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APPROXIMATE PERMUTABILITY OF TRACES ON SEMIGROUPS OF MATRICES

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Abstract. It is known that if trace is permutable on a semigroup $\mathcal{S}$ of complex matrices, i.e., $\operatorname{tr}(ABC) = \operatorname{tr}(BAC)$ for all A, B, C in $\mathcal{S}$, then $\mathcal{S}$ is triangularizable. We study an approximate version of this condition: $|\operatorname{tr}(ABC - BAC)| \leqslant \varepsilon\rho(A)\rho(B)\rho(C)$ for all A, B, C in $\mathcal{S}$, where ρ is the spectral radius. We show that this condition with $\varepsilon < 3$ yields commutativity for compact groups and triangularizabilty for certain groups including connected ones. For general semigroups additional assumptions are needed. Moreover, we show that any property on semigroups of matrices that satisfies certain pretriangularizing conditions, yields similar conclusions.

1. Introduction

If $\mathcal{G}$ is a compact group of complex matrices, certain conditions on $\mathcal{G}$ are known to imply commutativity. One such condition is permutability of trace, that is the assumption that ABC and BAC have the same trace for all A, B, and C in $\mathcal{G}$. We are interested in weaker, approximate, versions of these hypotheses. To start with, consider the question: is there an $\varepsilon > 0$ such that if

$$|\operatorname{tr}(ABC - BAC)| < \varepsilon$$

for all A, B, C in $\mathcal{G}$, then $\mathcal{G}$ is abelian? The answer turns out to be yes with a perhaps surprisingly large ε, i.e., $\varepsilon = 3$.

For an arbitrary subgroup of $\mathrm{GL}_n(\mathbb{C})$ or, more generally, a multiplicative semigroup $\mathcal{S}$ in $M_n(\mathbb{C})$, a trace condition can be expected to yield not commutativity, but possibly (simultaneous) triangularizability. Of course, the inequality has to be normalized to take the "size" of A, B and C into account. Thus one is led to consider the condition

$$|\operatorname{tr}(ABC - BAC)| \leqslant \varepsilon\rho(A)\rho(B)\rho(C) \text{ for all } A, B, C \in \mathcal{S}, \tag{1}$$

where ρ denotes the spectral radius. Any triangularizability result deduced from this condition would extend the result in [4, Corollary 2.2.2] that a semigroup with permutable trace is triangularizable.

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For connected groups we show that the condition (1) with $\varepsilon < 3$ yields triangularizability. For a general group $\mathcal{G}$, the condition implies triangularizability for the commutator subgroup of $\mathcal{G}$, but only reducibility results for $\mathcal{G}$ itself. For a general semigroup no $\varepsilon > 0$ works unless we make additional assumptions, e.g., on the minimal rank r of the semigroup. We show that if $r \geqslant 2$ for a totally reducible semigroup $\mathcal{S}$, then $\mathcal{S}$ has a chain of invariant subspaces of length r.

In all the results mentioned above the number 3 is proved to be sharp.

Another approximate condition was studied by two of the authors in [2], where the inequality $\rho(AB-BA) \leqslant \varepsilon\rho(A)\rho(B)$ was proved to yield triangularizability results with $\varepsilon < \sqrt{3}$. Most of the results of that paper hold also for semigroups of matrices satisfying (1) for $\varepsilon < 3$ and the proofs are almost identical. This should not be surprising since, as we shall see later, both conditions are what we call pretriangularizing. Such conditions on semigroups of matrices imply the same conclusions as stated above for the property (1), possibly with different bounds for ε.

2. The compact group case

First we consider an example which will have an important role in our proofs.

EXAMPLE 2.1. Suppose that p and q are two prime numbers, which can be equal. (We assume the primeness of p and q so that the subgroup constructed below is irreducible.) We consider a subgroup $\mathcal{G} = \mathcal{G}(p,q)$ of $p \times p$ matrices generated by

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \theta_1 & 0 & \dots & 0 \\ 0 & \theta_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \theta_p \end{pmatrix},$$

where $\theta_j^q = 1$ for all j and B is a nonscalar matrix. Note that $\mathcal{G}$ depends on B. It is a subgroup of the group of all unitary matrices and so $\rho(C) = 1$ for all $C \in \mathcal{G}$. For any p and q there is a triple C, D, E such that

$$|\mathrm{tr}(CDE - DCE)| \geqslant 3.$$

Proof. We denote by $[C,D]$ the multiplicative commutator $CDC^{-1}D^{-1}$. If we take $E = D^{-1}C^{-1}$ then we have $(CD - DC)D^{-1}C^{-1} = I - [D,C]$.

Let $\omega_1, \dots, \omega_p$ be the eigenvalues of $[A,B]$ and note that at least two of them are different from 1 (since $[A,B] \neq I$ and $1 = \det([A,B]) = \omega_1 \dots \omega_p$). We consider two cases. Assume first that exactly two of the eigenvalues, say α and $\overline{\alpha}$ are distinct from 1. Since $[A,B]^j = [A,B^j]$ we have $|\mathrm{tr}(I-[A,B^j])| = 2-\alpha^j-\overline{\alpha}^j = 2-2\mathrm{Re}(\alpha^j)$ and since for some integer j we must have $2\mathrm{Re}(\alpha^j) \leqslant -1$, it follows that $\left|\mathrm{tr}(I-[A,B^j])\right| \geqslant 3$ for some j. So we are done in this case.

