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SEMI–FREDHOLM SINGULAR INTEGRAL OPERATORS WITH PIECEWISE CONTINUOUS COEFFICIENTS ON WEIGHTED VARIABLE LEBESGUE SPACES ARE FREDHOLM

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(*communicated by A. Böttcher*)

Abstract. Suppose Γ is a Carleson Jordan curve with logarithmic whirl points, ϱ is a Khvedelidze weight, $p:\Gamma\to(1,\infty)$ is a continuous function satisfying $|p(\tau)-p(t)|\leqslant -\mathrm{const}/\log|\tau-t|$ for $|\tau-t|\leqslant 1/2$, and $L^{p(\cdot)}(\Gamma,\varrho)$ is a weighted generalized Lebesgue space with variable exponent. We prove that all semi-Fredholm operators in the algebra of singular integral operators with $N\times N$ matrix piecewise continuous coefficients are Fredholm on $L_N^{p(\cdot)}(\Gamma,\varrho)$.

1. Introduction

Let X be a Banach space and $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators on X. An operator $A\in\mathcal{B}(X)$ is said to be n-normal (resp. d-normal) if its image $\operatorname{Im}A$ is closed in X and the defect number $n(A;X):=\dim\operatorname{Ker}A$ (resp. $d(A;X):=\dim\operatorname{Ker}A^*$) is finite. An operator A is said to be semi-Fredholm on X if it is n-normal or d-normal. Finally, A is said to be Fredholm if it is simultaneously n-normal and d-normal. Let N be a positive integer. We denote by X_N the direct sum of N copies of X with the norm

$$\|f\|=\|(f_1,\dots,f_N)\|:=(\|f_1\|^2+\dots+\|f_N\|^2)^{1/2}.$$

Let Γ be a Jordan curve, that is, a curve that is homeomorphic to a circle. We suppose that Γ is rectifiable. We equip Γ with Lebesgue length measure $|d\tau|$ and the counter-clockwise orientation. The *Cauchy singular integral* of $f\in L^1(\Gamma)$ is defined by

$$(Sf)(t):=\lim_{R\to 0}\frac{1}{\pi i}\int_{\Gamma\setminus\Gamma(t,R)}\frac{f(\tau)}{\tau-t}d\tau\quad(t\in\Gamma),$$

where $\Gamma(t,R):=\{\tau\in\Gamma:|\tau-t|<R\}$ for $R>0$. David [7] (see also [3, Theorem 4.17]) proved that the Cauchy singular integral generates the bounded operator

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S on the Lebesgue space $L^p(\Gamma)$, $1 < p < \infty$, if and only if Γ is a Carleson (Ahlfors-David regular) curve, that is,

$$\sup_{t\in\Gamma}\sup_{R>0}\frac{|\Gamma(t,R)|}{R} < \infty,$$

where $|\Omega|$ denotes the measure of a measurable set $\Omega \subset \Gamma$. We can write $\tau - t = |\tau - t|e^{i\arg(\tau-t)}$ for $\tau \in \Gamma\setminus\{t\}$, and the argument can be chosen so that it is continuous on $\Gamma\setminus\{t\}$. It is known [3, Theorem 1.10] that for an arbitrary Carleson curve the estimate

$$\arg(\tau - t) = O(-\log|\tau - t|) \quad (\tau \to t)$$

holds for every $t \in \Gamma$. One says that a Carleson curve Γ satisfies the *logarithmic whirl condition* at $t \in \Gamma$ if

$$\arg(\tau - t) = -\delta(t)\log|\tau - t| + O(1) \quad (\tau \to t) \tag{1}$$

with some $\delta(t) \in \mathbb{R}$. Notice that all piecewise smooth curves satisfy this condition at each point and, moreover, $\delta(t) \equiv 0$. For more information along these lines, see [2], [3, Chap. 1], [4].

Let $t_1, \ldots, t_m \in \Gamma$ be pairwise distinct points. Consider the Khvedelidze weight

$$\varrho(t) := \prod_{k=1}^{m}|t - t_k|^{\lambda_k} \quad (\lambda_1, \ldots, \lambda_m \in \mathbb{R}).$$

Suppose $p : \Gamma \to (1,\infty)$ is a continuous function. Denote by $L^{p(\cdot)}(\Gamma,\varrho)$ the set of all measurable complex-valued functions f on Γ such that

$$\int_\Gamma |f(\tau)\varrho(\tau)/\lambda|^{p(\tau)}|d\tau| < \infty$$

for some $\lambda = \lambda(f) > 0$. This set becomes a Banach space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{p(\cdot),\varrho} := \inf\left\{\lambda > 0 : \int_\Gamma |f(\tau)\varrho(\tau)/\lambda|^{p(\tau)}|d\tau| \leqslant 1\right\}.$$

If p is constant, then $L^{p(\cdot)}(\Gamma,\varrho)$ is nothing else than the weighted Lebesgue space. Therefore, it is natural to refer to $L^{p(\cdot)}(\Gamma,\varrho)$ as a *weighted generalized Lebesgue space with variable exponent* or simply as weighted variable Lebesgue spaces. This is a special case of Musielak-Orlicz spaces [24]. Nakano [25] considered these spaces (without weights) as examples of so-called modular spaces, and sometimes the spaces $L^{p(\cdot)}(\Gamma,\varrho)$ are referred to as weighted Nakano spaces.

If S is bounded on $L^{p(\cdot)}(\Gamma,\varrho)$, then from [13, Theorem 6.1] it follows that Γ is a Carleson curve. The following result is announced in [16, Theorem 7.1] and in [18, Theorem D]. Its full proof is published in [20].

THEOREM 1.1. *Let Γ be a Carleson Jordan curve and $p : \Gamma \to (1,\infty)$ be a continuous function satisfying*

$$|p(\tau) - p(t)| \leqslant -A_\Gamma/\log|\tau - t| \quad \textit{whenever} \quad |\tau - t| \leqslant 1/2, \tag{2}$$

where A_Γ *is a positive constant depending only on* Γ. *The Cauchy singular integral operator* S *is bounded on* $L^{p(\cdot)}(\Gamma,\varrho)$ *if and only if*

$$0 < 1/p(t_k)+\lambda_k < 1 \quad \textit{for all} \quad k\in\{1,\dots,m\}. \tag{3}$$

We define by $PC(\Gamma)$ as the set of all $a\in L^\infty(\Gamma)$ for which the one-sided limits

$$a(t\pm 0) := \lim_{\tau\to t\pm 0} a(\tau)$$

exist and finite at each point $t\in\Gamma$; here $\tau\to t-0$ means that τ approaches t following the orientation of Γ, while $\tau\to t+0$ means that τ goes to t in the opposite direction. Functions in $PC(\Gamma)$ are called piecewise continuous functions.

The operator S is defined on $L_N^{p(\cdot)}(\Gamma,\varrho)$ elementwise. We let stand $PC_{N\times N}(\Gamma)$ for the algebra of all $N\times N$ matrix functions with entries in $PC(\Gamma)$. Writing the elements of $L_N^{p(\cdot)}(\Gamma,\varrho)$ as columns, we can define the multiplication operator aI for $a\in PC_{N\times N}(\Gamma)$ as multiplication by the matrix function a. Let $\operatorname{alg}(S,PC;L_N^{p(\cdot)}(\Gamma,\varrho))$ denote the smallest closed subalgebra of $\mathcal{B}(L_N^{p(\cdot)}(\Gamma,\varrho))$ containing the operator S and the set $\{aI : a\in PC_{N\times N}(\Gamma)\}$.

