the same proof) for two-weight estimates of Hilbert transform.

More precisely, suppose that $w_1(\cdot)$ and $w_2(\cdot)$ are two nonnegative functions (weights) and $E = \operatorname{supp} w_2 = \overline{E}$ is a topological support of w_2. Then the two-weight inequality

$$\int_{\mathbb{R}} |Hf(x)|^2 \cdot w_1(x)dx \leqslant K_2 \int_{\mathbb{R}} |f(x)|^2 \cdot w_2(x)dx \tag{2.39}$$

implies the estimate

$$\sup_{\mathcal{I}} \left(\frac{1}{|\mathcal{I} \cap E|} \int_{\mathcal{I}} w_1(t)dt \right) \left(\frac{1}{|\mathcal{I} \cap E|} \int_{\mathcal{I}} \left(\frac{1}{w_2(t)} \right) dt \right) < \infty. \tag{2.40}$$

In turn, inequality (2.40) yields

$$\operatorname{esssup}_{t\in E}[w_1(x) \cdot w_2(x)^{-1}] = C < \infty. \tag{2.41}$$

In fact, inequalities (2.40) and (2.41) are not equivalent, that is (2.40) is stronger than (2.41).

In what follows, $w_2^{-1}(\cdot)$ stands for the quasi-inverse of $w_2(\cdot)$, that is

$$w_2^{-1}(x) = \begin{cases} w_2^{-1}(x), & \text{if} \quad w_2(x) \neq 0 \\ 0, & \text{if} \quad w_2(x) = 0. \end{cases} \tag{2.42}$$

Note, that two-weight estimate (2.39) is equivalent to boundedness of the operator $w_1^{1/2} H w_2^{-1/2}$ in $L^2(\mathbb{R})$.

Following [54] we mention one more consequence of two-weight estimate (2.39).

PROPOSITION 2.8. *Let $w_1,\ w_2 \geqslant 0$ be two nonnegative measurable functions on $\mathbb{R}$. Then for the two-weight estimate (2.39) to be valid it is necessary that*

$$\sup_{\lambda\in\mathbb{C}_+} \left(\mathcal{P}_\lambda(w_1) \cdot \mathcal{P}_\lambda(w_2^{-1})\right) = C < \infty. \tag{2.43}$$

D. Sarason has conjectured that the converse is also true, that is condition (2.43) is also sufficient for the two-weight estimate to be hold. Later on F. Nazarov (see [54]) showed that it is false.

It is easily seen (and well known) that condition (2.43) is stronger than (2.40). Indeed, if x is the midpoint of $\mathcal{I}$, $y = |\mathcal{I}|/2$ and $\lambda = x + iy$, then $|\mathcal{I}|^{-1}\chi_{\mathcal{I}}(t) \leqslant \pi P_y(x-t)$ (cf. [19, Theorem VI.1.2]). Hence for any nonnegative $\varphi \in L^1_{loc}(\mathbb{R})$

$$\frac{1}{|\mathcal{I}|} \int_{\mathcal{I}} \varphi(t)dt \leqslant \int_{\mathcal{I}} P_y(x-t)\varphi(t)dt = \mathcal{P}_\lambda(\varphi). \tag{2.44}$$

Also we will use the following result.

PROPOSITION 2.9. (cf. Theorem 4 in [25]) *Let $\{t_j\}_{j=1}^N$ be a finite set of real numbers. Assume that a (positive) weight function $w(\cdot)$, $t \in \mathbb{R}$, has the following properties:*

$$w(t) \asymp t^{\alpha_\infty}, \quad |t| \to \infty, \quad \textit{where} \quad -1 < \alpha_\infty < 1, \tag{2.45}$$

$$w(t) \asymp |t-t_j|^{\alpha_j}, \quad t \to t_j, \quad \textit{where} \quad -1 < \alpha_j < 1,\ j \in \{1,\dots,N\}, \tag{2.46}$$

$$w(t) \asymp 1, \quad t \to t_0, \quad \textit{for any} \ \ t_0 \in \mathbb{R} \setminus \{t_j\}_{j=1}^N. \tag{2.47}$$

Then $w \in (A_2)$, i.e., the weight $w(\cdot)$ satisfies (2.38).

Proof. A letter C will be used to denote a positive constant not necessarily the same at each occurrence.

If $w \notin (A_2)$, then there exists a sequence of intervals $\mathcal{I}_n = [a_n, b_n]$, $n \in \mathbb{N}$, with the following properties:

(S1) $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are monotone;

(S2) there exist limits $a = \lim a_n, \quad b = \lim b_n, \quad -\infty \leqslant a \leqslant b \leqslant +\infty$;

(S3) $\lim\limits_{n\to\infty} \left(\frac{1}{|\mathcal{I}_n|} \int_{\mathcal{I}_n} w(t)dt \right) \left(\frac{1}{|\mathcal{I}_n|} \int_{\mathcal{I}_n} \frac{1}{w(t)} dt \right) = \infty$.

Let us suppose now that assumptions (2.45)–(2.47) hold true and let the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ have properties (S1), (S2). We will prove that property (S3) does not hold in this case, i.e.,

$$\mathfrak{P}_n := \left(\frac{1}{|\mathcal{I}_n|} \int_{\mathcal{I}_n} w(t)dt \right) \left(\frac{1}{|\mathcal{I}_n|} \int_{\mathcal{I}_n} \frac{1}{w(t)} dt \right) < C \quad \text{for all} \quad n \in N. \tag{2.48}$$

First note that assumptions (2.45)–(2.47) yield that $w(\cdot) \in L^1_{loc}(\mathbb{R})$ and $\frac{1}{w(\cdot)} \in L^1_{loc}(\mathbb{R})$. Hence it suffices to show (2.48) for sufficiently large n.

We will consider 7 cases.

Case 1. Let $a = b = +\infty$ (the case $a = b = -\infty$ is similar).

By (2.45), $w(t) < C|t|^{\alpha_\infty}$ and $\frac{1}{w(t)} < C|t|^{-\alpha_\infty}$ for sufficiently large $t > 0$. Hence, for n large enough, we have

$$\mathfrak{P}_n = \frac{1}{(b_n - a_n)^2} \int_{a_n}^{b_n} w(t)dt \int_{a_n}^{b_n} \frac{1}{w(t)} dt < C \frac{1}{(b_n - a_n)^2} \int_{a_n}^{b_n} t^{\alpha_\infty} dt \int_{a_n}^{b_n} t^{-\alpha_\infty} dt.$$

Since $\alpha_\infty \in (-1, 1)$, we have

$$\mathfrak{P}_n < C \frac{(b_n^{1+\alpha_\infty} - a_n^{1+\alpha_\infty})(b_n^{1-\alpha_\infty} - a_n^{1-\alpha_\infty})}{(b_n - a_n)^2 (1+\alpha_\infty)(1-\alpha_\infty)} < C \frac{b_n^2 + a_n^2 - b_n^{1-\alpha_\infty} a_n^{1+\alpha_\infty} - b_n^{1+\alpha_\infty} a_n^{1-\alpha_\infty}}{b_n^2 + a_n^2 - 2 b_n a_n}$$

(it is assumed that $a_n, b_n > 0$). By the Cauchy inequality,

$$b_n^{1-\alpha_\infty} a_n^{1+\alpha_\infty} + b_n^{1+\alpha_\infty} a_n^{1-\alpha_\infty} > 2 b_n a_n.$$

Thus $\mathfrak{P}_n < C$ for n large enough.

Case 2. Let $a = -\infty$, $b = +\infty$.

By (2.45), there exist a constants $a_0 < 0$ and $b_0 > 0$ such that

$$w(t) < C|t|^{\alpha_\infty} \quad \text{and} \quad \frac{1}{w(t)} < C|t|^{-\alpha_\infty} \qquad \text{for} \quad t \in (-\infty, a_0) \cup (b_0, +\infty).$$

Therefore,

$$\mathfrak{P}_n < C \frac{1}{(b_n - a_n)^2} \left(\int_{a_n}^{a_0} |t|^{\alpha_\infty} dt + \int_{a_0}^{b_0} w(t)dt + \int_{b_0}^{b_n} t^{\alpha_\infty} dt \right) \cdot$$
$$\cdot \left(\int_{a_n}^{a_0} |t|^{-\alpha_\infty} dt + \int_{a_0}^{b_0} \frac{1}{w(t)} dt + \int_{b_0}^{b_n} t^{-\alpha_\infty} dt \right)$$

for n large enough. Taking into account the fact that $\int_{a_0}^{b_0} w(t)dt < \infty$ and $\int_{a_0}^{b_0} \frac{1}{w(t)}dt < \infty$, we get

$$\mathfrak{P}_n < C\frac{\left(|a_n|^{1+\alpha_\infty}-|a_0|^{1+\alpha_\infty}+C+b_n^{1+\alpha_\infty}-b_0^{1+\alpha_\infty}\right)\left(|a_n|^{1-\alpha_\infty}-|a_0|^{1-\alpha_\infty}+C+b_n^{1-\alpha_\infty}-b_0^{1-\alpha_\infty}\right)}{(b_n-a_n)^2}$$
$$< C\frac{\left(|a_n|^{1+\alpha_\infty}+b_n^{1+\alpha_\infty}\right)\left(|a_n|^{1-\alpha_\infty}+b_n^{1-\alpha_\infty}\right)}{(b_n-a_n)^2} < C.$$

Case 3. Let $-\infty < a = b < +\infty$, $a_n \uparrow a$, and $b_n \downarrow a(=b)$.
By (2.46)–(2.47), there exist $\alpha \in (-1,1)$ such that

$$w(t) \asymp |t-a|^{\alpha}, \quad \frac{1}{w(t)} \asymp |t-a|^{-\alpha}, \qquad t \to a.$$

So, for n large enough,

$$\mathfrak{P}_n < C\frac{1}{(b_n-a_n)^2}\left(\int_{a_n}^{a}|t-a|^{\alpha}dt+\int_{a}^{b_n}(t-a)^{\alpha}dt\right)\left(\int_{a_n}^{a}|t-a|^{-\alpha}dt+\int_{a}^{b_n}(t-a)^{-\alpha}dt\right)$$
$$< C\,\frac{\left((a-a_n)^{1+\alpha}+(b_n-a)^{1+\alpha}\right)\left((a-a_n)^{1-\alpha}+(b_n-a)^{1-\alpha}\right)}{\left((b_n-a)+(a-a_n)\right)^2}$$
$$= C\,\frac{(a-a_n)^2+(b_n-a)^2+(a-a_n)^{1-\alpha}(b_n-a)^{1+\alpha}+(a-a_n)^{1+\alpha}(b_n-a)^{1-\alpha}}{(a-a_n)^2+(b_n-a)^2+2(a-a_n)(b_n-a)}$$
$$< C + C\,\frac{(a-a_n)^{1-\alpha}(b_n-a)^{1+\alpha}+(a-a_n)^{1+\alpha}(b_n-a)^{1-\alpha}}{\max\{(a-a_n)^2,(b_n-a)^2\}} < C.$$

Case 4. Let $-\infty < a < b = +\infty$ and $a_n \downarrow a$ (the case $-\infty = a < b < +\infty$, $b_n \uparrow b$ is similar). By (2.45)–(2.47),

$$w(t) < Ct^{\alpha_\infty}, \quad \frac{1}{w(t)} < Ct^{-\alpha_\infty} \qquad \text{for} \quad t \in (b_0,+\infty), \tag{2.49}$$

where b_0 is a certain positive constant. Since

$$\int_{a_n}^{b} w(t)dt \leqslant \int_{a}^{b} w(t)dt < C \qquad \text{and} \qquad \int_{a_n}^{b}\frac{1}{w(t)}dt \leqslant \int_{a}^{b}\frac{1}{w(t)}dt < C$$

for all $n \in N$, we clearly have

$$\mathfrak{P}_n < \frac{1}{(b_n-a_n)^2}\left(\int_{a_n}^{b_0} w(t)dt + \int_{b_0}^{b_n} w(t)dt\right)\left(\int_{a_n}^{b_0}\frac{1}{w(t)}dt+\int_{b_0}^{b_n}\frac{1}{w(t)}dt\right)$$
$$< C\frac{1}{(b_n-a_n)^2}\left(C+\int_{b}^{b_n} t^{\alpha_\infty}dt\right)\left(C+\int_{b}^{b_n} t^{-\alpha_\infty}dt\right)$$
$$< C\frac{(b_n^{1+\alpha_\infty}-b_0^{1+\alpha_\infty})(b_n^{1-\alpha_\infty}-b_0^{1+\alpha_\infty})}{b_n^2-2b_na_n+a_n^2}.$$

It follows from $\lim b_n = +\infty$ that $\mathfrak{P}_n < C$ for $n \in N$.

In the same way one can treat the following cases:

Case 5. $-\infty < a = b < +\infty$, $a_n \downarrow a$, and $b_n \downarrow a (= b)$ (the case $a_n \uparrow a$, $b_n \uparrow a$ is similar);

Case 6. $-\infty < a < b = +\infty$, $a_n \uparrow a$ (the case $-\infty = a < b < +\infty$, $b_n \downarrow b$ is analogous);

Case 7. $-\infty < a < b < +\infty$.

Thus property (S3) does not hold. This shows that $w \in (A_2)$. $\square$

2.6.2. The Smirnov class

We denote by $\mathcal{N}^+(\mathbb{C}_+)$ the Smirnov class on $\mathbb{C}_+$. Recall that $\mathcal{N}^+(\mathbb{C}_+)$ consists of holomorphic on $\mathbb{C}_+$ functions $U(\cdot)$ admitting the following factorization

$$U(z) = c\,B(z)\,F(z)\,S(z), \quad z \in \mathbb{C}_+,$$

where $B(\cdot)$ is a Blaschke product, $F(\cdot)$ is an outer function, $S(\cdot)$ is a singular function, c is a constant, $|c| = 1$ (see [19, Corollary II.5.6 and Theorem II.5.5]).

The following lemmas are well known.

LEMMA 2.10. *If $f, g \in \mathcal{N}^+(\mathbb{C}_+)$, then $f + g \in \mathcal{N}^+(\mathbb{C}_+)$.*

LEMMA 2.11. *Let $\{t_j\}_{j=1}^N$ be a finite set of real numbers. Let $U(z)$ be a holomorphic function on $\mathbb{C}_+$ such that*

$$\begin{gathered} U(z) = O(z^{\alpha_\infty}), \quad z \to \infty, \\ U(z - t_j) \asymp |z - t_j|^{\alpha_j}, \quad z \to t_j, \qquad j \in \{1, \dots, N\}, \\ U(z - z_0) = O(1), \quad z \to z_0, \quad z_0 \in (\mathbb{C}_+ \cup \mathbb{R}) \setminus \{t_j\}_{j=1}^N, \end{gathered}$$

where $\alpha_\infty \in \mathbb{R}_+$, $\alpha_j \in \mathbb{R}_-$, $j \in \{1, \dots, N\}$. Then $U(\cdot) \in \mathcal{N}^+(\mathbb{C}_+)$.

3. Similarity conditions

3.1. Characteristic functions and similarity

Let S be a symmetric operator in a Hilbert space $\mathfrak{H}$ with finite deficiency indices (n,n), $n \in \mathbb{N}$. Let T be a proper extension of S. Then by Definition 2.10 (see also 2.5. and [15]) there exists a boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for S^* such that $\operatorname{dom} T = \ker(\Gamma_1 - B\Gamma_0)$ with some $B \in [\mathcal{H}]$, that is $T = S_B$. Let $M(\cdot)$ be the Weyl function corresponding to the boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$. The characteristic function $\theta_T(\cdot)$ of almost solvable extension $T (\in \operatorname{Ext}_S)$ is determined and investigated in [14], [15] (see also [64]). In the sequel we need only the following formula for $\theta_T(\cdot)$ obtained in [14]. It express the $\theta_T(\cdot)$ by means of a boundary operator B and the corresponding Weyl function $M(\cdot)$. One may consider it as a definition of $\theta_T(\cdot)$.

THEOREM 3.1. ([14]) *Let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be a boundary triple for* S^*, $M(\cdot)$ *the corresponding Weyl function,* $B \in [\mathcal{H}]$, *and* E *an auxiliary Hilbert space. Then for any factorization* $B_I := (B - B^*)/2i = K\mathcal{J}K^*$ *of* B_I *with* $K \in [E, \mathcal{H}]$ *and* $\mathcal{J} = \mathcal{J}^* = \mathcal{J}^{-1} \in [E]$, *the characteristic function* $\theta(\lambda) := \theta_{A_B}(\lambda)$ *of the extension* $A_B (\in \mathrm{Ext}_S)$, $\mathrm{dom}\, S_B = \ker(\Gamma_1 - B\Gamma_0)$, *admits the following representation*

$$\theta_T(\lambda) = I + 2iK^*\big(B^* - M(\lambda)\big)^{-1} K\mathcal{J}. \tag{3.1}$$

It is shown in [14] that if $\ker(B - B^*) = \{0\}$, then

$$\theta_T(\lambda) = (B - M(\lambda))\,(B^* - M(\lambda))^{-1}.$$

It is well known that the characteristic function $\theta_T(\lambda)$ satisfys the following properties ($\mathcal{J}$-properties):

$$\begin{cases} \omega_\theta(\lambda) := \mathcal{J} - \theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda) \;>\; 0, & \lambda \in \mathbb{C}_+, \\ \omega_\theta(\lambda) := \mathcal{J} - \theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda) \;<\; 0, & \lambda \in \mathbb{C}_-. \end{cases} \tag{3.2}$$

The second $\mathcal{J}$-form $\omega_{\theta^*}(\lambda) := \mathcal{J} - \theta_T^*(\lambda)\mathcal{J}\theta_T(\lambda)$ has the same properties.

Next we recall some (sufficient) conditions for the similarity to a selfadjoint operator in terms of the characteristic function $\theta_T(\lambda)$ and the corresponding $\mathcal{J}$-forms $\omega_\theta(\cdot)$ and $\omega_{\theta^*}(\cdot)$.

THEOREM 3.2. ([47]) *Let* T *be a solvable extension of* S, *that is* $\mathrm{dom}\, T = \ker(\Gamma_1 - B\Gamma_0)$, *with* $B \in [\mathcal{H}]$, $B_I := (B - B^*)/2i = K\mathcal{J}K^*$ *where* $\mathcal{J} := \mathrm{sgn}\, B_I$ *and* $\pi_\pm := (I \pm \mathcal{J})/2$. *Suppose that* $\sigma(T) \subset \mathbb{R}$ *and at least one of the following two conditions is satisfied*

$$(i)\;\max\left\{ \sup_{\lambda\in\mathbb{C}_-} \|\pi_+\theta_T^*(\lambda)\mathcal{J}\theta_T(\lambda)\pi_+\|,\; \sup_{\lambda\in\mathbb{C}_+} \|\pi_-\theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda)\pi_-\| \right\} <\; \infty. \tag{3.3}$$

$$(ii)\;\max\left\{ \sup_{\lambda\in\mathbb{C}_+} \|\pi_-\theta_T^*(\lambda)\mathcal{J}\theta_T(\lambda)\pi_-\|,\; \sup_{\lambda\in\mathbb{C}_-} \|\pi_+\theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda)\pi_+\| \right\} <\; \infty. \tag{3.4}$$

Then T *is similar to a selfadjoint operator* T_0. *Moreover, if* T *is completely non-selfadjoint then* T_0 *has purely absolutely continuous spectrum.*

The next result has originally been obtained in [60]. It is immediate from Theorem 3.2, other proofs can be found in [50, 46, 47].

THEOREM 3.3. ([60]) *Let* T *be a quasi-selfadjoint extension of* S *and the spectrum* $\sigma(T)$ *is real,* $\sigma(T) \subset \mathbb{R}$. *If*

$$\sup_{\lambda\in\mathbb{C}_+\cup\mathbb{C}_-} \|\theta_T(\lambda)\| < \infty, \tag{3.5}$$

then T *is similar to a selfadjoint operator* T_0. *Moreover, if* T *is completely non-selfadjoint then* T_0 *has purely absolutely continuous spectrum.*

According to the B.S. Nagy and C. Foias result (see [62]) condition (3.5) is also necessary for a dissipative operator T to be similar to a selfadjoint operator.

To the best of our knowledge the weakest sufficient conditions for the similarity of a non-dissipative operator to a selfadjoint one in terms of characteristic functions, is contained in Theorem 3.2. Some previous results in this direction can be found in [62], [60], [46], and [47] (see also references in [47]). We mention also recent publication [41] and [30].

Note that under the conditions of all mentioned results a completely nonselfadjoint part of T is similar to a selfadjoint operator $T_0 = T_0^*$ with absolutely continuous spectrum. In this connection we mention that V. Kapustin [30] found some sufficient conditions for an almost unitary operator T to be similar to an operator $U_{ac} \oplus T_s$ where U_{ac} is an absolutely continuous unitary operator and T_s is some singular almost unitary operator. Recall, that T is called an almost unitary operator, if $\sigma(T) \not\supset \mathbb{D}$ and (at least one of) non-unitary defects $I - T^*T$ and $I - TT^*$ are trace class operators.

DEFINITION 3.1. Let T be a closed operator on $\mathfrak{H}$ with real spectrum. We say that a point $a \in \mathbb{R} \cup \{\infty\}$ is a spectral singularity of T if at least one of the $\mathcal{J}$-forms $\omega_\theta(\cdot)$ and $\omega_{\theta^*}(\cdot)$ of the characteristic function $\theta_T(\cdot)$ is unbounded in any neighborhood of a, that is for any ε

$$\sup_{\lambda\in\mathbb{D}_\varepsilon(a)} \left(\|\mathcal{J} - \theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda)\| + \|\mathcal{J} - \theta_T^*(\lambda)\mathcal{J}\theta_T(\lambda)\|\right) = \infty,$$

where $\mathbb{D}_\varepsilon(a) = \{\lambda \in \mathbb{C}_+ \cup \mathbb{C}_- : |\lambda - a| < \varepsilon\}$ for $a \in \mathbb{R}$ and $\mathbb{D}_\varepsilon(\infty) = \{\lambda \in \mathbb{C}_+ \cup \mathbb{C}_- : |\lambda| > 1/\varepsilon\}$.

Next we present sufficient condition for an operator T with a finite number of singularities to be similar to a selfadjoint one with absolutely continuous spectrum.

PROPOSITION 3.4. *Let a closed operator T on $\mathfrak{H}$ be similar to a selfadjoint operator $T_0 = T_0^*$, $VTV^{-1} = T_0$, and let $E_{T_0}(\cdot)$ be the spectral measure of T_0. Then*

(i) For any Borel subset $\delta \subset \mathbb{R}$ the subspace $\mathfrak{H}_T(\delta) := V^{-1}\mathfrak{H}_{T_0}(\delta)$, where $\mathfrak{H}_{T_0}(\delta) := E_{T_0}(\delta)\mathfrak{H}$ is a regularly and ultra-invariant (see definitions in [62]) subspace for T;

(ii) The operator $T(\delta) := T\lceil\mathfrak{H}_T(\delta)$, $\operatorname{dom} T(\delta) = V^{-1}\operatorname{dom} T_0(\delta)$ is similar to the operator $T_0(\delta) := E_{T_0}(\delta)T$;

(iii) Suppose additionally that T is completely non-selfadjoint, $\sigma_p(T) = \emptyset$ and there exists a closed at most countable set $\{a_j\}_1^N \subset \mathbb{R}$, $N \leqslant \infty$, such that for any domain $\mathcal{D} := \cup_1^N \mathbb{D}_{\varepsilon_j}(a_j) \cup \mathbb{D}_{\varepsilon_\infty}(\infty)$ with sufficiently small ε_∞, ε_1, ε_2, $\ldots$, the following inequality holds

$$\sup_{\lambda\in\mathbb{C}_+\cup\mathbb{C}_-\setminus\mathcal{D}} \|\omega_\theta(\lambda)\| = \sup_{\lambda\in\mathbb{C}_+\cup\mathbb{C}_-\setminus\mathcal{D}} \|\mathcal{J} - \theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda)\| < \infty. \quad (3.6)$$

Then the spectrum of T_0 is purely absolutely continuous, that is T is similar to the selfadjoint operator T_0 with absolutely continuous spectrum.

Proof. (i) It is clear that $\mathfrak{H}_T(\delta) \in \operatorname{Lat} T$, that is $\mathfrak{H}_T(\delta)$ is invariant for T. Moreover, $\mathfrak{H}_T(\delta) \in \operatorname{Lat} T$ is regularly invariant, that is $(T - \lambda)^{-1}\mathfrak{H}_T(\delta) = \mathfrak{H}_T(\delta)$

since

$$\begin{aligned} E_{T_0}(\delta)\mathfrak{H} &= (T_0-\lambda)^{-1}E_{T_0}(\delta)\mathfrak{H} = V(T-\lambda)^{-1}V^{-1}E_{T_0}(\delta)\mathfrak{H} \\ &= V(T-\lambda)^{-1}\mathfrak{H}_T(\delta). \end{aligned} \tag{3.7}$$

The last statement is a partial case of Proposition 5.1 from [62], part II.

