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INVERTIBILITY FOR SPECTRAL TRIANGLES

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Abstract. A spectral inclusion for block triangles is extended to "spectral" triangles.

0. INTRODUCTION. Suppose G is a ring, with identity I and invertible group G^{-1}: we recall

$$G^{-1} = G^{-1}_{\text{left}} \cap G^{-1}_{\text{right}} \tag{0.1}$$

where

$$G^{-1}_{\text{left}} = \{T \in G : I \in GT\} \ , \ G^{-1}_{\text{right}} = \{T \in G : I \in TG\} \ . \tag{0.2}$$

We are interested here in rings with a simple block structure [5]:

$$G = \begin{pmatrix} A & M \\ N & B \end{pmatrix} = \{\begin{pmatrix} a & m \\ n & b \end{pmatrix} : (a,m,n,b) \in A \times M \times N \times B\} \tag{0.3}$$

with the standard addition and multiplication for two-by-two matrices. This means that A and B are themselves rings with identity, while M and N are bimodules over A and B: writing out a matrix product reveals all. In such a ring G we distinguish "spectral triangles":

1. DEFINITION. $T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in G$ *is called a spectral upper triangle if there is inclusion*

$$1 - Mn \subseteq A^{-1} \textit{ and } 1 - nM \subseteq B^{-1} \ , \tag{1.1}$$

and a spectral lower triangle if instead

$$1 - mN \subseteq A^{-1} \textit{ and } 1 - Nm \subseteq B^{-1} \ . \tag{1.2}$$

By an old lemma of Jacobson ([3] Theorem 7.2.3) both components of (1.1) are equivalent, and similarly for (1.2): as a sample of the eight verifications needed we observe

$$c(1 - mn) = 1 \in A \Longrightarrow (1 + ncm)(1 - nm) = 1 \in B \ . \tag{1.3}$$

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The most obvious way for $T \in G$ to be a spectral upper triangle is for it to be an "upper triangle", with $n = 0$; dually a "lower triangle" has $m = 0$. Conversely if for example $A = M = N = B$ are all the same ring then the condition (1.1) says ([3] Theorem 7.2.3) that $n \in N$ is a "radical element", belonging to the Jacobson radical of $N = A$; thus if also A is "semi simple" then the distinction between triangles and spectral triangles evaporates.

The simple observation about spectral triangles is the following triple property, analagous to restrictions and quotients of operators ([3] Theorem 3.11.3):

2. THEOREM. *If* $T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in G$ *is a spectral triangle then each two of the following three conditions implies the third:*

$$a \in A^{-1}\ ;\ b \in B^{-1}\ ;\ T \in G^{-1}\ . \tag{2.1}$$

We will do this for upper triangles, and do it by considering left and right invertibility separately, but together:

3. THEOREM. *If* $T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in G$ *is a spectral upper triangle then there is implication*

$$\left(a \in A^{-1}_{\text{left}} \textit{ and } b \in B^{-1}_{\text{left}}\right) \Longrightarrow T \in G^{-1}_{\text{left}} \Longrightarrow a \in A^{-1}_{\text{left}}\ ; \tag{3.1}$$

$$\left(a \in A^{-1}_{\text{right}} \textit{ and } T \in G^{-1}_{\text{left}}\right) \Longrightarrow b \in B^{-1}_{\text{left}}\ ; \tag{3.2}$$

$$\left(a \in A^{-1}_{\text{right}} \textit{ and } b \in B^{-1}_{\text{right}}\right) \Longrightarrow T \in G^{-1}_{\text{right}} \Longrightarrow b \in B^{-1}_{\text{right}}\ ; \tag{3.3}$$

$$\left(b \in B^{-1}_{\text{left}} \textit{ and } T \in G^{-1}_{\text{right}}\right) \Longrightarrow a \in A^{-1}_{\text{right}}\ . \tag{3.4}$$

Proof. If $T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in G^{-1}_{\text{left}}$ then there is $T' = \begin{pmatrix} a' & m' \\ n' & b' \end{pmatrix} \in G$ for which

$$T'T = \begin{pmatrix} a' & m' \\ n' & b' \end{pmatrix} \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I\ , \tag{3.5}$$

giving in particular, using (1.1),

$$a'a = 1 - m'n \in A^{-1}\ , \Longrightarrow a \in A^{-1}_{\text{left}}\ : \tag{3.6}$$

this is the second implication of (3.1). Conversely if $a'a = 1 \in A$ and $b'b = 1 \in B$ then

$$\begin{pmatrix} (1 - a'mb'n)^{-1} & 0 \\ -b'n(1 - a'mb'n)^{-1} & 1 \end{pmatrix} \begin{pmatrix} a' & -a'mb' \\ 0 & b' \end{pmatrix} \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ , \tag{3.7}$$

giving the first; the inverse in the first factor exists by (1.1). Towards (3.2) suppose that (3.5) holds together with $aa'' = 1 \in A$: then

$$n'a = -b'n \Longrightarrow n' = -b'na'' \Longrightarrow b'b = 1 + na''m \in B^{-1}\ . \tag{3.8}$$