For the case of piecewise Lyapunov curves Γ and constant exponent p, a Fredholm criterion for an arbitrary operator $A\in\operatorname{alg}(S,PC;L_N^p(\Gamma,\varrho))$ was obtained by Gohberg and Krupnik [10] (see also [11] and [22]). Spitkovsky [29] established a Fredholm criterion for the operator $aP+Q$, where $a\in PC_{N\times N}(\Gamma)$ and

$$P := (I+S)/2, \quad Q := (I-S)/2,$$

on the space $L_N^p(\Gamma,w)$, where Γ is a smooth curve and w is an arbitrary Muckenhoupt weight. He also proved that if $aP+Q$ is semi-Fredholm on $L_N^p(\Gamma,w)$, then it is automatically Fredholm on $L_N^p(\Gamma,w)$. These results were extended to the case of an arbitrary operator $A\in\operatorname{alg}(S,PC;L_N^p(\Gamma,w))$ in [12]. The Fredholm theory for singular integral operators with piecewise continuous coefficients on Lebesgue spaces with arbitrary Muckenhoupt weights on arbitrary Carleson curves curves was accomplished in a series of papers by Böttcher and Yu. Karlovich. It is presented in their monograph [3] (see also the nice survey [4]).

The study of singular integral operators with discontinuous coefficients on generalized Lebesgue spaces with variable exponent was started in [17, 19]. The results of [3] are partially extended to the case of weighted generalized Lebesgue spaces with variable exponent in [13, 14, 15]. Suppose Γ is a Carleson curve satisfying the logarithmic whirl condition (1) at each point $t\in\Gamma$, ϱ is a Khvedelidze weight, and p is a variable exponent as in Theorem 1.1. Under these assumptions, a Fredholm criterion for an arbitrary operator A in the algebra $\operatorname{alg}(S,PC;L_N^{p(\cdot)}(\Gamma,\varrho))$ is obtained in [14, Theorem 5.1] by using the Allan-Douglas local principle [5, Section 1.35] and the two projections theorem [9]. However, this approach does not allow us to get additional information about semi-Fredholm and Fredholm operators in this algebra. For instance, to obtain an index formula for Fredholm operators in this algebra, we need other means (see, e.g., [15, Section 6]). Following the ideas of [10, 29, 12], in this paper we present a self-contained proof of the following result.

THEOREM 1.2. *Let Γ be a Carleson Jordan curve satisfying the logarithmic whirl condition* (1) *at each point $t \in \Gamma$, let $p : \Gamma \to (1,\infty)$ be a continuous function satisfying* (2)*, and let ϱ be a Khvedelidze weight satisfying* (3)*. If an operator in the algebra $\operatorname{alg}(S, PC; L_N^{p(\cdot)}(\Gamma,\varrho))$ is semi-Fredholm, then it is Fredholm.*

The paper is organized as follows. Section 2 contains general results on semi-Fredholm operators. Some auxiliary results on singular integral operators acting on $L^{p(\cdot)}(\Gamma,\varrho)$ are collected in Section 3. In Section 4, we prove a criterion guaranteeing that $aP+Q$, where $a \in PC(\Gamma)$, has closed image in $L^{p(\cdot)}(\Gamma,\varrho)$. This criterion is intimately related with a Fredholm criterion for $aP+Q$ proved in [14]. Notice that we are able to prove both results for Carleson Jordan curves which satisfy the additional condition (1). Section 5 contains the proof of the fact that if the operator $aP+bQ$ is semi-Fredholm on $L_N^{p(\cdot)}(\Gamma,\varrho)$, then the coefficients a and b are invertible in the algebra $L^\infty_{N\times N}(\Gamma)$. In Section 6, we prove that the semi-Fredholmness and Fredholmness of $aP+bQ$ on $L_N^{p(\cdot)}(\Gamma,\varrho)$, where a and b are piecewise continuous matrix functions, are equivalent. In Section 7, we extend this result to the sums of products of operators of the form $aP+bQ$ by using the procedure of linear dilation. Since these sums are dense in $\operatorname{alg}(S, PC; L_N^{p(\cdot)}(\Gamma,\varrho))$, Theorem 1.2 follows from stability properties of semi-Fredholm operators.

2. General results on semi-Fredholm and Fredholm operators

2.1. The Atkinson and Yood theorems

For a Banach space X, let $\Phi(X)$ be the set of all Fredholm operators on X and let $\Phi_+(X)$ (resp. $\Phi_-(X)$) denote the set of all n-normal (resp. d-normal) operators $A \in \mathcal{B}(X)$ such that $d(A;X) = +\infty$ (resp. $n(A;X) = +\infty$).

THEOREM 2.1. *Let X be a Banach space and K be a compact operator on X.*

(a) *If $A, B \in \Phi(X)$, then $AB \in \Phi(X)$ and $A+K \in \Phi(X)$.*
(b) *If $A, B \in \Phi_\pm(X)$, then $AB \in \Phi_\pm(X)$ and $A+K \in \Phi_\pm(X)$.*
(c) *If $A \in \Phi(X)$ and $B \in \Phi_\pm(X)$, then $AB \in \Phi_\pm(X)$ and $BA \in \Phi_\pm(X)$.*

Part (a) is due to Atkinson, parts (b) and (c) were obtained by Yood. For a proof, see e.g. [11, Chap. 4, Sections 6 and 15].

THEOREM 2.2. (see e.g. [11], Chap. 4, Theorem 7.1) *Let X be a Banach space. An operator $A \in \mathcal{B}(X)$ is Fredholm if and only if there exists an operator $R \in \mathcal{B}(X)$ such that $AR - I$ and $RA - I$ are compact.*

2.2. Stability of semi-Fredholm operators

THEOREM 2.3. (see e.g. [11], Chap. 4, Theorems 6.4, 15.4) *Let X be a Banach space.*

(a) *If $A \in \Phi(X)$, then there exists an $\varepsilon = \varepsilon(A) > 0$ such that $A+D \in \Phi(X)$ whenever $\|D\|_{\mathcal{B}(X)} < \varepsilon$.*

(*b*) *If* $A \in \Phi_\pm(X)$, *then there exists an* $\varepsilon = \varepsilon(A) > 0$ *such that* $A + D \in \Phi_\pm(X)$ *whenever* $\|D\|_{\mathcal{B}(X)} < \varepsilon$.

LEMMA 2.4. *Let* X *be a Banach space. Suppose* A *is a semi-Fredholm operator on* X *and* $\|A_n - A\|_{\mathcal{B}(X)} \to 0$ *as* $n \to \infty$. *If the operators* A_n *are Fredholm on* X *for all sufficiently large* n, *then* A *is Fredholm, too.*

Proof. Assume A is semi-Fredholm, but not Fredholm. Then either $A \in \Phi_-(X)$ or $A \in \Phi_+(X)$. By Theorem 2.3(b), either $A_n \in \Phi_-(X)$ or $A_n \in \Phi_+(X)$ for all sufficiently large n. That is, A_n are not Fredholm. This contradicts the hypothesis. □

We refer to the monograph by Gohberg and Krupnik [11] for a detailed presentation of the theory of semi-Fredholm operators on Banach spaces.

2.3. Semi-Fredholmness of block operators

Let a Banach space X be represented as the direct sum of its subspaces $X = X_1 \dot{+} X_2$. Then every operator $A \in \mathcal{B}(X)$ can be written in the form of an operator matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{ij} \in \mathcal{B}(X_j, X_i)$ and $i, j = 1, 2$. The following result is stated without proof in [27]. Its proof is given in [28] (see also [23, Theorem 1.12]).

THEOREM 2.5.

(*a*) *Suppose* A_{21} *is compact. If* A *is* n*-normal (*d*-normal), then* A_{11} *(resp.* A_{22}*) is* n*-normal (resp.* d*-normal).*

(*b*) *Suppose* A_{12} *or* A_{21} *is compact. If* A_{11} *(resp.* A_{22}*) is Fredholm, then* A_{22} *(resp.* A_{11}*) is* n*-normal,* d*-normal, Fredholm if and only if* A *has the corresponding property.*

3. Singular integrals on weighted variable Lebesgue spaces

3.1. Duality of weighted variable Lebesgue spaces

Suppose Γ is a rectifiable Jordan curve and $p : \Gamma \to (1, \infty)$ is a continuous function. Since Γ is compact, we have

$$1 < \underline{p} := \min_{t \in \Gamma} p(t), \quad \overline{p} := \max_{t \in \Gamma} p(t) < \infty.$$

Define the conjugate exponent p^* for the exponent p by

$$p^*(t) := \frac{p(t)}{p(t) - 1} \quad (t \in \Gamma).$$

Suppose ϱ is a Khvedelidze weight. If $\varrho \equiv 1$, then we will write $L^{p(\cdot)}(\Gamma)$ and $\|\cdot\|_{p(\cdot)}$ instead of $L^{p(\cdot)}(\Gamma, 1)$ and $\|\cdot\|_{p(\cdot),1}$, respectively.