(ii) It follows from the identity $VTV^{-1}=T_0$ that $V(T-\lambda)^{-1}V^{-1}=(T_0-\lambda)^{-1}$. Introducing block matrix representations of the operators V, $T(\delta)$ and $T_0(\delta)$ with respect to the orthogonal decompositions $\mathfrak{H}=\mathfrak{H}_T(\delta)\oplus\mathfrak{H}_T(\delta)^{\perp}=\mathfrak{H}_{T_0}(\delta)\oplus\mathfrak{H}_{T_0}(\mathbb{R}\setminus\delta)$ we rewrite the above identity in the block-matrix form

$$\begin{aligned} &\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}\cdot\begin{pmatrix} \big(T(\delta)-\lambda\big)^{-1} & T_{12} \\ 0 & T_{22} \end{pmatrix} \\ &\qquad = \begin{pmatrix} \big(T_0(\delta)-\lambda\big)^{-1} & 0 \\ 0 & \big(T_0(\mathbb{R}\setminus\delta)-\lambda\big)^{-1} \end{pmatrix}\cdot\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \end{aligned} \tag{3.8}$$

where $V_{ij}=P_iV\lceil\mathfrak{H}_j,\ i,j\in\{1,2\}$, P_1 is the orthoprojection in $\mathfrak{H}$ onto $\mathfrak{H}_T(\delta)$ and $P_2:=I-P_1$. Hence $V_{11}\big(T(\delta)-\lambda\big)^{-1}=\big(T_0(\delta)-\lambda\big)^{-1}V_{11}$. To complete the proof it remains to note that $\operatorname{dom}V_{11}=\mathfrak{H}_T(\delta)$, $\operatorname{ran}V_{11}=\mathfrak{H}_{T_0}(\delta)$ and $\ker V_{11}=\{0\}$ by definition of V_{11}.

(iii) First we prove that the operator $T_2:=P_2T\lceil\mathfrak{H}_T(\delta)^{\perp}$ is similar to the operator $T_0(\mathbb{R}\setminus\delta)$. Note that $T_2^*=T^*\lceil\mathfrak{H}_T(\delta)^{\perp}$ and

$$(V^{-1})^*T^*V^*=T_0=T_0^*. \tag{3.9}$$

By statement (ii) the operator T_2^* is similar the operator $T_0(\mathbb{R}\setminus\delta)$ since $\mathfrak{H}_T(\delta)^{\perp}=V^*\mathfrak{H}_{T_0}(\mathbb{R}\setminus\delta)\in\operatorname{Lat}T^*$. Hence T_2 is similar to the operator $T_0(\mathbb{R}\setminus\delta)=T_0^*(\mathbb{R}\setminus\delta)$ too.

Now, let (a,b) be any component interval of the (open) set $\mathbb{R}\setminus\{a_j\}_1^N$ and $\delta=(a+\varepsilon,b-\varepsilon)$, $\varepsilon>0$. It is clear that T is a coupling (see [6, 14, 15]) of $T_1=T(\delta)$ and $T_2=P_2T\lceil\mathfrak{H}_T(\delta)^{\perp}$. Therefore $\theta_T(\cdot)$ admits a factorization (see [14, 15])

$$\theta_T(\lambda)=\theta_{T_1}(\lambda)\cdot\theta_{T_2}(\lambda)=:\theta_1(\lambda)\cdot\theta_2(\lambda),\qquad \lambda\in\mathbb{C}_+\cup\mathbb{C}_-, \tag{3.10}$$

where $\theta_j(\cdot):=\theta_{T_j}(\cdot)$ is the corresponding characteristic function of the operator $T_j, j\in\{1,2\}$. Since T_2 is similar to $T_0(\mathbb{R}\setminus\delta)$, then $\theta_2(\cdot)=\theta_{T_2}(\cdot)$ admits a holomorphic continuation through $(a+\varepsilon,b-\varepsilon)$.

It easily follows from (3.10) and the first $\mathcal{J}$-property of θ_{T_1} and θ_{T_2} (see (3.2)) that

$$\begin{aligned} \omega_\theta(\lambda) &= \mathcal{J}-\theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda) \\ &= \mathcal{J}-\theta_{T_1}(\lambda)\mathcal{J}\theta_{T_1}^*(\lambda)+\theta_{T_1}(\lambda)\cdot(\mathcal{J}-\theta_{T_2}(\lambda)\mathcal{J}\theta_{T_2}^*(\lambda))\cdot\theta_{T_1}^*(\lambda) \\ &\geqslant \mathcal{J}-\theta_{T_1}(\lambda)\mathcal{J}\theta_{T_1}^*(\lambda)\geqslant 0 \quad \text{for} \quad \lambda\in\mathbb{C}_+\cup(a+\varepsilon,b-\varepsilon). \end{aligned} \tag{3.11}$$

In turn, it follows from (3.6) that $\omega_\theta(\cdot)$ is bounded in a small neighborhood G_δ^+ $(\subset \mathbb{C}_+)$ of $\delta = (a+\varepsilon, b-\varepsilon)$. Therefore (3.11) yields the estimate $\sup\limits_{\lambda\in G_\delta^+} \|\omega_{\theta_1}(\lambda)\| \leqslant \sup\limits_{\lambda\in G_\delta^+} \|\omega_{\theta_T}(\lambda)\| < \infty$.

On the other hand, $\theta_1(\lambda) = \theta_{T_1}(\lambda)$ is bounded at infinity since T_1 is bounded. Therefore $C_+ := \sup\limits_{\lambda\in\mathbb{C}_+} \|\omega_{\theta_1}(\lambda)\| < \infty$.

Similarly, starting with (3.10) and using the second $\mathcal{J}$-property (3.2) of θ_{T_1} and θ_{T_2} we get

$$\begin{aligned} \omega_\theta(\lambda) &= \mathcal{J} - \theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda) \\ &= \mathcal{J} - \theta_{T_1}(\lambda)\mathcal{J}\theta_{T_1}^*(\lambda) + \theta_{T_1}(\lambda)\cdot(\mathcal{J} - \theta_{T_2}(\lambda)\mathcal{J}\theta_{T_2}^*(\lambda))\cdot\theta_{T_1}^*(\lambda) \\ &\leqslant \mathcal{J} - \theta_{T_1}(\lambda)\mathcal{J}\theta_{T_1}^*(\lambda) \leqslant 0 \quad \text{for} \quad \lambda \in \mathbb{C}_- \cup (a+\varepsilon, b-\varepsilon). \end{aligned} \tag{3.12}$$

By (3.6) $\omega_\theta(\cdot)$ is bounded in a small neighborhood G_δ^- $(\subset \mathbb{C}_-)$ of $\delta = (a+\varepsilon, b-\varepsilon)$ and due to (3.12) so is $\omega_{\theta_1}(\cdot)$. Since $\theta_1(\lambda)$ is bounded at infinity we have $C_- := \sup\limits_{\lambda\in\mathbb{C}_-} \|\omega_{\theta_1}(\lambda)\| < \infty$. Summing up we get

$$\sup_{\lambda\in\mathbb{C}_+\cup\mathbb{C}_-} \|\theta_{T_1}(\lambda)\mathcal{J}\theta_{T_1}^*(\lambda)\| < \infty. \tag{3.13}$$

Note that T_1 is completely nonselfadjoint because so is T. Since $T_1 = T(\delta)$ is completely nonselfadjoint and it is similar to the selfadjoint operator $T_0(\delta)$, then condition (3.13) imply absolute continuity of the operator $T_0(\delta)$ (see [47], Theorem 1.4). Since (a,b) is any component interval of $\mathbb{R}\setminus\{a_j\}_1^N$, $\delta = (a+\varepsilon, b-\varepsilon)$, and $\varepsilon > 0$ is arbitrary, then the singular spectrum $\sigma_s(T_0)$ of T_0 is supported on $\{a_j\}_1^N$, that is $\sigma_s(T_0) \subset \{a_j\}_1^N$. Thus, $\sigma_s(T_0)$ is at most countable, hence $\sigma_s(T_0) = \sigma_p(T_0)$. But according to our assumption $\sigma_p(T_0) = \emptyset$ and T_0 is purely absolutely continuous. $\square$

COROLLARY 3.5. *Let a closed operator T on $\mathfrak{H}$ be similar to a selfadjoint operator $T_0 = T_0^*$. Suppose additionally that T is completely non-selfadjoint, $\sigma_p(T) = \emptyset$ and there exists a closed at most countable set $\{a_j\}_1^N \subset \mathbb{R}$, $N \leqslant \infty$, such that for any domain $\mathcal{D} := \cup_1^N \mathbb{D}_{\varepsilon_j}(a_j) \cup \mathbb{D}_{\varepsilon_\infty}(\infty)$ with sufficiently small ε_∞, ε_1, ε_2, ..., the following inequality holds*

$$\sup_{\lambda\in\mathbb{C}_+\cup\mathbb{C}_-\setminus\mathcal{D}} \|\theta_T(\lambda)\| < \infty. \tag{3.14}$$

Then T_0 is purely absolutely continuous, that is T is similar to the selfadjoint operator T_0 with absolutely continuous spectrum.

REMARK 3.1. It is shown in [47] that conditions (3.3) and (3.4) are equivalent to each other and even are equivalent to similar conditions obtaining by dropping the corresponding orthoprojections $\pi_\pm$. Note, however that in general condition (3.13) is weaker than each of the (equivalent) conditions (3.3), (3.4) and it is not sufficient for similarity to a selfadjoint operator (cf. [47]).

3.2. Characteristic functions and similarity of J-selfadjoint operators

In the case of J-selfadjoint operators conditions (3.3), (3.4) and (3.5) can be weakened. The following two results are immediate from Theorem 3.2 and Theorem 3.3 respectively.

PROPOSITION 3.6. *Suppose additionally to the conditions of Theorem 3.2 that T is a J-selfadjoint operator. Assume also that $\sigma(T) \subset \mathbb{R}$ and at least one of the following four conditions is satisfied*

$$(i) \quad C_1 := \sup_{\lambda \in \mathbb{C}_+} \|\theta_T^*(\lambda)\mathcal{J}\theta_T(\lambda)\| < \infty, \tag{3.15}$$

$$(ii) \quad C_2 := \sup_{\lambda \in \mathbb{C}_-} \|\theta_T^*(\lambda)\mathcal{J}\theta_T(\lambda)\| < \infty,$$

$$(iii) \quad C_3 := \sup_{\lambda \in \mathbb{C}_-} \|\theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda)\| < \infty, \tag{3.16}$$

$$(iv) \quad C_4 := \sup_{\lambda \in \mathbb{C}_+} \|\theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda)\| < \infty,$$

Then T is similar to a selfadjoint operator T_0. Moreover, if T is completely non-selfadjoint then T_0 has purely absolutely continuous spectrum.

Proof. If two operators T_1 and T_2 are unitarily equivalent, then any characteristic function $\theta_{T_1}(\cdot)$ of T_1 is at the same time the characteristic function of T_2.

We prove only that conditions (i) and (iii) are equivalent and $C_1 = C_3$. The equivalence $(ii) \iff (iv)$ and the equality $C_2 = C_4$ can be proved in just the same way.

Since T is J-selfadjoint it is unitarily equivalent to T^*, $\quad T^* = JTJ^{-1}$. Hence $\theta_T(\lambda) = \theta_{T^*}(\lambda)$. On the other hand, it easily follows from (3.1), that

$$\theta_T^*(\overline{\lambda}) = \mathcal{J}\theta_{T^*}(\lambda)\mathcal{J} \left(= \mathcal{J}\theta_T(\lambda)^{-1}\mathcal{J}\right), \qquad \lambda \in \rho(T).$$

This relation yields

$$\mathcal{J}\theta_T^*(\overline{\lambda})\mathcal{J}\theta_T(\overline{\lambda})\mathcal{J} = \theta_{T^*}(\lambda)\mathcal{J}\theta_{T^*}^*(\lambda) = \theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda). \tag{3.17}$$

It follows that $C_1 = C_3$. To complete the proof it suffices to apply Theorem 3.2. $\square$

COROLLARY 3.7. *Suppose additionally to the conditions of Theorem 3.3 that T is a J-selfadjoint operator. If $\sigma(T) \subset \mathbb{R}$ and*

$$\sup_{\lambda \in \mathbb{C}_+} \|\theta_T(\lambda)\| < \infty, \tag{3.18}$$

then T is similar to a selfadjoint operator T_0. Moreover, if T is completely non-selfadjoint then T_0 has purely absolutely continuous spectrum.

REMARK 3.2. Note, that four conditions (i), (ii), (iii), (iv) in Proposition 3.6 are equivalent. This statement is implied by combining identity (3.17) with Proposition 1.4 from [47].

In fact, it can be proved using some reasonings from [47] based on the resolvent criterion (see below) that for J-selfadjoint operator T only "half" of either conditions (3.3) or conditions (3.4) is sufficient for T to be similar to a selfadjoint operator. Say, the condition $\sup_{\lambda\in\mathbb{C}_-} \|\pi_+\theta_T^*(\lambda)\mathcal{J}\theta_T(\lambda)\pi_+\| < \infty$ is sufficient for T to be similar to a selfadjoint operator.

Next combining Proposition 3.4 with Proposition 3.6 we arrive at the following result showing that in the case of J-selfadjointness of the operator T condition (3.6) can also be weaken.

PROPOSITION 3.8. *Let a closed J-selfadjoint operator T on $\mathfrak{H}$ be similar to a selfadjoint operator $T_0 = T_0^*$. Suppose additionally that T is completely non-selfadjoint, $\sigma_p(T) = \emptyset$ and there exists a closed at most countable set $\{a_j\}_1^N \subset \mathbb{R}$, $N \leqslant \infty$, such that for any domain $\mathcal{D} := \cup_1^N \mathbb{D}_{\varepsilon_j}(a_j) \cup \mathbb{D}_{\varepsilon_\infty}(\infty)$ with sufficiently small ε_∞, ε_1, ε_2, ..., the following inequality holds*

$$\sup_{\lambda\in\mathbb{C}_+\setminus\mathcal{D}} \|\omega_\theta(\lambda)\| = \sup_{\lambda\in\mathbb{C}_+\setminus\mathcal{D}} \|\mathcal{J} - \theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda)\| < \infty. \tag{3.19}$$

Then T_0 is purely absolutely continuous, that is T is similar to the selfadjoint operator T_0 with absolutely continuous spectrum.

Proof. Since T is $\mathcal{J}$-selfadjoint, then combining condition (3.13) with identity (3.17) we get

$$\sup_{\lambda\in\mathbb{C}_-\setminus\mathcal{D}} \|\omega_{\theta^*}(\lambda)\| = \sup_{\lambda\in\mathbb{C}_-\setminus\mathcal{D}} \|\mathcal{J} - \theta_T^*(\lambda)\mathcal{J}\theta_T(\lambda)\| < \infty, \tag{3.20}$$

Following [47] it can easily be shown that both conditions (3.19) and (3.20) together yield condition (3.6). It remains to apply Proposition 3.4. □

The following proposition is immediate from Proposition 2.5 and formula (3.1).

PROPOSITION 3.9. *Let $S := A_{\min}$ be a (minimal) symmetric operator defined by (2.4) and $A = JL$. Suppose that conditions of Proposition 2.5 are satisfied and $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then*

(i) $B_I = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} =: \mathcal{J}$ and the characteristic function $\theta_A(\cdot)$ of the operator A admits the following representation

$$\theta_A(\lambda) = \frac{1}{M_-(\lambda) - M_+(\lambda)} \begin{pmatrix} M_+(\lambda) + M_-(\lambda) & 2M_+(\lambda)M_-(\lambda) \\ 2 & M_+(\lambda) + M_-(\lambda) \end{pmatrix} \tag{3.21}$$

(ii) The corresponding $\mathcal{J}$-forms are

$$\begin{aligned} \omega_\theta(\lambda) \ := \ & \mathcal{J} - \theta_A(\lambda)\mathcal{J}\theta_A^*(\lambda) = \mathcal{J} - \frac{1}{|M_+ - M_-|^2} \cdot \\ & \cdot \begin{pmatrix} 4\cdot \operatorname{Im}\left(\overline{M_+M_-}\cdot(M_+ + M_-)\right) & 4iM_+M_- - i|M_+ + M_-|^2 \\ i|M_+ + M_-|^2 - 4i\overline{M_+M_-} & 4\cdot\operatorname{Im}\overline{(M_+ + M_-)} \end{pmatrix}, \end{aligned} \tag{3.22}$$

$$\omega_{\theta^*}(\lambda) := \mathcal{J} - \theta_A^*(\lambda)\mathcal{J}\theta_A(\lambda) = \mathcal{J} - \frac{1}{|M_+ - M_-|^2} \cdot \tag{3.23}$$
$$\cdot \begin{pmatrix} 4\cdot \operatorname{Im}\overline{(M_+ + M_-)} & 4iM_+M_- - i|M_+ + M_-|^2 \\ i|M_+ + M_-|^2 - 4i\overline{M_+M_-} & 4\cdot \operatorname{Im}\left(\overline{M_+M_-}\cdot (M_+ + M_-)\right) \end{pmatrix}.$$

(*iii*) *The determinant* $\det \theta_A(\lambda)$ *defined originally on* $\rho(A^*)$, *admits holomorphic continuation to the complex plane* $\mathbb{C}$ *and*

$$\det \theta_A(\lambda) = 1, \qquad \lambda \in \mathbb{C}. \tag{3.24}$$

REMARK 3.3. Here we briefly explain a general abstract scheme allowing us to consider the operator (1.1) in the framework of extension theory.

Let $S_\pm$ be symmetric operators in $\mathfrak{H}_\pm$ with deficiency indices $n_\pm(S_\pm) = 1$. Let also $\Pi_+ := \{\mathbb{C}_+, \Gamma_0^+, \Gamma_1^+\}$ and $\Pi'_- := \{\mathbb{C}, \Gamma'_0, \Gamma'_1\}$ be boundary triplets for S_+^* and S_-^* respectively and $m_\pm(\cdot)$ the corresponding Weyl functions. Setting $\Gamma_0^- := -\Gamma'_0$ and $\Gamma_1^- = \Gamma'_1$, we obtain a boundary triplet $\Pi_- = \{\mathbb{C}, \Gamma_0^-, \Gamma_1^-\}$ for the operator $-S_-^*$. The corresponding Weyl function is $M_-(\lambda) = -m_-(-\lambda)$. It is easily seen that $\Pi := \Pi_+ \oplus \Pi_- = \{\mathbb{C}^2, \Gamma_0^+ \oplus \Gamma_0^-, \Gamma_1^+ \oplus \Gamma_1^+\} =: \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $S^* = S_+^* \oplus (-S_-^*)$ and the corresponding Weyl function is $M(\cdot) = \operatorname{diag}\left(-M_+^{-1}(\cdot),\ M_-(\cdot)\right)$, where $M_+(\cdot) := m_+(\cdot)$. Define a quasi-selfadjoint extension A of $S = S_+ \oplus (-S_-)$ by setting

$$A = S^*|\operatorname{dom} A, \quad \operatorname{dom} A = \ker(\Gamma_1 - B\Gamma_0), \quad \text{where} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.25}$$

If $\mathfrak{H}_\pm = L^2(\mathbb{R}_\pm)$, $S_\pm = L_{\min}^\pm$, and $\Gamma_0^+ f = f(+0)$, $\Gamma_1^+ f = f'(+0)$, $\Gamma_0^- f = f'(-0)$, $\Gamma_1^- f = -f(-0)$, then according to Proposition 2.27 the operator A defined by (3.25) coincides with that defined by (1.1). It is natural to call A a nonselfadjoint coupling of two symmetric operators S_+ and S_-.

We emphasize that under the assumption $M_+(\cdot) \not\equiv M_-(\cdot)$, Proposition 3.9, as well as the biggest part of the results of Sections 3-6, remain valid for the operator A defined by (3.25).

Combining Theorem 3.2 with Proposition 3.9 and Remark 3.3 we arrive at the following statement.

COROLLARY 3.10. *Let* A, $S_\pm$ *and* $M_\pm$ *be as in Remark 3.3 and let* $S_\pm$ *be simple. Then the operator* A *is similar to a selfadjoint operator with absolutely continuous spectrum if the following two conditions hold*

$$(a)\ \sup_{\lambda\in\mathbb{C}_+} \frac{\operatorname{Im}\left(M_+(\lambda)+M_-(\lambda)\right)+|M_+(\lambda)|^2\cdot \operatorname{Im} M_-(\lambda)+|M_-(\lambda)|^2\cdot \operatorname{Im} M_+(\lambda)}{|M_+(\lambda)-M_-(\lambda)|^2} < \infty, \tag{3.26}$$

$$(b)\ \sup_{\lambda\in\mathbb{C}_+} \frac{\operatorname{Im} M_+(\lambda)\cdot \operatorname{Im} M_-(\lambda)}{|M_+(\lambda) - M_-(\lambda)|^2} < \infty. \tag{3.27}$$

Proof. Note that $\pi_\pm := (I \pm \mathcal{J})/2 = \frac{1}{2}\begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix}$. Setting for brevity $\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \mathcal{J} - \omega_{\theta^*}(\lambda)$ and noting that $\mathcal{J} - \omega_\theta(\lambda) = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$ we easily get

$$\begin{aligned} \pi_+ \omega_{\theta^*}(\lambda) \pi_+ &= \pi_+ - \frac{1}{4}\begin{pmatrix} k_+ & -ik_+ \\ ik_+ & k_+ \end{pmatrix}, \\ \pi_- \omega_\theta(\lambda) \pi_- &= -\pi_- - \frac{1}{4}\begin{pmatrix} k_- & ik_- \\ -ik_- & k_- \end{pmatrix}, \end{aligned} \tag{3.28}$$

where $k_+ = a - ic + ib + d$ and $k_- = a + ic - ib + d$. Hence both k_+ and k_- are bounded in $\mathbb{C}_+$ if and only if so are $a + d = k_+ + k_-$ and $b - c = i(k_- - k_+)$. Note that

$$\frac{c-b}{2i} = \frac{|M_+(\lambda) + M_-(\lambda)|^2 - 4\mathrm{Re}\,\big(M_+(\lambda)\cdot M_-(\lambda)\big)}{|M_+(\lambda) - M_-(\lambda)|^2} = 1 + \frac{8\mathrm{Im}\,M_+(\lambda)\cdot \mathrm{Im}\,M_-(\lambda)}{|M_+(\lambda) - M_-(\lambda)|^2},$$

$$\mathrm{Im}\,\big(M_+(\lambda)\cdot M_-(\lambda)\cdot \overline{(M_+(\lambda) + M_-(\lambda))}\big) = |M_+(\lambda)|^2\cdot \mathrm{Im}\,M_-(\lambda) + |M_-(\lambda)|^2\cdot \mathrm{Im}\,M_+(\lambda).$$

Next we show that A is completely non-selfadjoint. First we note that $S = S_+ \oplus (-S_-)$ is simple because so are S_+ and S_-. If A is not completely non-selfadjoint, then $A = T_0 \oplus T_1$, where $T_0 = T_0^*$. Hence, $A^* = T_0 \oplus T_1^*$. Since $\mathrm{dom}\,(A) \cap \mathrm{dom}\,(A^*) = \mathrm{dom}\,(S)$ due to the definition given by (3.25), we see that $T_0 \subset S$. Thus, $\mathfrak{H}_0 := \overline{\mathrm{dom}\,(T_0)}$ reduces the operator S, and $T_0 = T_0^* = S|\mathfrak{H}_0$. This contradicts the fact that S is simple.

To complete the proof it remains to apply Theorem 3.2. □

REMARK 3.4. (i) A weaker sufficient condition for the similarity is implied by Theorem 3.3. Namely, combining Theorem 3.3 with formula (3.21) we conclude that the condition

$$\max\left\{ \sup_{\lambda\in\mathbb{C}_+} \frac{|M_+(\lambda)+M_-(\lambda)|}{|M_-(\lambda)-M_+(\lambda)|}, \sup_{\lambda\in\mathbb{C}_+} \frac{1}{|M_-(\lambda)-M_+(\lambda)|}, \sup_{\lambda\in\mathbb{C}_+} \frac{|M_+(\lambda)M_-(\lambda)|}{|M_-(\lambda)-M_+(\lambda)|} \right\} < \infty \tag{3.29}$$

is sufficient for the operator A to be similar to a selfadjoint operator with absolutely continuous spectrum.

(ii) It is interesting to note that due to asymptotic behavior (2.10) of $M_\pm(\cdot)$ neither conditions (3.29) nor (weaker) conditions (3.26)–(3.27) are satisfied for any operator A of the form (1.1). Indeed, (2.10) yields

$$\begin{gathered} \mathrm{Im}\,\big(M_+(i\rho) + M_-(i\rho)\big) = \rho^{-1/2} + O(\rho^{-1}), \\ |M_+(i\rho) - M_-(i\rho)|^2 = \rho^{-1} + O(\rho^{-3/2}) \quad \text{as} \quad \rho \to \infty. \end{gathered}$$

So, neither Theorem 3.2 nor Theorem 3.3 can be applied to operators A of the form (1.1), though some of them are similar to selfadjoint ones.

