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HADAMARD DUALS, RETRACTABILITY AND OPPENHEIM'S INEQUALITY

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Abstract. Oppenheim's determinantal inequality was originally proved for positive semidefinite matrices and has produced many interesting consequences and applications. Positive semidefinite matrices were a natural class to consider partly because they are closed under Hadamard (or entry-wise) multiplication. Since Oppenheim's original contribution, others have considered similar inequalities for M-matrices, inverse M-matrices and totally nonnegative matrices. We attempt to unify many of these existing results dealing with Oppenheim's inequality, and our approach relies on two major themes: retractions and Hadamard duals. Retractions are a type of diagonal perturbation and the Hadamard dual is a maximal collection of matrices with a closure property under Hadamard multiplication. These notions are applied to yield results that generalize Oppenheim's original result.

1. Introduction

The *Hadamard product* of two m-by-n matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is defined and denoted by

$$A \circ B = [a_{ij}b_{ij}].$$

The Hadamard product plays a substantial role within matrix analysis and in its applications (see, for example, [11, Chap. 5] and consult the comprehensive reference [9]).

Some classes of matrices, such as the positive definite matrices, are closed under Hadamard multiplication (see [10, pg. 458]), and given such closure, inequalities involving the Hadamard product, usual product, determinants and eigenvalues, etc. may be considered. For example, Oppenheim's inequality states that

$$\det(A \circ B) \geqslant \prod_{i=1}^{n} a_{ii} \det B,$$

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for any two n-by-n positive definite matrices $A = [a_{ij}]$ and B (see [10, pg. 480]). Since Hadamard's inequality

$$\det A \leqslant \prod_{i=1}^{n} a_{ii},$$

also holds for positive definite matrices $A = [a_{ij}]$, it follows from Oppenheim that

$$\det(A \circ B) \geqslant \det(AB),$$

i.e., the Hadamard product dominates the usual product in determinant.

Our interest here lay in studying the Hadamard product of two matrices from a fixed subclass of the P-matrices (i.e., real square matrices with positive principal minors). We are interested in the following key positivity classes of matrices: positive (semi)-definite (PD, (PSD)); P-matrices (positive principal minors) (P); P_0-matrices (nonnegative principal minors) (P_0); M-matrices (nonpositive off-diagonal entries and positive principal minors) (M); inverse M-matrices (inverses of M-matrices) (IM); totally nonnegative matrices (all minors nonnegative) (TN); totally positive matrices (all minors positive) (TP); completely positive matrices (matrices of the form BB^T with B entry-wise nonnegative) (CP); doubly nonnegative matrices (positive semidefinite and entry-wise nonnegative) (DN). See reference [14] for a broad survey of related closure issues for various positivity classes of matrices.

Unfortunately, the class of totally nonnegative matrices is not in general closed under Hadamard multiplication. Consider the following simple example:

$$W = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad W^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \tag{1}$$

W (and, thus, W^T) is TN, but

$$W \circ W^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

is not. Similarly, TP is not Hadamard closed. Not surprisingly, then, inequalities such as Oppenheim's do not generally hold for TP or TN matrices. However, there has been interest in significant subclasses of the TP or TN matrices that are Hadamard closed, i.e., are such that arbitrary Hadamard products from them are TP or TN, see [1, 4, 8, 19].

For M-matrices, the issue of closure under Hadamard multiplication does not make much sense because of the sign-pattern restriction on M-matrices. However, by considering comparison matrices such closure questions have been addressed. A more interesting situation deals with the issue of the Hadamard product of an M-matrix and an inverse M-matrix. In [12] it was verified that the Hadamard product of an M-matrix and an inverse M-matrix is again an M-matrix.

Closure under Hadamard multiplication for the class of inverse M-matrices is quite different from the previous cases, and was only recently resolved. The resolution is as follows: For $n \leqslant 3$ the Hadamard product of two inverse M-matrices is again

IM; however for $n \geqslant 4$ this is no longer true (see [20] for more details). See also [3, 16, 18, 20] for other work on Hadamard products of IM matrices.

Taking into account all of the above situations, we define a unifying notion that we refer to as the *Hadamard dual* for a class of matrices.

DEFINITION 1. Let $\mathcal{C}$ be a subclass of the P_0-matrices. Then the Hadamard dual of $\mathcal{C}$ is defined to be

$$\mathcal{C}^{(\mathrm{D})} = \{A : B \in \mathcal{C} \implies A \circ B \in \mathcal{C}\}.$$

One of our main objectives is to describe the Hadamard dual for the key subclasses of the P_0-matrices such as PSD, PD, M, IM, TN, and P itself. See [4], in which this set is referred to as the *Hadamard core* in the case of totally nonnegative matrices.

For an m-by-n matrix A, $\alpha \subseteq \{1,2,\ldots,m\}$, and $\beta \subseteq \{1,2,\ldots,n\}$, the submatrix of A lying in rows indexed by α and the columns indexed by β will be denoted by $A[\alpha|\beta]$. Similarly, $A(\alpha|\beta)$ is the submatrix obtained from A by deleting the rows indexed by α and columns indexed by β. If A is square and $\alpha = \beta$, then the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$, and the complementary principal submatrix to $A(\alpha)$. For brevity, if $\alpha = \{i\}$, then $A(\alpha)$ is denoted by $A(i)$, or A_{ii}. The $(i,j)^{th}$ standard basis matrix, that is the m-by-n matrix whose only nonzero entry is in the $(i,j)^{th}$ position and this entry is a one, will be denoted by E_{ij}. The m-by-n matrix of all ones is denoted by J.

The remainder of the paper is organized as follows: In section 2 we give conditions for subclasses of P-matrices so that they enjoy Oppenheim's inequality. In this section we introduce the notion of retractability and re-visit Hadamard duality. In section 3, we elaborate on retractability and investigate this property for certain well known subclasses of P-matrices. In the final section we investigate the Hadamard dual for the same important subclasses of P-matrices.