THEOREM 3.1. (see [21], Theorem 2.1) *If* $f \in L^{p(\cdot)}(\Gamma)$ *and* $g \in L^{p^*(\cdot)}(\Gamma)$, *then* $fg \in L^1(\Gamma)$ *and*

$$\|fg\|_1 \leqslant (1 + 1/\underline{p} - 1/\overline{p})\, \|f\|_{p(\cdot)} \|g\|_{p^*(\cdot)}.$$

The above Hölder type inequality in the more general setting of Musielak-Orlicz spaces is contained in [24, Theorem 3.13].

THEOREM 3.2. *The general form of a linear functional on* $L^{p(\cdot)}(\Gamma, \varrho)$ *is given by*

$$G(f) = \int_\Gamma f(\tau)\overline{g(\tau)}\, |d\tau| \quad (f \in L^{p(\cdot)}(\Gamma, \varrho)),$$

where $g \in L^{p^*(\cdot)}(\Gamma, \varrho^{-1})$. *The norms in the dual space* $[L^{p(\cdot)}(\Gamma, \varrho)]^*$ *and in the space* $L^{p^*(\cdot)}(\Gamma, \varrho^{-1})$ *are equivalent.*

The above result can be extracted from [24, Corollary 13.14]. For the case $\varrho = 1$, see also [21, Corollary 2.7].

3.2. Smirnov classes and Hardy type subspaces

Let Γ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$. We denote by D_+ and D_- the bounded and unbounded components of $\mathbb{C} \setminus \Gamma$, respectively. We orient Γ counter-clockwise. Without loss of generality we assume that $0 \in D_+$. A function f analytic in D_+ is said to be in the Smirnov class $E^q(D_+)$ ($0 < q < \infty$) if there exists a sequence of rectifiable Jordan curves Γ_n in D_+ tending to the boundary Γ in the sense that Γ_n eventually surrounds each compact subset of D_+ such that

$$\sup_{n \geqslant 1} \int_{\Gamma_n} |f(z)|^q |dz| < \infty. \tag{4}$$

The Smirnov class $E^q(D_-)$ is the set of all analytic functions in $D_- \cup \{\infty\}$ for which (4) holds with some sequence of curves Γ_n tending to the boundary in the sense that every compact subset of $D_- \cup \{\infty\}$ eventually lies outside Γ_n. We denote by $E_0^q(D_-)$ the set of functions in $E^q(D_-)$ which vanish at infinity. The functions in $E^q(D_\pm)$ have nontangential boundary values almost everywhere on Γ (see, e.g. [8, Theorem 10.3]). We will identify functions in $E^q(D_\pm)$ with their nontangential boundary values. The next result is a consequence of the Hölder inequality.

LEMMA 3.3. *Let* Γ *be a rectifiable Jordan curve. Suppose* $0 < q_1, \dots, q_r < \infty$ *and* $f_j \in E^{q_j}(D_\pm)$ *for all* $j \in \{1, 2, \dots, r\}$. *Then* $f_1 f_2 \dots f_r \in E^q(D_\pm)$, *where*

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_r}.$$

Let $\mathcal{R}$ denote the set of all rational functions without poles on Γ.

THEOREM 3.4. *Let* Γ *be a rectifiable Jordan curve and* $0 < q < \infty$. *If* f *belongs to* $E^q(D_\pm) + \mathcal{R}$ *and its nontangential boundary values vanish on a subset* $\gamma \subset \Gamma$ *of positive measure, then* f *vanishes identically in* $D_\pm$.

This result follows from the Lusin-Privalov theorem for meromorphic functions (see, e.g. [26, p. 292]).

We refer to the monographs by Duren [8] and Privalov [26] for a detailed exposition of the theory of Smirnov classes over domains with rectifiable boundary.

LEMMA 3.5. *Let Γ be a Carleson Jordan curve, let $p:\Gamma\to(1,\infty)$ be a continuous function satisfying* (2)*, and let ϱ be a Khvedelidze weight satisfying* (3)*. Then $P^2=P$ and $Q^2=Q$ on $L^{p(\cdot)}(\Gamma,\varrho)$.*

This result follows from Theorem 1.1 and [13, Lemma 6.4].

In view of Lemma 3.5, the Hardy type subspaces $PL^{p(\cdot)}(\Gamma,\varrho)$, $QL^{p(\cdot)}(\Gamma,\varrho)$, and $QL^{p(\cdot)}(\Gamma,\varrho)\dot{+}\mathbb{C}$ of $L^{p(\cdot)}(\Gamma,\varrho)$ are well defined. Combining Theorem 1.1 and [13, Lemma 6.9] we obtain the following.

LEMMA 3.6. *Let Γ be a Carleson Jordan curve, let $p:\Gamma\to(1,\infty)$ be a continuous function satisfying* (2)*, and let ϱ be a Khvedelidze weight satisfying* (3)*. Then*

$$
\begin{aligned}
E^1(D_+)\cap L^{p(\cdot)}(\Gamma,\varrho) &= PL^{p(\cdot)}(\Gamma,\varrho),\\
E^1_0(D_-)\cap L^{p(\cdot)}(\Gamma,\varrho) &= QL^{p(\cdot)}(\Gamma,\varrho),\\
E^1(D_-)\cap L^{p(\cdot)}(\Gamma,\varrho) &= QL^{p(\cdot)}(\Gamma,\varrho)\dot{+}\mathbb{C}.
\end{aligned}
$$

3.3. Singular integral operators on the dual space

For a rectifiable Jordan curve Γ we have $d\tau=e^{i\Theta_\Gamma(\tau)}|d\tau|$ where $\Theta_\Gamma(\tau)$ is the angle between the positively oriented real axis and the naturally oriented tangent of Γ at τ (which exists almost everywhere). Let the operator H_Γ be defined by $(H_\Gamma\varphi)(t)=e^{-i\Theta_\Gamma(t)}\overline{\varphi(t)}$ for $t\in\Gamma$. Note that H_Γ is additive but $H_\Gamma(\alpha\varphi)=\overline{\alpha}H_\Gamma\varphi$ for $\alpha\in\mathbb{C}$. Evidently, $H_\Gamma^2=I$.

From Theorem 1.1 and [13, Lemma 6.6] we get the following.

LEMMA 3.7. *Let Γ be a Carleson Jordan curve, let $p:\Gamma\to(1,\infty)$ be a continuous function satisfying* (2)*, and let ϱ be a Khvedelidze weight satisfying* (3)*. The adjoint operator of $S\in\mathcal{B}(L^{p(\cdot)}(\Gamma,\varrho))$ is the operator $-H_\Gamma SH_\Gamma\in\mathcal{B}(L^{p^*(\cdot)}(\Gamma,\varrho^{-1}))$.*

LEMMA 3.8. *Let Γ be a Carleson Jordan curve, let $p:\Gamma\to(1,\infty)$ be a continuous function satisfying* (2)*, and let ϱ be a Khvedelidze weight satisfying* (3)*. Suppose $a\in L^\infty(\Gamma)$ and $a^{-1}\in L^\infty(\Gamma)$.*

(*a*) *The operator $aP+Q$ is n-normal on $L^{p(\cdot)}(\Gamma,\varrho)$ if and only if the operator $a^{-1}P+Q$ is d-normal on $L^{p^*(\cdot)}(\Gamma,\varrho^{-1})$. In this case*

$$
n\big(aP+Q;L^{p(\cdot)}(\Gamma,\varrho)\big)=d\big(a^{-1}P+Q;L^{p^*(\cdot)}(\Gamma,\varrho^{-1})\big). \tag{5}
$$

(*b*) *The operator $aP+Q$ is d-normal on $L^{p(\cdot)}(\Gamma,\varrho)$ if and only if the operator $a^{-1}P+Q$ is n-normal on $L^{p^*(\cdot)}(\Gamma,\varrho^{-1})$. In this case*

$$
d\big(aP+Q;L^{p(\cdot)}(\Gamma,\varrho)\big)=n\big(a^{-1}P+Q;L^{p^*(\cdot)}(\Gamma,\varrho^{-1})\big).
$$

