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PROJECTIONS AND THE KADISON–SINGER PROBLEM

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Abstract. We prove some new equivalences of the paving conjecture and obtain some estimates on the paving constants. In addition we give a new family of counterexamples to one of the Akemann-Anderson conjectures.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space and let $B(\mathcal{H})$ denote the bounded, linear operators on $\mathcal{H}$. By a *MASA* we mean a maximal, abelian subalgebra of $B(\mathcal{H})$. R. Kadison and I. Singer studied [18] whether or not pure states on a MASA extend uniquely to states on $B(\mathcal{H})$. In their original work on this subject [18], it was shown that this question has a negative answer if the MASA had any continuous part. The remaining case, whether or not pure states on discrete MASA's have unique extensions to states on $B(\mathcal{H})$, has come to be known as the *Kadison-Singer problem.* The statement that pure states on discrete MASA's have unique extensions has come to be known as the *Kadison-Singer conjecture,* in spite of the fact that neither Kadison nor Singer made this conjecture and quite possibly believed the opposite.

The work of Kadison-Singer showed that their problem was equivalent to certain questions about "paving" operators by projections. J. Anderson[2] developed this idea significantly into a series of so-called "paving" conjectures, which are true if and only if the Kadison-Singer conjecture is true. Since the time of Anderson's work, there has been a great deal of research on these paving conjectures [1], [5], [6], [8], [10], [12], [14], [15] and [16].

In this paper, we begin by restating some of these paving conjectures and add a few new equivalent paving conjectures.

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2. Some New Equivalences of the Paving Conjecture

Let us begin with the familiar.

Given $A \subseteq I$, where I is some index set, we let $Q_A \in B(\ell^2(I))$ denote the diagonal projection defined by $Q_A = (q_{i,j}), q_{i,i} = 1, i \in A, q_{i,i} = 0, i \notin A$ and $q_{i,j} = 0, i \neq j$.

DEFINITION 1. An operator $T \in B(\ell^2(I))$ is said to have an (r,ϵ)-paving if there is a partition of I into r subsets $\{A_j\}_{j=1}^r$ such that $\|Q_{A_j}TQ_{A_j}\| \leqslant \epsilon$. A collection of operators $\mathcal{C}$ is said to be (r,ϵ)-pavable if each element of $\mathcal{C}$ has an (r,ϵ)-paving.

Note that in this definition, we do *not* require that the diagonal entries of the operator be 0.

Some classes that will play a role are:

- $\mathcal{C}_\infty = \{T = (t_{i,j}) \in B(\ell^2(\mathbb{N})) : \|T\| \leqslant 1,\ t_{i,i} = 0,\ \forall i \in \mathbb{N}\}$,
- $\mathcal{C} = \cup_{n=2}^\infty \{T = (t_{i,j}) \in M_n : \|T\| \leqslant 1,\ t_{i,i} = 0,\ i = 1,...,n\}$,
- $\mathcal{S}_\infty = \{T \in \mathcal{C}_\infty : T = T^*\}$,
- $\mathcal{S} = \{T \in \mathcal{C} : T = T^*\}$,
- $\mathcal{R}_\infty = \{T \in \mathcal{S}_\infty : T^2 = I\}$,
- $\mathcal{R} = \{T \in \mathcal{S} : T^2 = I\}$,
- $\mathcal{P}_{1/2}^\infty = \{T = (t_{i,j}) \in B(\ell^2(\mathbb{N})) : T = T^* = T^2,\ t_{i,i} = 1/2,\ \forall i \in \mathbb{N}\}$,
- $\mathcal{P}_{1/2} = \cup_{n=2}^\infty \{T = (t_{i,j}) \in M_n : T = T^* = T^2,\ t_{i,i} = 1/2,\ i = 1,...,n\}$.

Note that the operators satisfying, $R = R^*$, $R^2 = I$ are *reflections* and that for such an operator, $\sigma(R) = \{-1,+1\}$. Since the traces of our matrices are 0, in the finite dimensional case these types of reflections can only exist in even dimensions. If the space is $2n$-dimensional, then there exists an n-dimensional subspace that is fixed by R and such that for any vector x orthogonal to the subspace $Rx = -x$.

J. Anderson's [2] remarkable contribution follows.

THEOREM 2. (Anderson)*The following are equivalent:*

(1) *the Kadison-Singer conjecture is true,*

(2) *for each $T \in \mathcal{C}_\infty$, there exists (r,ϵ) (depending on T) $\epsilon < 1$, such that T is (r,ϵ)-pavable,*

(3) *there exists $(r,\epsilon), \epsilon < 1$, such that $\mathcal{C}_\infty$ is (r,ϵ)-pavablle,*

(4) *there exists $(r,\epsilon), \epsilon < 1$, such that $\mathcal{C}$ is (r,ϵ)-pavable,*

(5) *for each $T \in \mathcal{S}_\infty$, there exists $(r,\epsilon), \epsilon < 1$ (depending on T), such that T is (r,ϵ)-pavable,*

(6) *there exists $(r,\epsilon), \epsilon < 1$, such that $\mathcal{S}_\infty$ is (r,ϵ)-pavable,*

(7) *there exists $(r,\epsilon), \epsilon < 1$, such that $\mathcal{S}$ is (r,ϵ)-pavable.*

Generally, when people talk about the paving conjecture they mean one of the above equivalences of the Kadison-Singer problem. Also, generally, when one looks at operators on an infinite dimensional space, it is enough to find (r,ϵ) depending on the operator, but for operators on finite dimensional spaces it is essential to have a uniform (r,ϵ), for all operators of norm one. Finally, since $\mathcal{S}_\infty \subset \mathcal{C}_\infty$, people looking for counterexamples tend to study $\mathcal{C}_\infty$, while people trying to prove the theorem is true, study $\mathcal{S}_\infty$ or $\mathcal{S}$. However, by the above equivalences, if a counterexample exists in one set then it must exist in the other as well.

In this spirit, we prove that the following smaller sets with "more structure" are sufficient for paving.

THEOREM 3. *If* $\epsilon < 1$, *then the following are equivalent:*

(1) *the set* $\mathcal{S}_\infty$ *can be* (r_1, ϵ)*-paved,*
(2) *the set* $\mathcal{R}_\infty$ *can be* (r_1, ϵ)*-paved,*
(3) *the set* $\mathcal{P}_{1/2}^\infty$ *can be* $(r_2, \frac{1+\epsilon}{2})$*-paved,*
(4) *the set* $\mathcal{S}$ *can be* (r_1, ϵ)*-paved,*
(5) *the set* $\mathcal{R}$ *can be* (r_1, ϵ)*-paved,*
(6) *the set* $\mathcal{P}_{1/2}$ *can be* $(r_2, \frac{1+\epsilon}{2})$*-paved.*

Proof. Since the reflections are a subset of the self-adjoint matrices, it is clear that (1) implies (2) and that (4) implies (5).