2. Oppenheim's Inequality for General Sets of P_0-Matrices

One proof of Oppenheim's inequality for PSD matrices (see [10]) requires four key facts. The first is Hadamard closure, and the second fact is that the property of being positive semidefinite is inherited by principal submatrices. The third fact is Fischer's determinantal inequality (see [10]): $\det A \leqslant \det A[\alpha] \det A[\alpha^c]$ for any index set $\alpha \subset \{1,2,\ldots,n\}$. Specifically, the type of inequality needed here is of the form $\det A \leqslant a_{11} \det A(1)$. The final fact is that for any positive semidefinite matrix A the matrices $A - \gamma E_{11}$, are positive semidefinite for every $\gamma \in [0, \det A/ \det A(1)]$. If $\det A(1) = 0$, then the interval is defined to be the singleton $\{0\}$. We call this property *retractability*. Observe that if $\gamma = \det A/ \det A(1)$, then $A - \gamma E_{11}$ is necessarily singular. This leads us to our first definition concerning retractions.

DEFINITION 2. Let $\mathcal{C}$ denote a given subclass of the P_0-matrices, and suppose A is an n-by-n matrix. Then we define:

(i) $A^R = \{A - tE_{11} : t \in [0, \frac{\det A}{\det A(1)}]\}$ – "the set of retractions of A",

(ii) $\mathcal{C}^{(R)} = \{A \in \mathcal{C}\colon\ A^R \subseteq \mathcal{C}\}$ – "the retractable subset of $\mathcal{C}$",

(iii) $\mathcal{C}_R = \cup_{A \in \mathcal{C}} A^R$ – "the set of all retractions of matrices in $\mathcal{C}$".

We now state and prove a general statement regarding the validity of Oppenheim's inequality for general subclasses of P_0-matrices.

THEOREM 3. *Let $\mathcal{C} \subset P_0$ and $\mathcal{D} \subset P_0$ be two subclasses of the P_0-matrices that are closed under extraction of principal submatrices. Assume that Fischer's inequality holds for all $A \in \mathcal{C}$. Suppose $A = [a_{ij}]$ and $B = [b_{ij}]$ are two n-by-n matrices. Then*

$$\det(A \circ B) \geqslant \det A \prod_{i=1}^{n} b_{ii}, \tag{2}$$

if for each n either

1. *$A \in \mathcal{C}^{(\mathrm{R})}$ and $B \in \mathcal{C}^{(\mathrm{D})}$, or*
2. *$A \in \mathcal{C}$, $B \in \mathcal{D}$, and $\mathcal{C}_{\mathrm{R}} \subset \mathcal{D}^{(\mathrm{D})}$*

Proof. If A is singular, then there is little to show, since $\det(A \circ B) \geqslant 0$ (in case 2. observe that $\mathcal{C} \subset \mathcal{C}_{\mathrm{R}} \subset \mathcal{D}^{(\mathrm{D})}$). Assume A is nonsingular. If $n = 1$, then the inequality is trivial. Suppose, by induction, that (2) holds for all $(n-1)$-by-$(n-1)$ matrices A and B under either situation. Then by induction

$$\det(A(1) \circ B(1)) \geqslant \det A(1) \prod_{i=2}^{n} b_{ii}.$$

Suppose we are in case 1. Since A is nonsingular, and $A \in \mathcal{C}^{(\mathrm{R})}$, we have that $A \in \mathcal{C}$ and thus by Fischer's inequality $A(1)$ is nonsingular. Consider the matrix $\tilde{A} = A - xE_{11}$, where $x = \frac{\det A}{\det A(1)}$. Then $\det \tilde{A} = 0$, and $\tilde{A} \in \mathcal{C}$. Therefore $B \circ \tilde{A}$ is in $\mathcal{C}$ since $B \in \mathcal{C}^{(\mathrm{D})}$, and in particular, $\det(B \circ \tilde{A}) \geqslant 0$. Observe that $\det(B \circ \tilde{A}) = \det(A \circ B) - xb_{11} \det(A(1) \circ B(1)) \geqslant 0$. Thus

$$\begin{aligned} \det(A \circ B) &\geqslant xb_{11} \det(A(1) \circ B(1)) \\ &\geqslant xb_{11} \det A(1) \prod_{i=2}^{n} b_{ii} \\ &= \det A \prod_{i=1}^{n} b_{ii}. \end{aligned}$$

On the other hand in case 2 if A is nonsingular, then $A \in \mathcal{C}$ and satisfies Fischer's inequality, so $A(1)$ is nonsingular. Proceeding as in case 1, consider the matrix $\tilde{A} = A - xE_{11}$, where $x = \frac{\det A}{\det A(1)}$. Then $\tilde{A} \in \mathcal{C}_{\mathrm{R}}$. Since $B \in \mathcal{D}$, and $\mathcal{C}_{\mathrm{R}} \subset \mathcal{D}^{(\mathrm{D})}$ it follows that $\det(B \circ \tilde{A}) \geqslant 0$. The remainder of the proof is identical to case 1. $\square$

Since for PSD matrices we have $\mathrm{PSD} = \mathrm{PSD}^{(\mathrm{D})} = PSD_R$ (see sections 3 and 4), Oppenheim's Inequality is an immediate consequence of Theorem 3.

