Operators
and
Matrices
Volume 1, Number 3 (2007), 385–389

LOCALIZATIONS OF THE KLEINECKE–SHIROKOV THEOREM

JANKO BRAČIČ AND BOJAN KUZMA

(*communicated by M. Omladič*)

Abstract. A local version of the Kleinecke-Shirokov theorem is proved. The results easily extend to bounded linear derivations on Banach algebras. In case of algebraic elements, an improved bound on the nilindex of a commutator is obtained as a consequence.

1. Localizations of the Kleinecke-Shirokov Theorem

Let $\mathcal{X}$ be a complex Banach space and $\mathcal{B}(\mathcal{X})$ be the Banach algebra of all bounded linear operators on $\mathcal{X}$. If $A,\ B \in \mathcal{B}(\mathcal{X})$ are operators such that the commutant $[A,B] := AB - BA$ commutes with A, then the Kleinecke-Shirokov Theorem [7, 12] (see also [4, Problem 184], and [1, 2, 6, 8, 10] for some generalizations of the theme) asserts that $[A,B]$ is a quasinilpotent operator. Actually the Kleinecke-Shirokov theorem holds for any Banach algebra. It follows that the local spectral radius of $[A,B]$ at any vector $x \in \mathcal{X}$ is zero, that is

$$r_{[A,B]}(x) := \limsup_{n\to\infty} \|[A,B]^n x\|^{1/n} = 0 \qquad (x \in \mathcal{X}).$$

Now, assume that A and $[A,B]$ commute only locally, that is, there is a closed subspace $\mathcal{Y}$ of $\mathcal{X}$ such that $[A,[A,B]]y = 0$ for all $y \in \mathcal{Y}$. Do there exist vectors $0 \neq x \in \mathcal{X}$ at which the local spectral radius of $[A,B]$ is zero? We shall give a positive answer for spaces related to the kernel and the range of A.

For $T \in \mathcal{B}(\mathcal{X})$, let $\operatorname{Lat} T$ be the lattice of all closed T-invariant subspaces of $\mathcal{X}$. We start with the following simple observation.

PROPOSITION 1.1. *Let $A, B \in \mathcal{B}(\mathcal{X})$ and assume that $\mathcal{Y} \in \operatorname{Lat} A \cap \operatorname{Lat} B$. If $[A,[A,B]]\mathcal{Y} = \{0\}$, then $r_{[A,B]}(y) = 0$ for all $y \in \mathcal{Y}$.*

Proof. Let $\widetilde{A} := A|_{\mathcal{Y}}$ and $\widetilde{B} := B|_{\mathcal{Y}}$. These are bounded operators on $\mathcal{Y}$ and it follows from $[A,[A,B]]\mathcal{Y} = \{0\}$ that $[\widetilde{A},[\widetilde{A},\widetilde{B}]] = 0$. By Kleinecke-Shirokov theorem, the commutant $[\widetilde{A},\widetilde{B}]$ is quasinilpotent, which gives

Mathematics subject classification (2000): 47B47, 47B48, 46H05.

Key words and phrases: Banach algebra, Kleinecke-Shirokov Theorem, nilindex.

Paper No. 01-22

$$
\begin{aligned}
r_{[A,B]}(y) &= \limsup_{n\to\infty} \|[A,B]^n y\|^{1/n} = \limsup_{n\to\infty} \|[\widetilde{A},\widetilde{B}]^n y\|^{1/n} \\
&\leqslant \limsup_{n\to\infty} \|[\widetilde{A},\widetilde{B}]^n\|^{1/n} \cdot \|y\|^{1/n} = 0
\end{aligned}
$$

for any $y \in \mathcal{Y}$. $\square$

Let $A \in \mathcal{B}(\mathcal{X})$. In the next proposition we will show that sometimes there are non-trivial proper subspaces $\mathcal{Y}$ in $\operatorname{Lat} A$ such that $[A,[A,B]]\mathcal{Y} = \{0\}$ forces $\mathcal{Y} \in \operatorname{Lat} B$. We introduce the necessary notation.