Proof. By Theorem 3.2, the space $L^{p^*(\cdot)}(\Gamma,\varrho^{-1})$ may be identified with the dual space $[L^{p(\cdot)}(\Gamma,\varrho)]^*$. Let us prove part (a). The operator $aP+Q$ is n-normal on $L^{p(\cdot)}(\Gamma,\varrho)$ if and only if its adjoint $(aP+Q)^*$ is d-normal on the dual space $L^{p^*(\cdot)}(\Gamma,\varrho^{-1})$ and

$$n\big(aP+Q;L^{p(\cdot)}(\Gamma,\varrho)\big)=d\big((aP+Q)^*;L^{p^*(\cdot)}(\Gamma,\varrho^{-1})\big). \tag{6}$$

From Theorem 3.2 it follows that

$$(aI)^*=H_\Gamma aH_\Gamma. \tag{7}$$

Combining Lemma 3.7 and (7), we get

$$(aP+Q)^*=H_\Gamma(P+QaI)H_\Gamma. \tag{8}$$

On the other hand, taking into account Lemma 3.5, it is easy to check that

$$P+QaI=(I+Pa^{-1}Q)(a^{-1}P+Q)(I-Qa^{-1}P)aI, \tag{9}$$

where $I+Pa^{-1}Q$, $I-Qa^{-1}P$, and aI are invertible operators on $L^{p^*(\cdot)}(\Gamma,\varrho^{-1})$. From (8) and (9) it follows that $(aP+Q)^*$ and $a^{-1}P+Q$ are d-normal on the space $L^{p^*(\cdot)}(\Gamma,\varrho^{-1})$ only simultaneously and

$$d\big((aP+Q)^*;L^{p^*(\cdot)}(\Gamma,\varrho^{-1})\big)=d\big(a^{-1}P+Q;L^{p^*(\cdot)}(\Gamma,\varrho^{-1})\big). \tag{10}$$

Combining (6) and (10), we arrive at (5). Part (a) is proved. The proof of part (b) is analogous. □

Denote by $L^\infty_{N\times N}(\Gamma)$ the algebra of all $N\times N$ matrix functions with entries in the space $L^\infty(\Gamma)$.

LEMMA 3.9. *Let Γ be a Carleson Jordan curve, let $p:\Gamma\to(1,\infty)$ be a continuous function satisfying* (2)*, and let ϱ be a Khvedelidze weight satisfying* (3)*. Suppose $a\in L^\infty_{N\times N}(\Gamma)$ and a^T is the transposed matrix of a. Then the operator $P+aQ$ is n-normal (resp. d-normal) on $L^{p(\cdot)}_N(\Gamma,\varrho)$ if and only if the operator a^TP+Q is d-normal (resp. n-normal) on $L^{p^*(\cdot)}_N(\Gamma,\varrho^{-1})$.*

Proof. In view of Theorem 3.2, the space $L^{p^*(\cdot)}_N(\Gamma,\varrho^{-1})$ may be identified with the dual space $[L^{p(\cdot)}_N(\Gamma,\varrho)]^*$, and the general form of a linear functional on $L^{p(\cdot)}_N(\Gamma,\varrho)$ is given by

$$G(f)=\sum_{j=1}^N\int_\Gamma f_j(\tau)\overline{g_j(\tau)}\,|d\tau|,$$

where $f=(f_1,\dots,f_N)\in L^{p(\cdot)}_N(\Gamma,\varrho)$ and $g=(g_1,\dots,g_N)\in L^{p^*(\cdot)}_N(\Gamma,\varrho^{-1})$, and the norms in $[L^{p(\cdot)}_N(\Gamma,\varrho)]^*$ and in $L^{p^*(\cdot)}_N(\Gamma,\varrho^{-1})$ are equivalent. It is easy to see that $(aI)^*=H_\Gamma a^TH_\Gamma$, where H_Γ is defined on $L^{p^*(\cdot)}_N(\Gamma,\varrho^{-1})$ elementwise.

From Lemma 3.7 it follows that $P^* = H_\Gamma Q H_\Gamma$ and $Q^* = H_\Gamma P H_\Gamma$ on $L_N^{p^*(\cdot)}(\Gamma, \varrho^{-1})$. Then

$$(P + aQ)^* = H_\Gamma(Pa^T I + Q)H_\Gamma. \tag{11}$$

On the other hand, it is easy to see that

$$Pa^T I + Q = (I + Pa^T Q)(a^T P + Q)(I - Qa^T P), \tag{12}$$

where the operators $I + Pa^T Q$ and $I - Qa^T P$ are invertible on $L_N^{p^*(\cdot)}(\Gamma, \varrho^{-1})$. From (11) and (12) it follows that $(P + aQ)^*$ and $a^T P + Q$ are n-normal (resp. d-normal) on $L_N^{p^*(\cdot)}(\Gamma, \varrho^{-1})$ only simultaneously. This implies the desired statement. □

4. Closedness of the image of $aP + Q$ in the scalar case

4.1. Functions in $L^{p(\cdot)}(\Gamma, \varrho)$ are better than integrable if S is bounded

LEMMA 4.1. *Suppose Γ is a Carleson Jordan curve and $p : \Gamma \to (1, \infty)$ is a continuous function satisfying* (2). *If ϱ is a Khvedelidze weight satisfying* (3), *then there exists an $\varepsilon > 0$ such that $L^{p(\cdot)}(\Gamma, \varrho)$ is continuously embedded in $L^{1+\varepsilon}(\Gamma)$.*

Proof. If (3) holds, then there exists a number $\varepsilon > 0$ such that

$$0 < (1/p(t_k) + \lambda_k)(1 + \varepsilon) < 1 \quad \text{for all} \quad k \in \{1, \ldots, m\}.$$

Hence, by Theorem 1.1, the operator S is bounded on $L^{p(\cdot)/(1+\varepsilon)}(\Gamma, \varrho^{1+\varepsilon})$. In that case the operator $\varrho^{1+\varepsilon} S \varrho^{-1-\varepsilon} I$ is bounded on $L^{p(\cdot)/(1+\varepsilon)}(\Gamma)$. Obviously, the operator V defined by $(Vg)(t) = tg(t)$ is bounded on $L^{p(\cdot)/(1+\varepsilon)}(\Gamma)$, and

$$((AV - VA)g)(t) = \frac{\varrho^{1+\varepsilon}(t)}{\pi i} \int_\Gamma \frac{g(\tau)}{\varrho^{1+\varepsilon}(\tau)}\, d\tau.$$

Since $AV - VA$ is bounded on $L^{p(\cdot)/(1+\varepsilon)}(\Gamma)$, there exists a constant $C > 0$ such that

$$\left| \int_\Gamma \frac{g(\tau)}{\varrho^{1+\varepsilon}(\tau)}\, d\tau \right| \|\varrho^{1+\varepsilon}\|_{p(\cdot)/(1+\varepsilon)} = \left\| \varrho^{1+\varepsilon} \int_\Gamma \frac{g(\tau)}{\varrho^{1+\varepsilon}(\tau)}\, d\tau \right\|_{p(\cdot)/(1+\varepsilon)} \leqslant C \|g\|_{p(\cdot)/(1+\varepsilon)}$$

for all $g \in L^{p(\cdot)/(1+\varepsilon)}(\Gamma)$. Since $\varrho(\tau) > 0$ a.e. on Γ, we have $\|\varrho^{1+\varepsilon}\|_{p(\cdot)/(1+\varepsilon)} > 0$. Hence

$$\Lambda(g) = \int_\Gamma \frac{g(\tau)}{\varrho^{1+\varepsilon}(\tau)} e^{i\Theta_\Gamma(\tau)}\, |d\tau|$$

is a bounded linear functional on $L^{p(\cdot)/(1+\varepsilon)}(\Gamma)$. From Theorem 3.2 it follows that $\varrho^{-1-\varepsilon} \in L^{[p(\cdot)/(1+\varepsilon)]^*}(\Gamma)$, where

$$\left(\frac{p(t)}{1 + \varepsilon} \right)^* = \frac{p(t)}{p(t) - (1 + \varepsilon)}$$

is the conjugate exponent for $p(\cdot)/(1 + \varepsilon)$. By Theorem 3.1,

$$\int_\Gamma |f(\tau)|^{1+\varepsilon} |d\tau| \leqslant C_{p(\cdot),\varepsilon} \left\| |f|^{1+\varepsilon} \varrho^{1+\varepsilon} \right\|_{p(\cdot)/(1+\varepsilon)} \|\varrho^{-1-\varepsilon}\|_{[p(\cdot)/(1+\varepsilon)]^*}. \tag{13}$$