Thus $b \in B_{\text{left}}^{-1}$ as required by (3.2). The arguments for (3.3) and (3.4) are identical. $\square$

Theorem 2 follows:

4. COROLLARY. *If* $T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in G$ *is an upper spectral triangle then each of the following conditions implies its successor:*

$$T \in G^{-1} \text{ and } \left(a \in A_{\text{right}}^{-1} \text{ or } b \in B_{\text{left}}^{-1}\right) ; \tag{4.1}$$

$$a \in A^{-1} \text{ and } b \in B^{-1} ; \tag{4.2}$$

$$T \in G^{-1} ; \tag{4.3}$$

$$a \in A_{\text{left}}^{-1} \text{ and } b \in B_{\text{right}}^{-1} . \tag{4.4}$$

Proof. This is left to the reader: the implication (4.1) $\Longrightarrow$ (4.2) uses all four parts of Theorem 3, while the rest is much simpler. $\square$

We leave it also to the reader to state and prove the analogue of Theorem 3 and Corollary 4 for lower spectral triangles, and to confirm that in each case Theorem 2 follows.

We recall ([4] Theorems 3.1, 3.2) that for upper triangles slightly more is true, involving also left and right one-one-ness and zero divisors. As for upper triangles [5] we can [8], [9] make explicit the gap between (4.3) and (4.4):

5. THEOREM. *If* $T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in G$ *then necessary and sufficient for* $T \in G^{-1}$ *is that there are* $a'' \in A$, $b'' \in B$ *and* $n'' \in N$ *for which*

$$a''a = 1 \in A \text{ and } bb'' = 1 \in B \tag{5.1}$$

with

$$1 - aa'' = mn'' \in A \text{ and } 1 - b''b = n''m \in B . \tag{5.2}$$

Proof. If $T' \in G$ is a two sided inverse for T satisfying (3.5) then (5.1) holds with

$$(a'', b'') = ((1 - m'n)^{-1}a', b'(1 - m'n)^{-1}) , \tag{5.3}$$

which also satisfy

$$\begin{pmatrix} 1 - aa'' \\ 1 - b''b \end{pmatrix} = \begin{pmatrix} 1 - a(1 - m'n)^{-1}a' \\ 1 - b'(1 - nm')^{-1}b \end{pmatrix} = \begin{pmatrix} 1 - aa'' + a(1 - (1 - m'n)^{-1})a' \\ 1 - b''b + b''(1 - (1 - nm')^{-1})b \end{pmatrix}$$

$$= \begin{pmatrix} mn' - am'n(1 - m'n)^{-1})a' \\ n'm - b'(1 - nm')^{-1})nm'b \end{pmatrix} = \begin{pmatrix} mn' + mb'n(1 - m'n)^{-1})a' \\ n'm + b'(1 - nm')^{-1})na'b \end{pmatrix} = \begin{pmatrix} mn'' \\ n''m \end{pmatrix}$$

with

$$n'' = n' + b'n(1 - m'n)^{-1}a' = n' + b'(1 - nm')^{-1}na' . \tag{5.4}$$

Conversely if (5.1) and (5.2) hold and

$$T'' = \begin{pmatrix} a'' & -a''mb'' \\ n'' & b'' \end{pmatrix} \tag{5.5}$$

then

$$T''T = \begin{pmatrix} 1 & 0 \\ n''a + b''n & 1 \end{pmatrix} = TT'' \tag{5.6}$$

is invertible. □

The condition (5.2) says that the idempotents $1 - aa'$ and $1 - b'b$ are "similar": when $A = B(X)$ and $B = B(Y)$ this means that their ranges are isomorphic. Indeed necessary and sufficient, for $a \in A$, $b \in B$ and "radical" $n \in N$, that there should exist $m \in M$ giving rise to an invertible triangle $T \in G$, is that there should be a left inverse $a' \in A$ and a right inverse $b' \in B$ for which the idempotents $1 - aa'$ and $1 - b'b$ are similar. In particular when $A = B(X)$ and $B = B(Y)$ this similarity reduces to isomorphism

$$X/a(X) \cong b^{-1}(0) . \tag{5.7}$$

Theorem 5 was done for Hilbert space $X = Y$ by Du and Pan [8], and extended to Banach spaces by Han, Lee and Lee [9].