(iii) A counter part of identity (3.24) for a discrete part $A_{\rm disc}$ of the operator A, $\det\theta_{A_{\rm disc}}(\lambda) = 1$, is immediate from symmetry of its spectrum (see Proposition 2.5(vi)). However, identity (3.24) is not predictable for operators with absolutely continuous spectrum. In the latter case $\theta_A(\cdot)$ is $\mathcal{J}$-outer function while $\det\theta_A(\lambda) = 1$.

Alongside the operator A we consider its "dissipative and accumulative parts". More precisely, we consider extensions $A_\pm$ of $S = A_{\min}$ determined by

$$\operatorname{dom}(A_\pm) := \{y \in \operatorname{dom}((S^*) : 2y'(+0) = y'(-0) \pm iy(+0), 2y(-0) = y(+0) \mp iy'(-0)\}. \tag{3.30}$$

PROPOSITION 3.11. *Let $S := A_{\min}$ be a (minimal) symmetric operator defined by (2.4) and let $M_\pm(\cdot)$ be defined by (2.7). Let also $\Pi = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ be a boundary triplet defined by (2.27). Then*

(i) *The operators $A_\pm$ defined by (2.2) are quasi-selfadjoint extensions of S and they are determined by*

$$A_\pm = S^*|\operatorname{dom} A_\pm, \qquad \operatorname{dom} A_\pm = \ker(\Gamma_1 - B_\pm\Gamma_0), \quad \textit{and}$$

$$B_\pm := \pi_\pm B = \frac{1}{2}\begin{pmatrix} \pm i & 1 \\ -1 & \pm i \end{pmatrix}, \tag{3.31}$$

that is $A_\pm = S_{B_\pm}$, where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{J} = -iB = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \textit{and}$$

$$\pi_\pm := (I \pm \mathcal{J})/2 = \frac{1}{2}\begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix}. \tag{3.32}$$

(ii) *Some of the characteristic functions of the operators $A_\pm$ are*

$$\theta_{A_\pm}(\lambda) = I - \frac{1 - M_+(\lambda)M_-(\lambda)}{\Delta_\pm(\lambda)}\begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix}, \tag{3.33}$$

where $\Delta_\pm(\lambda) := 1 - M_+(\lambda)M_-(\lambda) \mp 2iM_-(\lambda)$.

(iii) *The operator A_+ (resp A_-) is similar to a selfadjoint operator if and only if*

$$\inf_{\lambda\in\mathbb{C}_-} |1 - i\Phi(\lambda)| =: \varepsilon > 0, \tag{3.34}$$

where

$$\Phi(\cdot) := 2(M_-^{-1}(\cdot) - M_+(\cdot))^{-1} \in (R).$$

Proof. (i) This statement is obvious.

(ii) This statement is implied by combining formula (3.1) with (3.31) and (3.32).

(iii) First we note that by (3.34)

$$\sup_{\lambda\in\mathbb{C}_-}\left|\frac{1 - M_+(\lambda)M_-(\lambda)}{\Delta_\pm(\lambda)}\right| = \sup_{\lambda\in\mathbb{C}_-}\left|\frac{1}{1 - i\Phi(\lambda)}\right| = \frac{1}{\varepsilon} < \infty.$$

Therefore it follows from (3.33) that condition (3.34) is equivalent to the boundedness of the characteristic function $\theta_{A_+}(\cdot)$ in $\mathbb{C}_-$.

Now the result is immediate from the B.S. Nagy and Foias [62] criterion. $\square$

3.3. Resolvent criterion

It turns out, that in general conditions (3.5), (3.3), (3.4) are not satisfied for the operators of type (2.2), though such operators may be similar to a selfadjoint operator (see [47]).

Our approach is based on the resolvent similarity criterion obtained in [50] and [46] (under an additional assumption this criterion was obtained in [7], another proof has also been obtained in [24]).

THEOREM 3.12. ([50, 46]) *A closed operator T on a Hilbert space $\mathfrak{H}$ is similar to a selfadjoint operator if and only if $\sigma(T) \subset \mathbb{R}$ and for all $f \in \mathfrak{H}$ the inequalities*

$$\sup_{\varepsilon>0} \varepsilon \cdot \int_{\mathbb{R}} \|\mathcal{R}_T(\eta+i\varepsilon)f\|^2 \, d\eta \leqslant K_1 \|f\|^2,$$
$$\sup_{\varepsilon>0} \varepsilon \cdot \int_{\mathbb{R}} \|\mathcal{R}_{T^*}(\eta+i\varepsilon)f\|^2 \, d\eta \leqslant K_{1*} \|f\|^2, \tag{3.35}$$

hold with constants K_1 and K_{1*} independent of f.

The following proposition is immediate from Theorem 3.12.

PROPOSITION 3.13. *A J-selfadjoint operator T on a Hilbert space $\mathfrak{H}$ is similar to a selfadjoint operator if and only if $\sigma(T) \subset \mathbb{R}$ and the following inequality holds*

$$\sup_{\varepsilon>0} \varepsilon \cdot \int_{\mathbb{R}} \|\mathcal{R}_T(\eta+i\varepsilon)f\|^2 \, d\eta \leqslant K_1 \|f\|^2, \qquad f \in \mathfrak{H}, \tag{3.36}$$

with a constant K_1 independent of f.

Proof. If T is a J-selfadjoint operator, then $T^* = JTJ$ and the second inequality in (3.35) is equivalent to the first one. □

In the case of a bounded operator T we can slightly clarify Theorem 3.12 in the following way.

COROLLARY 3.14. *Let $T = T_1 + iT_2$ where $T_1 = T_1^*$ and $T_2 = T_2^* \in [\mathfrak{H}]$. Then T is similar to a selfadjoint operator if and only if $\sigma(T) \subset \mathbb{R}$ and for all $f \in \mathfrak{H}$ the inequalities*

$$\sup_{0<\varepsilon<2\|T_2\|} \varepsilon \int_{\mathbb{R}} \|\mathcal{R}_T(\eta+i\varepsilon)f\|^2 \, d\eta \leqslant K_1 \|f\|^2,$$
$$\sup_{0<\varepsilon<2\|T_2\|} \varepsilon \int_{\mathbb{R}} \|\mathcal{R}_{T^*}(\eta+i\varepsilon)f\|^2 \, d\eta \leqslant K_{1*} \|f\|^2, \tag{3.37}$$

hold with constants K_1 and K_{1*} independent of $f \in \mathfrak{H}$.

In particular, a bounded operator T on $\mathfrak{H}$ with $\sigma(T) \subset \mathbb{R}$ is similar to a selfadjoint operator if and only if inequalities (3.37) are valid with $2\|T_2\|$ replaced for any $\varepsilon_0 > 0$.

Proof. (i) It is clear that

$$(T-z)^{-1} = (T_1-z)^{-1} - (T_1-z)^{-1}\cdot T_2\cdot (T-z)^{-1}, \qquad z\in\mathbb{C}_+.$$

It follows that

$$\begin{aligned}\|(T-z)^{-1}f\|^2 &\leqslant 2\|(T_1-z)^{-1}f\|^2 + 2\|(T_1-z)^{-1}\cdot T_2\cdot (T-z)^{-1}f\|^2\\ &\leqslant 2\|(T_1-z)^{-1}f\|^2 + \frac{2\|T_2\|^2}{|\mathrm{Im}\, z|}\|(T-z)^{-1}f\|^2, \qquad z\in\mathbb{C}_+, \quad f\in\mathfrak{H}.\end{aligned}$$

In turn, this inequality yields $\|(T-z)^{-1}f\|^2 \leqslant 4\|(T_1-z)^{-1}f\|^2$ for $\mathrm{Im}\, z > (2\|T_2\|)^2$. Hence

$$\sup_{\varepsilon\geqslant 2\|T_2\|} \varepsilon\cdot\int_{\mathbb{R}} \|\mathcal{R}_T(\eta+i\varepsilon)f\|^2\, d\eta \leqslant 4\varepsilon\cdot\int_{\mathbb{R}} \|\mathcal{R}_{T_1}(\eta+i\varepsilon)f\|^2\, d\eta = 4\pi\,\|f\|^2. \quad (3.38)$$

Combining this inequality with the first of inequalities (3.35) we arrive at the first of inequalities (3.37). The second one can be proved similarly. □

REMARK 3.5. If T is a closed unbounded operator, then conditions (3.35) and (3.37) are not equivalent, in general. In fact, there exists an operator T such that:

(i) $\sigma(T)\subset\mathbb{R}$;

(ii) conditions (3.37) are fulfilled with any bounded C in place of $\|T_2\|$;

(iii) conditions (3.35) do not hold and, consequently, T is not similar to a selfadjoint operator.

Consider in $L^2(\mathbb{R})$ the operator $D_w = -i\frac{1}{w(x)}\frac{d}{dx}$ with the weight $w(\cdot)$ defined by

$$\begin{cases} w(x) = 1, & \text{if} \quad |x|\geqslant 1,\\ w(x) = -1, & \text{if} \quad |x|<1,\end{cases}$$

It is shown in [35] that the operator D_w has the properties (i) and (iii). It is not difficult to check that conditions (3.37) are fulfilled for D_w.

4. Eigenvalues and their multiplicities

Here we recall following [36, 37] the functional model of J-selfadjoint operator, which is a quasi-selfadjoint extension of a symmetric operator with deficiency indices $(2,2)$. The model is based on classical Sturm-Liouville spectral theory and the functional model of a symmetric operator described in Section 2.5.

Let $\Sigma_\pm$ be the spectral functions of $A_0^\pm$ (see (2.9)). It follows from (2.11) that they satisfy (2.31). Let $C_\pm := \int_{\mathbb{R}} \frac{t}{1+t^2} d\Sigma_\pm$. We denote $\widehat{\Gamma}_0^\pm := \Gamma_0^{\Sigma_\pm}$, $\widehat{\Gamma}_1^\pm := \Gamma_1^{\Sigma_\pm, C_\pm}$. From the definition of $C_\pm$ and (2.32), we get

$$\widehat{\Gamma}_1^\pm f = C_\pm\widehat{\Gamma}_0^\pm f + \int_{\mathbb{R}}\left(f(t) - \frac{t\,\widehat{\Gamma}_0^\pm f}{t^2+1}\right) d\Sigma_\pm(t) = \int_{\mathbb{R}} f(t)\, d\Sigma_\pm(t)$$

for $f \in \operatorname{dom}(\widehat{T}_{\Sigma_\pm})$. Consider the operator $\widehat{A}$ in $L^2(d\Sigma_+) \oplus L^2(d\Sigma_-)$ defined by

$$\begin{aligned} \widehat{A} &= \widehat{T}^*_{\Sigma_+} \oplus \widehat{T}^*_{\Sigma_-} \upharpoonright \operatorname{dom}(\widehat{A}), \qquad (4.1) \\ \operatorname{dom}(\widehat{A}) &= \{ f = f_+ + f_- \, : \, f_\pm \in \operatorname{dom}(\widehat{T}^*_{\Sigma_\pm}),\ \widehat{\Gamma}_0^+ f_+ = \widehat{\Gamma}_0^- f_-,\ \widehat{\Gamma}_1^+ f_+ = \widehat{\Gamma}_1^- f_- \} \end{aligned}$$

(for the definition of $\widehat{T}_{\Sigma_\pm}$ see Section 2.5.).

PROPOSITION 4.1. ([36, 37]) *The operator A of type (2.2) is unitary equivalent to the operator $\widehat{A}$. Moreover,*

$$(\mathcal{F}_- \oplus \mathcal{F}_+) A (\mathcal{F}_-^{-1} \oplus \mathcal{F}_+^{-1}) = \widehat{A}\,. \qquad (4.2)$$

Note that we can write the Weyl functions of A in the form

$$M_\pm(\lambda) = M_{\Sigma_\pm, C_\pm}(\lambda)\,, \quad \lambda \in \mathbb{C} \setminus \operatorname{supp} d\Sigma_\pm$$

(see (2.33) for the definition of $M_{\Sigma_\pm, C_\pm}$).

Now we classify eigenvalues of $\widehat{T}^*_\Sigma$. Let us introduce the following mutually disjoint sets:

$$\mathfrak{A}_0(\Sigma) = \left\{ \lambda \in \sigma_c(Q_\Sigma) \ : \ \int_{\mathbb{R}} |t-\lambda|^{-2} d\Sigma(t) = \infty \right\},$$

$$\mathfrak{A}_r(\Sigma) = \left\{ \lambda \notin \sigma_p(Q_\Sigma) \ : \ \int_{\mathbb{R}} |t-\lambda|^{-2} d\Sigma(t) < \infty \right\}, \qquad \mathfrak{A}_p(\Sigma) = \sigma_p(Q_\Sigma).$$

Observe that $\mathbb{C} = \mathfrak{A}_0(\Sigma) \cup \mathfrak{A}_r(\Sigma) \cup \mathfrak{A}_p(\Sigma)$ and

$$\begin{aligned} \mathfrak{A}_0(\Sigma) &= \{ \lambda \in \mathbb{C} \ : \ \ker(A_\Sigma^* - \lambda) = \{0\} \}, \\ \mathfrak{A}_r(\Sigma) &= \left\{ \lambda \in \mathbb{C} : \ \ker(A_\Sigma^* - \lambda) = \{ c(t-\lambda)^{-1},\ c \in \mathbb{C} \} \right\}, \qquad (4.3) \\ \mathfrak{A}_p(\Sigma) &= \left\{ \lambda \in \mathbb{C} \ : \ \ker(A_\Sigma^* - \lambda) = \{ c\chi_{\{\lambda\}}(t),\ c \in \mathbb{C} \} \right\}. \qquad (4.4) \end{aligned}$$

The following theorem gives a description of the point spectrum of $\widehat{A}$.

THEOREM 4.2. ([36, 37]) *Let $\widehat{A}$ be given by (4.1).*
(1) *If $\lambda \in \mathfrak{A}_0(\Sigma_+) \cup \mathfrak{A}_0(\Sigma_-)$, then $\lambda \notin \sigma_p(\widehat{A})$.*
(2) *If $\lambda \in \mathfrak{A}_p(\Sigma_+) \cap \mathfrak{A}_p(\Sigma_-)$, then*
(*i*) *λ is an eigenvalue of $\widehat{A}$; the geometric multiplicity of λ equals 1;*
(*ii*) *the eigenvalue λ is simple (i.e., the algebraic and geometric multiplicities are equal one) if and only if at least one of the following conditions is not fulfilled:*

$$\Sigma_-(\lambda+0) - \Sigma_-(\lambda-0) = \Sigma_+(\lambda+0) - \Sigma_+(\lambda-0), \qquad (4.5)$$

$$\int_{\mathbb{R}\setminus\{\lambda\}} \frac{1}{|t-\lambda|^2}\, d\Sigma_+(t) < \infty, \qquad (4.6)$$

$$\int_{\mathbb{R}\setminus\{\lambda\}} \frac{1}{|t-\lambda|^2}\, d\Sigma_-(t) < \infty; \qquad (4.7)$$

(iii) *if conditions (4.5), (4.6) and (4.7) hold true, then the algebraic multiplicity of* λ *equals the greatest number* k *(* $2 \leqslant k \leqslant \infty$ *) such that the following conditions*

$$\int_{\mathbb{R}\setminus\{\lambda\}} \frac{1}{|t-\lambda|^{2j}}\, d\Sigma_-(t) < \infty, \qquad \int_{\mathbb{R}\setminus\{\lambda\}} \frac{1}{|t-\lambda|^{2j}}\, d\Sigma_+(t) < \infty, \tag{4.8}$$

$$\int_{\mathbb{R}\setminus\{\lambda\}} \frac{1}{(t-\lambda)^{j-1}}\, d\Sigma_-(t) \;=\; \int_{\mathbb{R}\setminus\{\lambda\}} \frac{1}{(t-\lambda)^{j-1}}\, d\Sigma_+(t), \tag{4.9}$$

are fulfilled for all $j \in \mathbb{N} \cap [2, k-1]$.

(3) *Assume that* $\lambda \in \mathfrak{A}_r(\Sigma_+) \cap \mathfrak{A}_r(\Sigma_+)$. *Then* $\lambda \in \sigma_p(\widehat{A})$ *if and only if*

$$\int_{\mathbb{R}} \frac{1}{t-\lambda} d\Sigma_+(t) = \int_{\mathbb{R}} \frac{1}{t-\lambda} d\Sigma_-(t)\ . \tag{4.10}$$

If (4.10) holds true, then the geometric multiplicity of λ *is one and the algebraic multiplicity is the greatest number* k *(* $1 \leqslant k \leqslant \infty$ *) such that the following conditions*

$$\int_{\mathbb{R}} \frac{1}{|t-\lambda|^{2j}}\, d\Sigma_-(t) < \infty, \qquad \int_{\mathbb{R}} \frac{1}{|t-\lambda|^{2j}}\, d\Sigma_+(t) < \infty, \tag{4.11}$$

$$\int_{\mathbb{R}} \frac{1}{(t-\lambda)^{j}}\, d\Sigma_-(t) \;=\; \int_{\mathbb{R}} \frac{1}{(t-\lambda)^{j}}\, d\Sigma_+(t) \tag{4.12}$$

are fulfilled for all $j \in \mathbb{N} \cap [1, k]$.

(4) *If* $\lambda \in \mathfrak{A}_p(\Sigma_+) \cap \mathfrak{A}_r(\Sigma_-)$ *or* $\lambda \in \mathfrak{A}_p(\Sigma_-) \cap \mathfrak{A}_r(\Sigma_+)$, *then* $\lambda \notin \sigma_p(\widehat{A})$.

It follows from Theorem 4.2 (as well as from Proposition 2.5) that

$$\big\{\lambda \in \rho(Q_{\Sigma_+}) \cap \rho(Q_{\Sigma_-}) : M_+(\lambda) = M_-(\lambda)\big\} = \sigma(\widehat{A}) \cap \rho(Q_{\Sigma_+} \oplus Q_{\Sigma_-}) \subset \sigma_p(\widehat{A}). \tag{4.13}$$

It is easy to see that (4.13) and Theorem 4.2 yield the following description of the essential and discrete spectra.

PROPOSITION 4.3. ([36, 37])

(1) $\sigma_{\text{ess}}(\widehat{A}) = \sigma_{\text{ess}}(Q_{\Sigma_+}) \cup \sigma_{\text{ess}}(Q_{\Sigma_-})$;

(2) $\sigma_{\text{disc}}(\widehat{A}) = \big(\sigma_{\text{disc}}(Q_{\Sigma_+}) \cap \sigma_{\text{disc}}(Q_{\Sigma_-})\big) \cup \{\lambda \in \rho(Q_{\Sigma_+}) \cap \rho(Q_{\Sigma_-}) : M_+(\lambda) = M_-(\lambda)\}$*;*

(3) *the geometric multiplicity equals 1 for all eigenvalues of* $\widehat{A}$*;*

(4) *if* $\lambda_0 \in \big(\sigma_{\text{disc}}(Q_{\Sigma_+}) \cap \sigma_{\text{disc}}(Q_{\Sigma_-})\big)$, *then the algebraic multiplicity of* λ_0 *is equal to the multiplicity of* λ_0 *as a zero of the holomorphic function* $\dfrac{1}{M_+(\lambda)} - \dfrac{1}{M_-(\lambda)}$*;*

(5) *if* $\lambda_0 \in \rho(Q_{\Sigma_+}) \cap \rho(Q_{\Sigma_-})$ *then the algebraic multiplicity of* λ_0 *is equal to the multiplicity of* λ_0 *as zero of the holomorphic function* $M_+(\lambda) - M_-(\lambda)$.

PROPOSITION 4.4. *Let* A *be the operator defined by (2.2) and* $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$. *Then*

(i) $\rho(A) \neq \emptyset$ *and* $\lambda_0 \in \rho(A) \cap \mathbb{C}_\pm$ *if and only if* $M_+(\lambda_0) \neq M_-(\lambda_0)$.

(ii) *The resolvent of A has the following form*

$$\mathcal{R}_A(\lambda) f(\cdot) = \mathcal{R}_{A_0^- \oplus A_0^+}(\lambda) f(\cdot) + G^-(\lambda)\psi_-(\cdot,\lambda) + G^+(\lambda)\psi_+(\cdot,\lambda), \tag{4.14}$$

$$G^{-}(\lambda)=G^{+}(\lambda)=\frac{1}{M_{+}(\lambda)-M_{-}(\lambda)}\int_{\mathbb{R}}\frac{g^{-}(t)d\Sigma_{-}(t)-g^{+}(t)d\Sigma_{+}(t)}{t-\lambda}, \tag{4.15}$$

where $A_0^{\pm}$ *are defined by (2.12),* $g^{\pm}(t)=(\mathcal{F}_{\pm}f_{\pm})(t)$, $f_{\pm}:=P_{\pm}f\in L^2(\mathbb{R}_{\pm})$, *and* P_+ *and* P_- *are the orthoprojections in* $L^2(\mathbb{R})$ *onto* $L^2(\mathbb{R}_+)$ *and* $L^2(\mathbb{R}_-)$ *respectively.*

Proof. (i) This statement has already been proved in Proposition 2.5.

(ii) Let now $\lambda\in\rho(A)$ and $y(\cdot,\lambda)=(A-\lambda I)^{-1}f(\cdot)$. It means that $y\in\operatorname{dom}(A^*_{\min})$ and y is a solution of the equation

$$(\operatorname{sgn}x)(-y''(x)+q(x)y(x))-\lambda y(x)=f(x) \tag{4.16}$$

subject to “glue” boundary conditions

$$y(-0)=y(+0),\qquad y'(-0)=y'(+0)\,. \tag{4.17}$$

Hence,

$$y(x,\lambda)=(\mathcal{R}_{A_0^-\oplus A_0^+}(\lambda)f)(x)+G^{-}(\lambda)\psi_{-}(x,\lambda)+G^{+}(\lambda)\psi_{+}(x,\lambda),$$

where $G^{\pm}(\lambda)$ are the scalar functions. It is clear that

$$y(\pm0,\lambda)=(\mathcal{R}_{A_0^{\pm}}(\lambda)f_{\pm})(\pm0)+G^{\pm}(\lambda)\psi_{\pm}(0,\lambda).$$

By (2.8), we get

$$\psi_{\pm}(0,\lambda)=M_{\pm}(\lambda),\qquad \frac{d}{dx}\psi_{\pm}(0,\lambda)=-1. \tag{4.18}$$

Resolvent representation (2.14) yields

$$(\mathcal{R}_{A_0^{\pm}}(\lambda)f_{\pm})(\pm0)=\int_{\mathbb{R}}\frac{g^{\pm}(t)d\Sigma_{\pm}(t)}{t-\lambda}.$$

It follows from $\mathcal{R}_{A_0^{\pm}}(\lambda)f_{\pm}\in\operatorname{dom}(A_0^{\pm})$ and (2.12) that $\frac{d}{dx}(\mathcal{R}_{A_0^{\pm}}(\lambda)f_{\pm})_{x=\pm0}=0$. Taking into account (4.18), we see that conditions (4.17) take the form

$$\begin{cases}\displaystyle\int_{\mathbb{R}}\frac{g^{-}(t)d\Sigma_{-}(t)}{t-\lambda}+G^{-}(\lambda)M_{-}(\lambda)=\int_{\mathbb{R}}\frac{g^{+}(t)d\Sigma_{+}(t)}{t-\lambda}+G^{+}(\lambda)M_{+}(\lambda)\\ G^{-}(\lambda)=G^{+}(\lambda)\end{cases}.$$

Since $M_{+}(\lambda)\neq M_{-}(\lambda)$, problem (4.16)–(4.17) has the unique solution $y\in\operatorname{dom}(A^*_{\min})$ and it admits a representation (4.14)–(4.15). □

Next we clarify Proposition 4.4 in the case of J-nonnegative operator A, i.e., if $L\geqslant 0$.