For $\lambda \in \mathbb{C}$, let $\mathcal{N}_\lambda(A)$ be the closure of $\bigcup_{n=1}^\infty \ker(A-\lambda)^n$. If $A-\lambda$ has a finite *ascent*, that is, $a(A-\lambda) := \min\{n;\ \ker(A-\lambda)^n = \ker(A-\lambda)^{n+1}\}$ is a positive integer, then $\mathcal{N}_\lambda(A) = \ker(A-\lambda)^{a(A-\lambda)}$. The *cœur* of A (see [11], [9, C.12.2], and [3] for relevant set–theoretical properties) is a linear subspace $\operatorname{c\oe} A$ of $\mathcal{X}$ defined as follows. Let $\operatorname{im}_0 A := \mathcal{X}$, let $\operatorname{im}_{\alpha+1} A := A(\operatorname{im}_\alpha A)$, and let $\operatorname{im}_\alpha A := \cap_{\beta<\alpha} \operatorname{im}_\beta A$ for a limit (that is, without predecessor) ordinal α. The collection of these subspaces is decreasing, and forms a set. It can, therefore, be shown that there exists an ordinal ξ with $\operatorname{im}_\xi A = \operatorname{im}_{\xi+1} A$. Then $\operatorname{c\oe} A := \operatorname{im}_\xi A = \cap_{\alpha<\xi+1} \operatorname{im}_\alpha A$. The cœur of A, though not necessarily closed, is the maximal subspace of $\mathcal{X}$ that satisfies the condition $A(\operatorname{c\oe} A) = \operatorname{c\oe} A$ (see [11]). Let $\mathcal{R}_\lambda(A)$ be the closure of $\operatorname{c\oe}(A-\lambda)$. If the *descent* of $A-\lambda$, that is, $d(A-\lambda) := \min\{n;\ \operatorname{im}(A-\lambda)^n = \operatorname{im}(A-\lambda)^{n+1}\}$, is finite, then $\mathcal{R}_\lambda(A) = \overline{\operatorname{im}(A-\lambda)^{d(A-\lambda)}}$. Of course, $\mathcal{N}_\lambda(A)$ and $\mathcal{R}_\lambda(A)$ are in $\operatorname{Lat} A$.

The *inner derivation* on $\mathcal{B}(\mathcal{X})$ induced by A is a bounded liner map given by $\delta_A(B) := [A,B]$ $(B \in \mathcal{B}(\mathcal{X}))$. Note that $\delta_A^k(B) = \sum_{j=0}^k (-1)^j \binom{k}{j} A^{k-j} B A^j$, where we agreed upon $A^0 := \mathrm{Id}$.

PROPOSITION 1.2. *Let $A \in \mathcal{B}(\mathcal{X})$ and $\lambda \in \mathbb{C}$.*

(*i*) *If ${\delta_A}^k(B)\mathcal{N}_\lambda(A) = \{0\}$, for some $B \in \mathcal{B}(\mathcal{X})$ and some positive integer k, then $\mathcal{N}_\lambda(A) \in \operatorname{Lat} B$.*

(*ii*) *If ${\delta_A}^k(B)\mathcal{R}_\lambda(A) = \{0\}$, for some $B \in \mathcal{B}(\mathcal{X})$ and some positive integer k, then $\mathcal{R}_\lambda(A) \in \operatorname{Lat} B$.*

Proof. Since $\delta_A = \delta_{A-\lambda}$, there is no loss of generality if we assume that $\lambda = 0$.

To prove (i), choose an arbitrary vector $x \in \bigcup_{n=1}^\infty \ker A^n$. Then there exists a positive integer m such that $A^m x = 0$. Clearly, the vectors $A^i x$ are in $\mathcal{N}_0(A)$, for each $i \in \{0,1,\dots,m-1\}$ so, by the assumption, ${\delta_A}^k(B)A^i x = 0$. Hence, with $i := m-1$ we have

$$
0 = {\delta_A}^k(B)A^{m-1}x = \sum_{j=0}^k (-1)^j \binom{k}{j} A^{k-j} B A^{m-1+j} x = A^k B A^{m-1} x. \tag{1}
$$

The equality ${\delta_A}^k(B)A^{m-2}x = 0$ similarly gives $A^k B A^{m-2} x - k A^{k-1} B A^{m-1} x = 0$. If we multiply this equality by A and use (1), we get $A^{k+1} B A^{m-2} x = 0$. Using induction backwards, we are left with $A^{k+m} B x = 0$. Therefore, $Bx \in \bigcup_{n=1}^\infty \ker A^n$, and so $B\mathcal{N}_\lambda(A) \subseteq \mathcal{N}_\lambda(A)$.