If an upper spectral triangle has an upper spectral triangle for its inverse then both its diagonal elements must have inverses:

6. THEOREM. *If* (3.5) *holds for upper spectral triangles* T *and* T' *then* $a \in A$ *and* $b \in B$ *are left invertible.*

Proof. We already know from (3.1) that $a \in A$ is left invertible, since $n \in N$ has the "radical" property; but now also

$$b'b = 1 - n'm \in B^{-1} ,$$

by the radical property of $n' \in N$. □

The same argument shows that if (3.5) holds then $a' \in A$ and $b' \in B$ are right invertible, and hence if instead of (3.5) we have $TT' = I$ then $a \in A$ and $b \in B$ are both right invertible.

When G, and hence A and B, are complex linear algebras, then we can rewrite our conclusions in terms of the *spectrum*,

$$\sigma_G(T) = \sigma_G^{\text{left}}(T) \cup \sigma_G^{\text{right}}(T) , \tag{6.1}$$

where

$$\sigma_G^{\text{left}}(T) = \{\lambda \in \mathbf{C} : T - \lambda I \notin G_{\text{left}}^{-1}\} , \ \sigma_G^{\text{right}}(T) = \{\lambda \in \mathbf{C} : T - \lambda I \notin G_{\text{right}}^{-1}\} . \tag{6.2}$$

7. THEOREM. *If* $T = \begin{pmatrix} a & m \\ n & b \end{pmatrix}$ *is a spectral upper triangle then there is inclusion*

$$\sigma_A^{\text{left}}(a) \subseteq \sigma_G^{\text{left}}(T) \subseteq \sigma_A^{\text{left}}(a) \cup \sigma_B^{\text{left}}(b) ; \tag{7.1}$$

$$\sigma_B^{\text{left}}(b) \subseteq \sigma_G^{\text{left}}(T) \cup \sigma_A^{\text{right}}(a) \ ; \tag{7.2}$$

$$\sigma_B^{\text{right}}(b) \subseteq \sigma_G^{\text{right}}(T) \subseteq \sigma_A^{\text{right}}(a) \cup \sigma_B^{\text{right}}(b) \ ; \tag{7.3}$$

$$\sigma_A^{\text{right}}(a) \subseteq \sigma_G^{\text{right}}(T) \cup \sigma_A^{\text{left}}(a) \ . \tag{7.4}$$

Hence also

$$\sigma_A^{\text{left}}(a) \cup \sigma_B^{\text{left}}(b) \subseteq \sigma_G(T) \subseteq \sigma_A(a) \cup \sigma_B(b) \subseteq \sigma_G(T) \cup \left(\sigma_A^{\text{right}}(a) \cap \sigma_B^{\text{left}}(b)\right) . \tag{7.5}$$

Proof. The first part of this is a systematic rewriting of Theorem 3 with $T - \lambda I$ in place of $T \in G$; (7.5) is Corollary 4. $\square$

We also invite the reader to deploy Theorem 7 to see how each of

$$\sigma_A(a) \ , \ \sigma_B(b) \ , \ \sigma_G(T) \tag{7.6}$$

is contained in the union of the other two, the spectral version of Theorem 2.

Theorem 6 can also be expressed spectrally:

8. THEOREM. *If* $T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in G$ *is an upper spectral triangle then each of the following conditions is sufficient for equality*

$$\sigma_G(T) = \sigma_A(a) \cup \sigma_B(b) \ : \tag{8.1}$$

$$1 - MN \subseteq A^{-1} \textit{ and } 1 - NM \subseteq B^{-1} \ ; \tag{8.2}$$

$$\sigma_A(a) = \sigma_A^{\text{left}}(a) \ ; \tag{8.3}$$

$$\sigma_B(b) = \sigma_B^{\text{right}}(b) \ ; \tag{8.4}$$

$$\sigma_A^{\text{right}}(a) \cap \sigma_B^{\text{left}}(b) = \emptyset \ . \tag{8.5}$$

Proof. If (8.2) holds — equivalently, if either component of (8.2) holds — then every $T \in G$ is both an upper and a lower spectral triangle, and therefore Theorem 6 applies. It is immediately clear from the last inclusion of (7.5) that (8.5) is sufficient for (8.1), and immediately clear from the first inclusion that (8.3) and (8.4) are together sufficient for (8.1). To see that they each work separately combine the first and last inclusions of (7.5):

$$\sigma_A^{\text{left}}(a) \cup \sigma_B^{\text{right}}(b) \subseteq \sigma_G(T) \cup \left(\sigma_A^{\text{right}}(a) \cap \sigma_B^{\text{left}}(b) \setminus (\sigma_A^{\text{left}}(a) \cup \sigma_B^{\text{right}}(b))\right). \quad \square$$