COROLLARY 4. *Let A and B be two positive semidefinite matrices. Then*

$$\det(A \circ B) \geqslant \det B \prod_{i=1}^{n} a_{ii}.$$

In section 3 we show that $TN^{(R)} = TN_R = TN$. Using this and Theorem 3 (case 1.) we have

COROLLARY 5. [4] *Let A be an n-by-n matrix in $TN^{(\mathrm{D})}$, and suppose B is any n-by-n totally nonnegative matrix. Then*

$$\det(A \circ B) \geqslant \det B \prod_{i=1}^{n} a_{ii}.$$

If $A, B \in \mathrm{PSD}$, then it is known (by interchanging the roles of A and B, since $\mathrm{PSD} = \mathrm{PSD}^{(\mathrm{D})} = PSD_R$) that:

$$\det(A \circ B) \geqslant \max\left\{\det B \prod_{i=1}^{n} a_{ii},\ \det A \prod_{i=1}^{n} b_{ii}\right\}. \tag{3}$$

However, in the case in which $A \in TN^{(\mathrm{D})}$ and B is TN it is not true in general that $\det(A \circ B) \geqslant \det A \prod_{i=1}^{n} b_{ii}$. Consider the following example.

EXAMPLE 6. Let A be any 3-by-3 totally positive matrix in the Hadamard dual, and let $B = W = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Then since the (1,3) entry of A enters positively into $\det A$ it follows that $\det(A \circ B) < \det A = \det A \prod_{i=1}^{3} b_{ii}$.

The next example sheds some light on the necessity that A be in $TN^{(\mathrm{D})}$ in order for Oppenheim's inequality to hold. In particular, we show that if A and B are TN and $A \circ B$ is TN, then Oppenheim's inequality need not hold.

EXAMPLE 7. Let $A = \begin{bmatrix} 1 & .84 & .7 \\ .84 & 1 & .84 \\ 0 & .84 & 1 \end{bmatrix}$, and $B = A^T$. Then A (and hence B) is TN, and $\det A = \det B = .08272$. Now $A \circ B = \begin{bmatrix} 1 & .7056 & 0 \\ .7056 & 1 & .7056 \\ 0 & .7056 & 1 \end{bmatrix}$, and it is not difficult to verify that $A \circ B$ is TN with $\det(A \circ B) \approx .00426$. However, in this case

$$\det(A \circ B) \approx .00426 < .08272 = \begin{cases} \det A \prod_{i=1}^{3} b_{ii} \\ \det B \prod_{i=1}^{3} a_{ii}. \end{cases}$$

See [4] for more information on Oppenheim's inequality for TN matrices.

If $\mathcal{C}_{\mathrm{R}} \subset \mathcal{C}^{(\mathrm{D})}$ for a fixed subclass of the P_0-matrices, then we have the next result, which is a direct consequence of Theorem 3.

COROLLARY 8. *Let $\mathcal{C} \subset P_0$ be a subclass of the P_0-matrices that is closed under extraction of principal submatrices, and assume that Fischer's inequality is satisfied for* *all* $A \in \mathcal{C}$. *Suppose* $A = [a_{ij}]$ *and* $B = [b_{ij}]$ *are two* n*-by-*n *matrices. If* $\mathcal{C}_{\mathrm{R}} \subset \mathcal{C}^{(\mathrm{D})}$, *then*

$$\det(A \circ B) \geqslant \max\left\{\det B \prod_{i=1}^{n} a_{ii},\ \det A \prod_{i=1}^{n} b_{ii}\right\}.$$

For the case of M-matrices and inverse M-matrices, the situation is very interesting. We show that $M^{(R)} = M_R = M$ (see section 3), and that $IM \subset M^{(\mathrm{D})}$ (see Theorem 20). Finally, while the set of retractions of IM is still unclear we do establish that $IM_R \subset M^{(\mathrm{D})}$ (see Theorem 14). Taking the above facts into account along with Theorem 3, we have the next result.

COROLLARY 9. *Let* $A \in M$ *and* $B \in IM$ *be two* n*-by-*n *matrices. Then*

$$\det(A \circ B) \geqslant \max\left\{\det B \prod_{i=1}^{n} a_{ii},\ \det A \prod_{i=1}^{n} b_{ii}\right\}.$$

We can even push this idea a bit further to prove the next result.

THEOREM 10. *Let* $A \in M$ *and* $B \in IM$ *be two* n*-by-*n *matrices. Then*

$$\det(A \circ B) + \det A \det B \geqslant \det B \prod_{i=1}^{n} a_{ii} + \det A \prod_{i=1}^{n} b_{ii}. \tag{4}$$

Proof. The proof is by induction on the size of A and B. For $n = 1$, the equation (4) is easily seen to be valid. Thus assume that (4) holds for all such A and B of size at most $n-1$. Suppose $A \in M$ and $B \in IM$ are two n-by-n matrices. Since $A \in M$, if we let $\tilde{A} = A - tE_{11}$ where $t = \frac{\det A}{\det A(1)} = \frac{\det A}{\det A_{11}}$, then $\tilde{A} \in M$. Hence we can apply Oppenheim's inequality to $\tilde{A}$ and B (that is, we can use Corollary 9) and observe that

$$\left(a_{11} - \frac{\det A}{\det A_{11}}\right)\left(\prod_{i=2}^{n} a_{ii}\right)\det B \leqslant \det(\tilde{A} \circ B) \tag{5}$$

$$= \det(A \circ B) - \frac{\det A}{\det A_{11}} b_{11} \det(A_{11} \circ B_{11}), \tag{6}$$

where $A(1) = A_{11}$. Thus we have,

$$\det(A \circ B) \geqslant \prod_{i=1}^{n} a_{ii} \det B + \frac{\det A}{\det A_{11}}\left(b_{11} \det(A_{11} \circ B_{11}) - \prod_{i=2}^{n} a_{ii} \det B\right). \tag{7}$$