PROPOSITION 4.5. *If the operator* $L=-d^2/dx^2+q(x)$ *is nonnegative, then the spectrum of the operator* $A=JL$ *is real.*

Proof. Since $L \geqslant 0$ we have $A^{+}_{\min} = L^{+}_{\min} \geqslant 0$ and $A^{-}_{\min} = -L^{+}_{\min} \leqslant 0$. It is known that the Friedrichs extension $L^{\pm}_{F}$ of $L^{\pm}_{\min}$ is generated by the Dirichlet boundary value problem, that is

$$L^{\pm}_{F} = (L^{\pm}_{\min})^{*}\lceil \operatorname{dom}(L^{\pm}_{F}), \quad \operatorname{dom}(L^{\pm}_{F} = \{f \in \operatorname{dom}(L^{\pm}_{\min})^{*} : f(0) = 0\}. \tag{4.19}$$

Setting $\Gamma^{\pm}_{0} f = f(\pm 0)$ and $\Gamma^{\pm}_{1} f = \pm f'(\pm 0)$ we obtain the boundary triplets $\Pi_{\pm} = \{\mathbb{C}, \Gamma^{\pm}_{0}, \Gamma^{\pm}_{1}\}$ for $(L^{\pm}_{\min})^{*}$ such that $\ker \Gamma^{\pm}_{0} = \operatorname{dom}(L^{\pm}_{F})$. Therefore the corresponding Weyl function $m^{\pm}_{F}$ belongs to the Krein-Stielties class S^{-} (see [15]). Hence, it admits the following integral representation (see [29]).

$$m^{\pm}_{F}(\lambda) = C_{\pm} + \lambda \int_{0}^{\infty} \frac{d\sigma_{\pm}(t)}{t-\lambda}, \qquad \int_{0}^{\infty} \frac{d\sigma_{\pm}(t)}{1+t} < \infty, \tag{4.20}$$

with $C_{\pm} \leqslant 0$. On the other hand, it follows from definitions that

$$-M^{-1}_{+}(\lambda) = -m^{-1}_{+}(\lambda) = m^{+}_{F}(\lambda), \qquad M^{-1}_{-}(\lambda) = -m^{-1}_{-}(-\lambda) = m^{-}_{F}(-\lambda) \tag{4.21}$$

Combining (4.20) and (4.21) we get

$$\begin{aligned} M^{-1}_{-}(\lambda) - M^{-1}_{+}(\lambda) &= m^{-}_{F}(-\lambda) + m^{+}_{F}(\lambda) \qquad (4.22) \\ &= \lambda \left[\frac{C_{-} + C_{+}}{\lambda} + \int_{0}^{\infty} \frac{d\sigma_{+}(t)}{t-\lambda} - \int_{0}^{\infty} \frac{d\sigma_{-}(t)}{t+\lambda} \right] =: \lambda \widetilde{M}(\lambda), \end{aligned}$$

where $\widetilde{M}(\cdot) \in (R)$ since $C_{\pm} \leqslant 0$. To complete the proof it remains to note that

$$\begin{aligned} M_{+}(\lambda) - M_{-}(\lambda) &= M_{+}(\lambda) \cdot [M^{-1}_{-}(\lambda) - M^{-1}_{+}(\lambda)] \cdot M_{-}(\lambda) \qquad (4.23) \\ &= M_{+}(\lambda) \cdot \lambda \widetilde{M}(\lambda) \cdot M_{-}(\lambda) \neq 0 \end{aligned}$$

for $\lambda \in \mathbb{C}_{\pm}$, since $M_{\pm}, \widetilde{M} \in (R)$. $\square$

REMARK 4.1.
(i) Statement (i) of Proposition 4.4 is implied by (4.13). However, we presented an elementary proof based on Proposition 2.4.
(ii) Note that Proposition 4.5 follows immediately from Proposition 4.4 and Proposition 2.2. However, we presented another proof that is in a spirit of our paper and demonstrates applicability of Weyl function technic. Note also that in turn, Proposition 2.2 can be proved by using Weyl function technic similarly to the proof of Proposition 4.5.

5. Similarity conditions for the operator A. General case.

5.1. Similarity criterion in terms of Weyl functions.

In the sequel we write $\lambda = \eta + i\varepsilon$, that is $\eta = \operatorname{Re}\lambda$, $\varepsilon = \operatorname{Im}\lambda$.
Combining Proposition 3.13 and Proposition 4.4, we arrive at the following criterion.

THEOREM 5.1. *The operator $A = (\operatorname{sgn} x)(-d^2/dx^2 + q(x))$ is similar to a selfadjoint operator if and only if for all $\varepsilon > 0$ and $g^{\pm} \in L^2(\mathbb{R}, d\Sigma_{\pm})$ the following inequalities hold:*

$$\int_{\mathbb{R}} \frac{\operatorname{Im} M_{\pm}(\eta + i\varepsilon)}{|M_+(\eta + i\varepsilon) - M_-(\eta + i\varepsilon)|^2} \left| \int_{\mathbb{R}} \frac{g^-(t) d\Sigma_-(t)}{t - (\eta + i\varepsilon)} \right|^2 d\eta \leqslant K^- \|g^-\|^2_{L^2(d\Sigma_-)}, \quad (5.1)$$

$$\int_{\mathbb{R}} \frac{\operatorname{Im} M_{\pm}(\eta + i\varepsilon)}{|M_+(\eta + i\varepsilon) - M_-(\eta + i\varepsilon)|^2} \left| \int_{\mathbb{R}} \frac{g^+(t) d\Sigma_+(t)}{t - (\eta + i\varepsilon)} \right|^2 d\eta \leqslant K^+ \|g^+\|^2_{L^2(d\Sigma_+)}, \quad (5.2)$$

where $K^{\pm}$ are constants independent of $\varepsilon > 0$ and $g^{\pm}$.

Proof. It is known (see [59]) that for any selfadjoint $B = B^*$ with resolution of identity E_t^B the following identity holds:

$$\varepsilon \cdot \int_{\mathbb{R}} \|\mathcal{R}_B(\eta + i\varepsilon) f\|^2 \, d\eta = \pi \|f\|^2, \qquad \varepsilon > 0, \quad f \in \mathfrak{H}. \quad (5.3)$$

It follows from (4.14) that

$$\begin{aligned} \|\mathcal{R}_A(\lambda) f\|^2 - 2\|\mathcal{R}_{A_0^- \oplus A_0^+}(\lambda) f\|^2 &\leqslant 2\|G^-(\lambda)\psi_-(\lambda) + G^+(\lambda)\psi_+(\lambda)\|^2 \\ &\leqslant 4\|\mathcal{R}_A(\lambda) f\|^2 + 4\|\mathcal{R}_{A_0^- \oplus A_0^+}(\lambda) f\|^2. \end{aligned}$$

On the other hand, it follows with account of (5.3) and with B replaced by $A_0^- \oplus A_0^+$ that

$$\begin{aligned} &\frac{\varepsilon}{2} \int_{\mathbb{R}} \|\mathcal{R}_A(\eta + i\varepsilon) f\|^2 d\eta - \pi \|f\|^2 \\ &\qquad \leqslant \varepsilon \int_{\mathbb{R}} \|G^-(\eta + i\varepsilon)\psi_-(\eta + i\varepsilon) + G^+(\eta + i\varepsilon)\psi_+(\eta + i\varepsilon)\|^2 d\eta \\ &\qquad \leqslant 2\varepsilon \int_{\mathbb{R}} \|\mathcal{R}_A(\eta + i\varepsilon) f\|^2 d\eta + 2\pi \|f\|^2. \end{aligned} \quad (5.4)$$

Since $\psi_{\pm}(\cdot, \lambda) \in L^2(\mathbb{R}_{\pm}, dx)$ and $\|\psi_{\pm}(\cdot, \lambda)\|^2_{L^2(\mathbb{R}_{\pm})} = \operatorname{Im} M_{\pm}(\lambda)/\operatorname{Im} \lambda$ (see [45, Lemma 2.4.2]), we have

$$\begin{aligned} &\|G^-(\lambda)\psi_-(\cdot, \lambda) + G^+(\lambda)\psi_+(\cdot, \lambda)\|^2 \\ &\quad = |G^-(\lambda)|^2 \|\psi_-(\cdot, \lambda)\|^2 + |G^+(\lambda)|^2 \|\psi_+(\cdot, \lambda)\|^2 \\ &\quad = \frac{1}{|M_+(\lambda) - M_-(\lambda)|^2} \left| \int_{\mathbb{R}} \frac{g^-(t) d\Sigma_-(t) - g^+(t) d\Sigma_+(t)}{t - \lambda} \right|^2 \frac{\operatorname{Im} M_+(\lambda) + \operatorname{Im} M_-(\lambda)}{\operatorname{Im} \lambda}. \end{aligned}$$

Combining this relation with (5.4) one concludes that (3.36) is equivalent to the following condition

$$\int\limits_{\mathbb{R}} \frac{\operatorname{Im} M_+(\eta+i\varepsilon)+\operatorname{Im} M_-(\eta+i\varepsilon)}{|M_+(\eta+i\varepsilon)-M_-(\eta+i\varepsilon)|^2} \left| \int_R \frac{g^-(t)d\Sigma_-(t)-g^+(t)d\Sigma_+(t)}{t-(\eta+i\varepsilon)} \right|^2 d\eta \leqslant C_1\|f\|^2, \tag{5.5}$$

where C_1 is a constant independent of f and ε.

By definition, $\|g^{\pm}\|_{L^2(d\Sigma_{\pm})} = \|f_{\pm}\|_{L^2(\mathbb{R}_{\pm})}$, where $f_{\pm} = P_{\pm}f$. Thus, condition (5.5) holds if and only if both (5.2) and (5.1) are satisfied. □

5.2. Necessary conditions for the similarity in terms of the Weyl functions and Hilbert transforms.

Let $\Sigma_{\pm} = \Sigma_{ac\pm} + \Sigma_{s\pm} = \Sigma_{ac\pm} + \Sigma_{sc\pm} + \Sigma_{d\pm}$ be the Lebesgue decomposition of the measure $\Sigma_{\pm}$ into a sum of absolutely continuous, singular continuous, and pure point measures (see, for example, [58]).

Denote by $S'_{ac}(\Sigma_{\pm})$ and $S'_s(\Sigma_{\pm})$ mutually disjoint (not necessarily topological) supports of measures $\Sigma_{ac\pm}$ and $\Sigma_{s\pm}$, respectively.

Note that for almost all $t \in \mathbb{R}$ the nontangential limits

$$\lim_{\substack{\lambda \to t \\ \sphericalangle}} M_{\pm}(\lambda) =: M_{\pm}(t)$$

exist (see [19]). By the Luzin-Privalov uniqueness theorem (see e.g. [42]), the relation $M_+(\lambda) \not\equiv M_-(\lambda)$ for $\lambda \in \mathbb{C}_+$ yields

$$M_+(t) := M_+(t+i0) \neq M_-(t+i0) =: M_-(t) \qquad \text{for a.e. } t \in \mathbb{R}. \tag{5.6}$$

THEOREM 5.2. *Let the operator A be similar to a selfadjoint operator. Then, the following inequalities hold*

$$\int\limits_{\mathbb{R}} \frac{\operatorname{Im} M_{\pm}(t)}{|M_+(t)-M_-(t)|^2} \left|g^+(t)\Sigma'_{ac+}(t)+i\big(H(g^+\cdot d\Sigma_+)\big)(t)\right|^2 dt \leqslant K_1^+ \int_{\mathbb{R}} |g^+(t)|^2 d\Sigma_+(t), \tag{5.7}$$

$$\int\limits_{\mathbb{R}} \frac{\operatorname{Im} M_{\pm}(t)}{|M_+(t)-M_-(t)|^2} \left|g^-(t)\Sigma'_{ac-}(t)+i\big(H(g^-\cdot d\Sigma_-)\big)(t)\right|^2 dt \leqslant K_1^- \int_{\mathbb{R}} |g^-(t)|^2 d\Sigma_-(t), \tag{5.8}$$

with constants K_1^+ and K_1^- independent of $g^{\pm} \in L^2(\mathbb{R}, d\Sigma_{\pm})$.

Proof. Applying Fatou's theorem and using (2.36) we get

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \frac{g^{\pm}(t)}{t-(\eta+i\varepsilon)} d\Sigma_{\pm}(t) = \pi \cdot [g^{\pm}(\eta)\Sigma'_{\pm}(\eta) + iH(g^{\pm}d\Sigma_{\pm})(\eta)] \tag{5.9}$$

Passing to the limit in (5.1) (resp., (5.2)) as $\varepsilon \to 0$ and taking (5.9) into account we arrive at the inequality (5.7) (resp., (5.8). □

COROLLARY 5.3. *Let the operator A be similar to a selfadjoint operator. Then*

$$\frac{\operatorname{Im} M_\pm(t)}{M_+(t) - M_-(t)} \in L^\infty(\mathbb{R}) \,. \tag{5.10}$$

Proof. Let A be similar to a selfadjoint operator. Then inequalities (5.7) and (5.8) hold. By Fatou Theorem $\pi\Sigma'_{ac\pm}(t) = \operatorname{Im} M_\pm(t+i0) =: \operatorname{Im} M_\pm(t)$ for a.e. $t \in \mathbb{R}$. Taking this relation into account and substituting in (5.7) (resp. (5.8)) any real-valued g^+_{ac} (resp. g^-_{ac}) with $g^\pm_{ac}(t) = 0$ for $t \in S'_s(\Sigma_\pm)$, we easily get

$$\int_{\mathbb{R}} \frac{(\operatorname{Im} M_\pm(\eta))^2}{|M_+(\eta) - M_-(\eta)|^2} |g^\pm_{ac}(\eta)|^2 \cdot \Sigma'_{ac\pm}(\eta) d\eta \leqslant K_1^- \int_{\mathbb{R}} |g^\pm_{ac}(t)|^2 \cdot \Sigma'_{ac\pm}(t) dt.$$

Since this inequality holds for any $g^\pm_{ac} \in L^2(\mathbb{R}, d\Sigma_{ac\pm})$, we have

$$\frac{(\operatorname{Im} M_\pm(t))^2}{|M_+(t) - M_-(t)|^2} \in L^\infty(S'_{ac}(\Sigma_\pm)). \tag{5.11}$$

Inequality (5.11) yields (5.10) since $\operatorname{Im} M_\pm(t) = 0$ for a.a. $t \in \mathbb{R} \setminus S'_{ac}(\Sigma_\pm)$. $\square$

COROLLARY 5.4. *Let the operator A be similar to a selfadjoint operator. Then, for all*

$$h^\pm \in L^2(\mathbb{R}) \cap L^2\left(\frac{1}{\Sigma'_{ac\pm}(t)}, \mathbb{R}\right) \quad \textit{with} \quad h^\pm(t) = 0 \quad \textit{for} \quad t \in S'_s(\Sigma_\pm) \,,$$

the following inequalities hold:

$$\int_{\mathbb{R}} \frac{\operatorname{Im} M_\pm(t)}{|M_+(t) - M_-(t)|^2} \left|(Hh^+)(t)\right|^2 dt \leqslant K_1^+ \int_{\mathbb{R}} |h^+(t)|^2 \frac{1}{\operatorname{Im} M_+(t)} dt, \tag{5.12}$$

$$\int_{\mathbb{R}} \frac{\operatorname{Im} M_\pm(t)}{|M_+(t) - M_-(t)|^2} \left|(Hh^-)(t)\right|^2 dt \leqslant K_1^+ \int_{\mathbb{R}} |h^-(t)|^2 \frac{1}{\operatorname{Im} M_-(t)} dt, \tag{5.13}$$

where K_1^+ and K_1^- are constants independent of $h^\pm$.

Proof. Inequality (5.7) yields

$$\int_{\mathbb{R}} \frac{\operatorname{Im} M_\pm(t)}{|M_+(t) - M_-(t)|^2} \left|(H(g^+ \cdot d\Sigma_+))(t)\right|^2 dt \leqslant K_1^- \int_{\mathbb{R}} |g^+(t)|^2 d\Sigma_+(t), \tag{5.14}$$

Choosing any g^+_{ac} with $g^+_{ac}(t) = 0$ for $t \in S'_s(\Sigma_+)$, and setting in (5.14) $h^\pm := g^\pm \cdot (\Sigma'_{ac\pm})$ we arrive at the inequality (5.12). The inequality (5.13) is implied by (5.8) in just the same way. $\square$

COROLLARY 5.5. *Let $E_\pm = \operatorname{supp} d\Sigma_{ac\pm}$ be the topological supports of measures $\Sigma_{ac\pm}$. If the operator A is similar to a selfadjoint operator, then*

$$\sup_{\mathcal{I}} \left(\frac{1}{|\mathcal{I} \cap E_\pm|} \int_{\mathcal{I}} \frac{\operatorname{Im} M_+(t) + \operatorname{Im} M_-(t)}{|M_+(t) - M_-(t)|^2} dt\right) \cdot \left(\frac{1}{|\mathcal{I} \cap E_\pm|} \int_{\mathcal{I}} \operatorname{Im} M_\pm(t) dt\right) < \infty. \tag{5.15}$$

Proof. If A is similar to a selfadjoint operator, then by Corollary 5.4 two-weight estimates (5.12) and (5.13) for the Hilbert transform are valid. Due to (2.40) the result is immediate from (5.12) and (5.13). $\square$

Due to the Lebesgue theorem inequality (5.15) yields (5.10) and therefore gives another proof of Corollary 5.3. In fact, it gives a new necessary condition for the similarity to a selfadjoint operator and is stronger than (5.10).

The following corollary gives one more necessary condition for the similarity.

COROLLARY 5.6. *Let A be similar to a selfadjoint operator and let*

$$w_1(t) := \frac{\operatorname{Im} M_+(t) + \operatorname{Im} M_-(t)}{|M_+(t) - M_-(t)|^2}.$$

Then

$$\sup_{\lambda \in \mathbb{C}_+} \big(\mathcal{P}_\lambda(w_1) \cdot \operatorname{Im} M_{ac\pm}(\lambda)\big) = C_\pm < \infty, \tag{5.16}$$

where $M_{ac\pm}(\lambda) := \displaystyle\int_{\mathbb{R}} \frac{d\Sigma_{ac\pm}(t)}{t-\lambda}$, $\lambda \in \mathbb{C}_+$.

Proof. Note that $\operatorname{Im} M_\pm(t)$ is finite for a.e. $t \in \mathbb{R}$ and $\mathcal{P}_\lambda(\operatorname{Im} M_\pm) = \operatorname{Im} M_{ac\pm}(\lambda)$. We complete the proof by combining Corollary 5.4 with Proposition 2.8. $\square$

Inequality (2.44) shows that condition (5.16) is stronger than (5.15).

Finally we show that inequalities (5.7) and (5.8) are equivalent to (simpler) inequalities involved only two-weight estimates of the Hilbert transform.

LEMMA 5.7. *If A is similar to a selfadjoint operator, then the following inequalities hold*

$$\int_{\mathbb{R}} \frac{\operatorname{Im} M_+(t) + \operatorname{Im} M_-(t)}{|M_+(t) - M_-(t)|^2} \left|(H(g^+ \cdot d\Sigma_+))(t)\right|^2 dt \leqslant K_2^+ \int_{\mathbb{R}} |g^+(t)|^2 d\Sigma_+(t), \tag{5.17}$$

$$\int_{\mathbb{R}} \frac{\operatorname{Im} M_+(t) + \operatorname{Im} M_-(t)}{|M_+(t) - M_-(t)|^2} \left|(H(g^- \cdot d\Sigma_-))(t)\right|^2 dt \leqslant K_2^- \int_{\mathbb{R}} |g^-(t)|^2 d\Sigma_-(t), \tag{5.18}$$

where constants K_2^+ and K_2^- are independent of $g^\pm \in L^2(\mathbb{R}, d\Sigma_\pm)$.

Moreover, the inequalities (5.7) and (5.17) as well as the inequalities (5.8) and (5.18) are equivalent.

Proof. Substituting in (5.7) (resp. (5.8)) real-valued $g^+ \in L^2(\mathbb{R}, d\Sigma_+)$ (resp. $g^- \in L^2(\mathbb{R}, d\Sigma_-)$) we obtain (5.17) (resp. (5.18)) (first for real-valued, then for complex-valued $g^\pm$).

To prove the converse statement we put

$$w_{2\pm}(t) = \begin{cases} 1/\operatorname{Im} M_\pm(t), & \text{if} \quad \operatorname{Im} M_\pm(t) \neq 0, \\ 0, & \text{if} \quad \operatorname{Im} M_\pm(t) = 0. \end{cases} \tag{5.19}$$

By Corollaries 5.4 and 5.5 the inequality (5.17) implies the upper of the estimates (5.15) (with $\operatorname{Im} M_+$ in place of $\operatorname{Im} M_\pm$). In turn, it yields (see (2.41)) the inclusion $w_1(\cdot)w_{2+}(\cdot)^{-1} \in L^\infty(E_+)$, that is with some $C_0^+ > 0$ the following estimate holds

$$w_1(t) \leqslant C_0^+ w_{2+}(t) \qquad \text{for} \quad \text{a.e. } t \in E_+. \tag{5.20}$$

Hence, and taking into account (5.19), (5.20) and the equality $\pi\Sigma'_{ac+}(t) = \operatorname{Im} M_+(t)$, we have

$$\begin{aligned}
\int_{\mathbb{R}} \left|g^+(t)\Sigma'_{ac+}(t)\right|^2 w_1(t)dt &= \int_{E_+\cup E_-} \left|g^+(t)\Sigma'_{ac+}(t)\right|^2 w_1(t)dt \\
&= \int_{E_+} \left|g^+(t)\Sigma'_{ac+}(t)\right|^2 w_1(t)dt \\
&\leqslant C_0^+ \int_{E_+} \left|g^+(t)\Sigma'_{ac+}(t)\right|^2 w_{2+}(t)dt \\
&\leqslant \frac{C_0^+}{\pi} \int_{\mathbb{R}} |g^+(t)|^2 d\Sigma_+(t), \qquad g^+ \in L^2(\mathbb{R}, d\Sigma_+).
\end{aligned}$$

Combining this inequality with (5.17) we arrive at (5.7). The implication $(5.18) \Longrightarrow (5.8)$ is proved similarly. $\square$

We complete this subsection by the following conjecture.

CONJECTURE 5.1. *Suppose that* $\sigma_{\text{disc}}(A) = \emptyset$ *and both measures* $d\Sigma_+$ *and* $d\Sigma_-$ *are absolutely continuous,* $\Sigma_\pm = \Sigma_{ac\pm}$. *Then conditions (5.12) and (5.13) are sufficient for* A *to be similar to a selfadjoint operator.*

5.3. Sufficient conditions for the similarity in terms of Weyl functions.

Consider an operator $\widetilde{A}$ given by $\widetilde{A} = A^*_{\min} \upharpoonright \operatorname{dom}(\widetilde{A})$,

$$\operatorname{dom}(\widetilde{A}) = \{y \in \operatorname{dom}(A^*_{\min}) : \ y(+0) = y(-0), \ y'(+0) = -y'(-0)\}. \tag{5.21}$$

PROPOSITION 5.8. *The operator* $\widetilde{A}$ *is selfadjoint. For* $\lambda \in \mathbb{C} \setminus \mathbb{R}$ *the resolvent of* $\widetilde{A}$ *has the form*

$$\mathcal{R}_{\widetilde{A}}(\lambda)f = \mathcal{R}_{A_0^- \oplus A_0^+}(\lambda)f + \widetilde{G}^-(\lambda)\psi_-(\lambda) + \widetilde{G}^+(\lambda)\psi_+(\lambda), \tag{5.22}$$

$$\widetilde{G}^+(\lambda) = -\widetilde{G}^-(\lambda) = \frac{1}{M_+(\lambda) + M_-(\lambda)} \int_{\mathbb{R}} \frac{g^-(t)d\Sigma_-(t) + g^+(t)d\Sigma_+(t)}{t-\lambda}, \tag{5.23}$$

where $g^\pm(t) = (\mathcal{F}_\pm f_\pm)(t)$, $f_\pm := P_\pm f \in L^2(\mathbb{R}_\pm)$.

Proof. Let $\Pi = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $S^* := A^*_{\min}$ defined by (2.27). Clearly, the extension $\widetilde{A}$ of $A_{\min}$ determined by (5.21), admits the following representation

$$\widetilde{A} = S^*|\operatorname{dom}\widetilde{A}, \quad \operatorname{dom}\widetilde{A} = \ker(\Gamma_1 - B\Gamma_0), \quad \text{where} \quad B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{5.24}$$

Thus, $\widetilde{A}$ is selfadjoint since so is B.

The representation (5.22) for the resolvent $\mathcal{R}_{\widetilde{A}}(\lambda)$ can be obtained in just the same way as representation for $\mathcal{R}_A(\lambda)$ in Proposition 4.4. $\square$

THEOREM 5.9. *Suppose that*

$$\sup_{\lambda\in\mathbb{C}_+} \frac{|M_+(\lambda)+M_-(\lambda)|}{|M_+(\lambda)-M_-(\lambda)|} < \infty. \tag{5.25}$$

Then the operator A is similar to a selfadjoint operator.

Proof. Since $\widetilde{A}$ and $A_0^-\oplus A_0^+$ are selfadjoint operators, we obtain from (5.22) and (5.3)

$$\varepsilon\int_{\mathbb{R}} \|\widetilde{G}^-(\eta+i\varepsilon)\ \psi_-(\eta+i\varepsilon)+\widetilde{G}^+(\eta+i\varepsilon)\ \psi_+(\eta+i\varepsilon)\|^2 d\eta \leqslant 4\pi\|f\|^2. \tag{5.26}$$

On the other hand, it follows from (5.23) with $f=f_\pm$ that

$$\begin{aligned}\|\widetilde{G}^\pm(\eta+i\varepsilon)\psi_\pm(\eta+i\varepsilon)\|^2 \\ = \frac{\operatorname{Im}M_+(\lambda)+\operatorname{Im}M_-(\lambda)}{\operatorname{Im}\lambda\cdot|M_+(\lambda)+M_-(\lambda)|^2}\left|\int_{\mathbb{R}}\frac{g^\pm(t)d\Sigma_\pm(t)}{t-\lambda}\right|^2.\end{aligned} \tag{5.27}$$

Combining (5.26) with (5.27) we arrive at the following inequalities

$$\int_{\mathbb{R}} \frac{\operatorname{Im}M_\pm+\operatorname{Im}M_\mp(\lambda)}{|M_+(\lambda)+M_-(\lambda)|^2}\left|\int_{\mathbb{R}}\frac{g^\pm(t)d\Sigma_\pm(t)}{t-\lambda}\right|^2 d\eta \leqslant 4\pi\|f_\pm\|^2 = 4\pi\|g^\pm\|^2.$$

Combining these inequalities with (5.25) we arrive at estimates (5.1) and (5.2). Thus, by Theorem 5.1, A is similar to a selfadjoint operator. $\square$

REMARK 5.1. The condition (5.25) is not necessary for similarity to a selfadjoint operator (see Remark 8.1)).