To see that (2) implies (1), let $A \in \mathcal{S}_\infty$, and set

$$R = \begin{pmatrix} A & \sqrt{I - A^2} \\ \sqrt{I - A^2} & -A \end{pmatrix},$$

then $R \in \mathcal{R}_\infty$ and clearly any (r, ϵ)-paving of R yields an (r, ϵ)-paving of A.

Thus, (1) and (2) are equivalent and similarly, (4) and (5) are equivalent.

To see the equivalence of (2) and (3), note that $R \in \mathcal{R}_\infty$ (respectively, $\mathcal{R}$) if and only if $P = (I + R)/2 \in \mathcal{P}_{1/2}^\infty$ (respectively, $\mathcal{P}_{1/2}$). Also, if $\|Q_A R Q_A\| \leqslant \epsilon$, then $\|Q_A P Q_A\| \leqslant (1 + \epsilon)/2$. Thus, if $\mathcal{R}_\infty$ can be (r_1, ϵ)-paved, then $\mathcal{P}_{1/2}^\infty$ can be $(r_1, \frac{1+\epsilon}{2})$-paved.

Conversely, given $R \in \mathcal{R}_\infty$, let $P = (I + R)/2$. If $\|Q_A P Q_A\| \leqslant (1 + \epsilon)/2 = \beta$, then,

$$0 \leqslant Q_A P Q_A \leqslant \beta Q_A,$$

and since $R = 2P - I$, we have that

$$-Q_A \leqslant Q_A R Q_A \leqslant (2\beta - 1) Q_A = \epsilon Q_A.$$

Applying the same reasoning to the reflection $-R$, we get a new projection, $P_1 = (I - R)/2$, with a possibly different paving of P_1, such that $-Q_B \leqslant Q_B(-R)Q_B \leqslant \epsilon Q_B$. Thus, $-\epsilon Q_B \leqslant Q_B R Q_B$ and if $Q_C = Q_A Q_B$, we have that $-\epsilon Q_C \leqslant Q_C R Q_C \leqslant +\epsilon Q_C$. Therefore, we have that the set of all products of the Q_A's and Q_B's pave R. Thus, if $\mathcal{P}_{1/2}^\infty$ can be $(r_2, \frac{1+\epsilon}{2})$-paved, then $\mathcal{R}_\infty$ can be (r_2^2, ϵ)-paved.

The proof of the equivalence of (5) and (6), is identical.

Finally, (1) and (4) are equivalent by the standard limiting argument. In particular, see [10, Proposition 2.2] and the proof of [10, Theorem 2.3]. $\square$

COROLLARY 4. *The following are equivalent:*

(1) *the Kadison-Singer conjecture is true,*
(2) *for each* $R \in \mathcal{R}_\infty$ *there is a* $(r, \epsilon), \epsilon < 1$ *(depending on R) such that* R *can be* (r, ϵ)*-paved,*
(3) *there exists* $(r, \epsilon), \epsilon < 1$, *such that every* $R \in \mathcal{R}$ *can be* (r, ϵ)*-paved,*

(4) *for each* $P \in \mathcal{P}^{\infty}_{1/2}$ *there is a* $(r,\epsilon), \epsilon < 1$ *(depending on P) such that* P *can be* (r,ϵ) *-paved,*

(5) *there exists* $(r,\epsilon), \epsilon < 1,$ *such that every* $P \in \mathcal{P}_{1/2}$ *can be* (r,ϵ) *-paved.*

We will need some results from frame theory in this paper. We refer the reader to [13] for these. We will briefly give the definitions we will be using. If $\{f_i\}_{i\in I}$ is a set of vectors in a Hilbert space $\mathbb{H}$, the *analysis operator* of this set is the operator $T : \mathbb{H} \to \ell_2(I)$ given by $T(f) = \{\langle f, f_i\rangle\}_{i\in I}$, and the *synthesis operator* is the operator $T^*(\{a_i\}_{i\in I}) = \sum_{i\in I} a_i f_i$. If T is bounded, we call $\{f_i\}_{i\in I}$ a *Bessel sequence*. If T is also onto we call the set a *frame*. If T is bounded, onto and invertible the set is called a *Riesz basis*. A countable Riesz basis is, generally, called a *Riesz basic sequence.* A frame is *equal-norm* or *uniform* if the f_i all have the same norm and it is *equiangular* if there is a constant c so that $|\langle f_i, f_j\rangle| = c$ for all $i \neq j \in I$. A frame is called a *Parseval frame* if T is a partial isometry. In this case, the Gram matrix $(\langle f_i, f_j\rangle)_{i,j\in I}$ is an orthogonal projection of $\ell_2(I)$ onto the range of the analysis operator and this matrix takes e_i to $T(f_i)$ where $\{e_i\}_{i\in I}$ is the standard orthonormal basis of $\ell_2(I)$. We call a Parseval frame for a k-dimensional Hilbert space consisting of n vectors an *(n,k)-frame*.

A sort of meta-corollary of Theorem 3 is that the frame based conjectures that are known to be equivalent to the Kadison-Singer result can be reduced to the case of uniform Parseval frames of redundancy 2. Similarly, for most harmonic analysis analogues of paving, it is enough to consider say subsets $E \subseteq [0,1]$ of Lebesgue measure $1/2$. We state one such equivalence. The *Feichtinger Conjecture* in frame theory asserts that every unit norm Bessel sequence is a finite union of Riesz basic sequences. Casazza and Tremain [12] have shown that the Feichtinger conjecture is equivalent to the Kadison-Singer conjecture.

THEOREM 5. *The Feichtinger conjecture is true if and only if for each Parseval frame* $\{f_n\}_{n\in\mathbb{N}}$ *for a Hilbert space with* $\|f_n\|^2 = 1/2\ \forall n$ *there is an r (with r depending on the frame) and a partition of* $\mathbb{N}$ *into r disjoint subsets* $\{A_j\}_{j=1}^r$ *such that for each* $j, \{f_n\}_{n\in A_j}$ *is a Riesz basis for the space that it spans.*

Proof. Clearly, if the Feichtinger conjecture is true, then it is true for this special class of frames.

Conversely, assume that the above holds and let $P \in \mathcal{P}^{\infty}_{1/2}$. Then there exists a Parseval frame $\{f_n\}_{n\in\mathbb{N}}$ for some Hilbert space $\mathcal{H}$, such that $I - P = (\langle f_j, f_i\rangle)$ is their Grammian. Now let $\{A_k\}_{k=1}^r$ be the partition of $\mathbb{N}$ into r disjoint subsets as above and let $\mathcal{H}_k = \overline{\text{span}}\{f_n : n \in A_k\}$ denote the closed linear span.

Since $\{f_n : n \in A_k\}$ is a Riesz basis for $\mathcal{H}_k$, there exists an orthonormal basis, $\{e_n : n \in A_k\}$ for $\mathcal{H}_k$ and a bounded invertible operator, $S_k : \mathcal{H}_k \to \mathcal{H}_k$, with $S_k(e_n) = f_n$.