Applying the induction hypothesis to $\det(A_{11} \circ B_{11})$ (since $A_{11} \in M$, $B \in IM$) yields:

$$\det(A \circ B) \geqslant \prod_{i=1}^{n} a_{ii} \det B + \frac{\det A}{\det A_{11}} \cdot$$
$$\cdot \left(b_{11} \prod_{i=2}^{n} a_{ii} \det B_{11} + b_{11} \prod_{i=2}^{n} b_{ii} \det A_{11} - b_{11} \det A_{11} \det B_{11} - \prod_{i=2}^{n} a_{ii} \det B\right)$$

$$
\begin{aligned}
&= \prod_{i=1}^{n} a_{ii} \det B + \frac{\det A}{\det A_{11}} b_{11} \prod_{i=2}^{n} a_{ii} \det B_{11} + \prod_{i=1}^{n} b_{ii} \det A - b_{11} \det B_{11} \det A \\
&\qquad - \frac{\det A}{\det A_{11}} \prod_{i=2}^{n} a_{ii} \det B \\
&= \prod_{i=1}^{n} a_{ii} \det B + \frac{\det A}{\det A_{11}} b_{11} \prod_{i=2}^{n} a_{ii} \det B_{11} + \prod_{i=1}^{n} b_{ii} \det A - b_{11} \det B_{11} \det A \\
&\qquad - \frac{\det A}{\det A_{11}} \prod_{i=2}^{n} a_{ii} \det B + \det A \det B - \det A \det B \\
&= \prod_{i=1}^{n} a_{ii} \det B + \prod_{i=1}^{n} b_{ii} \det A - \det A \det B \\
&\qquad + \frac{\det A}{\det A_{11}} (\det B - b_{11} \det B_{11}) \left(\det A_{11} - \prod_{i=2}^{n} a_{ii} \right) \\
&\geqslant \prod_{i=1}^{n} a_{ii} \det B + \prod_{i=1}^{n} b_{ii} \det A - \det A \det B.
\end{aligned}
$$

The final inequality follows since $\det B \leqslant b_{11} \det B_{11}$ for all inverse M-matrices and since $\det A_{11} \leqslant \prod_{i=2}^{n} a_{ii}$ for all M-matrices (see [11]). $\square$

It is not difficult to deduce that Theorem 10 also holds if both A and B are PSD matrices.

We offer a final passing comment relating to Hadamard powers. If A is in $\mathcal{C}^{(\mathrm{D})}$, then certainly $A^{(k)} := A \circ A \circ \cdots \circ A$ (k times) is in $\mathcal{C}^{(\mathrm{D})}$. Furthermore, if $\mathcal{C}$ contains the matrix J and satisfies Fischer's determinantal inequality, then we have

$$\det A^{(k)} \geqslant \det A^{k},$$

for each positive integer k.

3. Retractability of Classes of P-Matrices

Of interest here are certain perturbations which leave a given class invariant. For example, if A is an n-by-n positive semidefinite, M-, P-, or inverse M-matrix, then $A+D$ (D a nonnegative diagonal matrix) is a positive semidefinite, M-, P-, or inverse M-matrix, respectively (see [10, 11]). It is an easy exercise to show this result does not hold in general for the class of totally nonnegative matrices, but a much weaker version does hold. In fact, if A is an m-by-n totally nonnegative matrix, then increasing the (1,1) or the (m,n) entry of A results in a totally nonnegative matrix.

Let $A = [a_{ij}]$ be fixed and define $A_\gamma = A - \gamma E_{11}$, for $\gamma \in [0, \det A / \det A_{11}]$. Further, assume that A is a P-matrix. Then the principal minors of A_γ are easily computed as follows:

$$\det A_\gamma[\alpha] = \begin{cases} \det A[\alpha] - \gamma \det A[\alpha \setminus \{1\}], & \text{if } 1 \in \alpha, \\ \det A[\alpha], & \text{if } 1 \notin \alpha, \end{cases}$$

A natural question to ask: Is A_γ a P-matrix for all $\gamma \in [0, \det A/\det A_{11}]$?

This is easily reformulated as: Is $\det A_{\gamma_0}[\alpha] > 0$ for $\gamma_0 = \det A/\det A_{11}$ (keeping in mind that $A \in \mathrm{P}$)? In other words is

$$\det A[\alpha] - \det A/\det A_{11} \det A[\alpha \setminus \{1\}] > 0?$$

The above inequality is equivalent to

$$\det A[\alpha] \det A_{11} > \det A \det A[\alpha \setminus \{1\}], \tag{8}$$

for each $\alpha \subset \{1, 2, \ldots, n\}$ with $1 \in \alpha$.

We have already mentioned the determinantal inequalities of Hadamard and Fischer. A third and more general determinantal inequality is attributed to Koteljanskii. For $A \in P_0$, and $\alpha, \beta \subset \{1, 2, \ldots, n\}$. Koteljanskii's inequality states

$$\det A[\alpha \cup \beta] \det A[\alpha \cap \beta] \leqslant \det A[\alpha] \det A[\beta]. \tag{9}$$

Observe that (8) is an example of a Koteljanskii type inequality (9) for $\alpha = \alpha$, $\beta = \{2, 3, \ldots, n\}$. Denote by K (K_0) the class of all $A \in P$ (P_0) satisfying Koteljanskii's inequality for all α, β. According to [2], $A \in K_0$ if and only if $A \in P_0$ and additionally satisfies

$$\det A[S|T] \det A[T|S] \geqslant 0,$$

whenever $|S \cap T| = |S \cup T| - 1$ (*almost principal minors*). In particular, M, IM, PSD, and TN, all satisfy Koteljanskii's determinantal inequality. Hence the set of all retractions of matrices in M, IM, PSD, and TN are all contained in P_0 (i.e., M_R, IM_R, PSD_R, and TN_R all lie in P_0 by (8)). One goal here is to prove much more precise statements about the set of all retractions for these (and other) subclasses of P and P_0. For example, if $A \in PSD$, then $A^R \subset P_0$, and since A^R contains only symmetric matrices it follows that $A^R \subset PSD$. Hence $PSD^{(R)} = PSD_R = PSD$. Similarly, $M^{(R)} = M_R = M$ and $DN^{(R)} = DN_R = DN$. The case of completely positive matrices (CP) is still unclear. Certainly for $n \leqslant 4$, $CP^{(R)} = CP_R = CP = DN$. However, for $n \geqslant 5$, CP_R and $CP^{(R)}$ have not been characterized.