REMARK 5.2. Note that sufficient condition (5.25) for the similarity is weaker than either conditions (3.26)–(3.27) or conditions (3.29) obtained from Theorem 3.2 and Theorem 3.3, respectively. However, the latter conditions guarantee a stronger result: similarity of A to an operator $B=B^*$ with absolutely continuous spectrum.

Finally, we apply Theorems 5.3 and 5.9 to the case of the operator A with a constant potential. Consider a family of such operators

$$A(a) := (\operatorname{sgn} x)(-d^2/dx^2+a), \qquad a\in\mathbb{R}, \tag{5.28}$$

depending on a parameter a.

PROPOSITION 5.10. (i) *The operator $A(a)$ is similar to a selfadjoint operator if and only if $a\geqslant 0$.*

(ii) *The operator $A(0)$ is similar to the multiplication operator $Q: f\to xf(\cdot)$ in $L^2(\mathbb{R})$.*

Proof. (i) In the case under consideration the functions $M_{\pm}(\lambda)$ are given by

$$M_{\pm}(\lambda) = \pm \frac{i}{\sqrt{\pm\lambda - a}}\,. \tag{5.29}$$

Since

$$M_{+}(\lambda) - M_{-}(\lambda) = \frac{i}{(\lambda - a)^{1/2}} + \frac{i}{(-\lambda - a)^{1/2}} \neq 0 \qquad \text{for} \quad \lambda \notin \mathbb{R},$$

Proposition 4.4 yields that the spectrum of $A(a)$ is real for any $a \in \mathbb{R}$ (see also [10]). It is clear that M_+ and M_- are holomorphic on $\mathbb{C} \setminus [a, +\infty)$ and $\mathbb{C} \setminus (-\infty, -a]$, respectively. Hence, by Proposition 2.5 (iv), we have $\sigma(A(a)) = (-\infty, -a] \cup [a, +\infty)$, that is $\sigma(A(a)) = \mathbb{R}$ for $a \leqslant 0$ and $\sigma(A(a)) = \mathbb{R} \setminus (-a, a)$ for $a > 0$.

If $a \geqslant 0$, then the function

$$\frac{M_{+}(\lambda) + M_{-}(\lambda)}{M_{+}(\lambda) - M_{-}(\lambda)}$$

is bounded in $\mathbb{C}_+$. Thus, by Theorem 5.9, A is similar to a selfadjoint operator.

Now let $a < 0$. Setting $\lambda = i\varepsilon$ and $i\varepsilon - a = \rho e^{i\phi}$ we get

$$M_{+}(i\varepsilon) - M_{-}(i\varepsilon) = i\rho^{-1/2} \cdot [e^{-i\phi/2} - e^{i\phi/2}] = 2\rho^{-1/2} \sin(\phi/2),$$

and

$$\operatorname{Im} M_{+}(i\varepsilon) = \operatorname{Im}(i\rho^{-1/2} e^{i\phi/2}) = \rho^{-1/2} \cos(\phi/2).$$

Hence

$$\operatorname{Im} M_{+}(i\varepsilon)(M_{+}(i\varepsilon) - M_{-}(i\varepsilon))^{-1} = 2^{-1} \cot(\phi/2)$$

is unbounded in any neighborhood of zero. Thus, by Corollary 5.3 the operator A is not similar to a selfadjoint operator.

(ii) Let now $A = A(0)$. Substituting expressions (5.29) in formula (3.21) for $\theta_A(\cdot)$ and using the relation $\sqrt{\lambda}/\sqrt{-\lambda} = -i$, we arrive at the following formula for the characteristic function

$$\theta_A(\lambda) = \begin{pmatrix} -i & (i-1)/\sqrt{-\lambda} \\ (i-1)\sqrt{\lambda} & -i \end{pmatrix}. \tag{5.30}$$

It follows that $\theta_A(\cdot)$ is unbounded only near zero and infinity. Since the operator A is completely nonselfadjoint (see the proof of Corollary 3.10) and has no eigenvalues, then by Proposition 3.4 (or by Corollary 3.5) it is similar to a selfadjoint operator $T_0 = T_0^*$ with absolutely continuous spectrum, $\sigma(T_0) = \sigma_{ac}(T_0) = \mathbb{R}$, $\sigma_s(T_0) = \sigma_p(T_0) = \emptyset$. It is easily seen that the multiplicity of spectrum of $A(0)$ is one. Therefore T_0 is unitarily equivalent to the multiplication operator Q. $\square$

REMARK 5.3. Using the Krein-Langer spectral theory of definitizable operators in Krein spaces Ćurgus and Langer [8] investigated the critical point ∞ of differential operators with an indefinite weight. Their results imply similarity of the operator $A(a)$ to a selfadjoint one if only $a > 0$.

The case $a = 0$ is more complicated since $A(0)$ has two critical points: zero and infinity. Similarity of $A(0)$ to a selfadjoint operator was established by Ćurgus and Najman [9] in the framework of Krein space approach.

Other proofs of the latter result have been obtained by several authors (see [32, 33, 16, 30]). In full generality statement (i) of Proposition 5.10 has originally been proved by one of the authors [34, 33], by using the resolvent criterion of similarity (see Theorem 3.12). The proof given above is similar to that contained in our short communication [38]. Proposition 3.4 makes it possible to prove statement (ii).

6. Restrictions of A to invariant subspaces corresponding to $\sigma_{\rm disc}(A)$ and $\sigma_{\rm ess}(A)$

In this section we consider the operator A of the form (1.1) under the following

ASSUMPTION 6.1. *Suppose that the set* $\sigma_{\rm disc}(A)$ *is finite.*

Under this assumption the problem of similarity of A to a normal operator is basically reduced to the same problem for $A_{\rm ess}$. Moreover, we show in Section 7. that Assumption 6.1 is fulfilled if a potential q is finite-zone.

Since $\operatorname{dist}(\sigma_{\rm ess}(A), \sigma_{\rm disc}(A)) > 0$, the theorem on spectral decomposition ([40, Theorem III.6.17]) implies that there exists a skew direct decomposition $L^2(\mathbb{R}) = \mathfrak{H} = \mathfrak{H}_e \dot{+} \mathfrak{H}_d$ such that

$$A = A_{\rm ess} \dot{+} A_{\rm disc}, \quad A_{\rm ess} = A \restriction (\operatorname{dom}(A) \cap \mathfrak{H}_e), \quad A_{\rm disc} = A \restriction (\operatorname{dom}(A) \cap \mathfrak{H}_d), \quad (6.1)$$
$$\text{and} \quad \sigma(A_{\rm disc}) = \sigma_{\rm disc}(A), \quad \sigma(A_{\rm ess}) = \sigma_{\rm ess}(A).$$

Denote by P_e and P_d the skew projections in $\mathfrak{H}$ onto $\mathfrak{H}_e$ and $\mathfrak{H}_d$, respectively.

Since $\sigma_{\rm disc}(A)$ is finite and each eigenvalue of A is of finite algebraic multiplicity, we see that $A_{\rm disc}$ is an operator on a finite dimensional space (i.e., $\dim \mathfrak{H}_d < \infty$). Jordan normal form of $A_{\rm disc}$ is described in Proposition 4.3 (3)–(5). By Proposition 4.3, we have $\sigma_{\rm ess}(A) = \sigma_{\rm ess}(A_2^-) \cup \sigma_{\rm ess}(A_2^+)$. Thus, $\sigma(A_{\rm ess}) \subset \mathbb{R}$.

Here we investigate similarity of $A_{\rm ess}$ to a selfadjoint operator.

PROPOSITION 6.1. *Let Assumption 6.1 be fulfilled. Let G_d be the closure of an open bounded set $G'_d (\subset \mathbb{C})$ such that $G_d \cap \sigma_{\rm ess}(A) = \emptyset$ and $\sigma_{\rm disc}(A) \subset G'_d$. Let $X^{\pm}$ be dense subsets in $L^2(\mathbb{R}, d\Sigma_{\pm})$. Then the following conditions are equivalent:*

(i) *the operator $A_{\rm ess}$ in $\mathfrak{H}_e$ is similar to a selfadjoint one;*

(ii) *the operator $A_{\rm ess} P_e$ in $\mathfrak{H} = L^2(\mathbb{R})$ is similar to a selfadjoint one;*

(iii) *the following inequality holds with some constant C_1^e*

$$\sup_{\varepsilon>0} \varepsilon \cdot \int_{\mathbb{R}} \|\mathcal{R}_A(\eta + i\varepsilon) f_e\|^2 d\eta \leqslant C_1^e \|f_e\|^2, \quad f_e \in \mathfrak{H}_e; \quad (6.2)$$

(iv) *for all $\varepsilon > 0$ and $g^{\pm} \in X^{\pm}$ the following inequalities hold:*

$$\int\limits_{\substack{\eta \in \mathbb{R} \\ \eta + i\varepsilon \notin G_d}} \frac{\operatorname{Im} M_+(\eta+i\varepsilon) + \operatorname{Im} M_-(\eta+i\varepsilon)}{|M_+(\eta+i\varepsilon) - M_-(\eta+i\varepsilon)|^2} \left| \int_{\mathbb{R}} \frac{g^{\pm}(t) d\Sigma_{\pm}(t)}{t - (\eta+i\varepsilon)} \right|^2 d\eta \leqslant C_2^{\pm} \|g^{\pm}\|^2_{L^2(d\Sigma_{\pm})}, \quad (6.3)$$

where $C_2^{\pm}$ *are constants independent of* ε *and* $g^{\pm}$.

Proof. The equivalence $(i) \Leftrightarrow (ii)$ is obvious.

Let us show the equivalence $(ii) \Leftrightarrow (iii)$. It can easily be checked that $A_{\text{ess}} P_e$ is a J-selfadjoint operator (see [43]). By Proposition 3.13, (ii) holds if and only if

$$\sup_{\varepsilon>0} \varepsilon \cdot \int_{\mathbb{R}} \|\mathcal{R}_{AP_e}(\eta+i\varepsilon)f\|^2 d\eta \leqslant C_1 \|f\|^2, \quad f \in \mathfrak{H}, \quad C_1 = const. \tag{6.4}$$

Clearly, (6.4) is equivalent to (6.2).

Now we prove the equivalence $(iii) \Leftrightarrow (iv)$. Let $f \in L^2(\mathbb{R})$, $f_e = P_e f$, $f_d = P_d f$. Clearly, there exist constants C_2, C_3 such that

$$\|\mathcal{R}_A f_d\| = \|\mathcal{R}_{A_{\text{disc}}} f_d\| \leqslant C_3 (1+|\lambda|)^{-1} \|f_d\|, \quad \lambda \in \mathbb{C} \setminus G_d,$$

and $\|\mathcal{R}_A f_e\| = \|\mathcal{R}_{A_{\text{ess}}} f_e\| \leqslant C_2 \|f_e\|$ for $\lambda \in G_d$. Therefore (6.4) is equivalent to

$$\varepsilon \cdot \int_{\eta+i\varepsilon \notin G_d} \|\mathcal{R}_A(\eta+i\varepsilon)f\|^2 d\eta \leqslant C_1 \|f\|^2, \quad f \in L^2(\mathbb{R}), \quad \varepsilon > 0. \tag{6.5}$$

Arguing as in the proof of Theorem 5.1, we see that condition (6.5) is fulfilled if and only if the inequalities (6.3) hold for all $g^{\pm} \in L^2(d\Sigma_{\pm})$ and $\varepsilon > 0$. We show that it suffices to check (6.3) only for dense subsets $X^{\pm}$.

Let $\varepsilon > 0$ be a fixed positive number, $\mathcal{I}$ an open bounded set in $\mathbb{R}$. Denote $\mathcal{I}_{\varepsilon} := \{\eta + i\varepsilon : \eta \in \mathcal{I}\}$. Assume that $\mathcal{I}_{\varepsilon} \cap G_d = \emptyset$. Then $(M_+(\lambda) - M_-(\lambda))^{-1}$ is holomorphic on $\mathcal{I}_{\varepsilon}$. By the Schwarz inequality, the operators

$$K_{\mathcal{I}_{\varepsilon}}^{\pm} : g^+ \mapsto \frac{(\operatorname{Im} M_{\pm}(\eta+i\varepsilon))^{1/2}}{M_+(\eta+i\varepsilon) - M_-(\eta+i\varepsilon)} \int_{\mathbb{R}} \frac{g^+(t) d\Sigma_+(t)}{t-(\eta+i\varepsilon)},$$

are bounded from $L^2(\mathbb{R}, d\Sigma_+)$ to $L^2(\mathcal{I}_{\varepsilon}, d\eta)$.

Suppose that X^+ is dense in $L^2(d\Sigma_+)$ and (6.3) is fulfilled for $g^{\pm} \in X^{\pm}$. Then $\|K_{\mathcal{I}_{\varepsilon}}^{\pm}\| \leqslant C_2^+$ for all $\varepsilon > 0$ and for all $\mathcal{I}$. This implies (6.3) for all $g^{\pm} \in L^2(d\Sigma_{\pm})$ and $\varepsilon > 0$. $\square$

Recall that $\sigma_{ac}(T)$ and $\sigma_s(T)$ stand for the absolutely continuous and singular spectra of a selfadjoint operator T. It is known (see [45, 51]) that the spectral functions $\Sigma_{\pm}(\cdot)$ completely characterize the spectra of the operators $A_0^{\pm}$. In particular,

$$\sigma_{ac}(A_0^{\pm}) = \operatorname{supp} d\Sigma_{ac\pm}, \qquad \sigma_s(A_0^{\pm}) = \operatorname{supp}(d\Sigma_{sc\pm} + d\Sigma_{d\pm}).$$

Note that $\sigma_{ac}(A_0^{\pm}) \subset \sigma_{\text{ess}}(A_0^{\pm})$. Therefore, by Proposition 4.3, we have

$$\operatorname{supp} d\Sigma_{ac-} \cup \operatorname{supp} d\Sigma_{ac+} = \sigma_{ac}(A_0^- \oplus A_0^+) \subset \sigma_{\text{ess}}(A). \tag{6.6}$$

PROPOSITION 6.2. *Let Assumption 6.1 be fulfilled. Suppose the operator* A_{ess} *is similar to a selfadjoint operator. Then*

$$\frac{\operatorname{Im} M_{ac\pm}(t)}{M_+(t) - M_-(t)} \in L^{\infty}(\mathbb{R}). \tag{6.7}$$

Proof. With account of (6.6), this result is immediate from Corollary 5.3. □

ASSUMPTION 6.2. *In what follows we assume additionally that both A_0^+ and A_0^- have no singular continuous spectrum, that is*

$$d\Sigma_- = d\Sigma_{ac-} + d\Sigma_{d-}, \quad \operatorname{supp} d\Sigma_{d-} = \{\theta_j^-\}_{j=1}^{N_\theta^-}, \quad N_\theta^- < \infty,$$

and

$$d\Sigma_+ = d\Sigma_{ac+} + d\Sigma_{d+}, \quad \operatorname{supp} d\Sigma_{d+} = \{\theta_j^+\}_{j=1}^{N_\theta^+}, \quad N_\theta^+ < \infty.$$

Then $M_\pm(\lambda) = M_{ac\pm}(\lambda) + M_{d\pm}(\lambda)$, where

$$M_{ac\pm}(\lambda) := \int_{\mathbb{R}} \frac{d\Sigma_{ac\pm}(t)}{t-\lambda}, \quad \text{and} \quad M_{d\pm}(\lambda) := \sum_{j=1}^{N_\theta^\pm} \frac{\Sigma_\pm(\theta_j^\pm + 0) - \Sigma_\pm(\theta_j^\pm - 0)}{\theta_j^\pm - \lambda}.$$

Let us introduce the sets

$$\{\widetilde{\theta}_j^\pm\}_1^{\widetilde{N}_\theta^\pm} := \{\theta_j^\pm\}_1^{N_\theta^\pm} \setminus \sigma_{\text{disc}}(A); \tag{6.8}$$

here $\widetilde{N}_\theta^\pm < \infty$ (these sets appear in Theorem 6.3).

Recall that we denote by $\mathcal{N}^+(\mathbb{C}_+)$ the Smirnov class on $\mathbb{C}_+$ (see Subsection 2.6.).

THEOREM 6.3. *Let Assumptions 6.1 and 6.2 be fulfilled and G_d the compact set from Proposition 6.1. Suppose additionally that there exist functions $U_\pm(\cdot)$ on $\mathbb{C}_+$ such that the following conditions hold with some constants $C_\pm^u$:*

$$\frac{\operatorname{Im} M_{ac\pm}(\lambda)}{|M_+(\lambda) - M_-(\lambda)|^2} \leqslant C_\pm^u |U_\pm(\lambda)|^2, \qquad \lambda \in \mathbb{C}_+ \setminus G_d, \tag{6.9}$$

$$U_\pm(\lambda) \in \mathcal{N}^+(C_+), \tag{6.10}$$

$$\frac{U_\pm(t)}{\theta_j^- - t} \in L^2(\mathbb{R}), \quad j \in \{1, \cdots, N_\theta^-\}; \quad \frac{U_\pm(t)}{\theta_j^+ - t} \in L^2(\mathbb{R}), \quad j \in \{1, \cdots, N_\theta^+\}. \tag{6.11}$$

Suppose additionally, that there exist functions $w_+(\cdot)$ and $w_-(\cdot)$ on $\mathbb{R}$, $w_\pm(t) > 0$ a.e., and constants $C_\pm^w$ such that the following conditions hold:

$$w_\pm(t) \leqslant C_\pm^w (\Sigma'_{ac\pm}(t))^{-1} \quad \textit{a.e. on } \operatorname{supp} d\Sigma_{ac\pm}, \tag{6.12}$$

$$w_+(t) \ \textit{and} \ w_-(t) \quad \textit{satisfy the } (A_2) \textit{ condition (see (2.38))}, \tag{6.13}$$

$$\frac{U_+^2(t)}{w_\pm(t)} \in L^\infty(\mathbb{R}), \qquad \frac{U_-^2(t)}{w_\pm(t)} \in L^\infty(\mathbb{R}). \tag{6.14}$$

Suppose also that for every point $\widetilde{\theta}_j^{\pm}$ *of the set* $\{\widetilde{\theta}_k^{\pm}\}_1^{\widetilde{N}_\theta^{\pm}}$ *, there exist a function* $U_j^{\pm}(\lambda) \in \mathcal{N}^+(C_+)$ *and a neighborhood* $D_j^{\pm}$ *of the point* $\widetilde{\theta}_j^{\pm}$ *such that with some constants* C_θ^u *and* C_θ^M *the following conditions hold:*

$$\frac{1}{|M_+(\lambda)-M_-(\lambda)|^2}\operatorname{Im}\frac{1}{\widetilde{\theta}_j^{\pm}-\lambda} \leqslant C_\theta^u|U_j^{\pm}(\lambda)|^2 \quad \text{for} \quad \lambda \in D_j^{\pm}\cap\mathbb{C}_+\,, \tag{6.15}$$

$$\frac{|U_j^{\pm}(t)|^2}{w_+(t)} \in L^\infty(\mathbb{R}), \qquad \frac{|U_j^{\pm}(t)|^2}{w_-(t)} \in L^\infty(\mathbb{R})\,, \tag{6.16}$$

$$\frac{1}{|\widetilde{\theta}_j^{\pm}-\lambda|\,|M_+(\lambda)-M_-(\lambda)|} \leqslant C_\theta^M \quad \text{for} \quad \lambda \in D_j^{\pm}\cap\mathbb{C}_+. \tag{6.17}$$

Then A_{ess} *is similar to a selfadjoint operator.*

Proof. Let us show that (6.3) holds.

Let $\lambda=\eta+i\varepsilon$, $\eta=\operatorname{Re}\lambda$, $\varepsilon=\operatorname{Im}\lambda$.

1) Denote

$$\mathcal{I}_{\pm}(\varepsilon) := \int\limits_{\substack{\eta\in\mathbb{R}\\ \eta+i\varepsilon\notin G_d}} \frac{\operatorname{Im}M_{ac\pm}(\lambda)}{|M_+(\lambda)-M_-(\lambda)|^2}\left|\int_{\mathbb{R}}\frac{g^+(s)d\Sigma_+(s)}{s-\lambda}\right|^2 d\eta, \quad g^+\in L^2(d\Sigma_+(t))\,.$$

Let

$$X_{ac}^+ := \{g^+\in L^2(\mathbb{R},d\Sigma_{ac+}(t)):\ (g^+\Sigma'_{ac+})\in L^2(\mathbb{R},dt)\}.$$

Then the set $X^+ := X_{ac}^+\oplus L^2(\mathbb{R},d\Sigma_{d+})$ is dense in $L^2(\mathbb{R},d\Sigma_+(t))$.

First we show that

$$\mathcal{I}_{\pm}(\varepsilon) \leqslant C_2^+\|g^+\|^2_{L^2(d\Sigma_+)} \quad \text{for } g^+\in X^+. \tag{6.18}$$

Let us denote

$$K_{\pm}^{ac}(\lambda) := U_{\pm}(\lambda)\int_{\mathbb{R}}\frac{g^+(t)d\Sigma_{ac+}(t)}{t-\lambda}, \qquad K_{\pm}^{d}(\lambda) := U_{\pm}(\lambda)\int_{\mathbb{R}}\frac{g^+(t)d\Sigma_{d+}(t)}{t-\lambda},$$
$$K_{\pm}(\lambda) := K_{\pm}^{ac}(\lambda)+K_{\pm}^{d}(\lambda) = U_{\pm}(\lambda)\int_{\mathbb{R}}\frac{g^+(t)d\Sigma_+(t)}{t-\lambda}.$$

By (6.9), we have

$$\mathcal{I}_{\pm}(\varepsilon) \leqslant \int_{\mathbb{R}}|K_{\pm}(\lambda)|^2 d\eta. \tag{6.19}$$

It follows from $U_{\pm}(\cdot)\in\mathcal{N}^+(\mathbb{C}_+)$ that $U_{\pm}(\cdot)$ are holomorphic in $\mathbb{C}_+$ and have the nontangential limits $U_{\pm}(\eta)=U_{\pm}(\eta+i0)$ for almost all $\eta\in\mathbb{R}$ (see [19]). Since $g^+\in X^+$, we have $g^+(\cdot)\Sigma'_{ac+}(\cdot)\in L^2(\mathbb{R},dt)$. Therefore,

$$\int_{\mathbb{R}}\frac{g^+(t)d\Sigma_{ac+}(t)}{t-\lambda}\in H^2(\mathbb{C}_+)\cdot$$

It follows from [19, Corollary II.5.6] and [19, Corollary II.5.7] that $K_{\pm}^{ac}(\lambda) \in \mathcal{N}^{+}(\mathbb{C}_{+})$. Note that $(\theta_j^{+} - \lambda)^{-1}$ are the outer functions in $\mathbb{C}_{+}$. Therefore [19, Corollary II.5.6] and Lemma 2.10 yield $K_{\pm}^{d}(\lambda) \in \mathcal{N}^{+}(\mathbb{C}_{+})$. Hence $K_{\pm}^{ac}(\lambda)$, $K_{\pm}^{d}(\lambda)$, and $K_{\pm}(\lambda)$ belong to $\mathcal{N}^{+}(\mathbb{C}_{+})$ and have the nontangential limits $K_{\pm}^{ac}(\eta)$, $K_{\pm}^{d}(\eta)$ and $K_{\pm}(\eta)$ for a.a. $\eta \in \mathbb{R}$. Note also that

$$K_{\pm}^{ac}(\eta) := \pi U_{\pm}(\eta)\left(g^{+}(\eta)\Sigma'_{ac+}(\eta) + H(g^{+}\Sigma'_{ac+})(\eta)\right) \quad \text{for a.e. } \eta \in \mathbb{R}. \tag{6.20}$$

Assume that the following inequality holds

$$\int_{\mathbb{R}} \|K_{\pm}(\eta)\|^2 d\eta \leqslant C_2^{+}\|g^{+}\|^2_{L^2(d\Sigma_{+})}. \tag{6.21}$$

Then, by [19, Section II.5], we have $K_{\pm}(\lambda) \in H^2(\mathbb{C}_{+})$ and for all $\varepsilon > 0$

$$\int_{\mathbb{R}} \|K_{\pm}(\eta + i\varepsilon)\|^2 d\eta \leqslant \|K_{\pm}(\lambda)\|^2_{H^2(\mathbb{C}_{+})} = \|K_{\pm}(\eta)\|^2_{L^2(\mathbb{R})} \leqslant C_2^{+}\|g^{+}\|^2_{L^2(d\Sigma_{+})}.$$

Combining this with (6.19), we see that (6.21) yields (6.18) with a constant C_2^{+} independent of $g^{+} \in X^{+}$.