We have that $Q_{A_k}(I-P)Q_{A_k} = (\langle f_j, f_j\rangle)_{i,j\in A_k} = (\langle S_k^* S_k e_j, e_i\rangle) \geqslant c_k Q_{A_k}$ where $S_k^* S_k \geqslant c_k Q_{A_k}$ for some constant $0 < c_k \leqslant 1$ since S_k is invertible. Hence, $Q_{A_k} P Q_{A_k} \leqslant (1-c_k) Q_{A_k}$ and we have that, $\max\{\|Q_{A_k} P Q_{A_k}\| : 1 \leqslant k \leqslant r\} < 1$.

Hence, condition (5) of Corollary 4 is met and so Kadison-Singer is true and thus, by [12, Theorem 5.3], the Feichtinger conjecture is true. $\square$

3. Some Paving Estimates

In this section we derive some estimates on paving constants that give some basic relationships between r and ϵ. In particular, we will prove that $\mathcal{P}_{1/2}$ cannot be $(2,\epsilon)$-paved for any $\epsilon < 1$.

We begin with a result on paving $\mathcal{R}$.

THEOREM 6. *Assume that* $\mathcal{R}$ *is* (r,ϵ)*-pavable. Then* $1 \leqslant r\epsilon^2$.

Proof. Recall that an $n \times n$ matrix C is a *conference matrix* if $C = C^*, c_{i,i} = 0, c_{i,j} = \pm 1, i \neq j$ and $C^2 = (n-1)I$. Such matrices exist for infinitely many n.

If we Set $A = \frac{1}{\sqrt{n-1}}C,$ then A is a unitary matrix with zero diagonal.

Assume that $\{1,...,n\} = B_1 \cup ... \cup B_r$ is a partition such that $\|Q_{B_i}AQ_{B_i}\| \leqslant \epsilon$. Let $d = \max\{\text{card}(B_i)\}$ and let B_j attain this max. Note that $d \geqslant \frac{n}{r}$. Setting $A_j = Q_{B_j}AQ_{B_j}$, the Schur product $A_j * A_j = \frac{1}{n-1}[J_d - I_d]$ where J_d denotes the matrix of all 1's. Hence, $\frac{d-1}{n-1} = \|A_j * A_j\| \leqslant \|A_j\|^2 \leqslant \epsilon^2$. Thus, $\frac{n/r-1}{n-1} \leqslant \epsilon^2$, and the result follows by letting $n \to +\infty$ $\square$

PROPOSITION 7. *If every projection* $P \in \mathcal{P}_{1/2}$ *can be* (r,ϵ)*-paved then every projection* $Q = (q_{i,j})$ *with*

$$\frac{1}{2} - \delta \leqslant q_{i,i} \leqslant \frac{1}{2} + \delta,$$

can be (r,β)*-paved, where*

$$\beta = (1+2\delta)\epsilon,$$

and so $\beta < 1$ *when* δ *is small enough.*

Proof. Let Q be a projection as above, let D be the diagonal of Q and set $B = Q - D$. Then

$$\|B\| \leqslant \frac{1+2\delta}{2}.$$

To see this note that for any vector x

$$0 \leqslant \langle Bx, x\rangle + \langle Dx, x\rangle \leqslant 1,$$

since Q is a projection. Hence,

$$-\langle Dx, x\rangle \leqslant \langle Bx, x\rangle \leqslant 1 - \langle Dx, x\rangle .$$

Hence,

$$\|B\| = \sup_{\|x\|=1} |\langle Bx, x\rangle| \leqslant \max\{|\langle Dx, x\rangle|, |1 - \langle Dx, x\rangle|\} \leqslant \frac{1}{2} + \delta = \frac{1+2\delta}{2}.$$

Let $R = R^*$ be the symmetry we get by dilating

$$\frac{2}{1+2\delta}B,$$

as in the proof of Theorem 3 and let $P = \frac{1}{2}(I+R)$ be the corresponding projection, which has $1/2's$ on the diagonal.

By assumption, we can (r,ϵ)-pave P, which yields an (r,ϵ)-paving of

$$\frac{1}{2}I + \frac{1}{1+2\delta}B,$$

since this is the upper left hand corner of P. Substituting $B = Q - D$ we have an (r,ϵ)-paving of

$$\frac{1}{1+2\delta}Q + \frac{1}{2}I - \frac{1}{1+2\delta}D = \frac{1}{1+2\delta}\left(Q + \frac{1+2\delta}{2}I - D\right).$$

Now, if $\{A_j\}_{j=1}^r$ are the sets that yield the paving, then for any $j = 1,2,\ldots,r$ since

$$\frac{1+2\delta}{2}I - D$$

is a positive operator,

$$\|Q_{A_j}QQ_{A_j}\| \leqslant \|Q_{A_j}(Q + \frac{1+2\delta}{2}I - D)Q_{A_j}\| \leqslant (1+2\delta)\epsilon = \beta,$$

and hence these same sets yield a (r,β)-paving of Q. $\square$

THEOREM 8. *If $\mathcal{P}_{1/2}$ can be (r,ϵ)-paved, then $\frac{r}{2(r-1)} \leqslant \epsilon$.*

Proof. Let $m > 2$ be an integer and consider a uniform, Parseval (n,k)-frame with $n = mr, k = m(r-1)+1$. This will give rise to a projection Q with diagonal entries, $\frac{m(r-1)+1}{mr} = \frac{1}{2} + \delta$, where $\delta = \frac{m(r-2)+3}{2mr}$. To see this, let

$$\begin{aligned}\delta &= \frac{m(r-1)+1}{mr} - \frac{1}{2}\\ &= \frac{2[m(r-1)+1] - mr}{2mr}\\ &= \frac{m(r-1)+2}{2mr}.\end{aligned}$$

By the above result, Q can be (r,β)-paved, where $\beta = (1+2\delta)\epsilon$.

However, for any r paving of Q, one of the blocks must be of size at least

$$n/r = m = n - k + 1,$$

by the choice of n and k. Since Q is a rank k projection, this block will have norm 1 by the eigenvalue inclusion principle or by the eigenvalue interlacing results. Hence $\beta \geqslant 1$. We solve for ϵ:

$$(1+2\delta)\epsilon \geqslant 1.$$

So

$$\begin{aligned}\epsilon &\geqslant \frac{1}{1+2\delta}\\ &= \frac{mr}{m(2r-2)+2}.\end{aligned}$$

Letting $m \to +\infty$ yields

$$\epsilon \geqslant \frac{r}{2(r-1)}.$$

□

COROLLARY 9. *The set* $\mathcal{P}_{1/2}$ *is not* 2*-pavable.*

Proof. When $r = 2$, the formula implies that $1 \leqslant \epsilon$ and hence 2-paving is impossible. □

COROLLARY 10. *The set* $\mathcal{R}$ *is not* 2*-pavable.*

We now generalize the results of the last theorem.