Assume now that at least three of the ω_j's are different from 1. Since $\omega_j^q = 1$ for all j it follows that $\sum_{k=1}^q \omega_j^k = 0$ for each $\omega_j \neq 1$. Therefore

$$\left|\sum_{k=1}^{q-1} \operatorname{tr}(I-[A,B]^k)\right| = \left|\sum_{k=1}^{q} \operatorname{tr}(I-[A,B]^k)\right| = \left|pq - \sum_{j=1}^{p}\sum_{k=1}^{q}\omega_j^k\right|$$
$$\geqslant pq - \left|\sum_{j=1}^{p}\sum_{k=1}^{q}\omega_j^k\right| \geqslant 3q.$$

This implies that $\left|\operatorname{tr}(I-[A,B^k])\right| > 3$ for at least one $k \in \{1,2,\ldots,q-1\}$. $\square$

LEMMA 2.2. *If a finite group G is not abelian, then there exists a nontrivial element $c \in G$ such that for every (positive) integer n, the element c^n is a commutator.*

Proof. Let H be a minimal nonabelian subgroup of G. By O.J. Schmidt's theorem (see [5]) H is solvable and hence its commutator subgroup $[H,H]$ is a proper subgroup and is therefore commutative. Let $a,b \in H$ be noncommuting elements. If a commutes with $c_1 := [a,b]$, then $c_1^n = [a^n,b]$ for every positive integer n. Otherwise $c_2 := [c_1,a] \neq e$ and since c_1 commutes with c_2 (they are both commutators) we have $c_2^n = [c_1^n,a]$. $\square$

THEOREM 2.3. *Let $\mathcal{G} \subset \mathrm{GL}_n(\mathbb{C})$ be a compact group and assume that*

$$|\operatorname{tr}(ABC-BAC)| < 3\,\rho(A)\rho(B)\rho(C) = 3 \tag{2}$$

for all $A,B,C \in \mathcal{G}$. Then $\mathcal{G}$ is abelian.

Proof. By applying a similarity if needed we may assume that $\mathcal{G}$ is a group of unitary matrices. Therefore $\rho(A) = 1$ for all $A \in \mathcal{G}$ which shows that the last equality in (2) holds. We now suppose that $\mathcal{G}$ is not abelian. Then by [1] $\mathcal{G}$ contains a finite nonabelian subgroup $\mathcal{H}$. By Lemma 2.2 there is an element $I \neq C \in \mathcal{H}$ such that for every integer n, the matrix C^n is a multiplicative commutator of matrices in $\mathcal{H}$. Let m be the smallest integer such that $C^m = I$. Let $\omega_1,\ldots,\omega_n$ be the eigenvalues of C and note that at least two of them are different from 1 (since $C \neq I$ and $1 = \det(C) = \omega_1\ldots\omega_n$). Now we use arguments similar to those in the Example 2.1. We consider two cases.

Case 1. Exactly two of the eigenvalues, say α and $\overline{\alpha}$ are distinct from 1. Since $C^n = EFE^{-1}F^{-1}$ for some E,F in $\mathcal{H}$ we have

$$I - C^n = FEE^{-1}F^{-1} - EFE^{-1}F^{-1} = FEG - EFG,$$

where $G = E^{-1}F^{-1}$. Then $|\operatorname{tr}(I-C^n)| = |2-\alpha^n-\overline{\alpha}^n| = |2-2\mathrm{Re}(\alpha^n)|$ and since for some integer n we must have $2\mathrm{Re}(\alpha^n) \leqslant -1$, it follows that the condition (2) does not hold, which is a contradiction and we are done in this case.

Case 2. At least three of C's eigenvalues are different from 1. Then by the same summation reasoning as in Example 2.1 we obtain

$$\sum_{n=1}^{m-1} |\mathrm{tr}(I - C^n)| \geqslant 3m.$$

It follows that for some n we must have $|\mathrm{tr}(I - C^n)| > 3$, which is again a contradiction and we are done in this case as well. $\square$

The following example shows that the bound 3 in Theorem 2.3 is sharp.

EXAMPLE 2.4. Consider again the group $\mathcal{G} = \mathcal{G}(2,3)$ of Example 2.1 generated by

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix},$$

where $\theta \neq 1$ and $\theta^3 = 1$. The derived subgroup of $\mathcal{G}$ is equal to

$$\mathcal{G}' = \left\{ \begin{pmatrix} \theta^j & 0 \\ 0 & \overline{\theta}^j \end{pmatrix}, j = 0,1,2 \right\}.$$

A short computation shows that for each $C, D, E \in \mathcal{G}$ there exist $j, k \in \{0,1,2\}$ such that

$$|\mathrm{tr}(CDE - DCE)| = |\mathrm{tr}\left((I - [D,C])CDE\right)| = \left|1 - \theta^j\right| \left|1 - \theta^k\right|.$$

Then it follows that $|\mathrm{tr}(CDE - DCE)| \leqslant 3$ and the equality holds if and only if $j, k \in \{1,2\}$. $\square$

Our final result on compact groups shows that even a much weaker form of condition (2) yields a strong structural result. In the proof of this result we use the following lemma which is implicit in the proof of Lemma 1.3 in [2].