Let us prove (6.21). By (6.11), we have

$$\begin{aligned} \|K_{\pm}^{d}(\eta)\|_{L^2(\mathbb{R})} &\leqslant C_3^{\pm}\sum_{j=1}^{N_\theta^{+}} g^{+}(\theta_j^{+})\left(\Sigma_{+}(\theta_j^{+}+0) - \Sigma_{+}(\theta_j^{+}-0)\right)^{1/2} \\ &\leqslant C_3^{\pm}\sqrt{N_\theta^{+}}\,\|g^{+}\|_{L^2(d\Sigma_{+})}, \end{aligned} \tag{6.22}$$

where

$$C_3^{\pm} = \max_{1\leqslant j\leqslant N_\theta^{+}}\left\{(\Sigma_{+}(\theta_j^{+}+0) - \Sigma_{+}(\theta_j^{+}-0))^{1/2}\left\|\frac{U_{\pm}(\eta)}{\theta_j^{+}-\eta}\right\|_{L^2(\mathbb{R})}\right\} < \infty.$$

It follows from (6.12) that

$$\|g^{+}(t)\Sigma'_{ac+}(t)\|^2_{L^2(w_{+}(t)dt)} \leqslant C_{+}^{w}\|g^{+}\|^2_{L^2(d\Sigma_{ac+})} \leqslant C_{+}^{w}\|g^{+}\|^2_{L^2(d\Sigma_{+})}. \tag{6.23}$$

Since $w_{+}(t) \in (A_2)$, we have

$$\|H(g^{+}\Sigma'_{ac+})(t)\|^2_{L^2(w_{+}(t)dt)} \leqslant C_1\|g^{+}(t)\Sigma'_{ac+}(t)\|^2_{L^2(w_{+}(t)dt)}, \tag{6.24}$$

where C_1 is a constant independent of g^{+}. It follows from (6.24) and (6.20) that

$$\begin{aligned} \int_{\mathbb{R}} |K_{\pm}^{ac}(\eta)|^2 d\eta &\leqslant \left\|\frac{U_{\pm}^2(\eta)}{w_{+}(\eta)}\right\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}} |g^{+}(\eta)\Sigma'_{ac+}(\eta) + \mathcal{H}(g^{+}\Sigma'_{ac+})(\eta)|^2 w_{+}(\eta)d\eta \\ &\leqslant 2(1+C_1)\left\|\frac{U_{\pm}^2(\eta)}{w_{+}(\eta)}\right\|_{L^\infty(\mathbb{R})}\|g^{+}(\eta)\Sigma'_{ac+}(\eta)\|^2_{L^2(w_{+}(\eta)d\eta)}. \end{aligned} \tag{6.25}$$

Combining (6.25), (6.14), and (6.23), we get

$$\int_{\mathbb{R}} |K_{\pm}^{ac}(\eta)|^2 d\eta \leqslant C_2 \|g^+\|^2_{L^2(d\Sigma_+)} ,$$

where the constant C_2 is independent of g^+. Taking (6.22) into account we obtain (6.21). In turn, (6.21) yields (6.18).

2) Denote

$$\mathcal{I}_{d\pm}(\varepsilon) := \int\limits_{\substack{\eta\in\mathbb{R}\\ \eta+i\varepsilon\notin G_d}} \frac{\operatorname{Im} M_{d\pm}(\lambda)}{|M_+(\lambda)-M_-(\lambda)|^2} \left| \int_{\mathbb{R}} \frac{g^+(s)d\Sigma_+(s)}{s-\lambda} \right|^2 d\eta.$$

Let us show that

$$\mathcal{I}_{d\pm}(\varepsilon) \leqslant C_3 \|g^+\|^2_{L^2(d\Sigma_+)} \quad \text{for} \quad g^+ \in X^+ \tag{6.26}$$

with a constant $C_3 > 0$. It suffices to prove inequality (6.26) for each summand, i.e.,

$$\int\limits_{\substack{\eta\in\mathbb{R}\\ \eta+i\varepsilon\notin G_d}} \frac{1}{|M_+(\lambda)-M_-(\lambda)|^2} \operatorname{Im} \frac{1}{\theta_j^{\pm}-\lambda} \left| \int_{\mathbb{R}} \frac{g^+(s)d\Sigma_+(s)}{s-\lambda} \right|^2 d\eta \leqslant C_4 \|g^+\|^2_{L^2(d\Sigma_+)} \tag{6.27}$$

for $j \in \{1, \ldots, N_\theta^{\pm}\}$.

Assume that $\theta_j^{\pm} \in \sigma_{\mathrm{disc}}(A)$. Then

$$\operatorname{Im} \frac{1}{\theta_j^{\pm}-\lambda} \leqslant C_5 \operatorname{Im} M_{ac\pm}(\lambda), \qquad \lambda \in \mathbb{C}_+ \setminus G_d .$$

Thus (6.27) follows from (6.18).

Assume $\theta_j^{\pm} \notin \sigma_{\mathrm{disc}}(A)$. In this case, $\theta_j^{\pm} \in \{\widetilde{\theta}_k^{\pm}\}_1^{\widetilde{N}_\theta}$. Let k be such that $\theta_j^{\pm} = \widetilde{\theta}_k^{\pm}$. By assumptions of the theorem, conditions (6.15), (6.16), and (6.17) hold. It is easy to see that

$$\operatorname{Im} \frac{1}{\widetilde{\theta}_k^{\pm}-\lambda} \leqslant C_6 \operatorname{Im} M_{ac\pm}(\lambda), \quad \lambda \in \mathbb{C}_+ \setminus D_k^{\pm} .$$

Therefore (6.18) implies

$$\int\limits_{\substack{\eta\in\mathbb{R}\\ \eta+i\varepsilon\notin(D_k^{\pm}\cup G_d)}} \frac{1}{|M_+(\lambda)-M_-(\lambda)|^2} \operatorname{Im} \frac{1}{\widetilde{\theta}_k^{\pm}-\lambda} \left| \int_{\mathbb{R}} \frac{g^+(s)d\Sigma_+(s)}{s-\lambda} \right|^2 d\eta \leqslant C_4 \|g^+\|^2_{L^2(d\Sigma_+)}. \tag{6.28}$$

By (6.17), we have

$$\int\limits_{\substack{\eta\in\mathbb{R}\\ \eta+i\varepsilon\in D_k^{\pm}}} \frac{1}{|M_+(\lambda)-M_-(\lambda)|^2}\,\mathrm{Im}\,\frac{1}{\widetilde{\theta}_k^{\pm}-\lambda}\left|\int_{\mathbb{R}}\frac{g^+(s)d\Sigma_{d+}(s)}{s-\lambda}\right|^2 d\eta$$

$$\leqslant \int\limits_{\substack{\eta\in\mathbb{R}\\ \eta+i\varepsilon\in D_k^{\pm}}} \frac{\varepsilon}{(\widetilde{\theta}_k^{\pm}-\eta)^2+\varepsilon^2}\,\frac{1}{|M_+(\lambda)-M_-(\lambda)|^2|\widetilde{\theta}_k^{\pm}-\lambda|^2}\left|(\widetilde{\theta}_k^{\pm}-\lambda)\int_{\mathbb{R}}\frac{g^+(s)d\Sigma_{d+}(s)}{s-\lambda}\right|^2 d\eta$$

$$\leqslant C_\theta^M \int\limits_{\substack{\eta\in\mathbb{R}\\ \eta+i\varepsilon\in D_k^{\pm}}} \frac{\varepsilon}{(\widetilde{\theta}_k^{\pm}-\eta)^2+\varepsilon^2}\left|(\widetilde{\theta}_k^{\pm}-\lambda)\int_{\mathbb{R}}\frac{g^+(s)d\Sigma_{d+}(s)}{s-\lambda}\right|^2 d\eta. \tag{6.29}$$

We may assume that

$$D_k^{\pm}\cap\left(\{\theta_j^+\}_1^{N_\theta^+}\cup\{\theta_j^-\}_1^{N_\theta^-}\right)=\{\widetilde{\theta}_k^{\pm}\}\,.$$

Therefore,

$$\left|(\widetilde{\theta}_k^{\pm}-\lambda)\int_{\mathbb{R}}\frac{g^+(s)d\Sigma_{d+}(s)}{s-\lambda}\right|\leqslant C_7\|g^+\|^2_{L^2(d\Sigma_+)},\quad \lambda\in D_k^{\pm}\,.$$

If we combine this with properties of Poisson kernel (see [19, Section I.3]), we get

$$\int\limits_{\substack{\eta\in\mathbb{R}\\ \eta+i\varepsilon\in D_k^{\pm}}} \frac{\varepsilon}{(\widetilde{\theta}_k^{\pm}-\eta)^2+\varepsilon^2}\left|(\widetilde{\theta}_k^{\pm}-\lambda)\int_{\mathbb{R}}\frac{g^+(s)d\Sigma_{d+}(s)}{s-\lambda}\right|^2 d\eta\leqslant\pi C_7\|g^+\|^2_{L^2(d\Sigma_+)}. \tag{6.30}$$

Using (6.30) and (6.29), we get

$$\int\limits_{\substack{\eta\in\mathbb{R}\\ \eta+i\varepsilon\in D_k^{\pm}}} \frac{1}{|M_+(\lambda)-M_-(\lambda)|^2}\,\mathrm{Im}\,\frac{1}{\widetilde{\theta}_k^{\pm}-\lambda}\left|\int_{\mathbb{R}}\frac{g^+(s)d\Sigma_{d+}(s)}{s-\lambda}\right|^2 d\eta\leqslant\pi C_\theta^M C_7\|g^+\|^2_{L^2(d\Sigma_+)}\,. \tag{6.31}$$

The inequality

$$\int\limits_{\substack{\eta\in\mathbb{R}\\ \eta+i\varepsilon\in D_k^{\pm}}} \frac{1}{|M_+(\lambda)-M_-(\lambda)|^2}\,\mathrm{Im}\,\frac{1}{\widetilde{\theta}_k^{\pm}-\lambda}\left|\int_{\mathbb{R}}\frac{g^+(s)d\Sigma_{ac+}(s)}{s-\lambda}\right|^2 d\eta\leqslant C_9\|g^+\|^2_{L^2(d\Sigma_+)} \tag{6.32}$$

follows from (6.15), (6.16), and (6.13) in the same way as (6.25) follows from (6.9), (6.11), and (6.13).

Combining (6.32), (6.31) and (6.28), we arrive at (6.27). Thus (6.26) is proved. So, inequalities (6.3) are proved and Proposition 6.1 yields similarity of A_{ess} to a selfadjoint operator. □

7. Indefinite Sturm-Liouville operators with finite-zone potentials

7.1. Spectral properties of A_{ess} and A_{disc}

Throughout this section $L = -d^2/dx^2 + q(x)$ is a Sturm-Liouville operator with a finite-zone potential q. We keep notations from Section 2.4.. In particular, $\{\xi_j\}_1^N$ and $\{\tau_j\}_0^N$ stand for the sets of zeros of the polynomials $P(\cdot)$ and $S(\cdot)$ (see (2.16) and (2.17)) respectively.

It follows from the form of the Weyl functions (2.20), (2.21) that in the case under consideration the spectra of the operators $A_0^{\pm}$ defined by (2.12) and (2.3), are described as follows

$$\sigma_{\mathrm{ess}}(A_0^{\pm}) = \sigma_{ac}(A_0^{\pm}), \qquad \sigma(A_0^{\pm}) = \sigma_{ac}(A_0^{\pm}) \cup \sigma_{\mathrm{disc}}(A_0^{\pm}), \tag{7.1}$$

$$\sigma_{ac}(A_0^{+}) = -\sigma_{ac}(A_0^{-}) = \sigma(L) = [\overset{r}{\mu}_0, \overset{l}{\mu}_1] \cup [\overset{r}{\mu}_1, \overset{l}{\mu}_2] \cup \cdots \cup [\overset{r}{\mu}_N, +\infty), \tag{7.2}$$

$$\sigma_{\mathrm{disc}}(A_0^{\pm}) = \{\pm\tau_j : \tau_j \notin \{\overset{r}{\mu}_k\}_0^N \cup \{\overset{l}{\mu}_k\}_1^N,\ Q(\tau_j) \pm i\sqrt{R(\tau_j)} \neq 0\} =: \{\theta_k^{\pm}\}_1^{N_\theta^{\pm}}. \tag{7.3}$$

It follows from (2.18), (2.20) and (2.21) that $\sigma_{\mathrm{disc}}(A_0^{\pm})$ can equivalently be described as

$$\sigma_{\mathrm{disc}}(A_0^{\pm}) = \{\pm\tau_j : \tau_j \notin \{\overset{r}{\mu}_k\}_0^N \cup \{\overset{l}{\mu}_k\}_1^N,\ Q(\tau_j) \mp i\sqrt{R(\tau_j)} = 0\} =: \{\theta_k^{\pm}\}_1^{N_\theta^{\pm}}. \tag{7.4}$$

Let $a = \bar{a}$ be an (isolated) algebraic branch point of order $n \geqslant 1$ for $M(\cdot)$ (for $n = 1$ we assume that a is a either a regular or a singular point of a single-valued function $M(\cdot)$). Then in a small deleted neighborhood of a the (multi-valued) function $M(\cdot)$ admits the following expansion

$$M(z) = \sum_{k=-\infty}^{+\infty} m_k (z-a)^{k/n}, \qquad 0 < |z-a| < r.$$

DEFINITION 7.1. Let $a = \bar{a}$ be a branch point of order $n \in \mathbb{N}$ for $M(\cdot)$ and let

$$p := \inf\{k : m_k \neq 0\} \qquad (p := -\inf\{k : m_k \neq 0\}).$$

We say that a is *a generalized zero (generalized pole)* for $M(\cdot)$ of the *generalized order* p/n.

We emphasize that according to Definition 7.1 the generalized order of a generalized zero (pole) may be negative (resp. positive).

Recall that the functions $M_{\pm}(\lambda)$ are holomorphic in $\rho(A_0^{\pm})$. For $\eta \in \sigma(A_0^{\pm})$, we set $M_{\pm}(\eta) := M_{\pm}(\eta + i0)$. It is known that in the case of a finite-zone potential q the Weyl functions $M_{\pm}(\lambda)$ admit meromorphic continuations on the Riemannian surfaces of the multifunction $\sqrt{R(\cdot)}$. Let us denote these continuations by $\widehat{M}_{\pm}(\lambda)$. Then

$\{\pm\overset{r}{\mu}_j\}_0^N \cup \{\pm\overset{l}{\mu}_j\}_1^N$ are the sets of branch points for $\widehat{M}_{\pm}(\lambda)$;

$\{\pm\xi_j\}_1^N \cap (\{\pm\overset{r}{\mu}_j\}_0^N \cup \{\pm\overset{l}{\mu}_j\}_1^N)$ are the sets of zeroes of the generalized order $1/2$ for $\widehat{M}_{\pm}(\lambda)$;

$\{\pm\tau_j\}_0^N \cap (\{\pm\overset{r}{\mu}_j\}_0^N \cup \{\pm\overset{l}{\mu}_j\}_1^N)$ are the sets of poles of the generalized order $1/2$ for $\widehat{M}_\pm(\lambda)$;

$\{\theta_j^\pm\}_1^{N_\theta^\pm}$ are the sets of poles of the first order for $\widehat{M}_\pm(\lambda)$;

$\{\pm\xi_j : \xi_j \notin \{\overset{r}{\mu}_k\}_0^N \cup \{\overset{l}{\mu}_k\}_1^N,\ Q(\xi_j) \mp i\sqrt{R(\xi_j)} \neq 0\}$ are the sets of zeroes of the first order for $\widehat{M}_\pm(\lambda)$.

We denote by $\overset{*}{M}_\pm(\cdot)$ the meromorphic continuations of $M_\pm(\cdot)$ from $\mathbb{C}_+$ on the domain

$$\mathbb{C} \setminus \left\{\lambda :\ \operatorname{Im}\lambda \leqslant 0,\ \operatorname{Re}\lambda \in \{\pm\overset{r}{\mu}_j\}_0^N \cup \{\pm\overset{l}{\mu}_j\}_1^N\right\}. \tag{7.5}$$

In other words, we consider the multifunction $\sqrt{R(\cdot)}$ on the complex plane $\mathbb{C}$ with cuts along vertical half-lines in $\mathbb{C}_-$ with vertexes at the points $\{\pm\overset{r}{\mu}_j\}_0^N \cup \{\pm\overset{l}{\mu}_j\}_1^N$, and choose the branch of $\sqrt{R(\cdot)}$ in such a way that $\sqrt{R(x_0 + i0)} > 0$ for some $x_0 \in (\overset{r}{\mu}_N, +\infty)$. Then we define $\overset{*}{M}_\pm(\cdot)$ by formulas (2.20) in the domain (7.5). Note, that both functions have non-negative imaginary parts in $\mathbb{C}_+$ and do not preserve Nevanlinna property in $\mathbb{C}_-$.

Note that that both Definition 7.1 and definition of $\overset{*}{M}_\pm(\cdot)$ are substantially used in the statement of Theorem 7.2 and in its proof as well.

THEOREM 7.1. *Let $L = -d^2/dx^2 + q(x)$ be a Sturm-Liouville operator with a finite-zone potential q and $A = (\operatorname{sgn} x)(-d^2/dx^2 + q(x))$. Then:*

(1) *The operator A has finite number of eigenvalues,*

$$\sigma_p(A) = \left(\{\theta_j^+\}_1^{N_\theta^+} \cap \{\theta_j^-\}_1^{N_\theta^-}\right) \cup \{\lambda \in \rho(A_2^+ \oplus A_2^-) : M_+(\lambda) = M_-(\lambda)\}. \tag{7.6}$$

(2) *The eigenvalues of A are isolated and have finite algebraic multiplicities. The geometric multiplicity of any eigenvalue is one.*

(3) *If $\lambda_0 \in \rho(A_0^+) \cap \rho(A_0^-)$, then the algebraic multiplicity of λ_0 is equal to the multiplicity of λ_0 as a zero of the holomorphic function $M_+(\cdot) - M_-(\cdot)$; if $\lambda_0 \in \{\theta_j^+\}_1^{N_\theta^+} \cap \{\theta_j^-\}_1^{N_\theta^-}$, then the algebraic multiplicity of λ_0 is equal to the multiplicity of λ_0 as a zero of the holomorphic function $M_+^{-1}(\cdot) - M_-^{-1}(\cdot)$.*

(4) *There exist a direct skew decomposition $L^2(\mathbb{R}) = \mathfrak{H}_e \dot{+} \mathfrak{H}_d$ such that*

$$A = A_{\text{ess}} \dot{+} A_{\text{disc}}, \quad A_{\text{ess}} = A \upharpoonright (\operatorname{dom}(A) \cap \mathfrak{H}_e), \quad A_{\text{disc}} = A \upharpoonright (\operatorname{dom}(A) \cap \mathfrak{H}_d),$$
$$\sigma(A_{\text{disc}}) = \sigma_{\text{disc}}(A), \quad \sigma(A_{\text{ess}}) = \sigma_{\text{ess}}(A). \tag{7.7}$$

Besides, $\mathfrak{H}_d$ is a finite-dimensional space.

Proof. It follows from (2.20) that the spectral functions $\Sigma_\pm(\cdot)$ of the operators $A_0^\pm$ have the following forms $\Sigma_\pm(t) = \Sigma_{ac\pm}(t) + \Sigma_{d\pm}(t)$, where

$$\Sigma'_{ac\pm}(t) = \begin{cases} \frac{\sqrt{R(\pm t)}}{S(\pm t)}, & t \in \pm \bigcup_{j=0}^{N} (\overset{r}{\mu}_j, \overset{l}{\mu}_{j+1}) \\ 0, & t \notin \pm \bigcup_{j=0}^{N} [\overset{l}{\mu}_j, \overset{r}{\mu}_j] \end{cases}. \tag{7.8}$$

Here the branch of multifunction $\sqrt{R(\pm\lambda)}$ is chosen such as it is explained in Subsection 2.4..

Consequently,

$$\Sigma'_{ac\pm}(t) \asymp 1 \quad (t \to t_0), \quad t_0 \in \pm \bigcup_{j=0}^{N} (\overset{r}{\mu}_j, \overset{l}{\mu}_{j+1}), \tag{7.9}$$

$$\Sigma'_{ac\pm}(t) \asymp |t - t_0|^{1/2} \chi_\pm(t - t_0), \quad t \to t_0, \quad t_0 \in \{\pm\overset{r}{\mu}_j\}_0^N \setminus \{\pm\tau_j\}_0^N, \tag{7.10}$$

$$\Sigma'_{ac\pm}(t) \asymp |t - t_0|^{1/2} \chi_\mp(t - t_0), \quad t \to t_0, \ t_0 \in \{\pm\overset{l}{\mu}_j\}_1^N \setminus \{\pm\tau_j\}_0^N, \tag{7.11}$$

$$\Sigma'_{ac\pm}(t) \asymp |t - t_0|^{-1/2} \chi_\pm(t - t_0), \quad t \to t_0, \ t_0 \in \{\pm\overset{r}{\mu}_j\}_0^N \cap \{\pm\tau_j\}_0^N, \tag{7.12}$$

$$\Sigma'_{ac\pm}(t) \asymp |t - t_0|^{-1/2} \chi_\mp(t - t_0), \quad t \to t_0, \ t_0 \in \{\pm\overset{l}{\mu}_j\}_1^N \cap \{\pm\tau_j\}_0^N. \tag{7.13}$$

Therefore,

$$\int\limits_{\mathbb{R}\setminus\{\eta_0\}} \frac{1}{|t-\eta_0|^2} d\Sigma_\pm(t) = \infty, \qquad \eta_0 \in \pm\left(\cup_{j=0}^{N} [\overset{r}{\mu}_j, \overset{l}{\mu}_{j+1}] \cup [\overset{r}{\mu}_N, +\infty) \right).$$

Combining this with Theorem 4.2 (1), we get

$$\sigma_p(A) \subset \mathbb{C} \setminus (\sigma_{ac}(A_2^+) \cup \sigma_{ac}(A_2^-)) = \mathbb{C} \setminus \sigma_{\mathrm{ess}}(A).$$

Thus, Proposition 4.3 yields (7.6).

Taking (2.20) into account we rewrite the equation $M_+(\lambda) = M_-(\lambda)$ in the form

$$\frac{P(\lambda)}{Q(\lambda) - i\sqrt{R(\lambda)}} = \frac{P(-\lambda)}{Q(-\lambda) + i\sqrt{R(-\lambda)}},$$

where P, Q, and R are polynomials. Thus the equation $M_+(\lambda) = M_-(\lambda)$ has a finite number of zeros. Therefore the set $\sigma_p(A)$ is finite. Statement (1) is proved.

Statements (2) and (3) follow from Statement (1) and Proposition 4.3. Statement (4) follows from statements (1), (2), and (6.1). $\square$

THEOREM 7.2. *Let $L = -d^2/dx^2 + q(x)$ be a Sturm-Liouville operator with a finite-zone potential, let $A = (\operatorname{sgn} x)(-d^2/dx^2 + q(x))$. Then the following statements are equivalent:*

(i) *The operator A_{ess} is similar to a selfadjoint operator;*

(ii) *The following conditions are satisfied*

$$\frac{\operatorname{Im} M_\pm(t)}{M_+(t) - M_-(t)} \in L^\infty(\mathbb{R}); \tag{7.14}$$

(iii) *The function $\overset{*}{M}_+(\lambda) - \overset{*}{M}_-(\lambda)$ has no generalized zeroes in*

$$(-\infty, -\overset{r}{\mu}_N) \cup (-\overset{l}{\mu}_N, -\overset{r}{\mu}_{N-1}) \cup \cdots \cup (-\overset{l}{\mu}_1, -\overset{r}{\mu}_0) \cup$$
$$\cup\, (\overset{r}{\mu}_0, \overset{l}{\mu}_1) \cup (\overset{r}{\mu}_1, \overset{l}{\mu}_2) \cup \cdots \cup (\overset{r}{\mu}_N, +\infty)\ ,$$

has no zeroes of the generalized order more than $1/2$ *in the set*

$$\left((\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N) \setminus \{\tau_j\}_0^N \right) \cup \left((\{-\overset{r}{\mu}_j\}_0^N \cup \{-\overset{l}{\mu}_j\}_1^N) \setminus \{-\tau_j\}_0^N \right),$$

has poles of generalized order greater than or equal to $1/2$ *at the points of the set*

$$\left((\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N) \cap \{\tau_j\}_0^N \right) \cup \left((\{-\overset{r}{\mu}_j\}_0^N \cup \{-\overset{l}{\mu}_j\}_1^N) \cap \{-\tau_j\}_0^N \right).$$

Combining Theorem 7.2 with Corollary 3.5 we arrive at the following result.