Returning to P-matrices, it is straightforward to construct $A \in P$, such that $A^R \not\subset P_0$. For example, let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Then $\det A/\det A_{11} = 3/2$, and hence A^R contains matrices with a negative (1,1) entry. Thus $A^R \not\subset P_0$. Of course $A \notin K$. Having noted this example, the set P_R, namely the set of all retractions of matrices in P, is not well understood in general, and we do not attempt to describe it here. However, the set $P^{(R)}$ is an interesting subset of P which we discuss now. Given the example above we know that $P^{(R)} \neq P$.

PROPOSITION 11. *The retractable subset of P contains the set of Koteljanskii matrices (i.e., $K \subset P^{(R)}$).*

Proof. Per the discussion above if $A \in K$, then $A^R \subset P$. Hence $K \subset P^{(R)}$. $\square$

If we wanted a desirable statement regarding the reverse containment we would need to generalize the definition of retraction to allow for retraction on any diagonal entry, and even further define retraction on each principal submatrix.

Here is an example of a P-matrix that is not a Koteljanskii matrix, but still is retractable. Consider

$$A = \begin{bmatrix} 1 & 1/2 & 1/4 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then A is a P-matrix and since the (2,3) and (3,2) entries have opposite sign it follows that $A \notin K$. However, since $\det A = 1$ and $\det A_{11} = 2$, it follows that $A \in P^{(R)}$. Furthermore, by considering $A[\{2,3\}]$ it is clear that retractability need not be an inherited property.

Finally, we note that the class of Koteljanskii matrices is not itself retractable, that is $K_R \not\subset K$. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

Then $A \in K$ and $A^R \in P$, but if we decrease the (1,1) entry by $x > 0$, then $\det A[12|13] = x$ and $\det A[13|12] = -1 + 2x$. Hence for $0 < x < 1/2$, A_x is not a Koteljanskii matrix, and so $K^{(R)} \not\subset K$.

Unfortunately, inverse M-matrices are not in general retractable (on any diagonal entry). Consider the following simple example. Let

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 4 & 1 \\ 1 & 2 & 4 \end{bmatrix}.$$

Then A is in IM and $\det A / \det A_{11} = 3.5$. However, $A - \gamma E_{11}$ is not an inverse M-matrix for any $\gamma \in (0, 3.5]$. Currently, we do not have a characterization of IM$^{(R)}$.

We will establish, however, that $IM_R \subset M^{(\mathrm{D})}$. To verify this, we need some definitions. An n-by-n $A = [a_{ij}]$ is said to be *diagonally dominant of its column entries* if for each i, $|a_{ii}| \geqslant |a_{ki}|$ for $k \neq i$. It is known that if $A = [a_{ij}]$ is *row diagonally dominant* (i.e., $|a_{ii}| \geqslant \sum_{k \neq i} |a_{ik}|$ for all i) and invertible, then A^{-1} is diagonally dominant of its column entries (see [11, Chap. 2]). Furthermore, if $A \in M$ and is invertible, then there exists a positive diagonal matrix D such that AD is row diagonally dominant (see [11, Chap. 2]). Define X to be the set of all n-by-n entry-wise nonnegative matrices A with the property that there exists a positive diagonal matrix D such that DA is diagonally dominant of its column entries. Given the comment above, we know that $IM \subset X$. The next result connects the set X with the set IM_R, but we need a lemma before we can prove this result. Recall that for a given n-by-n nonnegative matrix $A = [a_{ij}]$, a *cycle product* is any product of the form $a_{i_1,i_2} a_{i_2,i_3} \cdots a_{i_k,i_1}$, in which the indices $i_1, i_2, \ldots i_k$ are distinct, and the corresponding *cycle geometric mean* is equal to $\sqrt[k]{a_{i_1,i_2} a_{i_2,i_3} \cdots a_{i_k,i_1}}$.

LEMMA 12. [11, 5.7.21] *Suppose A is a nonnegative matrix with ones on the main diagonal, and let μ be the maximum (over all possible cycles) cycle geometric mean. Then there exists a positive diagonal matrix D such that all the entries of $D^{-1}AD$ are at most μ. Furthermore, we have*

$$\min_{D}\left(\max_{i,j} d_i a_{ij} d_j^{-1}\right) = \mu = \max_{i_1,\dots i_k} \sqrt[k]{a_{i_1,i_2}a_{i_2,i_3}\cdots a_{i_{k-1},i_k}a_{i_k,i_1}}.$$

From this lemma we deduce the next result immediately.

COROLLARY 13. *Suppose A is a nonnegative matrix with ones on the main diagonal. Then there exists a positive diagonal matrix D such that DA is column dominant of its entries if and only if the maximum (over all possible cycles) cycle product of A is at most one.*

Recall that any IM matrix $A = [a_{ij}]$ with ones on the main diagonal satisfies the so-called *path-product conditions* $a_{ij} \geqslant a_{ik}a_{kj}$, for any triple of indices i,j,k. These path-product inequalities can be realized by considering any 3-by-3 principal submatrix of such an IM matrix (see also [15]).