LEMMA 2.5. [2] *If a compact group $\mathcal{G}$ is not finite modulo its center, then for every prime p, $\mathcal{G}$ contains a finite nonabelian subgroup $\mathcal{H}$ whose derived subgroup $[\mathcal{H},\mathcal{H}]$ is a p-group.*

PROPOSITION 2.6. *If $\mathcal{G} \subset \mathrm{GL}_n(\mathbb{C})$ is a compact group such that*

$$|\mathrm{tr}(ABC - BAC)| < 4\,\rho(A)\rho(B)\rho(C) = 4 \tag{3}$$

for all $A, B, C \in \mathcal{G}$ then $\mathcal{G}$ is finite modulo its center.

Proof. Assume $\mathcal{G}$ is not finite modulo its center. Then by Lemma 2.5 for every prime p, $\mathcal{G}$ contains a finite minimal nonabelian subgroup $\mathcal{H}$ whose derived subgroup $[\mathcal{H},\mathcal{H}]$ is a p-group. By taking $p = 2$ we obtain a contradiction as desired. $\square$

Observe that the bound 4 in Proposition 2.6 is sharp, since in the case of $\mathcal{G} = SU_2(\mathbb{C})$ we have

$$|\mathrm{tr}(ABC - BAC)| \leqslant 4.$$

3. The semigroup case

In this section we study general semigroups of matrices. First we show that, in general, the condition (1) does not imply the reducibility of a semigroup.

EXAMPLE 3.1. Suppose that $\delta > 0$ and define a semigroup $\mathcal{S} = \mathcal{S}(\delta)$ in $M_n(\mathbb{C})$, $n \geqslant 2$, by

$$\mathcal{S} = \left\{ \alpha \begin{pmatrix} 1 & x^* \\ y & yx^* \end{pmatrix};\ y, x \in \mathbb{C}^{n-1}, \|x\| \leqslant \delta, \|y\| \leqslant \delta, \alpha \in \mathbb{C} \right\},$$

where $\|x\|$ denotes the Hilbert space norm of $x \in \mathbb{C}^{n-1}$. The semigroup $\mathcal{S}$ is irreducible, each nonzero element in $\mathcal{S}$ is of rank 1, and for each $\varepsilon > 0$ there is a $\delta > 0$ such that for all $A, B, C \in \mathcal{S}(\delta)$ we have

$$|\mathrm{tr}(ABC - BAC)| \leqslant \varepsilon\, \rho(A)\rho(B)\rho(C).$$

Proof. For $A = \alpha \begin{pmatrix} 1 & x^* \\ y & yx^* \end{pmatrix} \in \mathcal{S}$ we write $\alpha = \alpha_A$. The (possibly) nonzero eigenvalue of $\begin{pmatrix} 1 & x^* \\ y & yx^* \end{pmatrix}$ is $1 + x^*y$ and the corresponding eigenvector is $\begin{pmatrix} 1 \\ y \end{pmatrix}$. Thus

$$\rho(A) = |\alpha_A(1 + x^*y)|\,.$$

A straightforward calculation shows that

$$|\mathrm{tr}(ABC - BAC)| \leqslant 2\,|\alpha_A\alpha_B\alpha_C|\,\big((\delta^2+1)^3 - 1\big)$$

and

$$\rho(A)\rho(B)\rho(C) \geqslant |\alpha_A\alpha_B\alpha_C|\,\big(2 - (\delta^2+1)^3\big)\,. \qquad \square$$

Recall that a semigroup $\mathcal{S}$ of matrices is *totally reducible* if the underlying vector space decomposes as a direct sum of $\mathcal{S}$-invariant irreducible subspaces, or equivalently if the unital $\mathbb{C}$-algebra generated by $\mathcal{S}$ is semisimple. Note that in particular an irreducible semigroup is totally reducible. For a semigroup $\mathcal{S} \subseteq M_n(\mathbb{C})$ we call the closure (in the Euclidian topology) of the set $\mathbb{C}\mathcal{S} = \{\alpha S;\ \alpha \in \mathbb{C}, S \in \mathcal{S}\}$ the homogenized closure of $\mathcal{S}$. We denote this closure by $\overline{\mathbb{C}\mathcal{S}}$. We call the minimal rank of a nonzero element of $\mathcal{S}$ the minimal rank of $\mathcal{S}$.

In the proof of the main result of this section we use the following general fact.