COROLLARY 7.3. *Under the conditions (7.14) the operator* A_{ess} *is similar to a selfadjoint operator with absolutely continuous spectrum.*

Proof. Consider the decomposition (6.1) and note that the subspace $\mathfrak{H}_e$ in (6.1) is invariant for the operator A, $\mathfrak{H}_e \in \operatorname{Lat} A$. Alongside the skew decomposition (6.1) we consider the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_e \oplus \mathfrak{H}_e^{\perp}$. According to this decomposition the characteristic function $\theta_A(\cdot)$ of the operator A admits the factorization $\theta_A(\lambda) = \theta_1(\lambda) \cdot \theta_2(\lambda)$ where $\theta_1(\cdot)$ is the characteristic function of the operator $A_{\text{ess}} = A\lceil \mathfrak{H}_e$ and $\theta_2(\cdot)$ is the characteristic function of the operator $A_2 := P_2 A \lceil \mathfrak{H}_e^{\perp}$, where P_2 is the orthoprojection in $\mathfrak{H}$ onto $\mathfrak{H}_e^{\perp}$. Note, that $\theta_2(\cdot) = \theta_{A_2}(\cdot)$ is a finite Blaschke product since $\sigma(A_{\text{disc}})$ is finite.

It follows from (2.21) that $M_+(\cdot)$ (resp. $M_-(\cdot)$) admits a continuous extension $M_+(\cdot + i0)$ (resp. $M_+(\cdot + i0)$ from $\mathbb{C}_+$ to the real line with exception of the set of (real) zeros $\{\tau_k\}_0^N$ (resp. $\{-\tau_k\}_0^N$) of the polynomial $S(\lambda)$ (resp. $S(-\lambda)$). Moreover, it is clear from the formula (3.21) for the characteristic function $\theta_A(\cdot)$, that the real singularities (resp. poles) of $\theta_A(\cdot)$ coincide with the set of real (resp. non-real) roots of the function

$$F(\lambda) = P(\lambda)Q(-\lambda) + iP(\lambda)\sqrt{R(-\lambda)} - P(-\lambda)Q(\lambda) + iP(-\lambda)\sqrt{R(\lambda)}. \tag{7.15}$$

In particular, the numbers of real singularities and poles of $\theta_A(\cdot)$ are finite.

Note, that $A_2^* = A^* \lceil \mathfrak{H}_\theta^{\perp}$ and $\theta_2^{-1}(\lambda) = \theta_{A_2^*}(\lambda)$ is a finite Blaschke product too. Therefore the sets of real singularities of functions $\theta(\cdot)$ and $\theta_1(\cdot) = \theta(\cdot) \cdot \theta_2^{-1}(\cdot)$ coincide. In particular, $\theta_1(\cdot)$ may have only finite number of singularities. To apply Proposition 3.4 it suffices to note that the operator A is completely nonselfadjoint (see the proof of Corollary 3.10) because the minimal operator $A_{\min}$ is simple. Therefore, combining Theorem 7.2 with Proposition 3.4 we obtain that A_{ess} is similar to a selfadjoint operator with absolutely continuous spectrum. $\square$

COROLLARY 7.4. *Let* $L = -d^2/dx^2 + q(x)$ *be a nonnegative Sturm-Liouville operator with a finite-zone potential* q. *Then the operator* $A = (\operatorname{sgn} x)L$ *is similar to a selfadjoint operator, the operator* A_{ess} *is similar to a selfadjoint operator with absolutely continuous spectrum.*

Proof. By (7.8), we have $\operatorname{supp} d\Sigma_{ac\pm}(t) \subset \mathbb{R}_\pm$ and $\operatorname{Im} M_\pm(t) = \pi\Sigma'_{ac\pm}(t)$ for almost all $t \in \mathbb{R}$. Therefore,

$$\frac{|\Sigma'_{ac\pm}(t)|^2}{|M_+(t) - M_-(t)|^2} \leqslant \frac{|\Sigma'_{ac\pm}(t)|^2}{\pi^2|\Sigma'_{ac\pm}(t)|^2 + |\operatorname{Re} M_+(t) - \operatorname{Re} M_-(t)|^2} \leqslant \frac{1}{\pi^2},$$

for almost all $t \in \mathbb{R}$. Thus, by Theorem 7.2, A_{ess} is similar to a selfadjoint operator.

The operator A is J–nonnegative. Besides, L has absolutely continuous spectrum (see, for example, [44]). Hence, $\ker L = 0$. Combining this fact with Proposition 2.2, we get that all the eigenvalues of A are real and simple. Therefore, by Theorem 7.1, the operator A_{disc} is similar to a selfadjoint operator. Thus, by Corollary 7.3, A is similar to a selfadjoint operator and A_{ess} is similar to a selfadjoint operator with absolutely continuous spectrum. $\square$

7.2. Proof of Theorem 7.2

The implication $(i) \Rightarrow (ii)$ follows from Proposition 6.2.

$(ii) \Rightarrow (iii)$. It follows from (7.10) and (7.14) that there are no generalized zeroes of the function $\overset{*}{M}_+(\lambda) - \overset{*}{M}_-(\lambda)$ in the set $\cup_{j=0}^N(\overset{r}{\mu}_j, \overset{l}{\mu}_{j+1})$. Likewise, it follows from (7.10) and (7.14) that there are no generalized zeroes of the function $\overset{*}{M}_+(\lambda) - \overset{*}{M}_-(\lambda)$ in the set $\cup_{j=0}^N(-\overset{l}{\mu}_{j+1}, -\overset{r}{\mu}_j)$.

It follows from (7.11), (7.12), and (7.14) that the function $\overset{*}{M}_+(\lambda) - \overset{*}{M}_-(\lambda)$ has no zeroes of generalized order greater than $1/2$ in the sets

$$(\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N) \setminus \{\tau_j\}_0^N \quad \text{and} \quad (\{-\overset{r}{\mu}_j\}_0^N \cup \{-\overset{l}{\mu}_j\}_1^N) \setminus \{-\tau_j\}_0^N.$$

It follows from (7.13), (7.13), (7.14), and (7.14) that all the points of the sets

$$(\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N) \cap \{\tau_j\}_0^N \quad \text{and} \quad (\{-\overset{r}{\mu}_j\}_0^N \cup \{-\overset{l}{\mu}_j\}_1^N) \cap \{-\tau_j\}_0^N$$

are generalized poles of $\overset{*}{M}_+(\lambda) - \overset{*}{M}_-(\lambda)$. The generalized orders of these poles are greater than or equal to $1/2$.

$(iii) \Rightarrow (i)$. By Theorem 7.1 (1), Assumption (6.1) is fulfilled for the operator A. It follows from (7.1) that we can apply Theorem 6.3.

Let Statement (iii) be fulfilled. We construct the functions $U_\pm(\lambda)$, $w_\pm(t)$, $U_j^\pm$ and the sets G_d, $D_j^\pm$ such that all the conditions of Theorem 6.3 hold true.

Let G_d be any compact set such that $\sigma_{\mathrm{ess}}(A) \cap G_d = \emptyset$ and all the points of the set $\sigma_{\mathrm{disc}}(A)$ are interior points of G_d.

The set $\sigma_{\mathrm{disc}}(A) \cap \mathbb{C}_+$ is finite. Besides,

$$\sigma_{\mathrm{disc}}(A) \cap \mathbb{C}_+ = \{\lambda \in \mathbb{C}_+ \ : \ M_+(\lambda) - M_-(\lambda) = 0\}.$$

Let $B_{\mathbb{C}}(\lambda)$ be a finite Blaschke product (see [19]) with the same zeroes in $\mathbb{C}_+$ as $M_+(\lambda) - M_-(\lambda)$. Then $M_+(\lambda) - M_-(\lambda) = B_{\mathbb{C}}(\lambda) M_1(\lambda)$, the function $M_1(\lambda)$ being holomorphic on $\mathbb{C}_+$. Besides,

$$M_1(\cdot)^{-1} \in \operatorname{Hol}(\mathbb{C}_+) \quad \text{and} \quad M_1(\lambda) \asymp (M_+(\lambda) - M_-(\lambda)), \quad \lambda \in \mathbb{C}_+ \setminus G_d^+, \quad (7.16)$$

where G_d^+ is any compact subset of $G_d \cap \mathbb{C}_+$ such that all the points of the set $\sigma_{\mathrm{disc}}(A) \cap \mathbb{C}_+$ are interior points of G_d^+.

The set $\sigma_{\mathrm{disc}}(A_2^+) \cap \sigma_{\mathrm{disc}}(A_2^-) = \{\theta_j^+\}_1^{N_\theta^+} \cap \{\theta_j^-\}_1^{N_\theta^-}$ is finite. By Theorem 7.1, this set is a subset of $\sigma_{\mathrm{disc}}(A)$. Let

$$\{\theta_j\}_{j=1}^{N_\theta} := \sigma_{\mathrm{disc}}(A_2^+) \cap \sigma_{\mathrm{disc}}(A_2^-), \quad N_\theta < \infty.$$

Each point of the set $\{\theta_j\}_{j=1}^{N_\theta}$ is either a pole of the first order or a removable singularity of the function $M_+(\lambda) - M_-(\lambda)$. By κ_j denote the generalized order of a zero of $M_+(\lambda) - M_-(\lambda)$ at θ_j. Then $\kappa_j \in \{-1, 0\} \cup \mathbb{N}$, $j \in \{1, \ldots, N_\theta\}$. By Theorem 7.1, we have

$$\{\widetilde{\theta}_j^\pm\}_1^{\widetilde{N}_\theta^\pm} = \{\theta_j^\pm\}_1^{N_\theta^\pm} \setminus \{\theta_j\}_1^{N_\theta}$$

(the sets $\{\widetilde{\theta}_j^\pm\}_1^{\widetilde{N}_\theta^\pm}$ are defined by (6.8)).

Put

$$\{\widetilde{\theta}_j\}_1^{\widetilde{N}_\theta} = (\mathbb{R} \cap \sigma_{\mathrm{disc}}(A)) \setminus \{\theta_j\}_1^{N_\theta}.$$

The functions $M_\pm(\lambda)$ are regular at $\widetilde{\theta}_j$ and $M_+(\widetilde{\theta}_j) - M_-(\widetilde{\theta}_j) = 0$, $j \in \{1, \ldots, \widetilde{N}_\theta\}$. Let us denote generalized order of $\widetilde{\theta}_j$ as a zero of $M_+(\lambda) - M_-(\lambda)$ by $\widetilde{\kappa}_j$ (clearly, $\widetilde{\kappa}_j \in N$).

Put $M_2(\lambda) := M_1(\lambda)/B_\theta$, where

$$B_\theta(\lambda) := \frac{\prod_{j=1}^{N_\theta}(\lambda - \theta_j)^{\kappa_j+1}}{\prod_{j=1}^{N_\theta}(\lambda - (\theta_j - i\varepsilon_1))^{\kappa_j+1}} \; \frac{\prod_{j=1}^{\widetilde{N}_\theta}(\lambda - \widetilde{\theta}_j)^{\widetilde{\kappa}_j}}{\prod_{j=1}^{\widetilde{N}_\theta}\left(\lambda - (\widetilde{\theta}_j - i\varepsilon_1)\right)^{\widetilde{\kappa}_j}}.$$

Here and below ε_1 is an arbitrary fixed positive number. Taking (7.16) into account, we get

$$M_2(\cdot)^{-1} \in \mathrm{Hol}(\mathbb{C}_+) \quad \text{and} \quad M_2(\lambda) \asymp (M_+(\lambda) - M_-(\lambda)), \quad \lambda \in \mathbb{C}_+ \setminus G_d. \quad (7.17)$$

Denote

$$\rho_1 := \rho(L) \cup \rho(-L).$$

If λ_0 is a generalized zero of $\overset{*}{M}_+(\lambda) - \overset{*}{M}_-(\lambda)$ and $\lambda_0 \in \rho_1$, then $\lambda_0 \in \{\theta_j\}_{j=1}^{N_\theta} \cup \{\widetilde{\theta}_j\}_{j=1}^{\widetilde{N}_\theta}$. Moreover, it follows from Statement (iii) that the function $\overset{*}{M}_+(\lambda) - \overset{*}{M}_-(\lambda)$ has no generalized zeroes in the set $\sigma_1 := \sigma_1^+ \cup \sigma_1^-$, where

$$\sigma_1^\pm := \pm \bigcup_{j=0}^{N} (\overset{r}{\mu}_j, \overset{l}{\mu}_{j+1})$$

are the sets of interior points of the spectra $\sigma(\pm L)$. Therefore the definition of B_θ implies that

$$M_2^{-1}(\lambda) = O(1), \qquad \lambda \to \lambda_0, \qquad \lambda_0 \in \rho_1 \cup \sigma_1, \tag{7.18}$$
$$M_2(\lambda) \asymp (\theta_j^\pm - \lambda)^{-1}, \qquad \lambda \to \theta_j^\pm, \qquad j \in \{1, \dots, N_\theta^\pm\}. \tag{7.19}$$

Let us explain formula (7.19). If $\theta_j^\pm \in \{\theta_j\}_1^{N_\theta}$, the formula (7.19) follows from the definition of the function B_θ. If $\theta_j^\pm \in \{\widetilde{\theta}_j^\pm\}_1^{\widetilde{N}_\theta^\pm}$, the asymptotics

$$M_\pm(\lambda) \asymp (\widetilde{\theta}_j^\pm - \lambda)^{-1}, \quad M_\mp(\lambda) = (\widetilde{\theta}_j^\pm - \lambda)^{-1/2} \cdot O(1), \qquad \lambda \to \widetilde{\theta}_j^\pm,$$

and (7.16) imply (7.19).

Let us denote

$$\{\zeta_j^\pm\}_1^{N_\zeta^\pm} := \left\{ z \in \{\pm \overset{r}{\mu}_j\}_0^N \cup \{\pm \overset{l}{\mu}_j\}_1^N \ : \ z \text{ is a generalized zero of } \overset{*}{M}_+(\lambda) - \overset{*}{M}_-(\lambda) \right\}.$$

By Statement (iii), the generalized orders of all the zeroes $\zeta_j^\pm$ are equal to $1/2$. It follows from Statement (iii) and asymptotics for $M_\pm(\lambda)$ that

$$\{\zeta_j^\pm\} \subset (\{\pm \overset{r}{\mu}_j\}_0^N \cup \{\pm \overset{l}{\mu}_j\}_1^N) \setminus (\{\theta_j^\mp\}_1^{N_\theta^\mp} \cup \sigma_1). \tag{7.20}$$

Denote

$$\{\zeta_j\}_1^{N_\zeta} := \{\zeta_j^+\}_1^{N_\zeta^+} \cap \{\zeta_j^-\}_1^{N_\zeta^-}, \quad \{\widetilde{\zeta_j^\pm}_{\ 1}\}^{\widetilde{N_\zeta^\pm}} := \{\zeta_j^\pm\}_1^{N_\zeta^\pm} \setminus \{\zeta_j\}_1^{N_\zeta}. \tag{7.21}$$

Statement (iii) implies $\widetilde{\zeta_j^\pm} \notin \sigma_1$. Besides,

$$\widetilde{\zeta_j^\pm} \notin \{\zeta_j\}_1^{N_\zeta} = \{\zeta_j^\pm\}_1^{N_\zeta^\pm} \cap (\{\mp \overset{r}{\mu}_j\}_0^N \cup \{\mp \overset{l}{\mu}_j\}_1^N),$$

therefore $\widetilde{\zeta_j^\pm} \in \rho_1^\mp$, where

$$\rho_1^\pm := \pm \rho(L) \ (= \pm \bigcup_{j=0}^N (\overset{l}{\mu}_j, \overset{r}{\mu}_j)) \ .$$

Put

$$u_\pm(\lambda) := \frac{\sqrt{R(\pm\lambda)}}{S(\pm\lambda)} \; \frac{\prod_{j=1}^{N_\theta^\pm} (\lambda - \theta_j^\pm)}{\prod_{j=1}^{N_\theta^\pm} (\lambda - (\theta_j^\pm - i\varepsilon_1))} \; \frac{\prod_{j=1}^{\widetilde{N}_\zeta^\pm} (\lambda - \widetilde{\zeta}_j^\mp)}{\prod_{j=1}^{\widetilde{N}_\zeta^\pm} (\lambda - (\widetilde{\zeta}_j^\mp - i\varepsilon_1))} \, . \tag{7.22}$$

Now we define $U_\pm$ by

$$U_\pm := \frac{\sqrt{u_\pm(\lambda)}}{M_2(\lambda)}.$$

Let us check conditions (6.9), (6.10), and (6.11). All the asymptotics given below are considered on $\overline{\mathbb{C}_+}$, unless otherwise is specified.

LEMMA 7.5. *Let Statement (iii) be true. Then condition (6.9) is fulfilled, i.e.,*

$$\frac{\operatorname{Im} M_{ac\pm}(\lambda)}{|M_+(\lambda)-M_-(\lambda)|^2} \leqslant C^u_{\pm}|U_{\pm}(\lambda)|^2, \quad \lambda \in \mathbb{C}_+ \setminus G_d\,.$$

Proof. By (7.17), condition (6.9) is equivalent to

$$\operatorname{Im} M_{ac\pm}(\lambda) = O(1)\, u_{\pm}(\lambda), \qquad \lambda \in \overline{\mathbb{C}_+ \setminus G_d}. \tag{7.23}$$

Since

$$M_{ac\pm}(\lambda) \asymp M_{\pm}(\lambda) \asymp |\lambda|^{-1/2}, \qquad \lambda \to \infty \tag{7.24}$$

and

$$u_{\pm}(\lambda) \asymp |\lambda|^{-1/2}, \quad \lambda \to \infty, \tag{7.25}$$

we have

$$\frac{\operatorname{Im} M_{ac\pm}(\lambda)}{u_{\pm}(\lambda)} = O(1), \quad \lambda \to \infty.$$

If $\lambda_0 \in \overline{(\mathbb{C}_+ \setminus G_d)} \setminus \left(\{\pm\overset{r}{\mu}_j\}_0^N \cup \{\pm\overset{l}{\mu}_j\}_1^N \cup \{\widetilde{\zeta_j^{\mp}}\}_1^{\widetilde{N_\zeta^{\mp}}}\right)$, then

$$\operatorname{Im} M_{ac\pm}(\lambda) = O(1), \quad \lambda \to \lambda_0, \qquad u_{\pm}(\lambda) \asymp 1, \quad \lambda \to \lambda_0.$$

Let $\lambda_0 \in (\{\pm\overset{r}{\mu}_j\}_0^N \cup \{\pm\overset{l}{\mu}_j\}_1^N) \setminus \{\pm\tau_j\}_0^N$. Then (2.21) yields

$$\operatorname{Im} M_{ac\pm}(\lambda) \asymp \operatorname{Im} M_{\pm}(\lambda) = O(|\lambda-\lambda_0|^{1/2}), \quad \lambda \to \lambda_0;$$

besides,

$$u_{\pm}(\lambda) \asymp |\lambda-\lambda_0|^{1/2}, \quad \lambda \to \lambda_0.$$

Let $\lambda_0 \in (\{\pm\overset{r}{\mu}_j\}_0^N \cup \{\pm\overset{l}{\mu}_j\}_1^N) \cap \{\pm\tau_j\}_0^N$. Then (2.21) yields

$$\operatorname{Im} M_{ac\pm}(\lambda) \asymp \operatorname{Im} M_{\pm}(\lambda) = O(|\lambda-\lambda_0|^{-1/2}), \quad \lambda \to \lambda_0;$$

besides,

$$u_{\pm}(\lambda) \asymp |\lambda-\lambda_0|^{-1/2}, \quad \lambda \to \lambda_0.$$

Let $\lambda_0 \in \{\widetilde{\zeta_j^{\mp}}\}_0^{\widetilde{N_\zeta^{\mp}}}$. Then (7.20) and (2.9) yield

$$\operatorname{Im} M_{ac\pm}(\lambda) \asymp \operatorname{Im} M_{\pm}(\lambda) = O(\operatorname{Im}\lambda) = O(\lambda-\lambda_0), \quad \lambda \to \lambda_0\,.$$

On the other hand,

$$u_{\pm}(\lambda) \asymp |\lambda-\lambda_0|, \quad \lambda \to \lambda_0.$$

If we combine all these estimates, we get (7.23). Thus (6.9) is proved. □

LEMMA 7.6. *Condition (6.10) is fulfilled, i.e.,* $U_{\pm}(\lambda) \in \mathcal{N}^+(C_+)$.

Proof. The functions $U_\pm(\lambda)$ are holomorphic on $\mathbb{C}_+$ by definition. Since

$$M_+(\lambda) - M_-(\lambda) \asymp |\lambda|^{-1/2}, \quad \lambda \to \infty,$$

then (7.25) implies

$$U_\pm(\lambda) \asymp |\lambda|^{1/4}, \qquad \lambda \to \infty. \tag{7.26}$$

Condition (6.10) follows from (7.26) and Lemma 2.11. □

LEMMA 7.7. *Let Statement (iii) be true. Then condition (6.11) is fulfilled, i.e.,*

$$\frac{U_\pm(t)}{\theta_j^- - t} \in L^2(\mathbb{R}),\ j \in \{1, \cdots, N_\theta^-\}; \qquad \frac{U_\pm(t)}{\theta_j^+ - t} \in L^2(\mathbb{R}),\ j \in \{1, \cdots, N_\theta^+\}.$$

Proof. The definition of the polynomial $S(\lambda)$ yields

$$|u_\pm(\lambda)| \ = \ \frac{\prod\limits_{(\pm\lambda_0)\in(\{\overset{r}{\mu}_j\}_0^N\cup\{\overset{l}{\mu}_j\}_1^N)\setminus\{\tau_j\}_0^N} |\lambda-\lambda_0|^{1/2}}{\prod\limits_{(\pm\lambda_0)\in\{\tau_j\}_0^N\cap(\{\overset{r}{\mu}_j\}_0^N\cup\{\overset{l}{\mu}_j\}_1^N)} |\lambda-\lambda_0|^{1/2} \prod\limits_{j=1}^{N_\theta^\pm} |\lambda-(\theta_j^\pm - i\varepsilon_1)|} \cdot$$

$$\cdot\frac{\prod\limits_{j=1}^{\widetilde{N}_\zeta^\mp} |\lambda-\widetilde{\zeta}_j^\mp|}{\prod\limits_{j=1}^{\widetilde{N}_\zeta^\mp} |t-(\widetilde{\zeta}_j^\mp - i\varepsilon_1)|^{1/2}}. \tag{7.27}$$

It follows from (7.27), (7.18), (7.22), (7.19), Statement (iii), and the definition of $\{\widetilde{\zeta}_j^\mp\}_1^{\widetilde{N}_\zeta^\mp}$ that

$$U_\pm(\lambda) = O(1), \quad \lambda \to \lambda_0, \quad \lambda_0 \in \sigma_1^+ \cup \sigma_1^- \cup \rho_1^\pm \tag{7.28}$$

$$U_\pm(\lambda) \asymp (\lambda - \theta_j^\pm), \quad \lambda \to \theta_j^\pm, \quad j \in \{1, \ldots, N_\theta^\pm\}, \tag{7.29}$$

$$U_\pm(\lambda) = O(1)\,|\lambda-\theta_j^\mp|^{3/4}, \quad \lambda \to \theta_j^\mp, \quad j \in \{1, \ldots, N_\theta^\mp\}, \tag{7.30}$$

$$U_\pm(\lambda) = O(|\lambda-\lambda_0|^{-1/4}),\ \lambda \to \lambda_0,\ \lambda_0 \in \left(\{\pm\overset{r}{\mu}_j\}_{j=0}^N \cup \{\pm\overset{l}{\mu}_j\}_{j=1}^N\right) \setminus \{\pm\tau_j\}_{j=0}^N, \tag{7.31}$$

$$U_\pm(\lambda) = O(|\lambda-\lambda_0|^{1/4}),\ \lambda \to \lambda_0,\ \lambda_0 \in \left(\{\pm\overset{r}{\mu}_j\}_{j=0}^N \cup \{\pm\overset{l}{\mu}_j\}_{j=1}^N\right) \cap \{\pm\tau_j\}_{j=0}^N. \tag{7.32}$$

$$U_\mp(\lambda) = O(|\lambda-\lambda_0|^{1/4}),\ \lambda \to \lambda_0,\ \lambda_0 \in \left(\{\pm\overset{r}{\mu}_j\}_{j=0}^N \cup \{\pm\overset{l}{\mu}_j\}_{j=1}^N\right) \cap \{\pm\tau_j\}_{j=0}^N. \tag{7.33}$$

Therefore, $\dfrac{U_\pm(t)}{\theta_j^+ - t} \in L^2_{loc}(\mathbb{R})$ and $\dfrac{U_\pm(t)}{\theta_j^- - t} \in L^2_{loc}(\mathbb{R})$. Combining this with (7.26), we get (6.11). □

Let $w_\pm(t)$ be defined by

$$\frac{1}{w_\pm(t)} := \left| \frac{\sqrt{R(\pm t)}}{S(\pm t)} \frac{\prod_{j=1}^{N_\theta^\pm}(t-\theta_j^\pm)}{\prod_{j=1}^{N_\theta^\pm}(t-(\theta_j^\pm - i\varepsilon_1))} \frac{\prod_{j=1}^{\widetilde{N}_\zeta^\mp}(\lambda-\widetilde{\zeta}_j^\mp)^{1/2}}{\prod_{j=1}^{\widetilde{N}_\zeta^\mp}(\lambda-(\widetilde{\zeta}_j^\mp - i\varepsilon_1))^{1/2}} \right|.$$

Let us check conditions (6.12), (6.13), and (6.14).