It is easy to see that

$$\big\| |f|^{1+\varepsilon} \varrho^{1+\varepsilon} \big\|_{p(\cdot)/(1+\varepsilon)} = \|f\varrho\|_{p(\cdot)}^{1+\varepsilon} = \|f\|_{p(\cdot),\varrho}^{1+\varepsilon}. \tag{14}$$

From (13) and (14) it follows that $\|f\|_{1+\varepsilon} \leqslant C_{p(\cdot),\varepsilon,\varrho}\|f\|_{p(\cdot),\varrho}$ for all $f \in L^{p(\cdot)}(\Gamma,\varrho)$, where $C_{p(\cdot),\varepsilon,\varrho} := (C_{p(\cdot),\varepsilon}\|\varrho^{-1-\varepsilon}\|_{[p(\cdot)/(1+\varepsilon)]^*})^{1/(1+\varepsilon)} < \infty$. $\square$

4.2. Criterion for Fredholmness of $aP+Q$ in the scalar case

THEOREM 4.2. (see [14], Theorem 3.3) *Let Γ be a Carleson Jordan curve satisfying the logarithmic whirl condition* (1) *at each point $t \in \Gamma$, let $p : \Gamma \to (1,\infty)$ be a continuous function satisfying* (2)*, and let ϱ be a Khvedelidze weight satisfying* (3)*. Suppose $a \in PC(\Gamma)$. The operator $aP+Q$ is Fredholm on $L^{p(\cdot)}(\Gamma,\varrho)$ if and only if $a(t \pm 0) \neq 0$ and*

$$-\frac{1}{2\pi}\arg\frac{a(t-0)}{a(t+0)} + \frac{\delta(t)}{2\pi}\log\left|\frac{a(t-0)}{a(t+0)}\right| + \frac{1}{p(t)} + \lambda(t) \notin \mathbb{Z} \tag{15}$$

for all $t \in \Gamma$, where

$$\lambda(t) := \begin{cases} \lambda_k, & \textit{if } \ t = t_k, \quad k \in \{1,\dots,m\}, \\ 0, & \textit{if } \ t \notin \Gamma \setminus \{t_1,\dots,t_m\}. \end{cases}$$

The necessity portion of this result was obtained in [13, Theorem 8.1] for spaces with variable exponents satisfying (2) under the assumption that S is bounded on $L^{p(\cdot)}(\Gamma,w)$, where Γ is an arbitrary rectifiable Jordan curve and w is an arbitrary weight (not necessarily power). The sufficiency portion follows from [13, Lemma 7.1] and Theorem 1.1 (see [14] for details). The restriction (1) comes up in the proof of the sufficiency portion because under this condition one can guarantee the boundedness of the weighted operator $wSw^{-1}I$, where $w(\tau) = |(t-\tau)^\gamma|$ and $\gamma \in \mathbb{C}$. If Γ does not satisfy (1), then the weight w is not equivalent to a Khvedelidze weight and Theorem 1.1 is not applicable to the operator $wSw^{-1}I$, that is, a more general result than Theorem 1.1 is needed to treat the case of arbitrary Carleson curves. As far as we know, such a result is not known in the case of variable exponents. For a constant exponent p, the result of Theorem 4.2 (for arbitrary Muckenhoupt weights) is proved in [2] (see also [3, Proposition 7.3] for the case of arbitrary Muckenhoupt weights and arbitrary Carleson curves).

4.3. Criterion for the closedness of the image of $aP+Q$

THEOREM 4.3. *Let Γ be a Carleson Jordan curve satisfying the logarithmic whirl condition* (1) *at each point $t \in \Gamma$, let $p : \Gamma \to (1,\infty)$ be a continuous function satisfying* (2)*, and let ϱ be a Khvedelidze weight satisfying* (3)*. Suppose $a \in PC(\Gamma)$ has finitely many jumps and $a(t \pm 0) \neq 0$ for all $t \in \Gamma$. Then the image of $aP+Q$ is closed in $L^{p(\cdot)}(\Gamma,\varrho)$ if and only if* (15) *holds for all $t \in \Gamma$.*

Proof. The idea of the proof is borrowed from [3, Proposition 7.16]. The sufficiency part follows from Theorem 4.2. Let us prove the necessity part. Assume that $a(t\pm 0)\neq 0$ for all $t\in\Gamma$. Since the number of jumps, that is, the points $t\in\Gamma$ at which $a(t-0)\neq a(t+0)$, is finite, it is clear that

$$\begin{aligned}
&-\frac{1}{2\pi}\arg\frac{a(t-0)}{a(t+0)}+\frac{\delta(t)}{2\pi}\log\left|\frac{a(t-0)}{a(t+0)}\right|+\frac{1}{1+\varepsilon}\notin\mathbb{Z},\\
&-\frac{1}{2\pi}\arg\frac{a(t+0)}{a(t-0)}+\frac{\delta(t)}{2\pi}\log\left|\frac{a(t+0)}{a(t-0)}\right|+\frac{1}{1+\varepsilon}\notin\mathbb{Z}
\end{aligned}$$

for all $t\in\Gamma$ and all sufficiently small $\varepsilon>0$. By Theorem 4.2, the operators $aP+Q$ and $a^{-1}P+Q$ are Fredholm on the Lebesgue space $L^{1+\varepsilon}(\Gamma)$ whenever $\varepsilon>0$ is sufficiently small. From Lemma 4.1 it follows that we can pick $\varepsilon_0>0$ such that

$$L^{p(\cdot)}(\Gamma,\varrho)\subset L^{1+\varepsilon_0}(\Gamma),\quad L^{p^*(\cdot)}(\Gamma,\varrho^{-1})\subset L^{1+\varepsilon_0}(\Gamma)$$

and $aP+Q$, $a^{-1}P+Q$ are Fredholm on $L^{1+\varepsilon_0}(\Gamma)$. Then

$$n\big(aP+Q;L^{p(\cdot)}(\Gamma,\varrho)\big)\leqslant n\big(aP+Q;L^{1+\varepsilon_0}(\Gamma)\big)<\infty, \tag{16}$$

and taking into account Lemma 3.8(b),

$$\begin{aligned}
d\big(aP+Q;L^{p(\cdot)}(\Gamma,\varrho)\big) &= n\big(a^{-1}P+Q;L^{p^*(\cdot)}(\Gamma,\varrho^{-1})\big)\\
&\leqslant n\big(a^{-1}P+Q;L^{1+\varepsilon_0}(\Gamma)\big)<\infty.
\end{aligned}\tag{17}$$

If (15) does not hold, then $aP+Q$ is not Fredholm on $L^{p(\cdot)}(\Gamma,\varrho)$ in view of Theorem 4.2.