LEMMA 3.2. *Assume that $\mathcal{S} = \overline{\mathbb{C}\mathcal{S}} \subseteq M_n(\mathbb{C})$ is a totally reducible semigroup. If r is the minimal rank of $\mathcal{S}$ then there is an idempotent of rank r in $\mathcal{S}$.*

Proof. By [4, Lemma 3.8.6] the fact that $\mathcal{S} = \overline{\mathbb{C}\mathcal{S}}$ implies that either $\mathcal{S}$ contains an idempotent of the minimal rank r or all elements of rank r are nilpotent of index 2. Suppose the later. Let $\mathbb{C}^n = U_1 \oplus U_2 \oplus \cdots \oplus U_m$ be the direct sum decomposition to minimal invariant subspaces for $\mathcal{S}$. Suppose that $N \in \mathcal{S}$ is a nilpotent of rank r, $N^2 = 0$, and its block decomposition is

$$N = \begin{pmatrix} N_1 & 0 & \dots & 0 & 0 \\ 0 & N_2 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & N_{m-1} & 0 \\ 0 & 0 & \dots & 0 & N_m \end{pmatrix}. \tag{4}$$

Without loss we may assume that $N_1 \neq 0$. Then the ideal $\mathcal{I}_1$ generated by N_1 is a nonzero ideal in the irreducible semigroup $\mathcal{S}_1 = \mathcal{S}|_{U_1}$. All its elements are nilpotent by hypothesis. By Levitzki's Theorem [4, Thm 2.1.7] $\mathcal{I}_1$ is reducible, contradicting the irreducibility of $\mathcal{S}_1$. $\square$

THEOREM 3.3. *Assume that $\mathcal{S} = \overline{\mathbb{C}\mathcal{S}} \subseteq M_n(\mathbb{C})$ is a totally reducible semigroup and that*

$$|\mathrm{tr}(ABC - BAC)| < 3\,\rho(A)\rho(B)\rho(C)$$

for all $A, B, C \in \mathcal{S}$. If $r \geqslant 2$ is the minimal rank of $\mathcal{S}$, then $\mathcal{S}$ is reducible and there is a chain of length r of invariant subspaces for $\mathcal{S}$.

Proof. Lemma 3.2 implies that there is an idempotent E of rank r in $\mathcal{S}$. The restriction of $\mathcal{T} = E\mathcal{S}E - \{0\}$ to the range of E is a group [4, Lemma 3.1.6]. Our assumptions imply that the condition (2) holds for elements in $\mathcal{T}$. In fact, this restriction is simultaneously similar to $\mathbb{C}\mathcal{U}$, where $\mathcal{U}$ is a closed subgroup of unitary matrices [4, Lemma 3.1.6]. Theorem 2.3 implies that $\mathcal{U}$ is commutative and therefore diagonalizable. By [4, Lemma 8.2.10] it follows that there is a chain of invariant subspaces of length r for $\mathcal{S}$. $\square$

The following example shows that in general the length r of a chain of invariant subspaces in Theorem 3.3 is the best possible.

EXAMPLE 3.4. We consider again the semigroup $\mathcal{S}_1 = \mathcal{S}(\delta) \subseteq M_n(\mathbb{C})$, $n \geqslant 2$, as in Example 3.1. For a given $r \geqslant 2$ and $\varepsilon > 0$ we choose δ such that

$$|\mathrm{tr}(ABC - BAC)| \leqslant \frac{\varepsilon}{r}\,\rho(A)\rho(B)\rho(C)$$

for all $A, B, C \in \mathcal{S}_1$. Then all the nonzero elements in the semigroup

$$\mathcal{S}_r = \left\{ \begin{pmatrix} S & 0 & \dots & 0 & 0 \\ 0 & S & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & S & 0 \\ 0 & 0 & \dots & 0 & S \end{pmatrix} : S \in \mathcal{S}_1 \right\} \subseteq M_{rn}(\mathbb{C})$$

have rank equal to r. Our assumptions imply that

$$|\mathrm{tr}(ABC - BAC)| \leqslant \varepsilon\, \rho(A)\rho(B)\rho(C)$$

for all $A, B, C \in \mathcal{S}_r$. Observe that the maximal length of a chain of invariant subspaces of $\mathcal{S}_r$ is equal to r. □

LEMMA 3.5. *If $\mathcal{S} = \overline{\mathbb{C}\mathcal{S}} \subseteq M_n(\mathbb{C})$ is a totally reducible semigroup and there is an $\varepsilon > 0$ such that*

$$|\mathrm{tr}(ABC - BAC)| \leqslant \varepsilon\, \rho(A)\rho(B)\rho(C) \tag{5}$$

for all $A, B, C \in \mathcal{S}$, then there are no nonzero nilpotents in $\mathcal{S}$.

Proof. We assume that there is a nonzero nilpotent N in $\mathcal{S}$ and write it in the form (4) according to the decomposition $\mathbb{C}^n = U_1 \oplus U_2 \oplus \cdots \oplus U_m$. Without loss we may assume that $N_1 \neq 0$. Since N is nilpotent, the condition (5) implies that $\mathrm{tr}\,((SN - NS)A) = 0$ for all $A, S \in \mathcal{S}$. Let $\mathcal{A}$ be the subalgebra in $M_n(\mathbb{C})$ generated by $\mathcal{S}$. Since the trace is linear it follows that $\mathrm{tr}\,((SN - NS)A) = 0$ for all $S \in \mathcal{S}$ and all $A \in \mathcal{A}$.