THEOREM 11. *For each* $r, n \in \mathbb{N}$ *with* $r > 1$ *there is an* $\epsilon_n > 0$ *so that if* P *is a projection on* ℓ_2^n *with* $\frac{1}{r} \leqslant \langle Pe_i, e_i\rangle \leqslant 1 - \frac{1}{r}$ *for all* $i = 1, 2, \ldots, n$ *then* P *is* $(r, 1-\epsilon_n)$*-pavable.*

Moreover, for every $r \in \mathbb{N}$ *and* $\delta > 0$ *there is an* $n \in \mathbb{N}$ *and a projection* P *on* ℓ_2^{2n} *of rank* n *so that* $\frac{1}{r} - \delta \leqslant \langle Pe_i, e_i\rangle \leqslant 1 - \frac{1}{r} + \delta$ *for all* $i = 1, 2, \ldots, 2n$ *while* P *is not* (r, ϵ)*-pavable for any* $\epsilon < 1$.

Proof. Given our assumptions, we will check the Rado-Horn Theorem (see [11] and its references) to see that the row vectors of our projection can be divided into r linearly independent sets. Then the rest of the first part of the theorem follows by a compactness argument. For any $J \subset \{1, 2, \ldots, 2n\}$ let P_J be the orthogonal projection of ℓ_2^{2n} onto the span $\{Pe_i\}_{i\in J}$. Now,

$$\text{dim span } \{Pe_i\}_{i\in J} = \sum_{i=1}^{2n} \|P_J Pe_i\|^2 \geqslant \sum_{i\in J} \|Pe_i\|^2 \geqslant |J|\frac{1}{r}.$$

By the Rado-Horn Theorem we can now write $\{Pe_i\}_{i=1}^{2n}$ as a union of r-linearly independent sets.

For the moreover part, choose a $k \in \mathbb{N}$ so that

$$\frac{1}{r} - \delta < \frac{k}{rk+1} \leqslant \frac{1}{r} \leqslant 1 - \frac{1}{r}.$$

Now, choose an n so that

$$\frac{1}{r} - \delta \leqslant \frac{n - rk}{2n - (rk+1)} \leqslant 1 - \frac{1}{r} + \delta.$$

With $\{e_i\}_{i=1}^n$ the canonical basis for ℓ_2^n we can choose an equal norm Parseval frame $\{f_i\}_{i=1}^{rk+1}$ for the span of $\{e_i\}_{i=1}^k$. Next, choose an equal norm Parseval frame $\{f_i\}_{i=rk+2}^{2n}$ for the span of $\{e_i\}_{i=k+1}^n$. Now,

$$\frac{1}{r} - \delta \leqslant \|f_i\|^2 = \frac{k}{rk+1} \leqslant \frac{1}{r} \leqslant 1 - \frac{1}{r},$$

and

$$\frac{1}{r} - \delta \leqslant \frac{n-rk}{2n-(rk+1)} \leqslant 1 - \frac{1}{r} + \delta.$$

Taking the embedding of this Parseval frame with $2n$-elements for ℓ_2^n into ℓ_2^{2n} we get a projection P on ℓ_2^{2n} which has rank n and is given by the matrix

$$\begin{bmatrix} \|f_1\|^2 & b_{1,2} & \dots & b_{1,(rk+1)} & 0 & 0 & \dots & 0 \\ b_{21} & \|f_1\|^2 & \dots & b_{2,(rk+1)} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ b_{(rk+1),1} & b_{(rk+1),2} & \dots & \|f_1\|^2 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \|f_{rk+2}\|^2 & a_{(rk+2),(rk+3)} & \dots & a_{(rk+2),2n} \\ 0 & 0 & \dots & 0 & a_{(rk+3),(rk+2)} & \|f_{rk+2}\|^2 & \dots & a_{(rk+3),2n} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & a_{(2n),(rk+2)} & a_{(2n),(rk+3)} & \dots & \|f_{rk+2}\|^2 \end{bmatrix}.$$

For this projection, for any $J \subset \{1,2,\dots n\}$ with $|J| > r$ the family $\{Pe_i\}_{i\in J}$ is linearly dependent and so PQ_A has a zero eigenvalue. Hence, $(I-P)Q_A$ has one as an eigenvalue and hence $I-P$ is not ϵ-pavable for any $\epsilon > 0$.

Note that the diagonal entries of $I-P$ satisfy the same inequalities as P and hence the result follows. $\square$

4. Counterexamples to the Akemann-Anderson Conjecture

In [1] Akemann and Anderson introduce two paving conjectures, denoted *Conjecture A* and *Conjecture B*. They prove that Conjecture A implies Conjecture B and that Conjecture B implies Kadison-Singer, but it is not known if either of these implications can be reversed. Weaver[21] provides a set of counterexamples to Conjecture A. Thus, if these three statements were all equivalent then Weaver's counterexample would be the end of the story. However, it is generally believed that Conjecture A is strictly stronger than the Kadison-Singer conjecture.

In this section, we show that the Grammian projection matrices of *any* uniform, equiangular (n,k)-frame, with $n > 5k$ yield counterexamples to Conjecture A. It is known that infinitely many such frames exist for arbitrarily large n and k. The significance of our new set of counterexamples is that by the results of J. Bourgain and L. Tzafriri [8], there exists r and $\epsilon < 1$, such that the family of self-adjoint, norm one, 0 diagonal matrices obtained from these frames is (r,ϵ)-pavable.

Thus, these new examples drive an additional wedge between Conjecture A and Kadison-Singer.

We then turn our techniques to Conjecture B and derive some results that could lead to a counterexample to Conjecture B.

We now describe the Akemann-Anderson conjectures. Let $P = (p_{i,j}) \in M_n$ be the matrix of a projection and set $\delta_P = \max\{p_{i,i} : 1 \leqslant i \leqslant n\}$. By a *diagonal symmetry* we mean a diagonal matrix whose diagonal entries are ± 1, that is, S is a diagonal self-adjoint unitary.

CONJECTURE A [1, 7.1.1]. For any projection P there exists a diagonal symmetry S, such that $\|PSP\| \leqslant 2\delta_P$.

CONJECTURE B [1, 7.1.3]. There exists $\gamma, \epsilon > 0$ (and independent of n) such that for any P with $\delta_P < \gamma$ there exists a diagonal symmetry, S, such that $\|PSP\| < 1 - \epsilon$.

Weaver[21] states that a counterexample to Conjecture B would probably lead to a negative solution to Kadison-Singer. We believe that these two conjectures are really more closely related to 2-pavings and this is why we believe that counterexamples to Conjecture B should be close at hand.

Finally, note that Conjecture B is about paving projections with small diagonal. But our results show that Kadison-Singer is equivalent to paving projections with diagonal $1/2$. This would also seem to put further distance between these Akemann-Anderson conjectures and the Kadison-Singer conjecture.