From this fact and (16)–(17) we conclude that the image of $aP+Q$ is not closed in $L^{p(\cdot)}(\Gamma,\varrho)$, which contradicts the hypothesis. □

5. Necessary condition for semi-Fredholmness of $aP+bQ$. The matrix case

5.1. Two lemmas on approximation of measurable matrix functions

Let the algebra $L^\infty_{N\times N}(\Gamma)$ be equipped with the norm

$$\|a\|_{L^\infty_{N\times N}(\Gamma)}:=N\max_{1\leqslant i,j\leqslant N}\|a_{ij}\|_{L^\infty(\Gamma)}.$$

LEMMA 5.1. (see [23], Lemma 3.4) *Let Γ be a rectifiable Jordan curve. Suppose a is a measurable $N\times N$ matrix function on Γ such that $a^{-1}\notin L^\infty_{N\times N}(\Gamma)$. Then for every $\varepsilon>0$ there exists a matrix function $a_\varepsilon\in L^\infty_{N\times N}(\Gamma)$ such that $\|a_\varepsilon\|_{L^\infty_{N\times N}(\Gamma)}<\varepsilon$ and the matrix function $a-a_\varepsilon$ degenerates on a subset $\gamma\subset\Gamma$ of positive measure.*

LEMMA 5.2. (see [23], Lemma 3.6) *Let Γ be a rectifiable Jordan curve. If a belongs to $L^\infty_{N\times N}(\Gamma)$, then for every $\varepsilon>0$ there exists an $a_\varepsilon\in L^\infty_{N\times N}(\Gamma)$ such that $\|a-a_\varepsilon\|_{L^\infty_{N\times N}(\Gamma)}<\varepsilon$ and $a_\varepsilon^{-1}\in L^\infty_{N\times N}(\Gamma)$.*

5.2. Necessary condition for d-normality of $aP+Q$ and $P+aQ$

LEMMA 5.3. *Suppose Γ is a Carleson Jordan curve, $p:\Gamma\to(1,\infty)$ is a continuous function satisfying* (2)*, and ϱ is a Khvedelidze weight satisfying* (3)*. If $a\in L^\infty_{N\times N}(\Gamma)$ and at least one of the operators $aP+Q$ or $P+aQ$ is d-normal on $L^{p(\cdot)}_N(\Gamma,\varrho)$, then $a^{-1}\in L^\infty_{N\times N}(\Gamma)$.*

Proof. This lemma is proved by analogy with [23, Theorem 3.13]. For definiteness, let us consider the operator $P+aQ$. Assume that $a^{-1}\notin L^\infty_{N\times N}(\Gamma)$. By Lemma 5.1, for every $\varepsilon>0$ there exists an $a_\varepsilon\in L^\infty_{N\times N}(\Gamma)$ such that $\|a-a_\varepsilon\|_{L^\infty_{N\times N}(\Gamma)}<\varepsilon$ and a_ε degenerates on a subset $\gamma\subset\Gamma$ of positive measure. We have

$$\|(P+aQ)-(P+a_\varepsilon Q)\|_{\mathcal{B}(L^{p(\cdot)}_N(\Gamma,\varrho))}\leqslant\|a-a_\varepsilon\|_{L^\infty_{N\times N}(\Gamma)}\|Q\|_{\mathcal{B}(L^{p(\cdot)}_N(\Gamma,\varrho))}=O(\varepsilon)$$

as $\varepsilon\to0$. Hence there is an $\varepsilon>0$ such that $P+a_\varepsilon Q$ is d-normal together with $P+aQ$ due to Theorem 2.3. Since the image of the operator $P+a_\varepsilon Q$ is a subspace of finite codimension in $L^{p(\cdot)}_N(\Gamma,\varrho)$, it has a nontrivial intersection with any infinite-dimensional linear manifold contained in $L^{p(\cdot)}_N(\Gamma,\varrho)$. In particular, the image of $P+a_\varepsilon Q$ has a nontrivial intersection with linear manifolds M_j, $j\in\{1,\dots,N\}$, of those vector-functions, the j-th component of which is a polynomial of $1/z$ vanishing at infinity and all the remaining components are identically zero. That is, there exist

$$\psi_j^+\in PL^{p(\cdot)}_N(\Gamma,\varrho),\quad \psi_j^-\in QL^{p(\cdot)}_N(\Gamma,\varrho),\quad h_j\in M_j,\quad h_j\not\equiv0$$

such that $\psi_j^++a_\varepsilon\psi_j^-=h_j$ for all $j\in\{1,\dots,N\}$. Consider the $N\times N$ matrix functions

$$\Psi_+:=[\psi_1^+,\psi_2^+,\dots,\psi_N^+],\quad \Psi_-:=[\psi_1^-,\psi_2^-,\dots,\psi_N^-],\quad H:=[h_1,h_2,\dots,h_N],$$

where ψ_j^+, ψ_j^-, and h_j are taken as columns. Then $H-\Psi_+=a_\varepsilon\Psi_-$. Therefore,

$$\det(H-\Psi_+)=\det a_\varepsilon\det\Psi_-\quad\text{a.e. on}\quad\Gamma.$$

The left-hand side of this equality is a meromorphic function having a pole at zero of at least N-th order. Thus, it is not identically zero in D_+.

On the other hand, each entry of $H-\Psi_+$ belongs to

$$PL^{p(\cdot)}(\Gamma,\varrho)+\mathcal{R}\subset E^1(D_+)+\mathcal{R}$$

(see Lemma 3.6). Hence, by Lemma 3.3, the function $\det(H-\Psi_+)\in E^{1/N}(D_+)+\mathcal{R}$ and $\det(H-\Psi_+)$ degenerates on γ because a_ε degenerates on γ. In view of Theorem 3.4, $\det(H-\Psi_+)$ vanishes identically in D_+. This is a contradiction. Thus, a^{-1} belongs to $L^\infty_{N\times N}(\Gamma)$. $\square$

5.3. Necessary condition for semi-Fredholmness of $aP+bQ$

THEOREM 5.4. *Let Γ be a Carleson Jordan curve, let $p:\Gamma\to(1,\infty)$ be a continuous function satisfying* (2)*, and let ϱ be a Khvedelidze weight satisfying* (3)*. If the coefficients a and b belong to $L^\infty_{N\times N}(\Gamma)$ and the operator $aP+bQ$ is semi-Fredholm on $L^{p(\cdot)}_N(\Gamma,\varrho)$, then $a^{-1},b^{-1}\in L^\infty_{N\times N}(\Gamma)$.*

Proof. The proof is analogous to the proof of [23, Theorem 3.18]. Suppose $aP+bQ$ is d-normal on $L^{p(\cdot)}_N(\Gamma,\varrho)$. By Lemma 5.2, for every $\varepsilon>0$ there exist $a_\varepsilon\in L^\infty_{N\times N}(\Gamma)$ such that $a_\varepsilon^{-1}\in L^\infty_{N\times N}(\Gamma)$ and $\|a-a_\varepsilon\|_{L^\infty_{N\times N}(\Gamma)}<\varepsilon$. Since

$$\|(aP+bQ)-(a_\varepsilon P+bQ)\|_{\mathcal{B}(L^{p(\cdot)}_N(\Gamma,\varrho))}\leqslant\|a-a_\varepsilon\|_{L^\infty_{N\times N}(\Gamma)}\|P\|_{\mathcal{B}(L^{p(\cdot)}_N(\Gamma,\varrho))}=O(\varepsilon)$$

as $\varepsilon\to0$, from Theorem 2.3 it follows that $\varepsilon>0$ can be chosen so small that $a_\varepsilon P+bQ$ is d-normal on $L^{p(\cdot)}_N(\Gamma,\varrho)$, too. Since $a_\varepsilon^{-1}\in L^\infty_{N\times N}(\Gamma)$, the operator $a_\varepsilon I$ is invertible on $L^{p(\cdot)}_N(\Gamma,\varrho)$. From Theorem 2.1 it follows that the operator $P+a_\varepsilon^{-1}bQ=a_\varepsilon^{-1}(a_\varepsilon P+bQ)$ is d-normal. By Lemma 5.3, $b^{-1}a_\varepsilon$ belongs to $L^\infty_{N\times N}(\Gamma)$. Hence $b^{-1}=b^{-1}a_\varepsilon a_\varepsilon^{-1}\in L^\infty_{N\times N}(\Gamma)$.

Furthermore, $b^{-1}aP+Q=b^{-1}(aP+bQ)$ and the operator $b^{-1}aP+Q$ is d-normal on $L^{p(\cdot)}_N(\Gamma,\varrho)$. By Lemma 5.3, $a^{-1}b\in L^\infty_{N\times N}(\Gamma)$. Then $a^{-1}=a^{-1}bb^{-1}$ belongs to $L^\infty_{N\times N}(\Gamma)$. That is, we have shown that if $aP+bQ$ is d-normal on $L^{p(\cdot)}_N(\Gamma,\varrho)$, then $a^{-1},b^{-1}\in L^\infty_{N\times N}(\Gamma)$.