For each $i = 1, 2, \ldots, m$, the restriction $\mathcal{A}_i$ of $\mathcal{A}$ to U_i is an irreducible algebra, and so it is equal to the algebra of all linear transformations on U_i, by Burnside's Theorem. In view of [3, Theorem 1.5.1] on block triangularization of $\mathcal{A}$, we may assume that for some $k \in \{1, 2, \ldots, m\}$ the projection P on $U_1 \oplus U_2 \oplus \cdots \oplus U_k$ belongs to $\mathcal{A}$, and we have $A_1 = A_2 = \ldots = A_k$ for all $A \in \mathcal{A}$, where A_i is the restriction of A to U_i. Since $\mathrm{tr}((SN - NS)PAP) = 0$ for all $S \in \mathcal{S}$ and all $A \in \mathcal{A}$, we now conclude that $\mathrm{tr}((S_1N_1 - N_1S_1)A_1) = 0$ for all $S_1 \in \mathcal{S}_1$ and all $A_1 \in \mathcal{A}_1$, so that $S_1N_1 - N_1S_1 = 0$ for all $S_1 \in \mathcal{S}_1$. This implies that $\mathcal{S}_1$ is reducible, which is a contradiction. □

4. Pretriangularizing conditions

So far we have only considered semigroups $\mathcal{S}$ of matrices satisfying the following condition: for a positive $\varepsilon < 3$ we have

$$|\mathrm{tr}(ABC - BAC)| \leqslant \varepsilon\, \rho(A)\rho(B)\rho(C) \tag{6}$$

for all $A, B, C \in \mathcal{S}$. Most of the results we have obtained run in parallel to those in [2] where the condition: for a positive $\varepsilon < \sqrt{3}$ we have

$$\rho(AB - BA) \leqslant \varepsilon\, \rho(A)\rho(B) \tag{7}$$

for all $A, B \in \mathcal{S}$ was considered. This is not surprising as both conditions turn out to be *pretriangularizing* according to the following definition.

DEFINITION 4.1. A property $\mathcal{P}$ that a semigroup of complex matrices may possess is called pretriangularizing if the following holds:

1. $\mathcal{P}$ is similarity invariant.
2. $\mathcal{P}$ passes to subsemigroups, homogenized closures and semisimplifications.

3. If $\mathcal{S} \oplus 0$ has property $\mathcal{P}$ then so does $\mathcal{S}$.
4. Totally reducible semigroups with $\mathcal{P}$ have no non-zero nilpotents.
5. Finite groups with $\mathcal{P}$ are abelian.

It is immediate that the property (6) for a semigroup $\mathcal{S}$ satisfies the first three conditions of this definition; that it satisfies the last two was shown in the previous sections (Theorem 2.3 and Lemma 3.5). Therefore, (6) is an example of a pretriangularizing property. That (7) is pretriangularizing was shown in [2]. Let us also remark that any triangularizing property for matrix semigroups that satisfies the first three conditions (which is usually the case, see for instance various conditions studied in [4] like, e.g., sublinearity of spectrum) then necessarily satisfies also the last two.

We now proceed to prove some results on groups and semigroups satisfying a pretriangularizing property. Although the proofs mimic those in [2] and can be quoted almost verbatim, we nevertheless present them completely for the reader's benefit.

LEMMA 4.2. *Let $\mathcal{S} \subseteq M_n(\mathbb{C})$ be a totally reducible semigroup satisfying a pretriangularizing property $\mathcal{P}$. Let $P_1, \ldots, P_m$ denote a complete set of mutually orthogonal projections to minimal invariant subspaces of $\mathcal{S}$ and let E be any minimal idempotent in $\overline{\mathbb{C}\mathcal{S}}$. Then the rank of EP_i is either zero or one for all $i = 1, \ldots, m$. In particular, if $\mathcal{S}$ is irreducible, then there is a rank-one idempotent in $\overline{\mathbb{C}\mathcal{S}}$.*

Proof. Since the property $\mathcal{P}$ is preserved under homogenization and closure, we may assume $\mathcal{S}$ is closed and homogeneous. Since $\mathcal{S} = \overline{\mathbb{C}\mathcal{S}}$ and $\mathcal{S}$ is totally reducible we note that there exists an idempotent E in $\mathcal{S}$ of minimal rank and thus minimal by Lemma 3.2.

Now take any minimal idempotent $E \in \mathcal{S}$ and consider $E\mathcal{S}E$. By minimality it follows that the nonzero elements in $E\mathcal{S}E$ have constant rank, so the restriction of $E\mathcal{S}E$ to the range of E consists of scalar multiples of a compact group $\mathcal{U}$ (see [4, Lemma 3.1.6]). Since by assumptions every finite subgroup of $\mathcal{U}$ is abelian it follows that $\mathcal{U}$ is abelian by [1].

Next, let P_j be such that $P_jE \neq 0$. The semigroup $P_j\mathcal{S}P_j$ restricted to the range of P_j is by assumption irreducible. It follows that $P_jE\mathcal{S}EP_j$, restricted to the range of P_jE, is irreducible. Since it is abelian we have that the rank of P_jE is one as claimed. □

Keeping the same notation and hypotheses, we have the following immediate corollary to this lemma.