When G is a complex Banach algebra then the spectrum $\sigma_G(T)$ becomes a compact subset of the plane, and is also topologically constrained by the spectra of a and b:

9. THEOREM. *If* G *is a Banach algebra and* $T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in G$ *is an upper spectral triangle then there is inclusion*

$$\partial\left(\sigma_A(a) \cup \sigma_B(b)\right) \subseteq \sigma_G(T) \subseteq \sigma_A(a) \cup \sigma_B(b) \subseteq \eta\sigma_G(T) \ , \tag{9.1}$$

where [2], [3] *we write* ∂K *and* ηK *for the topological boundary and the "connected hull" of compact subsets* $K \subseteq \mathbf{C}$*:*

$$\mathbf{C} \setminus \eta K \text{ is the unbounded component of } \mathbf{C} \setminus K \ . \tag{9.2}$$

Proof. The first inclusion follows from the familiar ([3] Theorem 9.3.3)

$$\partial\big(\sigma_A(a) \cup \sigma_B(b)\big) \subseteq \partial\sigma_A(a) \cup \partial\sigma_B(b) \subseteq \sigma_A^{\text{left}}(a) \cup \sigma_B^{\text{right}}(b) \ , \tag{9.3}$$

the second is the second part of (7.5), and the third follows ([2]; [3] Theorem 7.10.3) from the first two. $\square$

From (9.1) follows a slight improvement to the end of (7.5):

$$\sigma_A(a) \cup \sigma_B(b) \subseteq \sigma_G(T) \cup \operatorname{int}\big(\sigma_A^{\text{right}}(a) \cap \sigma_B^{\text{left}}(b)\big) \ : \tag{9.4}$$

this is because

$$\partial\big(\sigma_A^{\text{right}}(a) \cap \sigma_B^{\text{left}}(b)\big) \subseteq \partial\sigma_A^{\text{right}}(a) \cup \partial\sigma_B^{\text{left}}(b) \subseteq \sigma_A^{\text{left}}(a) \cup \sigma_B^{\text{right}}(b) \ , \tag{9.5}$$

We noted (7.5) and (9.1) elsewhere [5] for triangles, and used it [7] in the extension of determinants, traces and the adjugates from finite dimensional to arbitrary Banach algebras. The condition (8.5) says ([3] (11.6.9.11)) that the multiplication operator

$$L_a - R_b : m \mapsto am - mb \ (M \to M) \tag{9.6}$$

is onto, since its defect spectrum is a subset of the algebraic difference between the right spectrum of $a \in A$ and the left spectrum of $b \in B$: it has been noticed by Radjavi and Rosenthal ([13] Corollary 0.15) that this makes all the upper triangles with diagonal (a,b) similar to their diagonal.

In fact the condition (9.6) is by itself sufficient for equality (8.1); we construct an auxiliary operator and use a joint spectrum argument:

10. THEOREM. *With*

$$S = \begin{pmatrix} a & am - mb \\ n & b \end{pmatrix} \ , \ P = \begin{pmatrix} 1 & m \\ 0 & 0 \end{pmatrix} \ , \tag{10.1}$$

there is equality

$$\sigma_G(S,P) = \big(\sigma_B(b) \times \{1\}\big) \cup \big(\sigma_A(a) \times \{0\}\big) \ , \tag{10.2}$$

and hence inclusion, necessarily equality,

$$\sigma_A(a) \cup \sigma_B(b) \subseteq \sigma_G(S) \ . \tag{10.3}$$

Proof. This holds separately for left and for right spectra: we argue

$$\begin{pmatrix} a' & m' \\ n' & b' \end{pmatrix}\begin{pmatrix} a & am-mb \\ n & b \end{pmatrix}+\begin{pmatrix} a'' & m'' \\ n'' & b'' \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 0 \end{pmatrix}$$
$$\equiv \begin{pmatrix} a'a+m'n+a'' & a'am-a'mb+m'b+a''m \\ n'a+b'n+n'' & n'am-n'mb+b'b+b''m \end{pmatrix}$$
$$=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Longleftrightarrow \begin{matrix} a''=1-a'a-m'n & a'am-a'mb+m'b+(1-a'a)m=0 \\ n''=-n'a-b'n & (b'-n'm)b=1-n'am+n'am \end{matrix},$$