PROPOSITION 12. *Let* $P = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ *be a projection, written in block-form with* A $m \times m, B$ $m \times (m+l), C$ $(m+l) \times (m+l)$, *where* $l \geqslant 0$. *Then there exists a* $m \times m$ *unitary* U_1 *and an* $(m+l) \times (m+l)$ *unitary* U_2 *such that,* $U_1^* A U_1 = D_1, U_1^* B U_2 = (D_2, 0), U_2^* C U_2 = \begin{pmatrix} D_3 & 0 \\ 0 & D_4 \end{pmatrix}$ *where each of the* D_i *'s is a diagonal matrix with non-negative entries,* D_1, D_2, D_3 *are all* $m \times m, D_4$ *is* $l \times l$ *with* 1 *'s and* 0 *'s for its diagonal entries and the* 0 *'s represent matrices of all zeroes that are either* $m \times l$ *or* $l \times m$.

Proof. First note that since P is a projection we have that $A^2 + BB^* = A, B^*B + C^2 = C$ and $AB + BC = B$. Also, since the rank of B is at most m, the matrix B^*B must have a kernel of dimension at least l.

Conjugating P by a unitary of the form $U = \begin{pmatrix} I_m & 0 \\ 0 & U_2 \end{pmatrix}$, we may diagonalize C and the new matrix, P_1, will still be a projection. Since $U_2^* B^* B U_2 = U_2^*(C - C^2)U_2$, we see that both sides of this equation are in diagonal form. Since at least l of the diagonal entries of $U_2^* B^* B U_2$ are zeroes, after applying a permutation if necessary, we may assume that,

$$U_2^* B^* B U_2 = \begin{pmatrix} D_2^2 & 0 \\ 0 & 0 \end{pmatrix}, U_2^* C U_2 = \begin{pmatrix} D_3 & 0 \\ 0 & D_4 \end{pmatrix},$$

where D_2, D_3, D_4 are as claimed.

Now we may polar decompose the $m \times (m+l)$ matrix $BU_2 = W|BU_2| = W \begin{pmatrix} D_2 & 0 \\ 0 & 0 \end{pmatrix}$, where W is a $m \times (m+l)$ partial isometry whose initial space is the range of $|BU_2|$. Thus, $W = (W_1, 0)$ where W_1 is an $m \times m$ partial isometry. Hence, we may extend W_1 to an $m \times m$ unitary U_1 with $W_1D_2 = U_1D_2$ and $BU_2 = (U_1, 0)\begin{pmatrix} D_2 & 0 \\ 0 & 0 \end{pmatrix} = (U_1D_2, 0)$.

Conjugating P_1 by the unitary $\begin{pmatrix} U_1 & 0 \\ 0 & I_{m+l} \end{pmatrix}$ we arrive at a new projection of the form,

$$\begin{pmatrix} U_1^*AU_1 & D_2 & 0 \\ D_2 & D_3 & 0 \\ 0 & 0 & D_4 \end{pmatrix}.$$

Note that since this last matrix is a projection, $U_1^*AU_1D_2 + D_2D_3 = D_2$ and so, $U_1^*AU_1D_2$ is diagonal. If all of the entries of D_2 were non-zero, then this would imply that $U_2^*AU_2$ is diagonal. In general, this implies that $U_1^*AU_1$ (which is self-adjoint) is of the form a diagonal matrix direct sum with another matrix corresponding to the block where D_2 is 0. Conjugating $U_1^*AU_1$ by another unitary to diagonalize this lower block, yields the desired form.

Finally, note that since D_4 is a diagonal projection, all of its entries must be 0's or 1's. $\square$

LEMMA 13. *Let* $P = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ *be a non-zero projection with real entries and let* $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *Then* $\|PSP\| = |1 - 2c|$.

Proof. If P is rank 2 then $P = I$ and the result is trivial. So assume that P is rank one. We have that $PSP = \begin{pmatrix} a^2 - b^2 & ab - bc \\ ab - bc & b^2 - c^2 \end{pmatrix}$ and since P is a rank one projection, $a + c = 1, b^2 + c^2 = c$. A little calculation shows that the characteristic polynomial of PSP is $x^2 - \mathrm{Tr}(PSP)x + Det(PSP) = x^2 - (1 - 2c)x$, and hence the eigenvalues are 0 and 1-2c, from which the result follows. $\square$

Note that when S is a diagonal symmetry, then $-S$ is also a diagonal symmetry, and since $\|PSP\| = \|P(-S)P\|$, we may and do assume in what follows that the number of -1's in S is greater than or equal to the number of $+1$'s. Also, given a matrix A, we let $\sigma(A)$ denote the spectrum of A and set $\sigma'(A) \equiv \sigma(A) \backslash \{0\}$.

THEOREM 14. *Let* $P = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ *be an* $n \times n$ *projection and let* $S = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ *be a diagonal symmetry. Then* $\|PSP\| \geqslant \max\{|1 - 2\lambda| : \lambda \in \sigma'(C) \cup \sigma'(A)\}$.

Proof. Given any unitary of the type in the above Proposition, we have that $\|PSP\| = \|U^*PSPU\| = \|(U^*PU)(U^*SU)(U^*PU)\| = \|(U^*PU)S(U^*PU)\|$. Thus, we may and do assume that P has been replaced by U^*PU. But this reduces the norm calculation to the direct sum of a set of 2×2 matrices of the form of the lemma together with the diagonal projection D_4. Now if $\lambda \in \sigma'(C)$, then this 2×2 matrix is necessarily rank one and so the lemma applies. Note also that in this case the corresponding eigenvalue of D_1 is $1-\lambda$ and that $|1-2(1-\lambda)| = |-1+2\lambda| = |1-2\lambda|$ so the values of this function agree. When $\lambda = 0$, then this 2×2 matrix is either the 0 matrix or it is rank 1 and the corresponding eigenvalue of D_1 is 1. $\square$

We now provide a counterexample to Conjecture A.

THEOREM 15. *Let $\{f_1,...,f_n\}$ be a uniform equiangular Parseval frame for $\mathbb{C}^k$ with $n > 2k$ and let $P = (\langle f_i, f_j\rangle)$ be the correlation matrix. If there exists a diagonal symmetry, S, such that, $\|PSP\| \leqslant 2\delta_P = \frac{2k}{n}$, then $(k-1)n^2 \leqslant 4k^2(n-1)$.*

Proof. Without loss of generality we may assume that S is a diagonal symmetry with m diagonal entries that are $+1$ and $n-m$ diagonal entries that are -1 and, $m \leqslant n-m$. Putting P into the form of the Proposition, we see that since D_4 is a projection, if it is non-zero, then $\|PSP\| = 1$. So we may assume that $D_4 = 0$.

Similarly, if any of the diagonal entries of D_1 or D_3 are 1, then $\|PSP\| = 1$. Thus, when we put P into the form of the above Proposition, we obtain a direct sum of 2×2 rank 1 projections, together with some matrices of all 0's.