If $aP+bQ$ is n-normal on $L^{p(\cdot)}_N(\Gamma,\varrho)$, then arguing as above, we conclude that the operator $P+a_\varepsilon^{-1}bQ$ is n-normal on $L^{p(\cdot)}_N(\Gamma,\varrho)$. By Lemma 3.9, the operator $(a_\varepsilon^{-1}b)^TP+Q$ is d-normal on $L^{p^*(\cdot)}_N(\Gamma,\varrho^{-1})$. From Lemma 5.3 it follows that $[(a_\varepsilon^{-1}b)^T]^{-1}\in L^\infty_{N\times N}(\Gamma)$. Therefore, $b^{-1}=(a_\varepsilon^{-1})^{-1}a_\varepsilon^{-1}\in L^\infty_{N\times N}(\Gamma)$. Furthermore, $b^{-1}aP+Q=b^{-1}(aP+bQ)$ and the operator $b^{-1}aP+Q=b^{-1}(aP+bQ)$ is n-normal on $L^{p(\cdot)}_N(\Gamma,\varrho)$. From Lemma 3.9 we get that the operator $P+(b^{-1}a)^TQ$ is d-normal on $L^{p^*(\cdot)}_N(\Gamma,\varrho^{-1})$. Applying Lemma 5.3 to the operator $P+(b^{-1}a)^TQ$ acting on $L^{p^*(\cdot)}_N(\Gamma,\varrho^{-1})$, we obtain $a^{-1}b\in L^\infty_{N\times N}(\Gamma)$. Thus $a^{-1}=a^{-1}bb^{-1}\in L^\infty_{N\times N}(\Gamma)$. □

6. Semi-Fredholmness and Fredholmness of $aP+bQ$ are equivalent

6.1. Decomposition of piecewise continuous matrix functions

Denote by $PC^0(\Gamma)$ the set of all piecewise continuous functions a which have only a finite number of jumps and satisfy $a(t-0)=a(t)$ for all $t\in\Gamma$. Let $C_{N\times N}(\Gamma)$ and $PC^0_{N\times N}(\Gamma)$ denote the sets of $N\times N$ matrix functions with continuous entries and with entries in $PC^0(\Gamma)$, respectively. A matrix function $a\in PC_{N\times N}(\Gamma)$ is said to be nonsingular if $\det a(t\pm0)\neq0$ for all $t\in\Gamma$.

LEMMA 6.1. (see [6], Chap. VII, Lemma 2.2) *Suppose Γ is a rectifiable Jordan curve. If a matrix function $f\in PC^0_{N\times N}(\Gamma)$ is nonsingular, then there exist an upper-triangular nonsingular matrix function $g\in PC^0_{N\times N}(\Gamma)$ and nonsingular matrix functions $c_1,c_2\in C_{N\times N}(\Gamma)$ such that $f=c_1gc_2$.*

6.2. Compactness of commutators

LEMMA 6.2. *Let Γ be a Carleson Jordan curve, let $p:\Gamma\to(1,\infty)$ be a continuous function satisfying* (2)*, and let ϱ be a Khvedelidze weight satisfying* (3)*. If c belongs to $C_{N\times N}(\Gamma)$, then the commutators $cP-PcI$ and $cQ-QcI$ are compact on $L_N^{p(\cdot)}(\Gamma,\varrho)$.*

This statement follows from Theorem 1.1 and [13, Lemma 6.5].

6.3. Equivalence of semi-Fredholmness and Fredholmness of $aP+bQ$

THEOREM 6.3. *Let Γ be a Carleson Jordan curve satisfying the logarithmic whirl condition* (1) *at each point $t\in\Gamma$, let $p:\Gamma\to(1,\infty)$ be a continuous function satisfying* (2)*, and let ϱ be a Khvedelidze weight satisfying* (3)*. If $a,b\in PC^0_{N\times N}(\Gamma)$, then $aP+bQ$ is semi-Fredholm on $L_N^{p(\cdot)}(\Gamma,\varrho)$ if and only if it is Fredholm on $L_N^{p(\cdot)}(\Gamma,\varrho)$.*

Proof. The idea of the proof is borrowed from [29, Theorem 3.1]. Only the necessity portion of the theorem is nontrivial. If $aP+bQ$ is semi-Fredholm, then a and b are nonsingular by Theorem 5.4. Hence $b^{-1}a$ is nonsingular. In view of Lemma 6.1, there exist an upper-triangular nonsingular matrix function $g\in PC^0_{N\times N}(\Gamma)$ and continuous nonsingular matrix functions c_1, c_2 such that $b^{-1}a=c_1gc_2$. It is easy to see that

$$aP+bQ=bc_1\big[(gP+Q)(Pc_2I+Qc_1^{-1}I)+g(c_2P-Pc_2I)+(c_1^{-1}Q-Qc_1^{-1}I)\big]. \quad (18)$$

From Lemma 6.2 it follows that the operators c_2P-Pc_2I and $c_1^{-1}Q-Qc_1^{-1}I$ are compact on $L_N^{p(\cdot)}(\Gamma,\varrho)$ and

$$(Pc_2I+Qc_1^{-1}I)(c_2^{-1}P+c_1Q)=I+K_1,\quad (c_2^{-1}P+c_1Q)(Pc_2I+Qc_1^{-1}I)=I+K_2,$$

where K_1 and K_2 are compact operators on $L_N^{p(\cdot)}(\Gamma,\varrho)$. In view of these equalities, by Theorem 2.2, the operator $Pc_2I+Qc_1^{-1}I$ is Fredholm on $L_N^{p(\cdot)}(\Gamma,\varrho)$. Obviously, the operator bc_1I is invertible because bc_1 is nonsingular. From (18) and Theorem 2.1 it follows that $aP+bQ$ is n-normal, d-normal, Fredholm if and only if $gP+Q$ has the corresponding property.

Let g_j, $j\in\{1,\dots,N\}$, be the elements of the main diagonal of the upper-triangular matrix function g. Since g is nonsingular, all g_j are nonsingular, too. Assume for definiteness that $gP+Q$ is n-normal on $L_N^{p(\cdot)}(\Gamma,\varrho)$. By Theorem 2.5 (a), the operator g_1P+Q is n-normal on $L^{p(\cdot)}(\Gamma,\varrho)$. Hence the image of g_1P+Q is closed. From Theorem 4.3 it follows that (15) is fulfilled with g_1 in place of a. Therefore, the operator g_1P+Q is Fredholm on $L^{p(\cdot)}(\Gamma,\varrho)$ due to Theorem 4.2. Applying Theorem 2.5(b), we deduce that the operator $g^{(1)}P+Q$ is n-normal on $L_{N-1}^{p(\cdot)}(\Gamma,\varrho)$, where $g^{(1)}$ is the $(N-1)\times(N-1)$ upper-triangular nonsingular matrix function obtained from g by deleting the first column and the first row. Arguing as before with $g^{(1)}$ in place of g, we conclude that g_2P+Q is Fredholm on $L^{p(\cdot)}(\Gamma,\varrho)$ and $g^{(2)}P+Q$ is n-normal on $L_{N-2}^{p(\cdot)}(\Gamma,\varrho)$, where $g^{(2)}$ is the $(N-2)\times(N-2)$ upper-triangular nonsingular matrix function obtained from $g^{(1)}$ by deleting the first column and the first row. Repeating this procedure N times, we can show that all operators g_jP+Q, $j \in \{1,\dots,N\}$, are Fredholm on $L^{p(\cdot)}(\Gamma,\varrho)$.

If the operator $gP+Q$ is d-normal, then we can prove in a similar fashion that all operators g_jP+Q, $j \in \{1,\dots,N\}$, are Fredholm on $L^{p(\cdot)}(\Gamma,\varrho)$. In this case we start with g_N and delete the last column and the last row of the matrix $g^{(j-1)}$ on the j-th step (we assume that $g^{(0)}=g$).