COROLLARY 4.3. *If the semigroup $\mathcal{S} = \overline{\mathbb{C}\mathcal{S}}$ contains a set $E_1, \ldots, E_l$ of mutually orthogonal minimal idempotents of ranks r_i respectively, then the lattice of invariant subspaces of $\mathcal{S}$ contains a chain of length at least $r_1 + \cdots + r_l$. In particular, if $E_1 + \cdots + E_l = I$, then $\mathcal{S}$ is diagonalizable.*

Proof. We only need to show that if $P_jE_i \neq 0$ for some i,j, then $P_jE_k = 0$ for all $k \neq i$. So assume that there exists $k \neq i$ such that $P_jE_k \neq 0$. The irreducibility of $P_j\mathcal{S}P_j$, restricted to the range of P_j, therefore forces $P_jE_i\mathcal{S}E_kP_j$ to be nonzero. On the other hand $E_i\mathcal{S}E_k$ consists of nilpotents and is hence zero by assumptions which gives the desired contradiction. □

We now consider groups of invertible matrices satisfying a pretriangularizing condition. Our main result is the following theorem.

THEOREM 4.4. *Let $\mathcal{P}$ be a pretriangularizing property and let $\mathcal{G} \subseteq GL_n(\mathbb{C})$ be a group satisfying $\mathcal{P}$. Then $\mathcal{G}$ is solvable and the following hold:*

1. *If $n \leqslant 3$, then $\mathcal{G}$ is triangularizable and if $n \geqslant 4$, then the lattice of invariant subspaces of $\mathcal{G}$ contains a chain of length at least three.*
2. *The derived subgroup $\mathcal{G}'$ is triangularizable.*
3. *For each $A \in \mathcal{G}'$ we have $\sigma(A) \subseteq \{z \in \mathbb{C} : |z| = 1\}$.*

In addition, if $\sigma(A) \subseteq \{z \in \mathbb{C} : |z| = \rho(A)\}$ holds for every $A \in \mathcal{G}$, then $\mathcal{G}$ is triangularizable.

Proof. With no loss of generality $\mathcal{G} = \mathbb{C}^*\mathcal{G}$. Assume first that $\mathcal{G}$ is totally reducible and consider $\mathcal{S} = \overline{\mathcal{G}}$. Let $P_1, \ldots, P_m$ be a complete set of mutually orthogonal projections to invariant subspaces of $\mathcal{G}$. Observe that $\mathcal{S}$ satisfies the property $\mathcal{P}$ as well. Now by Lemma 4.2 we know that there exists a minimal idempotent $E \in \mathcal{S}$. If $E = I$, then by the previous Corollary $\mathcal{S}$ (and hence $\mathcal{G}$) is diagonalizable so there is nothing more to prove. So we assume $E \neq I$. Consider a sequence $G_n \to E$, $G_n \in \mathcal{G}$. By passing to a subsequence we may assume

$$\lim_{n\to\infty} \frac{G_n^{-1}}{||G_n^{-1}||} = A \neq 0\,.$$

It is immediate that $AE = EA = 0$. Now, in the homogenized closure of the semigroup generated by A there exists an idempotent F, since by assumptions there are no nonzero nilpotents in $\mathcal{S}$. This forces any maximal set of mutually orthogonal minimal idempotents $E_1, \ldots, E_k$ to have at least two elements.

Let $\{E_1, \ldots, E_k\}$ be such a set. By Corollary 4.3 we know that the rank of P_iE_j is either zero or one. We claim that if the rank of P_iE_j is one, then the rank of P_i is one as well. So assume not. Now the fact that $P_i\mathcal{S}P_i$ is irreducible implies that there exists $A = E_jA \in \mathcal{S}$ such that $A(I - E_j) \neq 0$. Let $G_n \to E_j$, $G_n \in \mathcal{G}$, and consider the sequence AG_n^{-1}. If this is bounded we may assume, after passing to a suitable subsequence, that $\lim_{n\to\infty} AG_n^{-1} = T \in \mathcal{S}$. We now have $0 \neq A = (AG_n^{-1})G_n$ and on the other hand $\lim_{n\to\infty}(AG_n^{-1})G_n = TE_j$ and therefore $TE_j = A$ which implies $AE_j = A$, a contradiction. Hence the sequence AG_n^{-1} is unbounded. Again, by passing to a suitable subsequence we may assume that $\lim_{n\to\infty} ||AG_n^{-1}|| = \infty$ and

$$\lim_{n\to\infty} \frac{A(G_n^{-1} - I)}{||AG_n^{-1}||} = \lim_{n\to\infty} \frac{AG_n^{-1}}{||AG_n^{-1}||} = S \neq 0, S \in \mathcal{S}\,.$$

We now have

$$SE_j = \lim_{n\to\infty} \frac{A(G_n^{-1} - I)G_n}{||AG_n^{-1}||} = \lim_{n\to\infty} \frac{A(I - G_n)}{||AG_n^{-1}||} = 0\,.$$

Since $S = E_jS$ this shows that $S \in \mathcal{S}$ is a nonzero nilpotent which is again a contradiction. We have thus shown that the rank of P_i is one as claimed. In particular, since the idempotents P_i commute with the elements of $\mathcal{S}$, we have that P_iE_j is either zero or $P_iE_j = P_i$.