Since the points $\theta_j^\pm$, $\zeta_j^\mp$ belong to $\rho_1^\pm (= \mathbb{R} \setminus \operatorname{supp} d\Sigma_{ac\pm})$, formulae (7.10)–(7.13) imply (6.12).

LEMMA 7.8. *Condition (6.13) is fulfilled, i.e., the weights w_+ and w_- satisfy the (A_2) condition.*

We give two proofs. They are based on the Hunt-Muckenhoupt-Wheeden theorem and the Helson-Szegö theorem respectively. Note, that [25, Theorem 4] can also be applied.

The First Proof of Lemma 7.8. It is clear that the functions $w_\pm$ satisfy all the conditions of Proposition 2.9. Thus, $w_\pm \in (A_2)$. □

The Second Proof of Lemma 7.8. The Helson-Szegö condition (see (2.37)) is equivalent to the (A_2) condition. Let us prove that condition (2.37) is satisfied for w_+.

Obviously,

$$w_+(t) = \frac{\prod\limits_{\lambda_0 \in \{\tau_j\}_0^N \cap (\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N)} |t-\lambda_0|^{1/2} \prod_{j=1}^{N_\theta^+} |t-(\theta_j^+ - i\varepsilon_1)| \prod_{j=1}^{\widetilde{N}_\zeta^-} |t-(\widetilde{\zeta}_j^- - i\varepsilon_1)|^{1/2}}{\prod\limits_{\lambda_0 \in (\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_0^N) \setminus \{\tau_j\}_0^N} |t-\lambda_0|^{1/2} \prod_{j=1}^{\widetilde{N}_\zeta^-} |t-\widetilde{\zeta}_j^-|^{1/2}}. \tag{7.34}$$

Consequently,

$$\begin{aligned} \log w_+(t) &= \frac{1}{2} \sum_{\lambda_0 \in \{\tau_j\}_0^N \cap (\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N)} \log|t-\lambda_0| + \sum_{j=1}^{N_\theta^+} \log|t-(\theta_j^+ - i\varepsilon_1)| \\ &\quad + \frac{1}{2} \sum_{j=1}^{\widetilde{N}_\zeta^-} \log|t-(\widetilde{\zeta}_j^- - i\varepsilon_1)| - \frac{1}{2} \sum_{\lambda_0 \in (\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_0^N) \setminus \{\tau_j\}_0^N} \log|t-\lambda_0| \\ &\quad - \frac{1}{2} \sum_{j=1}^{\widetilde{N}_\zeta^-} \log|t-\widetilde{\zeta}_j^-| = (Hv_+)(t) + c_1, \end{aligned} \tag{7.35}$$

where c_1 is a constant, H is the Hilbert transform (see Subsection 2.6.), and

$$\begin{aligned} v_+(t) = \frac{1}{2} &\sum_{\lambda_0 \in (\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_0^N) \setminus \{\tau_j\}_0^N} \arg(t-\lambda_0) + \frac{1}{2}\sum_{j=1}^{\widetilde{N}_\zeta^-} \arg(t-\widetilde{\zeta}_j^-) \\ &- \frac{1}{2} \sum_{\lambda_0 \in \{\tau_j\}_0^N \cap (\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N)} \arg(t-\lambda_0) - \sum_{j=1}^{N_\theta^+} \arg(t-(\theta_j^+ - i\varepsilon_1)) \\ &- \frac{1}{2}\sum_{j=1}^{\widetilde{N}_\zeta^-} \arg(t-(\widetilde{\zeta}_j^- - i\varepsilon_1)). \end{aligned}$$

Note, that the branch of $\arg z$ is fixed by $\arg z \in (-\pi, \pi]$, $z \in \mathbb{C}$.

The function v_+ is bounded and piecewise smooth; the set of jumps of v_+ is

$$\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N \cup \{\widetilde{\widetilde{\zeta}_j^-}\}_1^{\widetilde{\widetilde{N}_\zeta^-}} .$$

The absolute values of all the jumps are equal to $\pi/2$. Moreover,

$$v_+(t) \asymp \arctan\frac{1}{t} \asymp \frac{1}{t} \ \ (t \to +\infty), \quad \text{and} \quad v_+(t) + \frac{\pi}{2} \asymp \arctan\frac{1}{|t|} \asymp \frac{1}{|t|} \ \ (t \to -\infty),$$

$v_+(t)$ monotonically increases on $t \in (-\infty, \overset{r}{\mu}_0)$ and $(\overset{r}{\mu}_N, +\infty)$. Therefore, v_+ admits a representation

$$v_+(t) = v_1(t) + v_2(t) - \pi/4,$$

where v_1 is a piecewise continuous function such that

$$\|v_1(t)\|_{L^\infty} < \pi/2, \tag{7.36}$$

$$v_1 \text{ has jumps at the points } \quad \{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N \cup \{\widetilde{\zeta_j^+}\}_1^{\widetilde{N}_\zeta^+} ,$$

v_2 is a C^1 function on $\mathbb{R}$ such that

$$v_2(t) = 0 \quad \text{for } t \notin [\overset{r}{\mu}_0 - \delta_2, \overset{r}{\mu}_N + \delta_2]; \tag{7.37}$$

here δ_2 is a specified positive number.

It follows from (7.37) that

$$(Hv_2)(t) \asymp |t|^{-1} \quad (|t| \to \infty) .$$

Next, it follows from $v_2 \in C^1(\mathbb{R})$ that $v_2 \in \mathrm{Lip}^\alpha(\mathcal{I})$ for any compact interval $\mathcal{I} \subset \mathbb{R}$ and any $\alpha \in (0,1)$. If we combine this with Privalov's theorem (see [42]) and (7.37), we get $Hv_2 \in \mathrm{Lip}^\alpha(\mathcal{I})$, $0 < \alpha < 1$. Hence, Hv_2 is a continuous function on $\mathbb{R}$ and (7.37) implies $Hv_2 \in L^\infty(\mathbb{R})$. Taking into account (7.35), we get $\log w_+(t) = (Hv_1)(t) + (Hv_2)(t) + c_1$, where $\|v_1\|_{L^\infty} < \pi/2$, $Hv_2 + c_1 \in L^\infty(\mathbb{R})$.

In other words, w_+ satisfies the Helson-Szegö condition and (6.13) is proved for w_+. Condition (6.13) for w_- is proved similarly. $\square$

LEMMA 7.9. *Let Statement (iii) be true. Then condition (6.14) is fulfilled, i.e.,*

$$\frac{U_+^2(t)}{w_\pm(t)} \in L^\infty(\mathbb{R}), \qquad \frac{U_-^2(t)}{w_\pm(t)} \in L^\infty(\mathbb{R}).$$

Proof. Note that

$$w_+^{-1}(t) \asymp |t|^{-1/2} \quad (|t| \to \infty). \tag{7.38}$$

It follows from (7.26), (7.28)–(7.33), (7.34), Statement (iii), and (7.17) that

$$U_+^2(t)w_+^{-1}(t) \in L^\infty(\mathbb{R}) \quad \text{and} \quad U_-^2(t)w_+^{-1}(t) = O(1) \quad (t \to t_0)$$

for

$$t_0 \in \{-\infty\} \cup \{+\infty\} \cup \rho_1^- \cup \sigma_1^- \cup \sigma_1^+ \cup \{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N. \tag{7.39}$$

Note that

$$\mathbb{R} \setminus \left(\rho_1^- \cup \sigma_1^- \cup \sigma_1^+ \cup \{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N\right) = (\{-\overset{r}{\mu}_j\}_0^N \cup \{-\overset{l}{\mu}_j\}_1^N) \cap \rho_1^+. \tag{7.40}$$

If λ_0 is a generalized zero of $\overset{*}{M}_+(\lambda) - \overset{*}{M}_-(\lambda)$ and $\lambda_0 \in (\{-\overset{r}{\mu}_j\}_0^N \cup \{-\overset{l}{\mu}_j\}_1^N) \cap \rho_1^+$, the definition of the set $\{\widetilde{\zeta_j^-}\}_1^{\widetilde{N_\zeta^-}}$ imply that $\lambda_0 \in \{\widetilde{\zeta_j^-}\}_1^{\widetilde{N_\zeta^-}}$ and the generalized order of λ_0 equals $1/2$. Thus, by (7.17) and the definitions of w_+, U_-, we have

$$U_-^2(t)w_+^{-1}(t) = O(1) \quad (t \to t_0), \quad t_0 \in (\{-\overset{r}{\mu}_j\}_0^N \cup \{-\overset{l}{\mu}_j\}_1^N) \cap \rho_1^+ .$$

Taking into account (7.39) and (7.40), we get

$$U_-^2(t)w_+^{-1}(t) \in L^\infty(\mathbb{R}).$$

One can prove $U_\pm^2(t)w_-^{-1}(t) \in L^\infty(\mathbb{R})$ in the same way. Thus (6.14) is proved. $\square$

Let $\widetilde{\theta}_j^\pm$ be a point of the set $\{\widetilde{\theta}_k^\pm\}_1^{\widetilde{N}_\theta^\pm}$. Let $D_j^\pm$ be a sufficiently small neighborhood of $\widetilde{\theta}_j^\pm$ such that

$$D_j^\pm \cap \left(\{\theta_k^\pm\}_1^{N_\theta^\pm} \cup \{\pm\overset{r}{\mu}_k\}_0^N \cup \{\pm\overset{l}{\mu}_k\}_1^N\right) = \widetilde{\theta}_j^\pm.$$

Put

$$U_\theta^\pm := \frac{\sqrt{u_\theta^\pm(\lambda)}}{M_2(\lambda)}, \quad \text{where} \quad u_\theta^\pm(\lambda) := \frac{\sqrt{R(\pm\lambda)}}{S(\pm\lambda)} \frac{\prod_{j=1}^{\widetilde{N}_\zeta^\pm}(\lambda - \widetilde{\zeta}_j^\mp)}{\prod_{j=1}^{\widetilde{N}_\zeta^\pm}(\lambda - (\widetilde{\zeta}_j^\mp - i\varepsilon_1))}. \tag{7.41}$$

We define $U_j^{\pm}$ as $U_j^{\pm} := U_\theta^{\pm}$ for all $j = 1, \dots, \widetilde{N}_\theta^{\pm}$.

Lemma 2.11 imply that $U_\theta^{\pm} \in \mathcal{N}^+(\mathbb{C}_+)$.

LEMMA 7.10. *Let Statement* (iii) *be true. Then conditions (6.15), (6.16), and (6.17) are fulfilled. That is, for every* $\widetilde{\theta}_j^{\pm} \in \{\widetilde{\theta}_k^{\pm}\}_1^{\widetilde{N}_\theta^{\pm}}$, *the following conditions hold:*

$$\frac{1}{|M_+(\lambda) - M_-(\lambda)|^2} \operatorname{Im} \frac{1}{\widetilde{\theta}_j^{\pm} - \lambda} \leqslant C_\theta^u |U_\theta^{\pm}(\lambda)|^2 \quad \text{for } \lambda \in D_j^{\pm} \cap \mathbb{C}_+ ,$$

$$\frac{|U_\theta^{\pm}(t)|^2}{w_+(t)} \in L^\infty(\mathbb{R}), \qquad \frac{|U_\theta^{\pm}(t)|^2}{w_-(t)} \in L^\infty(\mathbb{R}) ,$$

$$\frac{1}{|\widetilde{\theta}_j^{\pm} - \lambda| \cdot |M_+(\lambda) - M_-(\lambda)|} \leqslant C_\theta^M \qquad \text{for } \lambda \in D_j^{\pm} \cap \mathbb{C}_+,$$

where C_θ^u *and* C_θ^M *are constants.*

Proof. Note that

$$M_2(\lambda) \asymp M_+(\lambda) - M_-(\lambda), \quad \lambda \to \widetilde{\theta}_j^{\pm}.$$

Therefore (6.15) is equivalent to

$$\operatorname{Im} \frac{1}{\widetilde{\theta}_j^{\pm} - \lambda} \leqslant C_1 |u_\theta^{\pm}(\lambda)| \quad \text{for } \lambda \in D_j^{\pm} \cap \mathbb{C}_+ . \tag{7.42}$$

By (7.20) and (7.21), it follows that $\widetilde{\theta}_j^{\pm} \notin \{\widetilde{\zeta}_k^{\mp}\}_1^{\widetilde{N}_\zeta^{\pm}}$. Taking into account (7.41) and (7.3), we see that $u_\theta^{\pm}(\lambda)$ has a pole of the first order at $\widetilde{\theta}_j^{\pm}$. This implies (7.42). Thus (6.15) is proved.

Lemma 7.9 and the definitions of $u^{\pm}$ and $u_\theta^{\pm}$ imply

$$\frac{|U_\theta^+(t)|^2}{|w_\pm(t)|} \leqslant C_2 \quad \text{for } t \in \mathbb{R} \setminus \bigcup_{k=1}^{\widetilde{N}_\theta^+} D_k^+ . \tag{7.43}$$

Hence, to check condition (6.16) for U_θ^+, it suffices to show that

$$\frac{|U_\theta^+(t)|^2}{|w_\pm(t)|} \leqslant C_2 \quad \text{for } t \in D_k^+, \quad k = 1, \dots, \widetilde{N}_\theta^+ . \tag{7.44}$$

It is easy to see that

$$M_2(\lambda) \asymp (\lambda - \widetilde{\theta}_k^{\pm})^{-1}, \quad u_\theta^{\pm}(\lambda) \asymp (\lambda - \widetilde{\theta}_k^{\pm})^{-1} \quad (t \to \widetilde{\theta}_k^{\pm}) \tag{7.45}$$

$$\frac{1}{w_\pm(t)} = O(1)\,(t - \widetilde{\theta}_k^+)^{1/2} \quad (t \to \widetilde{\theta}_k^+) .$$

Combining these formulae, we obtain (7.44). Thus (6.16) for U_θ^+ is proved. The proof of (6.16) for U_θ^- is similar. Condition (6.17) follows from (7.45). □

Since all the conditions of Theorem 6.3 are fulfilled, we see that A_{ess} is similar to a selfadjoint operator. Theorem 7.2 is proved.

8. Examples

Let $L = -d^2/dx^2 + q$ be a Sturm-Liouville operator with a finite-zone potential q. Put

$$A := JL = (\operatorname{sgn} x)(-d^2/dx^2 + q)\ .$$

DEFINITION 8.1. We say that a point $a \in \sigma_{\text{ess}}(A) \cup \{\infty\}$ is *a strong spectral singularity of* A_{ess} if the function

$$\frac{\operatorname{Im} M_+(t) + \operatorname{Im} M_-(t)}{M_+(t) - M_-(t)},$$

is not essentially bounded in any neighborhood of a.

It follows from (3.21)–(3.23) that if a is a strong spectral singularity of A_{ess} then the characteristic function $\theta_A(\cdot)$ as well as the corresponding $\mathcal{J}$-forms $\omega_\theta(\cdot)$ and $\omega_{\theta^*}(\lambda)$ are not bounded in any neighborhood of a too. Therefore a is a spectral singularity in a classical sense and in the sense of Definition 3.1 as well. Thus Definition 8.1 is compatible with the both Definition 3.1 and the classical one.

By Theorem 7.2, A_{ess} is similar to a selfadjoint operator if and only if A_{ess} has no strong spectral singularities. Combining Theorems 7.2 and 7.1, we arrive at the following result.

THEOREM 8.1. *The indefinite Sturm-Liouville operator A with a finite-zone potential is similar to a normal operator if and only if*

(i) A_{ess} *is similar to a selfadjoint operator;*

(ii) *all the eigenvalues of A are simple.*

Denote by $L(\xi, q)$ the Sturm-Liouville operator with a finite-zone potential $q(x)+\xi$,

$$L(\xi, q) := -d^2/dx^2 + q(x) + \xi,$$

where ξ is a real constant. Put

$$A(\xi, q) := JL(\xi, q) = (\operatorname{sgn} x)(-d^2/dx^2 + q(x) + \xi)\ .$$

Let $A_{\text{ess}}(\xi, q)$ be the part of $A(\xi, q)$ on $\mathfrak{H}_e$.

EXAMPLE 8.1. Consider the following periodic one-zone potential

$$q_1(x) = (1-k^2)(2\operatorname{sn}^2(x,k') - 1), \ \ k \in (0,1), \ \ k' = \sqrt{1-k^2}, \tag{8.1}$$

where $\operatorname{sn}(x,k')$ is the Jacobi elliptic function. Then $L(\xi, q_1)$ is a one-zone periodic operator with the gaps $(-\infty, \xi)$ and $(k^2+\xi, 1+\xi)$.

The corresponding Weyl functions $M_\pm(\lambda)$ are

$$M_+(\lambda) = -M_-(-\lambda) = i\frac{\lambda - (\xi+1)}{\sqrt{(\lambda-\xi)(\lambda-(\xi+k^2))}}\ , \quad 0 < k^2 < 1,$$

(cf. [2, Appendix II, Subsection 5]). By Theorem 7.2 $A_{\text{ess}}(\xi, q_1)$ is similar to a selfadjoint operator if and only if

$$\xi \in [-1, -k^2] \cup [0, \infty).$$

Note that for $\xi \in [-1, -k^2]$ the operator $L(\xi, q_1)$ is not nonnegative. If

$$\xi \in (-1+\sqrt{1-k^2}, -1-\sqrt{1-k^2}),$$

then $A(\xi, q_1)$ has exactly two eigenvalues

$$\pm\sqrt{(\xi+1)^2-(1-k^2)}\,;$$

these eigenvalues are simple and nonreal.

It is interesting to metion that for sufficiently small $\xi \geqslant 0$, the potential $q_1(x)+\xi$ is not nonnegative, while $L(\xi, q_1) \geqslant 0$. Note that the same fact is valid for any nonnegative one zone Sturm-Liouville operator and it is implied by the corresponding trace formula.

Spectral properties of $A(\xi, q_1)$ are given in more details in the following table. The abbreviations 'S-A' ('Norm') in the column 'Similarity' means that $A(\xi, q_1)$ is similar to a selfadjoint (normal) operator. 'NonSim' in the column 'Similarity' means that $A(\xi, q_1)$ is not similar to a normal operator. We put $\lambda_\pm(\xi) := \pm\sqrt{(\xi+1)^2-(1-k^2)}$.

Spectral properties of the operator $A(\xi, q_1)$

Intervals	Strong spectral singularities	Eigenvalues	Similarity
$\xi \in [0, +\infty)$	No	$\lambda_\pm(\xi)$	S-A
$\xi \in (-\frac{k^2}{2}, 0)$	0	$\lambda_\pm(\xi)$	NonSim
$\xi = -\frac{k^2}{2}$	0	No	NonSim
$\xi \in (-1+\sqrt{1-k^2}, -\frac{k^2}{2})$	$0,\ \lambda_\pm(\xi)$	No	NonSim
$\xi = -1+\sqrt{1-k^2}$	0	No	NonSim
$\xi \in (-k^2, -1+\sqrt{1-k^2})$	0	$\lambda_\pm(\xi)$	NonSim
$\xi \in [-1, -k^2]$	No	$\lambda_\pm(\xi)$	Norm
$\xi \in (-1-\sqrt{1-k^2}, -1)$	0	$\lambda_\pm(\xi)$	NonSim
$\xi = -1-\sqrt{1-k^2}$	0	No	NonSim
$\xi \in (-\infty, -1-\sqrt{1-k^2})$	$0,\ \lambda_\pm(\xi)$	No	NonSim

Table 7.1

REMARK 8.1. Example 8.1 shows that condition (5.25) is not necessary for similarity of A to a self-adjoint operator. Indeed, let $\xi > 0$. Then $A(\xi, q_1)$ is similar to a selfadjoint operator, while the function $\dfrac{M_+(\lambda)+M_-(\lambda)}{M_+(\lambda)-M_-(\lambda)}$, is unbounded in neighborhoods of the eigenvalues $\lambda_\pm := \pm\sqrt{(\xi+1)^2-(1-k^2)}$. Namely, the functions $M_{\pm}$ are holomorphic at points $\lambda_{\pm}$ that are zeroes of $M_{+}(\cdot) - M_{-}(\cdot)$. On the other hand, it is easy to check that

$$M_{+}(\lambda_{+}) < 0, \quad M_{-}(\lambda_{+}) < 0, \quad M_{+}(\lambda_{-}) > 0 \quad M_{-}(\lambda_{-}) > 0,$$

and hence, $M_{+}(\lambda_{\pm}) + M_{-}(\lambda_{\pm}) \neq 0$.

EXAMPLE 8.2. Consider even periodic potential

$$q_2 = -2k^2\big(1 - (1-k^2)\mathrm{sn}^2(x,k')\big)^{-1} + 1 + k^2, \qquad k \in (0,1), \quad k' = \sqrt{1-k^2}.$$

The operator $L(\xi, q_2)$ is a one-zone operator with gaps $(-\infty, \xi)$ and $(k^2+\xi, 1+\xi)$. The corresponding Weyl functions $M_{\pm}(\lambda)$ have the forms (cf. [2, Appendix II, Subsection 5])

$$M_{+}(\lambda) = -M_{-}(-\lambda) = i\frac{\lambda - (\xi + k^2)}{\sqrt{(\lambda - \xi)(\lambda - (\xi + 1))}}, \quad 0 < k^2 < 1.$$

The operator $A(\xi, q_2)$ has no eigenvalues for all $\xi \in \mathbb{R}$. Hence, $A_{\mathrm{ess}}(\xi, q_2) = A(\xi, q_2)$.

Let $0 < k^2 \leqslant \frac{1}{2}$. Using Theorem 7.2, we get the following result: The operator $A(\xi, q_2)$ is similar to a selfadjoint operator if and only if $\xi \in [-\frac{1}{2}, -k^2] \cup [0, \infty)$. Spectral properties of $A(\xi, q_2)$ are described in the following table.

Spectral properties of $A(\xi, q_2)$, the case $k^2 \in (0, 1/2]$

Intervals ξ	Strong spectral singularities	Similarity
$\xi \in [0, +\infty)$	No	S-A
$\xi \in (-k^2, 0)$	0	NonSim
$\xi \in [-\frac{1}{2}, -k^2]$	No	S-A
$\xi \in [-1, -\frac{1}{2})$	$\pm\sqrt{(\xi + k^2)^2 + k^2(1-k^2)}$	NonSim
$\xi \in (-\infty, -1)$	$0, \pm\sqrt{(\xi + k^2)^2 + k^2(1-k^2)}$	NonSim

Table 7.2

Assume $k^2 > \frac{1}{2}$. Then $A(\xi, q_2)$ is similar to a selfadjoint operator if and only if $\xi \geqslant 0$. In other words, $A(\xi, q_2)$ is similar to a selfadjoint operator if and only if $L(\xi, q_2) \geqslant 0$. A description of the spectral properties of the operator $A(\xi, q_2)$ in this case is given in the following table.

Spectral properties of $A(\xi, q_2)$, the case $k^2 \in (1/2, 1)$

Intervals	Strong spectral singularities	Similarity
$\xi \in [0, +\infty)$	No	S-A
$\xi \in [-\frac{1}{2}, 0)$	0	NonSim
$\xi \in (-k^2, -\frac{1}{2})$	$0,\ \pm\sqrt{(\xi+k^2)^2+k^2(1-k^2)}$	NonSim
$\xi \in [-1, -k^2]$	$\pm\sqrt{(\xi+k^2)^2+k^2(1-k^2)}$	NonSim
$\xi \in (-\infty, -1)$	$0,\ \pm\sqrt{(\xi+k^2)^2+k^2(1-k^2)}$	NonSim

Table 7.3

EXAMPLE 8.3. Let q_3 be a periodic potential of the form (8.1) with $k^2 = 1/2$ and let $\xi \in [-1, -1/2)$. Then, by Theorem 2.1 $A(\xi, q_3)$ is not definitizable. On the other hand, according to the Table 7.1 the operator $A_{\text{ess}}(\xi, q_3)$ is similar to a selfadjoint operator and $A(\xi, q_3)$ is similar to a normal operator. The nonreal spectrum of $A(\xi, q_3)$ consists of two simple eigenvalues $\lambda_\pm(\xi) := \pm\sqrt{(\xi+1)^2-(1-k^2)}$. The operator $A(\xi, q_3)$ has no real eigenvalues. Note that in the case $\xi = -1/2$ the operator $A(-\frac{1}{2}, q_3)$ is definitizable due to Theorem 2.1. Moreover, it is similar to a normal operator.

Acknowledgement. We are highly indebted to B. Ćurgus, A. Kostenko and the anonymous referee for carefully reading the preprint [39] and the present manuscript, respectively, and indicating numerous misprints and suggesting many improvements.

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(Received November 16, 2006)

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