We proceed to prove (ii) with transfinite induction. It is trivial that $B(\operatorname{cœ} A) \subseteq \operatorname{im}_0 A$. Pick an ordinal α, and assume that we have $B(\operatorname{cœ} A) \subseteq \operatorname{im}_\beta A$ for each ordinal $\beta < \alpha$. Consequently, if α is a limit ordinal then $B(\operatorname{cœ} A) \subseteq \cap_{\beta<\alpha} \operatorname{im}_\beta A = \operatorname{im}_\alpha A$.

Suppose lastly α is a nonlimit ordinal, say $\alpha = \alpha' + 1$. Let $x \in \operatorname{cœ} A$ be arbitrary. Since $\operatorname{cœ} A = A(\operatorname{cœ} A) = \cdots = A^k(\operatorname{cœ} A)$ there exists a vector $y \in \operatorname{cœ} A$ such that $x = A^k y$. It follows from ${\delta_A}^k(B)(\operatorname{cœ} A) = \{0\}$ that

$$Bx = BA^k y = -\left((-1)^k A^k By + (-1)^{k-1}\binom{k}{1}A^{k-1}BAy \pm \cdots - \binom{k}{k-1}ABA^{k-1}y\right). \quad (2)$$

Now, vectors $y, Ay, \ldots, A^{k-1}y$ are in $\operatorname{cœ} A$ and therefore, by the induction hypothesis, $By, BAy, \ldots, BA^{k-1}y$ are all in $B(\operatorname{cœ} A) \subseteq \operatorname{im}_{\alpha'} A$. Since $A^k(\operatorname{im}_{\alpha'} A) \subseteq A(\operatorname{im}_{\alpha'} A)$ for each $k \geqslant 1$, we conclude from (2) that $Bx \in A^k(\operatorname{im}_{\alpha'} A) + \cdots + A(\operatorname{im}_{\alpha'} A) = A(\operatorname{im}_{\alpha'} A) = \operatorname{im}_{\alpha'+1} A = \operatorname{im}_\alpha A$. Hence, $B(\operatorname{cœ} A) \subseteq \operatorname{im}_\alpha A$.

By transfinite induction, $B(\operatorname{cœ} A) \subseteq \operatorname{cœ} A$ and consequently $\mathcal{R}_0(A) = \overline{\operatorname{cœ} A} \in \operatorname{Lat} B$. $\square$

THEOREM 1.3. *Let $A \in \mathcal{B}(\mathcal{X})$ and let $\mathcal{Y}$ be the closure of finite sum of spaces $\mathcal{R}_\lambda(A)$ and $\mathcal{N}_\mu(A)$, for instance, let*

$$\mathcal{Y} = \overline{\mathcal{R}_{\lambda_1}(A) + \cdots + \mathcal{R}_{\lambda_m}(A) + \mathcal{N}_{\mu_1}(A) + \cdots + \mathcal{N}_{\mu_n}(A)}, \quad (3)$$

where $\lambda_1, \ldots, \lambda_m$, $\mu_1, \ldots, \mu_n$ are arbitrary complex numbers. If $B \in \mathcal{B}(\mathcal{X})$ is such that $[A,[A,B]]\mathcal{Y} = \{0\}$ then $\mathcal{Y}$ is invariant for B and $r_{[A,B]}(y) = 0$ for every $y \in \mathcal{Y}$.

Proof. Assume that $\mathcal{Y}$ is of the form (3). It follows from $[A,[A,B]]\mathcal{Y} = \{0\}$ that

$$[A,[A,B]]\mathcal{R}_{\lambda_i}(A) = \{0\} \quad \text{and} \quad [A,[A,B]]\mathcal{N}_{\mu_j}(A) = \{0\} \quad \text{for all } 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n.$$

Thus, all spaces $\mathcal{R}_{\lambda_i}(A)$ and $\mathcal{N}_{\mu_j}(A)$ are in $\operatorname{Lat} B$, which gives $\mathcal{Y} \in \operatorname{Lat} B$. Now the assertion follows by Proposition 1.1. $\square$