Let $0 < \lambda_1 \leqslant ... \leqslant \lambda_t < 1$, denote the non-zero diagonal entries of D_1, so that the corresponding diagonal entries of D_3 are $1-\lambda_1, ..., 1-\lambda_t$, and the remaining entries of D_3 are 0's. Since P is a rank k projection, we have that $k = \mathrm{Tr}(P) = \mathrm{Tr}(D_1) + \mathrm{Tr}(D_3) = t$.

By the above Theorem, we have that $\|PSP\| = \max\{|1-2\lambda_1|, |1-2\lambda_k|\} = \max\{1-2\lambda_1, 2\lambda_k - 1\}$. Since $\|PSP\| \leqslant \frac{2k}{n}$, we have, $\frac{n-2k}{2n} \leqslant \lambda_1$ and $\lambda_k \leqslant \frac{n+2k}{2n}$.

Since $\mathrm{Tr}(D_1) = \mathrm{Tr}(A) = mk/n$, we have that $0 < \lambda_1 \leqslant m/n \leqslant \lambda_k$. Hence, $\frac{n-2k}{2n} \leqslant m/n \leqslant \frac{n+2k}{2n}$ and the left inequality yields $n \leqslant 2k+2m$. Note that by the choice of m we have that $2m \leqslant n$, so that the inequality $m/n \leqslant \frac{n+2k}{2n}$ is automatically satisfied.

If we let, $\mu_1, ..., \mu_k$ be the corresponding entries of D_2, then since each matrix, $\begin{pmatrix} \lambda_i & \mu_i \\ \mu_i & 1-\lambda_i \end{pmatrix}$ is a rank one projection, we have that $\mu_i^2 = \lambda_i(1-\lambda_i)$.

Since P is the correlation matrix of a uniform equiangular (n,k)-frame, by [17], we have that every off-diagonal entry of P is of constant modulus, $c = \sqrt{\frac{k(n-k)}{n^2(n-1)}}$. This yields,

$$\sum_{i=1}^{k} \mu_i^2 = \mathrm{Tr}(B^*B) = m(n-m)c^2 \leqslant \frac{n^2c^2}{4} = \frac{k(n-k)}{4(n-1)}.$$

Now observe that the function $t(1-t)$ is increasing on [0,1/2] and decreasing on [1/2,1]. Thus, we have that $\min\{\lambda_1(1-\lambda_1), \lambda_k(1-\lambda_k)\} = \min\{\mu_1^2, ..., \mu_k^2\} \leqslant \mathrm{Tr}(B^*B)/k \leqslant \frac{n-k}{4(n-1)}$.

However, since $\frac{n-2k}{2n} \leqslant \lambda_1$, we have $\frac{n-2k}{2n}(1-\frac{n-2k}{2n}) = \frac{n^2-4k^2}{4n^2} \leqslant \lambda_1(1-\lambda_1)$. Similarly, using the fact that $1/2 < \frac{n+2k}{2n}$, one sees that $\frac{n+2k}{2n}(1-\frac{n+2k}{2n}) = \frac{n^2-4k^2}{4n^2} \leqslant \lambda_k(1-\lambda_k)$.

Combining these inequalities, yields $\frac{n^2-4k^2}{4n^2} \leqslant \frac{n-k}{4(n-1)}$. Cross-multiplying and canceling like terms yields the result. $\square$

Note that the above inequality, for n and k large becomes asymptotically, $n \leqslant 4k$. Thus, any uniform, equiangular (n,k)-frame with $n/k >> 4$, and n sufficiently large will yield a counterexample to Conjecture A.

COROLLARY 16. *There exist uniform, equiangular Parseval frames whose projection matrices are counterexamples to Conjecture A.*

Proof. In [7, Example 6.4] a real uniform, equiangular (276, 23)-frame is exhibited and these values satisfy $(k-1)n^2 > 4k^2(n-1)$. In [19], uniform, equiangular (n,k)-frames are constructed using Singer difference sets of size,

$$n = \frac{q^{m+1}-1}{q-1}, k = \frac{q^m-1}{q-1},$$

where $q = p^r$ with p a prime. Note that $n/k > q-1$. Since Singer difference sets are known to exist for infinitely large q, these frames give a whole family of counterexamples. □

We now turn our attention to Conjecture B. We let $\gamma, \epsilon > 0$ be as in the statement of the conjecture. For each partition of $\{1,...,n\} = R \cup T$ into two disjoint sets, R, T, we let Q_R, Q_T denote the corresponding diagonal projections.

THEOREM 17. *Let $\gamma, \epsilon > 0$ be fixed, let $\{f_1,...,f_n\}$ be a uniform Parseval frame for $\mathbb{R}^k$ with $k/n < \min\{\gamma, \epsilon/2, 1/2\}$ and let $P = (\langle f_i, f_j\rangle)$ be the correlation matrix. If Conjecture B is true for the pair (γ, ϵ), then there exists a partition $\{1,...,n\} = R \cup T$ such that $\mathrm{Tr}(Q_RPQ_TPQ_R) \geqslant \frac{k\epsilon(2-\epsilon)}{4}$.*

Proof. Each such partition defines a diagonal symmetry as before and corresponding to such a partition we write $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$. Note that $Q_RPQ_T = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ so that $\mathrm{Tr}(Q_RPQ_TPQ_R) = \mathrm{Tr}(BB^*)$.

We have that $\delta_P = k/n < \gamma$. We repeat the proof above, with $m = \min\{|R|, |T|\}$.

Letting λ_1 be the minimum non-zero eigenvalue and λ_k the largest eigenvalue of A as before, we have $1-\epsilon \geqslant \|PSP\| \geqslant \max\{|1-2\lambda_1|, |1-2\lambda_k|\}$ and, hence, $\lambda_1 \geqslant \epsilon/2$ and $1-\lambda_k \geqslant \epsilon/2$.

Using the properties of the function $t \to t(1-t)$ and the fact that $\sum_{i=1}^k \lambda_i(1-\lambda_i) = \mathrm{Tr}(B^*B)$, we have that $\epsilon/2(1-\epsilon/2) \leqslant \min\{\lambda_1(1-\lambda_1), \lambda_k(1-\lambda_k)\} \leqslant 1/k\,\mathrm{Tr}(B^*B)$, which yields the result. □

Using equiangular frames we can obtain a relation between γ and ϵ in Conjecture B.