Let $\mathcal{M}$ be the span of ranges of a maximal set of mutually orthogonal minimal idempotents $E_1, \dots, E_k$ in $\mathcal{S}$. We have shown that $\mathcal{M}$ is invariant for $\mathcal{S}$, the restriction of $\mathcal{S}$ to $\mathcal{M}$ is diagonal and $\mathcal{M}$ has dimension at least two. If $\mathcal{M}$ is the whole space, then $\mathcal{S}$, and thus $\mathcal{G}$, is abelian and there is nothing more to prove. So assume $\mathcal{M} \neq \mathbb{C}^n$. We now turn our attention to $\mathcal{G}'$. Observe that $\mathcal{G}'$ acts trivially on $\mathcal{M}$ and consider its action on $\mathcal{N} = (I - \oplus E_i)\mathbb{C}^n$. We will show that $\mathcal{G}'$ is simultaneously similar to a unitary group (i.e., a group of unitary matrices). Just note that $\rho(A) = 1$ for every $A \in \mathcal{G}'$. Otherwise, $\overline{\mathbb{C}\mathcal{G}'}$ would contain nonzero operators whose null space contains $\mathcal{M}$. This would result in the existence of a nonzero idempotent whose range is contained in $\mathcal{N}$ and whose null space contains $\mathcal{M}$ contradicting the maximality of the set $\{E_1, \dots, E_k\}$. Since $\mathcal{G}'$, being a normal subgroup in $\mathcal{G}$, is totally reducible it follows that $\mathcal{G}'$ is bounded and thus similar to a unitary group (see, e.g., [4, Lemma 3.1.4, Theorem 3.1.5]). By assumptions, every finite subgroup of $\mathcal{G}'$ is abelian but then so is $\mathcal{G}'$ by [1]. This shows that $\mathcal{G}$ is solvable.

Similarly, if $\sigma(A) \subseteq \{z \in \mathbb{C} : |z| = \rho(A)\}$ for every $A \in \mathcal{G}$, then by the same argument the group $\mathcal{G}_1 = \{A \in \mathcal{G} : \rho(A) = 1\}$ is bounded and by the same reasoning as above abelian. Consequently, this holds also for $\mathcal{G}$ since every element of $\mathcal{G}$ is a positive multiple of an element of $\mathcal{G}_1$, which proves the theorem for the totally reducible case.

For the general case, let $\mathcal{C}$ be any maximal chain of invariant subspaces for $\mathcal{G}$ with the corresponding projections $P_1, P_1 \oplus P_2, \dots, P_1 \oplus P_2 \oplus \cdots \oplus P_l$. Then the group

$$\mathcal{G}_s = \{P_1AP_1 \oplus \cdots \oplus P_lAP_l \,:\, A \in \mathcal{G}\}$$

which is a semisimplification of $\mathcal{G}$ is totally reducible. By assumptions, $\mathcal{G}_s$ satisfies $\mathcal{P}$ so the desired conclusions for $\mathcal{G}$ now follow from those for $\mathcal{G}_s$ since the map $\mathcal{G} \to \mathcal{G}_s$ is a surjective group homomorphism with a solvable kernel. $\square$

We have the following immediate corollary to this theorem.

COROLLARY 4.5. *A connected group satisfying a pretriangularizing property is triangularizable. The same also holds if the Zariski closure of a group $\mathcal{G}$, that satisfies this property, is connected (in the Zariski topology on $GL_n(\mathbb{C})$).*

Proof. This follows immediately by observing that solvability passes to closures in either Euclidian or Zariski topology, and from the fact that connected solvable groups are triangularizable. $\square$

The following example shows that connectivity is essential in the hypothesis above. In particular it shows that, in general, Theorem 4.4 cannot be improved. We take condition (6) for the pretriangularizing property.

EXAMPLE 4.6. [2] Let $\mathcal{G} \subseteq GL_4(\mathbb{C})$ be a group generated by the following two matrices

$$U = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Observe that we have $|\mathrm{tr}(ABC - BAC)| \leqslant 4$ for all $A, B, C \in \mathcal{G}$. Now let $A, B \in \mathcal{G}$ be given and consider their expressions as a word (with integer exponents) in U and V. If the exponents of the U's occurring in A and B add up to zero respectively, then A and B commute and so the condition (6) is satisfied in this case. If not, then for at least one of them the spectral radius is at least 2, thus the condition (6) is satisfied in this case as well. □

Returning to semigroups we can now use Theorem 4.4 and Corollary 4.5 to get a sharper reducibility result. Let us fix some notation. Given a maximal chain $\mathcal{C}$ of invariant subspaces of $\mathcal{S}$ with the corresponding projections $P_1, P_1 \oplus P_2, \ldots, P_1 \oplus P_2 \cdots \oplus P_l$ let $\mathcal{S}_s$ denote the semisimplification of $\mathcal{S}$

$$\mathcal{S}_s = \{P_1 S P_1 \oplus \cdots \oplus P_l S P_l : S \in \mathcal{S}\}.$$

(Note that by the Wedderburn-Malcev theorem $\mathcal{S}_s$ is, up to conjugation, independent of $\mathcal{C}$).