2. Jacobson's Lemma

If $\mathcal{X}$ is a finite dimensional vector space, then the Kleinecke-Shirokov Theorem reduces to the Jacobson's Lemma [5, Lemma 2], which says that $[A,B]$ is nilpotent if $A, B \in \mathcal{B}(\mathcal{X})$ are such that $[A,[A,B]] = 0$. The original proof [5, Lemma 2], and its extension [6], bound the nilindex of $[A,B]$ above by $2^n - 1$ where n is the degree of the minimal polynomial for A. Arguments run as follows: Let $'$ be a derivation such that A' commutes with A and let f be the minimal polynomial of A. Differentiating $f(A) = 0$ gives $f'(A)A' = 0$, which is the case $k = 1$ of $f^{(k)}(A)(A')^{2^k-1} = 0$. Differentiating produces $f^{(k+1)}(A)A'(A')^{2^k-1} + f^{(k)}(A)(A''(A')^{2^k-2} + A'A''(A')^{2^k-3} + \cdots + (A')^{2^k-2}A'') = 0$. Now, premultiply with $(A')^{2^k-1}$ to get the induction step. We remark that if A'' commutes with A', similar arguments would bound nilindex above by $2n - 1$.

We shall use the results from the previous section to improve the estimate on the upper bound of the nilindex of $[A,B]$ (see Theorem 2.5 below).

PROPOSITION 2.4. *Let $\mathcal{X}$ be a complex Banach space.*

(i) If $A \in \mathcal{B}(\mathcal{X})$ is a nilpotent operator with nilindex $n \geqslant 1$, then the inner derivation δ_A is a nilpotent operator on $\mathcal{B}(\mathcal{X})$ with nilindex $2n-1$.

(ii) Let $A \in \mathcal{B}(\mathcal{X})$ be a nilpotent operator with nilindex $n \geqslant 1$ and let $B \in \mathcal{B}(\mathcal{X})$ be such that ${\delta_A}^2(B) = 0$. Then $\big(\delta_A(B)\big)^{2n-1} = 0$.

Proof. (i) For a nilpotent operator A with nilindex n, we have

$$ {\delta_A}^{2n-1}(T) = \sum_{j=0}^{2n-1} (-1)^j \binom{2n-1}{j} A^{2n-1-j} T A^j = 0 \qquad (T \in \mathcal{B}(\mathcal{X})), $$

which shows that ${\delta_A}^{2n-1} = 0$.

On the other hand, let $x \in \mathcal{X}$ and $T \in \mathcal{B}(\mathcal{X})$ be such that $A^{n-1}x \neq 0$ and $TA^{n-1}x = x$. Then,

$$ \begin{aligned} {\delta_A}^{2n-2}(T)x &= \sum_{j=0}^{2n-2} (-1)^j \binom{2n-2}{j} A^{2n-2-j} T A^j x = (-1)^{n-1} \binom{2n-2}{n-1} A^{n-1} T A^{n-1} x \\ &= (-1)^{n-1} \binom{2n-2}{n-1} x \neq 0 \end{aligned} $$

gives ${\delta_A}^{2n-2} \neq 0$.

(ii) The classical proof of Kleinecke-Shirokov [4, Solution 184] shows that ${\delta_A}^2(B) = 0$ implies ${\delta_A}^{2n-1}\big(B^{2n-1}\big) = (2n-1)!\big(\delta_A(B)\big)^{2n-1}$. By the first part of this proposition, δ_A is a nilpotent operator with nilindex $2n-1$. Thus, $\big(\delta_A(B)\big)^{2n-1} = 0$. $\square$

Assume that the ascent of $A \in \mathcal{B}(\mathcal{X})$ is a positive integer m. That is, $\mathcal{N}_0(A) = \overline{\bigcup_{n=1}^{\infty} \ker A^n} = \ker A^m$. If $[A,[A,B]]\mathcal{N}_0(A) = \{0\}$, for some $B \in \mathcal{B}(\mathcal{X})$, then, by (i) of Proposition 1.2, $\mathcal{N}_0(A)$ is invariant for B. Let $\tilde{A}$ and $\tilde{B}$ be the restrictions of A and B to $\mathcal{N}_0(A)$. Then $\tilde{A}$ is nilpotent with nilindex m and we have $[\tilde{A},[\tilde{A},\tilde{B}]] = 0$. It follows, by Proposition 2.4, that $[\tilde{A},\tilde{B}]^{2m-1} = 0$, which gives $[A,B]^{2m-1}\mathcal{N}_0 = \{0\}$. Thus, the local nilindex of $[A,B]$ on $\mathcal{N}_0(A)$ is $2m-1$.