THEOREM 14. *The set of all retractions of matrices in IM is contained in X (that is $IM_R \subset X$).*

Proof. The proof is by induction on n. For $n \leqslant 2$ the result is trivial, since IM matrices can be scaled to be diagonally dominant of their column entries. Suppose the result is true for IM matrices of size at most $n-1$. Let A be an IM matrix of size n and assume, without loss of generality, that A has ones on the main diagonal. Let $\tilde{A} = A - tE_{11}$, where $t = \det A/\det A_{11}$. Now assume that $\tilde{A}$ has been diagonally scaled so as to have ones on its main diagonal. Let $B = A(n)$ and let $\tilde{B} = \tilde{A}(n)$. Since A is a Koteljanskii matrix, we have that $\tilde{B}$ is still an entry-wise nonnegative matrix, and hence by Corollary 13 and the induction hypothesis, we may conclude that the maximum cycle product in $\tilde{B}$ is at most one. For example if the cycle of interest is $(1,2,\dots,n-1)$, then the corresponding cycle product in $\tilde{B}$ is given by

$$\frac{a_{1,2}}{1-t}a_{2,3}\cdots a_{n-2,n-1}a_{n-1,1} \leqslant 1.$$

Since A is IM, A satisfies the path-product conditions, and in particular $a_{n-1,1} \geqslant a_{n-1,n}a_{n,1}$. Hence we have

$$\frac{a_{1,2}}{1-t}a_{2,3}\cdots a_{n-2,n-1}a_{n-1,n}a_{n,1} \leqslant 1.$$

Similar arguments can be applied to any such cycle product in $\tilde{B}$. In other words, all cycle products of $\tilde{A}$ are at most one, and hence by Corollary 13, $\tilde{A}$ can be scaled to be column dominant of its entries, and hence A^R is contained in X. This completes the proof. $\square$

We remark here that if a subclass $\mathcal{C}$ is closed under arbitrary permutation similarity, then there is no distinction between the (1,1) entry and other main diagonal entries.

However, for the class TN (which is not closed under arbitrary permutation similarity) this is essential. We include the proof here for completeness (see [7] for more details).

THEOREM 15. [7] *Let A be an n-by-n totally nonnegative matrix with* $\det A(1) \neq 0$. *Then $A - xE_{11}$ is totally nonnegative for all $x \in [0, \frac{\det A}{\det A(1)}]$.*

Proof. Firstly, observe that for every value $x \in [0, \frac{\det A}{\det A(1)}]$, $\det(A - xE_{11}) \geqslant 0$. Recall that A admits a UL-factorization (follows from the LU-factorization result and reversal) into totally nonnegative matrices (see [5]). Partition A as follows,

$$A = \begin{bmatrix} a_{11} & a_{12}^T \\ a_{21} & A(1) \end{bmatrix},$$

where a_{11} is a scalar. Partition L and U conformally with A. Then

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12}^T \\ a_{21} & A(1) \end{bmatrix} = UL = \begin{bmatrix} u_{11} & u_{12}^T \\ 0 & U(1) \end{bmatrix} \begin{bmatrix} l_{11} & 0 \\ l_{21} & L(1) \end{bmatrix} \\ &= \begin{bmatrix} u_{11}l_{11} + u_{12}^T l_{21} & u_{12}^T L(1) \\ U(1)l_{21} & U(1)L(1) \end{bmatrix}. \end{aligned}$$

Consider the matrix $A - xE_{11}$, with $x \in [0, \frac{\det A}{\det A(1)}]$. Then

$$\begin{aligned} A - xE_{11} &= \begin{bmatrix} u_{11}l_{11} + u_{12}^T l_{21} - x & u_{12}^T L(1) \\ U(1)l_{21} & U(1)L(1) \end{bmatrix} \\ &= \begin{bmatrix} u_{11} - \frac{x}{l_{11}} & u_{12}^T \\ 0 & U(1) \end{bmatrix} \begin{bmatrix} l_{11} & 0 \\ l_{21} & L(1) \end{bmatrix} = U'L, \end{aligned}$$

if $l_{11} \neq 0$. Note that if $l_{11} = 0$, then L, and hence A, is singular. In this case $x = 0$ is the only allowed value for x. But in this case the desired result is trivial. Thus we assume that $l_{11} > 0$. To show that $A - xE_{11}$ is totally nonnegative it is enough to verify that $u_{11} - x/l_{11} \geqslant 0$. Since $l_{11} > 0$ and $\det A(1) > 0$ it follows that L and $U(1)$ are nonsingular. Hence $0 \leqslant \det(A - xE_{11}) = (u_{11} - x/l_{11}) \det U(1) \det L$, from which it follows that $u_{11} - x/l_{11} \geqslant 0$. $\square$

COROLLARY 16. *Let TN denote the class of all n-by-n totally nonnegative matrices. Then $TN^{(R)} = TN_R = TN$.*

4. Hadamard Duals of Classes of P_0-Matrices

We begin by presenting a couple of basic properties about the Hadamard dual. Observe that J is the Hadamard identity, that is $A \circ J = J \circ A = A$.

PROPOSITION 17. *Let $\mathcal{C}$ denote a given subclass of the P_0-matrices that contains the matrix J. Then $\mathcal{C}^{(\mathrm{D})} \subset \mathcal{C}$.*

Proof. Since J is in $\mathcal{C}$ it follows that for any A in $\mathcal{C}^{(\mathrm{D})}$, we have $A \circ J = A$ is in $\mathcal{C}$. $\square$

By sequential application of the definition of $\mathcal{C}^{(\mathrm{D})}$ we have

PROPOSITION 18. *Let $\mathcal{C}$ denote a given subclass of the P_0-matrices Suppose A and B are two matrices in $\mathcal{C}^{(\mathrm{D})}$. Then $A \circ B$ is in $\mathcal{C}^{(\mathrm{D})}$.*

From Proposition 17 we have that $PSD^{(\mathrm{D})} = PSD$. Similar arguments can be used to ascertain that $DN^{(\mathrm{D})} = DN$ and $CP^{(\mathrm{D})} = CP$. There is only one subtle observation to make and that is if A, B are two CP matrices, then $A \circ B$ is a CP matrix. This follows since the Hadamard product of two matrices written as a sum of symmetric rank one matrices can be realized as a sum of symmetric rank one matrices (see [10, Chap. 7]).