Since all operators g_jP+Q are Fredholm on $L^{p(\cdot)}(\Gamma,\varrho)$, from Theorem 2.5(b) we obtain that the operator $gP+Q$ is Fredholm on $L^{p(\cdot)}_N(\Gamma,\varrho)$. Hence $aP+bQ$ is Fredholm on $L^{p(\cdot)}_N(\Gamma,\varrho)$, too. $\square$

7. Semi-Fredholmness and Fredholmness are equivalent for arbitrary operators in $\operatorname{alg}(S,PC,L^{p(\cdot)}_N(\Gamma,\varrho))$

7.1. Linear dilation

The following statement shows that the semi-Fredholmness of an operator in a dense subalgebra of $\operatorname{alg}(S,PC,L^{p(\cdot)}_N(\Gamma,\varrho))$ is equivalent to the semi-Fredholmness of a simpler operator $aP+bQ$ with coefficients of a,b of larger size.

LEMMA 7.1. *Suppose Γ is a Carleson Jordan curve, $p:\Gamma\to(1,\infty)$ is a continuous function satisfying* (2)*, and ϱ is a Khvedelidze weight satisfying* (3)*. Let*

$$A=\sum_{i=1}^{k} A_{i1}A_{i2}\dots A_{ir},$$

where $A_{ij}=a_{ij}P+b_{ij}Q$ and all a_{ij},b_{ij} belong to $PC^0_{N\times N}(\Gamma)$. Then there exist functions $a,b\in PC^0_{D\times D}(\Gamma)$, where $D:=N(k(r+1)+1)$, such that A is n-normal (d-normal, Fredholm) on $L^{p(\cdot)}_N(\Gamma,\varrho)$ if and only if $aP+bQ$ is n-normal (resp. d-normal, Fredholm) on $L^{p(\cdot)}_D(\Gamma,\varrho)$.

Proof. The idea of the proof is borrowed from [10] (see also [1, Theorem 12.15]). Denote by O_s and I_s the $s\times s$ zero and identity matrix, respectively. For $\ell=1,\dots,r$, let B_ℓ be the $kN\times kN$ matrix

$$B_\ell=\operatorname{diag}(A_{1\ell},A_{2\ell},\dots,A_{k\ell}),$$

then define the $kN(r+1)\times kN(r+1)$ matrix Z by

$$Z=\begin{bmatrix} I_{kN} & B_1 & O_{kN} & \dots & O_{kN}\\ O_{kN} & I_{kN} & B_2 & \dots & O_{kN}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ O_{kN} & O_{kN} & O_{kN} & \dots & B_r\\ O_{kN} & O_{kN} & O_{kN} & \dots & I_{kN}\end{bmatrix}.$$

Put

$$X := \mathrm{column}(\underbrace{O_N, \dots, O_N}_{kr}, \underbrace{-I_N, \dots, -I_N}_{k}), \quad Y := (\underbrace{I_N, \dots, I_N}_{k}, \underbrace{O_N, \dots, O_N}_{kr}).$$

Define also $M_0 = (\underbrace{I_N, \dots, I_N}_{k})$ and for $\ell \in \{1, \dots, r\}$, let

$$M_\ell := (A_{11}A_{12}\dots A_{1\ell}\,,\, A_{21}A_{22}\dots A_{2\ell}\,,\, \dots\,,\, A_{k1}A_{k2}\dots A_{k\ell}).$$

Finally, put

$$W := (M_0, M_1, \dots, M_r).$$

It can be verified straightforwardly that

$$\begin{bmatrix} I_{kN(r+1)} & O \\ W & I_N \end{bmatrix} \begin{bmatrix} I_{kN(r+1)} & O \\ O & A \end{bmatrix} \begin{bmatrix} Z & X \\ O & I_N \end{bmatrix} = \begin{bmatrix} Z & X \\ Y & O_N \end{bmatrix}. \tag{19}$$

It is clear that the outer terms on the left-hand side of (19) are invertible. Hence the middle factor of (19) and the right-hand side of (19) are n-normal (d-normal, Fredholm) only simultaneously in view of Theorem 2.1. By Theorem 2.5(b), the operator A is n-normal (d-normal, Fredholm) if and only if the middle factor of (19) has the corresponding property. Finally, note that the left-hand side of (19) has the form $aP + bQ$, where $a, b \in PC^0_{D\times D}(\Gamma)$. $\square$

7.2. Proof of Theorem 1.2

Obviously, for every $f \in PC(\Gamma)$ there exists a sequence $f_n \in PC^0(\Gamma)$ such that $\|f - f_n\|_{L^\infty(\Gamma)} \to 0$ as $n \to \infty$. Therefore, for each operator $\alpha P + \beta Q$, where $\alpha = (\alpha_{rs})_{r,s=1}^N$, $\beta = (\beta_{rs})_{rs=1}^N$ and $\alpha_{rs}, \beta_{rs} \in PC(\Gamma)$ for all $r, s \in \{1, \dots, N\}$, there exist sequences $\alpha^{(n)} = (\alpha_{rs}^{(n)})_{r,s=1}^N$, $\beta^{(n)} = (\beta_{rs}^{(n)})_{r,s=1}^N$ with $\alpha_{rs}^{(n)}, \beta_{rs}^{(n)} \in PC^0(\Gamma)$ for all $r, s \in \{1, \dots, N\}$ such that

$$\begin{aligned} &\|(\alpha P + \beta Q) - (\alpha^{(n)}P + \beta^{(n)}Q)\|_{\mathcal{B}(L_N^{p(\cdot)}(\Gamma,\varrho))} \\ &\leqslant N \max_{1\leqslant r,s\leqslant N} \|\alpha_{rs} - \alpha_{rs}^{(n)}\|_{L^\infty(\Gamma)} \|P\|_{\mathcal{B}(L_N^{p(\cdot)}(\Gamma,\varrho))} \\ &\quad + N \max_{1\leqslant r,s\leqslant N} \|\beta_{rs} - \beta_{rs}^{(n)}\|_{L^\infty(\Gamma)} \|Q\|_{\mathcal{B}(L_N^{p(\cdot)}(\Gamma,\varrho))} = o(1) \end{aligned}$$

as $n \to \infty$.

Let $A \in \mathrm{alg}\,(S, PC; L_N^{p(\cdot)}(\Gamma, \varrho))$. Then there exists a sequence of operators $A^{(n)}$ of the form $\sum_{i=1}^k A_{i1}^{(n)} A_{i2}^{(n)} \dots A_{ir}^{(n)}$, where $A_{ij}^{(n)} = a_{ij}^{(n)}P + b_{ij}^{(n)}Q$ and $a_{ij}^{(n)}, b_{ij}^{(n)}$ belong to $PC_{N\times N}(\Gamma)$, such that $\|A - A^{(n)}\|_{\mathcal{B}(L_N^{p(\cdot)}(\Gamma,\varrho))} \to 0$ as $n \to \infty$. In view of what has been said above, without loss of generality, we can assume that all matrix functions $a_{ij}^{(n)}, b_{ij}^{(n)}$ belong to $PC^0_{N\times N}(\Gamma)$.

If A is semi-Fredholm, then for all sufficiently large n, the operators $A^{(n)}$ are semi-Fredholm by Theorem 2.3. From Lemma 7.1 it follows that for every semi-Fredholm operator $\sum_{i=1}^{k} A_{i1}^{(n)} A_{i2}^{(n)} \dots A_{ir}^{(n)}$ there exist $a^{(n)}, b^{(n)} \in PC^{0}_{D\times D}(\Gamma)$, where $D := N(k(r+1)+1)$, such that $a^{(n)}P + b^{(n)}Q$ is semi-Fredholm on $L_D^{p(\cdot)}(\Gamma, \varrho)$. By Theorem 6.3, $a^{(n)}P + b^{(n)}Q$ is Fredholm on $L_D^{p(\cdot)}(\Gamma, \varrho)$. Applying Lemma 7.1 again, we conclude that $\sum_{i=1}^{k} A_{i1}^{(n)} A_{i2}^{(n)} \dots A_{ir}^{(n)}$ is Fredholm on $L_N^{p(\cdot)}(\Gamma, \varrho)$. Thus, for all sufficiently large n, the operators $A^{(n)}$ are Fredholm. Lemma 2.4 yields that A is Fredholm. □

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