THEOREM 2.5. *Let $A \in \mathcal{B}(\mathcal{X})$ be an algebraic operator with the minimal polynomial $q_A(z) = (z-\lambda_1)^{m_1}\cdots(z-\lambda_k)^{m_k}$. If $[A,[A,B]] = 0$ for $B \in \mathcal{B}(\mathcal{X})$, then $[A,B]$ is a nilpotent operator with nilindex at most $2\cdot\max\{m_1,\dots,m_k\}-1$.*

Proof. For each $1 \leqslant i \leqslant k$, let $\mathcal{M}_i := \ker(A-\lambda_i)^{m_i}$ (thus $\mathcal{M}_i = \mathcal{N}_{\lambda_i}(A)$ in the notation used above). Then $\mathcal{X} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_k$. Since $[A-\lambda_i,[A-\lambda_i,B]] = [A,[A,B]] = 0$ we have $[A-\lambda_i,[A-\lambda_i,B]]\mathcal{M}_i = \{0\}$. The restriction of $A-\lambda_i$ to $\mathcal{M}_i$ is a nilpotent with nilindex m_i. It follows that the local nilindex of $[A-\lambda_i,B] = [A,B]$ on $\mathcal{M}_i$ is at most $2m_i-1$. Let $x = x_1 \oplus \cdots \oplus x_k$ be the decomposition of $x \in \mathcal{X}$ with $x_i \in \mathcal{M}_i$. Then, of course, $[A,B]^{2m-1}x = 0$, where $m = \max\{m_1,\dots,m_k\}$. $\square$

COROLLARY 2.6. *If A is a diagonalizable matrix then $[A,[A,B]] = 0$ implies $[A,B] = 0$.*

Proof. The minimal polynomial of A is a product of distinct linear factors. $\square$

Note that a diagonalizable matrix is similar to a diagonal, hence to a normal matrix. With this in mind, Corollary 2.6 can also be derived from Anderson's results [1] on range-kernel orthogonality of normal derivations; see also [10, Theorem 3].

REFERENCES

[1] J. ANDERSON, *On normal derivations*, Proc. Amer. Math. Soc. **38** (1973), 135–140.
[2] M. BREŠAR, A. FOŠNER, AND M. FOŠNER, *Jordan ideals revisited*, Monatsh. Math. **145** (2005), no. 1, 1–10.
[3] J. DUGUNDJI, *Topology*, Wm. C. Brown publishers, 1989.
[4] P. R. HALMOS, *A Hilbert Space Problem Book*, Springer, 1974.
[5] N. JACOBSON, *Rational methods in the theory of Lie algebras*, Ann. of Math. **36** (1935), 875–881.
[6] I. KAPLANSKY, *Jacobson's Lemma Revisited*, Journal of Algebra, **62** (1980), 473–476.
[7] D. C. KLEINECKE, On operator commutators, *Proc. Amer. Math. Soc.* **8** (1957) 535–536.
[8] M. MATHIEU, *Where to find the image of a derivation*, Functional analysis and operator theory (Warsaw, 1992), 237–249, Banach Center Publ., **30**, Polish Acad. Sci., Warsaw, 1994.
[9] V. MÜLLER, *Spectral Theory of Linear Operators*, Birkhäuser, 2003.
[10] M. ROITMAN, *Some commutativity results*, Bull. Sci. Math. **108** (1984), no. 3, 289–296.
[11] P. SAPHAR, *Contribution a l'étude des applications linéaires dans un espace de Banach*, Bull. Soc. Math. France, **92** 1964, 363–384.
[12] F. V. SHIROKOV, *Proof of a conjecture by Kaplansky*, Uspekhi Mat. Nauk **11** (1956), no. 4, 167–168 (in Russian).

(Received November 7, 2006)

Janko Bračič
University of Ljubljana, IMFM
Jadranska ul. 19
SI-1000 Ljubljana, Slovenia
e-mail: janko.bracic@fmf.uni-lj.si

Bojan Kuzma
University of Primorska
Cankarjeva 5
SI-6000 Koper, Slovenia

IMFM, Jadranska 19
1000 Ljubljana, Slovenia
e-mail: bojan.kuzma@pef.upr.si