The Hadamard dual of the positive definite matrices is the set of all positive semidefinite matrices with positive main diagonal. Observe that since the closure of positive definite matrices, the positive semidefinite matrices, contains the matrix J, it follows that $PD^{(\mathrm{D})} \subset PSD$. However, if B is a PSD matrix with a zero main diagonal entry, then $A \circ B$ is necessarily a singular PSD matrix, and therefore it will never be a PD matrix, for any PD matrix A. So $PD^{(\mathrm{D})}$ is contained among the PSD matrices with positive main diagonal. Let B be a PSD matrix with positive main diagonal entries. Then by applying Oppenheim's inequality it follows that $\det(A \circ B) > 0$, whenever A is PD. Hence $A \circ B$ is PD.

For M-matrices, we also have a complete characterization of the Hadamard dual, and it is worth noting that $M^{(\mathrm{D})}$ is not comparable to the class of M-matrices. Before we characterize $M^{(\mathrm{D})}$, we state some necessary facts. Recall that X was defined as the collection of n-by-n matrices A such that there exists a positive diagonal matrix D for which DA is diagonally dominant of its column entries.

LEMMA 19. [12] *If A is an M-matrix and B is an IM-matrix, then $A \circ B$ is an M-matrix. Furthermore, if A is an M-matrix and $B \in X$, then $A \circ B$ is an M-matrix*

We are now in a position to prove our main result on the Hadamard dual of the M-matrices.

THEOREM 20. *The Hadamard dual of the M-matrices coincides with the collection X of entry-wise nonnegative matrices A with positive main diagonal entries having the property that there exists a positive diagonal matrix D such that DA is diagonally dominant of its column entries.*

Proof. The fact that $X \subset M^{(\mathrm{D})}$ follows from Lemma 19. To establish the reverse containment we consider proof by contradiction. First note that any matrix in $M^{(\mathrm{D})}$ must be entry-wise nonnegative and must have positive main diagonal entries. So assume that for such a matrix A there does not exist a positive diagonal matrix D so that DA is diagonally dominant of its column entries. We can assume without loss of generality that A is normalized to have ones its main diagonal. Consider the matrix DAD^{-1}. Since DA is not diagonally dominant of its column entries, neither is the matrix DAD^{-1}, hence we have that

$$\max_{i,j} d_i a_{ij} d_j^{-1} > 1.$$

Therefore, since this is true for all such D, we have

$$\min_D \left(\max_{i,j} d_i a_{ij} d_j^{-1} \right) > 1.$$

So by Lemma 12, there exists a collection of distinct indices $i_1, i_2, \ldots i_k$ such that

$$a_{i_1,i_2} a_{i_2,i_3} \cdots a_{i_{k-1},i_k} a_{i_k,i_1} > 1.$$

Since permuting rows and columns simultaneously does not disturb any of the properties or the analysis above, we may assume that $\{1,2,\ldots,k\} = \{i_1, i_2, \ldots, i_k\}$. Now let B be the M-matrix of the form

$$B = \left[\begin{array}{c|c} I-(1-\varepsilon)P & 0 \\ \hline 0 & I \end{array} \right],$$

where P is the k-by-k cyclic shift matrix and $\varepsilon > 0$. Then for small enough ε, $B \circ A$ is not an M-matrix since the leading minor $(B \circ A)[1,2,\ldots,k]$ is negative. This completes the proof. $\square$

The Hadamard dual of the inverse M-matrices is still unknown in general, and in fact appears to be rather complicated to describe. Certainly $IM^{(\mathrm{D})}$ is contained in the entry-wise nonnegative matrices (in fact $IM^{(\mathrm{D})}$ is contained in the closure of IM), and $IM^{(\mathrm{D})} \neq IM$ for $n \geqslant 4$ (see [20]). In addition, it is straightforward to verify that matrices of the form $D + EJF$, where E and F are positive diagonal matrices, also belong to $IM^{(\mathrm{D})}$.

An entry-wise positive matrix $A = [a_{ij}]$ is called a *path-product matrix* (PP) if, for any triple of indices $i,j,k \in \{1,2,\ldots,n\}$,

$$\frac{a_{ij} a_{jk}}{a_{jj}} \leqslant a_{ik},$$

and *strict path product* (SPP) if it is PP and the above inequality is strict whenever $i = k$ (see [15]). If $A = [a_{ij}]$ is a path-product matrix with ones on its main diagonal, then the path-product inequalities become $a_{ij} a_{jk} \leqslant a_{ik}$. It is well-known, and can be easily verified by considering the 3-by-3 principal submatrix of A based on the indices $\{i,j,k\}$, that IM $\subset$ SPP (see also [15, 16, 17]). Furthermore, for $n \leqslant 3$, IM $=$ SPP (see [15]). We claim that $PP^{(\mathrm{D})} = PP$. To verify this note that since J is a PP matrix we have that $PP^{(\mathrm{D})} \subset PP$, and if A and B are two PP matrices then, clearly $A \circ B$ is a PP matrix. Knowing that the Hadamard dual of the PP matrices is precisely the PP matrices, we can deduce that the dual of the IM matrices is contained in the PP matrices (again consider arbitrary 3-by-3 principal submatrices).

Recall that adding to the main diagonal of an IM matrix yields an IM matrix (see [13]). In fact, we have the following sufficient condition for membership in IM$^{(\mathrm{D})}$.

THEOREM 21. *Suppose A is an n-by-n IM matrix, normalized in such a way that all of its main diagonal entries are equal to $n-2$. If $A-(n-3)I$ is a PP matrix, then A is in $IM^{(\mathrm{D})}$.*

Proof. Let B be any n-by-n IM matrix with ones on its main diagonal. Then $C = B \circ (A-(n-3)I)$ is a PP matrix with ones on its main diagonal. Hence it follows that $C+(n-3)I$ is an IM matrix (see [17]). Hence $A \circ B$ is an IM matrix, which implies that A is in $IM^{(\mathrm{D})}$. $\square$

For the class TN, we do not have a complete characterization of the Hadamard dual, but we refer the reader to the paper [4], that contains numerous results on the Hadamard dual of TN. We mention just two relevant results here for completeness.