COROLLARY 4.7. *Let $\mathcal{S} \subseteq M_n(\mathbb{C})$ be a semigroup satisfying a pretriangularizing property $\mathcal{P}$. Then $\mathcal{S}$ has a chain of invariant subspaces of length at least r, where r is the minimal rank in $\overline{\mathbb{C}\mathcal{S}} \backslash \{0\}$.*

Proof. If $\mathcal{S}$ is nil, then it is triangularizable by Levitzki's theorem and the claim follows. If not, then by [2, Lemma 2.9] there is a minimal idempotent $E \in \overline{\mathbb{C}\mathcal{S}_s}$ whose rank is at least r. Since $\overline{\mathbb{C}\mathcal{S}_s}$ satisfies the property $\mathcal{P}$, it follows by Corollary 4.3 that $\overline{\mathbb{C}\mathcal{S}_s}$ has a chain of invariant subspaces of length at least r and so does $\mathcal{S}$. □

5. On infinite dimensions

We will conclude this paper with a brief comment and an infinite-dimensional analogue of the above results. Taking the general setting of a Hilbert space, one may consider semigroups of bounded linear operators. By subspaces we shall mean closed subspaces. However, in this setting it is not possible to consider approximate conditions in a uniform way as in the previous section simply because they may not be meaningful for the same classes of operators. For example, condition (7) is meaningful for arbitrary semigroups of operators on a Banach space, but can be expected to give positive results only for semigroups of compact operators. On the other hand, if we want to consider condition (6) in infinite dimensions, we obviously must restrict our attention to trace class operators. These cannot be invertible, so we are limited to semigroup, and not group, analogues. Note that Example 3.1 can be trivially modified to give an irreducible semigroup of rank-one operators satisfying the condition (1): just replace "$\mathbb{C}^{n-1}$" with "ℓ^2" in the definition given there.

For a set $\mathcal{E}$ of trace class operators, $\overline{\mathcal{E}}$ denotes the closure of $\mathcal{E}$ in the trace class topology. In this case we have the following analogue of Corollary 4.7.

THEOREM 5.1. *Let $\mathcal{S}$ be a semigroup of trace class operators on a complex Hilbert space satisfying the condition* (6). *Let m be the minimal rank, possibly infinite, in $\overline{\mathbb{C}\mathcal{S}}\backslash\{0\}$. Then $\mathcal{S}$ has a chain of invariant subspaces of length at least m.*

Proof. Assume with no loss again that $\mathcal{S} = \overline{\mathbb{C}\mathcal{S}}$. If $m = \infty$, i.e., $\mathcal{S}$ has no finite rank members, then $\rho(S) = 0$ for all $S \in \mathcal{S}$ by [3, Lemma 2]. It now follows from Turovskii's Theorem [6] that $\mathcal{S}$ is triangularizable. Thus we can assume $m < \infty$. Let $\mathcal{C}$ be a maximal chain of invariant subspaces of $\mathcal{S}$. We must show that $\mathcal{C}$ has at least $m+1$ distinct members (counting the trivial ones). Suppose not. Then list the members of $\mathcal{C}$ as

$$\{0\} < \mathcal{M}_1 < \cdots < \mathcal{M}_k = \mathcal{X}$$

with $k < m$. For each i let $\mathcal{X}_i$ denote the quotient space $\mathcal{M}_i/\mathcal{M}_{i-1}$. Now form the Hilbert space

$$\mathcal{Y} = \bigoplus_i \mathcal{X}_i$$

where for $y = (x_1, \ldots, x_k)$ we set $||y|| = \sum_{i=1}^{k} ||x_i||$. There is an obvious homomorphism of $\mathcal{S}$ into a new semigroup $\mathcal{T}$ defined on $\mathcal{Y}$ whose members are of the form $\oplus A_i$ where A_i is the quotient operator on $\mathcal{X}_i$ induced by a member $A \in \mathcal{S}$. This homomorphism is a contraction and preserves spectra counting multiplicity (see, e.g., [4, Theorem 7.2.7]).

Observe that the proof of [2, Lemma 2.9] carries over to this situation (except that nilpotents are replaced by quasinilpotents and that Turovskii's result [6] is invoked in place of Levitzki's) showing that there is a minimal idempotent $E \in \overline{\mathbb{C}\mathcal{T}}$ of minimal nonzero finite rank l in $\overline{\mathbb{C}\mathcal{T}}$ where $l \geqslant m$. The semigroup $E\overline{\mathbb{C}\mathcal{T}}E$ restricted to the image of E consists therefore (after conjugation) of scalar multiples of elements of some unitary group by [4, Lemma 3.1.6] and is thus commutative by Theorem 2.3 and hence diagonalizable. By [4, Lemma 8.2.10] we have that $\mathcal{T}$ has a chain of invariant subspaces of length at least l which contradicts the fact that, by construction, every maximal chain of invariant subspaces of $\mathcal{T}$ has length $k < l$. $\square$

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