THEOREM 18. *Assume that Conjecture B is true for a pair (γ, ϵ) and let $\{f_1,...,f_n\}$ be a uniform, equiangular (n,k)-frame with $k/n \leqslant \gamma$. Then $\epsilon(2-\epsilon) \leqslant \frac{n-k}{n-1}$.*

Proof. By the above theorem, we have that there exists a partition with $|R| = m$, such that $\frac{k\epsilon(2-\epsilon)}{4} \leqslant \mathrm{Tr}(Q_RPQ_TPQ_R) = m(n-m)c^2 \leqslant \frac{n^2}{4}c^2 = \frac{k(n-k)}{4(n-1)}$. □

There are very few pairs (n,k) for which uniform, equiangular Parseval (n,k)-frames exist and, consequently, it is difficult to determine which real numbers can arise as limits of rational numbers of the form k/n for such pairs. If we have that infinitely many uniform, equiangular (n,k)-frames exist for which $n \to +\infty$ and $k/n \to \gamma$, then

$$\frac{n-k}{n-1} = \frac{1-k/n}{(1-1/n)} \to 1-\gamma,$$

and hence, $\epsilon(2-\epsilon) \leqslant 1-\gamma$. If for a given prime p, there are infinitely many Singer difference sets, with $q = p^r$, and we choose, $1/q \leqslant \gamma$ then we get that $\epsilon(2-\epsilon) < \frac{q-1}{q}$.

Unfortunately, there are no uniform Parseval (n,k) frames which violate the trace inequality in Theorem 17, so that finding a counter-example to Conjecture B is more subtle. We will show this below.

First, let us change the notation. If $\{f_i\}_{i=1}^n$ is a Parseval frame for l_2^k with analysis operator V then the frame operator is $S = V^*V = I$ and $P = VV^*$ is a projection on l_2^n onto the image of the analysis operator (which is now an isometry). Let $\{R,T\}$ be a partition of $\{1,2,\ldots,n\}$. If $x = \sum_{i=1}^n a_i e_i$ then

$$Q_R x = \sum_{i\in R} a_i e_i.$$

Next,

$$PQ_R x = \sum_{j=1}^n \langle \sum_{i\in R} a_i f_i, f_j\rangle\, e_j.$$

Finally,

$$Q_T P Q_R x = \sum_{j\in T} \langle \sum_{i\in R} a_i f_i, f_j\rangle\, e_j.$$

It follows that

$$Q_T P Q_R e_i = \sum_{i\in R}\sum_{j\in T} \langle f_i, f_j\rangle\, e_j.$$

Now we have:

LEMMA 19. *Given the conditions above we have*

$$\mathrm{Tr}(Q_R P Q_T P Q_R) = \sum_{i\in R}\sum_{j\in T} |\langle f_i, f_j\rangle|^2.$$

Proof. We compute:

$$\begin{aligned}
\sum_{i=1}^n \langle Q_R P Q_T P Q_R e_i, e_i\rangle &= \sum_{i=1}^n \langle Q_T P Q_R e_i, P Q_R e_i\rangle \\
&= \sum_{i=1}^n \langle Q_T P Q_R e_i, Q_T P Q_R e_i\rangle \\
&= \sum_{i=1}^n \|Q_T P Q_R e_i\|^2 \\
&= \sum_{i\in R}\sum_{j\in T} |\langle f_i, f_j\rangle|^2.
\end{aligned}$$

□

Now we need to recall a result of Berman, Halpern, Kaftal and Weiss [5].

THEOREM 20. *Let $(a_{ij})_{i,j=1}^n$ be a self-adjoint matrix with non-negative entries and with zero diagonal so that*

$$\sum_{m=1}^{n} a_{im} \leqslant B, \text{ for all } i = 1, 2, \ldots, n.$$

Then for every $r \in \mathbb{N}$ there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \ldots, n\}$ so that for every $j = 1, 2, \ldots, r$,

$$\sum_{m \in A_j} a_{im} \leqslant \sum_{m \in A_\ell} a_{im}, \text{ for every } i \in A_j \text{ and } \ell \neq j. \tag{1}$$

Now we are ready for our result.

PROPOSITION 21. *If $\{f_i\}_{i=1}^n$ is a uniform (n,k)-Parseval frame, then there is a partition $\{R, T\}$ of $\{1, 2, \ldots, n\}$ so that*

$$\mathrm{Tr}(Q_R P Q_T P Q_R) \geqslant \frac{k}{4}(1 - \frac{k}{n}).$$

In particular, if $\frac{k}{n}$ is small then the trace inequality of Theorem 17 holds.

Proof. Applying 20 to the matrix of values $(a_{ij})_{i,j=1}^n$ where $a_{ii} = 0$ and $a_{ij} = |\langle f_i, f_j\rangle|^2$ for $i \neq j$ we can find a partition $\{R, T\}$ of $\{1, 2, \ldots, n\}$ (and without loss of generality we may assume that $|R| \geqslant \frac{n}{2}$) satisfying for all $i \in R$:

$$\sum_{i \neq j \in R} |\langle f_i, f_j\rangle|^2 \leqslant \sum_{j \in T} |\langle f_i, f_j\rangle|^2.$$

It follows that for all $i \in R$:

$$\begin{aligned} \frac{k}{n} &= \sum_{j=1}^{n} |\langle f_i, f_j\rangle|^2 \\ &= \frac{k^2}{n^2} + \sum_{i \neq j \in R} |\langle f_i, f_j\rangle|^2 + \sum_{j \in T} |\langle f_i, f_j\rangle|^2 \\ &\leqslant \frac{k^2}{n^2} + 2\sum_{j \in T} |\langle f_i, f_j\rangle|^2. \end{aligned}$$

It follows that for all $i \in R$

$$\sum_{j \in T} |\langle f_i, f_j\rangle|^2 \geqslant \frac{1}{2}\left(\frac{k}{n} - \frac{k^2}{n^2}\right).$$

Now,

$$\begin{aligned}\sum_{i\in R}\sum_{j\in T}|\langle f_i, f_j\rangle|^2 &\geqslant |R|\frac{1}{2}\left(\frac{k}{n}-\frac{k^2}{n^2}\right)\\ &\geqslant \frac{n}{2}\frac{1}{2}\left(\frac{k}{n}-\frac{k^2}{n^2}\right)\\ &= \frac{k}{4}\left(1-\frac{k}{n}\right).\end{aligned}$$

Now, given $0<\epsilon<1$,

$$\frac{\epsilon}{2}(1-\frac{\epsilon}{2})<\frac{1}{4}$$

So the trace inequaltiy of Theorem 17 will hold provided

$$\frac{k}{4}(1-\frac{k}{n})\geqslant\frac{k\epsilon(2-\epsilon)}{4}),$$

which is true for k/n small enough. $\square$

In fact, as with the case of equiangular frames, we see that if $1-\gamma\geqslant\epsilon(2-\epsilon)$, then whenever $\frac{k}{n}\leqslant\gamma$, we have that

$$\mathrm{Tr}(Q_RPQ_TPQ_R)\geqslant\frac{k}{4}(1-\frac{k}{n})\geqslant\frac{k}{4}(1-\gamma)\geqslant\frac{k}{4}\epsilon(2-\epsilon).$$

5. A Family of Potential Counterexamples

It is still unknown if the paving conjectures are true even for a smaller family of operators known as the Laurent operators. In this section we introduce a family of Laurent operators that we believe are potential counterexamples to the paving conjecture. We also prove some results about these operators that lends credence to the belief that they might yield counterexamples. For the purposes of this section, it will be convenient to replace the countable index set $\mathbb{N}$ by $\mathbb{Z}$.