THEOREM 22. [4] *Let T be an n-by-n totally nonnegative tridiagonal matrix. Then T is in $TN^{(\mathrm{D})}$.*

For $n \leqslant 3$ we have the following characterization of the Hadamard dual for the TN matrices (see [4] for more details).

THEOREM 23. [4] *Let A be a 3-by-3 matrix. Then A is in $TN^{(\mathrm{D})}$ if and only if $A \circ W$ and $A \circ W^T$ are both totally nonnegative, in which* $W = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Finally, for the class of P-matrices (or P_0-matrices) the Hadamard dual seems rather unclear and we close this section with a suggestion for a description of the set $P^{(\mathrm{D})}$. Suppose that

$$A = \begin{bmatrix} A' & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_m(\mathbb{C}) \text{ and } B = \begin{bmatrix} b_{22} & b_{23} \\ b_{32} & B' \end{bmatrix} \in M_n(\mathbb{C}),$$

in which a_{22}, b_{22} are scalars. Then we call

$$C = \begin{bmatrix} A' & a_{12} & 0 \\ a_{21} & a_{22}+a_{22} & b_{23} \\ 0 & b_{32} & B' \end{bmatrix}$$

the 1-*subdirect sum* of A and B, which we denote by $A \oplus_1 B$. A basic fact of interest here is that if A and B are two P-matrices, then so is $A \oplus_1 B$ (see [6]).

Now suppose that D and E are two diagonal matrices with the property that DE is nonnegative. Then observe that for any P-matrix A, we have $A \circ DJE = D(A \circ J)E = DAE$ is also a P-matrix. Also adding a positive diagonal matrix to any P-matrix results in a P-matrix.

Let Y denote the collection of n-by-n matrices of the form $A_1 \oplus_1 A_2 \oplus_1 \cdots \oplus_1 A_k$, where each summand A_i can be written as $D_i + E_i J F_i$ in which D_i is a positive diagonal matrix and E_i, F_i are diagonal matrices that satisfy $E_i F_i$ is nonnegative.

Then, we conjecture that the Hadamard dual of the P-matrices (or P_0-matrices) is Y. In fact from the discussions above it is clear that Y is contained in the Hadamard dual of the P- or P_0-matrices. The reverse containment is true for $n \leqslant 3$, and we suspect that it may be true for $n \geqslant 4$.

REFERENCES

[1] B. A. ASNER, *On the Total Nonnegativity of the Hurwitz Matrix*, SIAM J. Appl. Math. **18** No. 2 (1970) 407–414.

[2] D. CARLSON, *Weakly, Sign-Symmetric Matrices and Some Determinantal Inequalities*, Colloq. Math. **17** (1967) 123–129.

[3] S. CHEN, *A property concerning the Hadamard powers of inverse M-matrices*, Linear Algebra Appl., **381** (2004) 53–60.

[4] A. S. CRANS, S. M. FALLAT AND C. R. JOHNSON, *The Hadamard core of the totally nonnegative matrices*, Linear Algebra Appl., **328** (2001) 203–222.

[5] S. M. FALLAT, *Bidiagonal Factorizations of Totally Nonnegative Matrices*, American Math. Monthly **109** (2001) 697–712.

[6] S. M. FALLAT AND C. R. JOHNSON, *Sub-direct Sums and Positivity Classes of Matrices*, Linear Algebra Appl. **288** (1999) 149–173.

[7] S. M. FALLAT, C. R. JOHNSON AND R. L. SMITH, *The General Totally Positive Matrix Completion Problem with Few Unspecified Entries*, Electron. J. Linear Algebra, **7** (2000) 1–20.

[8] J. GARLOFF AND D. G. WAGNER, *Hadamard Products of Stable Polynomials are Stable*, J. Math. Anal. Appl. **202** (1996) 797–809.

[9] R. A. HORN, *The Hadamard Product*, in Matrix Theory and Applications, C. R. Johnson, ed., Proc. Sympos. Appl. Math., **40** (1089) 87–169.

[10] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press, New York, 1985.

[11] R. A. HORN AND C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, New York, 1991.

[12] C. R. JOHNSON, *A Hadamard Product Involving M-Matrices*, Linear and Multilinear Algebra, **4** (1977) 261–264.

[13] C. R. JOHNSON, *Inverse M-matrices*, Linear Algebra Appl. **47** (1982) 195–216.

[14] C. R. JOHNSON, *Closure Properties of Certain Positivity Classes of Matrices under Various Algebraic Operations*, Linear Algebra Appl. **97** (1987) 243–247.

[15] C. R. JOHNSON AND R. L. SMITH, *Path Product Matrices*, Linear and Multilinear Algebra, **46** (1999) 177–191.

[16] C. R. JOHNSON AND R. L. SMITH, *Path Product matrices and Eventually Inverse M-matrices*, to appear in SIAM Journal on Matrix Anal. Appl.

[17] C. R. JOHNSON AND R. L. SMITH, *Positive, Path Product, and Inverse M-matrices*, to appear in Linear Algebra Appl.

[18] C. R. JOHNSON AND R. L. SMITH, *Monotonicity of Continuous Hadamard Powers of Inverse M-matrices*, to be submitted.

[19] D. G. WAGNER, *Total Positivity of Hadamard Products*, J. Math. Anal. Appl. **163** (1992) 459–483.

[20] B-Y. WANG, X. ZHANG AND F. ZHANG, *On the Hadamard product of inverse M-matrices*, Linear Algebra Appl. **305** (2000) 23–31.

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