possible if and only if $1\in Bb$;

$$\begin{pmatrix} a & am-mb \\ n & b \end{pmatrix}\begin{pmatrix} a' & m' \\ n' & b' \end{pmatrix}+\begin{pmatrix} 1 & m \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a'' & m'' \\ n'' & b'' \end{pmatrix}$$
$$\equiv \begin{pmatrix} aa'+amn'-mbn'+a''+mn'' & am'+amb'-mbb'+m''+mb'' \\ na'+bn' & nm'+bb' \end{pmatrix}$$
$$=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Longleftrightarrow$$
$$\begin{matrix} a''=1-aa'-amn-m(bn')-mn'' & m''=-am'-amb'+m(bb')-mb'' \\ bn'=-na' & bb'=1-nm'\in B^{-1} \end{matrix},$$

possible if and only if $1\in bB$;

$$\begin{pmatrix} a' & m' \\ n' & b' \end{pmatrix}\begin{pmatrix} a & am-mb \\ n & b \end{pmatrix}+\begin{pmatrix} a'' & m'' \\ n'' & b'' \end{pmatrix}\begin{pmatrix} 0 & -m \\ 0 & 1 \end{pmatrix}$$
$$\equiv \begin{pmatrix} a'a+m'n & a'am-a'mb+m'b-a''m+m'' \\ n'a+b'n & n'am-n'mb+b'b-n''m+b'' \end{pmatrix}$$
$$=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Longleftrightarrow$$
$$\begin{matrix} a'a=1-n'm\in A^{-1} & (a'a)m-a'mb+m'b-a''m+m''=0 \\ n'a=-b'n & (n'a)m-n'mb+b'b-n''m+b''=1 \end{matrix},$$

possible if and only if $1\in Aa$;

$$\begin{pmatrix} a & am-mb \\ n & b \end{pmatrix}\begin{pmatrix} a' & m' \\ n' & b' \end{pmatrix}+\begin{pmatrix} 0 & -m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a'' & m'' \\ n'' & b'' \end{pmatrix}$$
$$\equiv \begin{pmatrix} aa'+amn'-mbn'-mn'' & am'+amb'-mbb'-mb'' \\ na'+bn'+n'' & nm'+bb'+b'' \end{pmatrix}$$
$$=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Longleftrightarrow$$
$$\begin{matrix} a(a'+mn')=1-m(bn'+n'') & am'+amb'-m(bb'+b'')=0 \\ bn'+n''=0 & bb'+b''=1 \end{matrix},$$

possible if and only if $1\in aA$. $\square$

It is not necessary for the inclusion (10.3) that S commutes with P: we recall that in fact

$$SP=PSP\,, \tag{10.4}$$

and there is implication

$$SP = PS \in G \Longleftrightarrow mb = 0 \in M . \tag{10.5}$$

There no restriction on the operator $L_a - R_b$ of (9.6) in the operation of Theorem 10: the restriction is on the operator S. If however $L_a - R_b \in B(M)$ is onto, then an arbitrary spectral triangle $T \in G$ can be captured: take

$$T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} a & am' - m'b \\ n & b \end{pmatrix} = S' \text{ with } P' = \begin{pmatrix} 1 & m' \\ 0 & 0 \end{pmatrix} \tag{10.6}$$

where $(L_a - R_b)(m') = m \in M$.

When we specialize to the Banach algebra $G = B(X \times Y)$ of bounded operators on the product of Banach spaces X and Y, with the induced block structure, then it becomes sufficient for the condition (8.4) that

$$b \in B(Y) \text{ has the single valued extension property ,} \tag{10.7}$$

and sufficient for the condition (8.3) that

$$a^* \in B(X^*) \text{ has the single valued extension property .} \tag{10.8}$$

The "single valued extension property" was comprehensively discussed by Finch [1], who showed in particular that if it was satisfied by $b \in B(Y)$ then the spectrum of b coincided with its "defect spectrum", and a fortiori with its right spectrum. This shows that (9.7) implies (8.4), and the implication (9.8) $\Longrightarrow$ (8.3) proceeds by duality. The condition (8.1) is used [8], [10], [11], [12] in the attempt to extend the condition "Weyl's theorem holds" from $a \in A$ and $b \in B$ to the block triangle $T \in G$.

If in particular

$$X = \ell_p \text{ and } Y = \ell_q \text{ with } p \neq q \tag{10.9}$$

then the invertible group G^{-1} is not connected: a simple proof of this, due [14], [15] to Aiena and Gonzalez, can be explained [6] by noticing that in this situation the condition (8.2), and hence also the analogue for connected components of the identity, is satisfied.

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