Recall that a matrix, $A=(a_{i,j})_{i,j\in\mathbb{Z}}$ is called a *Laurent matrix* if it is constant on diagonals, i.e., $a_{i,j}=a(i-j)$ and that in this case A determines a bounded operator on $\ell^2(\mathbb{Z})$ if and only if there exists $f\in L^\infty[0,1]$ such that $a(n)=\hat{f}(n)\equiv\int_0^1 f(t)e^{-2\pi int}dt$ and in this case we set $A=L_f$ and call it the *Laurent operator with symbol f.* Indeed, the Laurent operator L_f is just the matrix representation of the operator of multiplication by f, M_f on the space $L^2[0,1]$ with respect to the orthonormal basis, $\{e^{2\pi int}\}_{n\in\mathbb{Z}}$. So, in particular, L_f is self-adjoint with diagonal 0 if and only if f is real-valued a.e. and $\int_0^1 f(t)dt=0$.

The problem of paving Laurent operators was first studied in [16] where it was shown that Laurent operators with Riemann integrable symbols can be paved. Further work on the relation between Laurent operators and the Feichtinger conjecture can be found in Bownik and Speegle [9].

Note that L_f is a projection if and only if $f=\chi_E$ for some measurable set E and L_f is a reflection if and only if $f=2\chi_E-1$, for some measurable set E. This reflection will have 0 diagonal when $m(E) = 1/2$, where m denotes Lebesgue measure. Thus, modulo the change from $\mathbb{N}$ to $\mathbb{Z}$, the family of Laurent operators corresponding to our set $\mathcal{R}$ is exactly the set of operators of the form, $L_f, f = 2\chi_E - 1, m(E) = 1/2$ and to $\mathcal{P}_{1/2}$ is the set of operators of the form $L_f, f = \chi_E, m(E) = 1/2$.

Hence, we are interested in the Laurent operators that arise from certain subsets E with $m(E) = 1/2$. It is known that for every $t, 0 < t < 1$, there exists a measurable set $E = E_t$ with $m(E) = t$, and such that for every $0 < a < b < 1, m(E \cap (a,b)) > 0$ and $m(E^c \cap (a,b)) > 0$, where $E^c = [0,1] \setminus E$. One way to construct such a set is as a countable union of fat Cantor sets.

We believe that the projections and reflections coming from such sets for $t = 1/2$, are good candidates for counterexamples to the paving conjectures and we outline our reasons below.

PROPOSITION 22. *Let E be a set as above for any $0 < t < 1$. If f_1,f_2 are continuous functions such that $f_1 \leqslant \chi_E \leqslant f_2$, a.e., then $f_1 \leqslant 0$ and $1 \leqslant f_2$.*

Proof. Since χ_E is zero on a set of positive measure in every interval, $f_1 \leqslant 0$. Similarly, χ_E is one on a set of positive measure in every interval and hence, $1 \leqslant f_2$. □

The above inequalities show that χ_E is far from Riemann integrable.

PROPOSITION 23. *Let $g,h \in L^\infty[0,1]$, with $0 \leqslant h \leqslant 1$. If for every $f_1,f_2 \in C[0,1]$, we have that $f_1 \leqslant g \leqslant f_2$, a.e., implies that $f_1 \leqslant 0, 1 \leqslant f_2$, then there exists a positive linear map, $\phi : L^\infty[0,1] \to L^\infty[0,1]$ such that $\phi(f) = f$ for every $f \in C[0,1]$ and $\phi(g) = h$.*

Proof. First define ϕ on the linear span of $C[0,1]$ and g by $\phi(f + \alpha g) = f + \alpha h$, and note that the inequalities imply that if $f + \alpha g \geqslant 0$, then $f + \alpha h \geqslant 0$. Hence, ϕ is a positive map. Now using the fact that $L^\infty[0,1]$ is an abelian, injective operator system, this map has a (completely) positive extension to all of $L^\infty[0,1]$. □

PROPOSITION 24. *Let $g,h \in L^\infty[0,1]$, with $0 \leqslant h \leqslant 1$. If for every $f_1,f_2 \in C[0,1]$, we have that $f_1 \leqslant g \leqslant f_2$, a.e., implies that $f_1 \leqslant 0, 1 \leqslant f_2$, then there exists a completely positive linear map, $\phi : B(\ell^2(\mathbb{Z})) \to B(\ell^2(\mathbb{Z}))$ such that $\phi(L_f) = L_f$ for every Laurent operator with continuous symbol, $f \in C[0,1]$ and $\phi(L_g) = L_h$.*

Proof. The identification of $L^\infty[0,1]$ with the space of Laurent operators is a complete order isomorphism. Hence, there exists a completely positive projection of $B(\ell^2(\mathbb{Z}))$ onto the space of Laurent operators. The remainder of the proof now follows from the last Proposition. □

THEOREM 25. *Let $E \subset [0,1]$ be a measurable set with $m(E) = 1/2$ such that for every $0 < a < b < 1, m(E \cap (a,b)) > 0$ and $m(E^c \cap (a,b)) > 0$ and let P denote the projection that is the Laurent operator with symbol χ_E. Then there exist completely positive maps, $\phi, \psi : B(\ell^2(\mathbb{Z})) \to B(\ell^2(\mathbb{Z}))$ such that $\phi(L_f) = \psi(L_f) = L_f$ for every Laurent operator with continuous symbol f, but $\phi(P) = 0, \psi(P) = I$.*

Proof. Apply the above Proposition with $h = 0$ and $h = 1$, respectively. □

Thus, for the Laurent reflection with 0 diagonal, $R = 2P - I$, we have that $\phi(R) = -I$, $\psi(R) = +I$ even though these maps fix all Laurent operators with continuous symbols. In this sense, the "value" of the diagonal of R is not very stationary under completely positive maps which fix all Laurent operators with continuous symbol. In fact, it follows from the theory of completely positive maps, that the maps ϕ and ψ constructed above are actually bimodule maps over the C*-algebra of Laurent operators with continuous symbol. That is, $\phi(L_{f_1}XL_{f_2}) = L_{f_1}\phi(X)L_{f_2}$, and $\psi(L_{f_1}XL_{f_2}) = L_{f_1}\psi(X)L_{f_2}$ for any continuous functions, f_1, f_2 and any $X \in B(\ell^2(\mathbb{Z}))$.

One suspects that the fact that the diagonal of R can be altered so dramatically, while fixing so many other operators, might be an obstruction to R being paved. It is also intriguing that for a suitable choice of the set E, one can actually compute the coefficients of the Laurent matrix for R, albeit as power series.

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