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CANONICAL STRUCTURES FOR PALINDROMIC MATRIX POLYNOMIALS

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Abstract. Spectral properties and canonical structures of palindromic matrix polynomials are studied in terms of their linearizations, standard triples, and unitary triples. These triples describe matrix polynomials via eigenvalues and Jordan chains. As an application of canonical structures and their properties, criteria are developed for stable boundedness of solutions of systems of linear differential equations with symmetries.

1. Introduction

The purpose of this paper is to study the spectral properties of palindromic matrix polynomials (definitions below). Motivation for this investigation and many interesting results can be found in works of Mackey et al. [15], [16], where practical examples and computational issues are discussed. Earlier results for palindromic polynomials of *even* degree can be found in the books of Gohberg, Lancaster, and Rodman, [5], [6]. Also, there is recent work [17], on polynomials of the first degree. A more comprehensive theory including both even and odd degree polynomials is the main objective of this paper. In particular, the unitary properties of such functions will be studied, including analysis of the appropriate choice of indefinite inner product.

We study $n \times n$ matrix polynomials

$$L(\lambda) = \sum_{j=0}^{\ell} A_j \lambda^j, \tag{1}$$

with complex $n \times n$ matrix coefficients A_j, $j = 0, 1, \dots, \ell$ and with $\det A_\ell \neq 0$. The *reverse* polynomial of $L(\lambda)$ is defined by $\operatorname{rev} L(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_{\ell-j} = \lambda^\ell L(\lambda^{-1})$. Now we make the formal definition:

DEFINITION 1. A matrix polynomial $L(\lambda)$ with the property that, for all $\lambda \in \mathbb{C}$, and a fixed $\mu = \pm 1$,

$$\operatorname{rev}(L(\lambda))^* = \mu L(\overline{\lambda}),$$

is said to be μ*-palindromic*.

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Thus, for a μ-palindromic polynomial we have $A_j^* = \mu A_{\ell-j}$ for $j = 0,1,\dots,\ell$. Furthermore, it is clear that, $L(\lambda)$ is μ-palindromic if and only if, with the same μ and any $\lambda \in \mathbb{C}\setminus\{0\}$,

$$L(\lambda)^* = \mu\,\overline{\lambda}^{\ell}\,L(\overline{\lambda}^{-1}). \tag{2}$$

Let us summarize the contributions made in this paper: Section 2 is largely a review of "linearizations" and especially those with symmetries. In particular, links between Hermitian and palindromic symmetries are established. Then Section 3 contains new material concerning the definition of indefinite inner products in which linearizations of palindromic polynomials have unitary properties. The notion of "standard triples" plays a central role in the theory of matrix polynomials, and this is developed further in Section 4 to characterize standard triples of μ-palindromic polynomials. For Hermitian matrix polynomials, the notion of standard triples which are "self-adjoint" plays the role of Jordan structures in the theory of matrices. For palindromic polynomials, the analogous "unitary triples" are required and are developed in Section 5. This theory depends on canonical forms for H-unitary matrices.

The theory developed here (and elsewhere) can be applied to the study of stability of the solutions of difference equations whose matrix coefficients form palindromic polynomials. In Section 6 this theory is extended to admit μ-palindromic polynomials with $\mu = \pm 1$. It contains a complete analysis of stability problems in the case of polynomials with even degree. Partial results are obtained in the case of odd degree, but the complete analogue of Theorem 14 for μ-palindromic polynomials of odd degree remains open.

Before beginning the general theory, it will be useful to record three observations concerning simple transformations of palindromic matrix polynomials:

PROPOSITION 1.

(a) *If* $L(\lambda) = \sum_{j=0}^{\ell}\lambda^j A_j$ *is* μ*-palindromic, then for every* $\alpha \in \mathbb{C}\setminus\{0\}$*, the matrix polynomial*

$$M_\alpha(\lambda) := (\lambda\alpha + \overline{\alpha})L(\lambda) \tag{3}$$

is μ*-palindromic as well.*

(b) *If* $L(\lambda)$ *is* μ*-palindromic and* ℓ *is even, then* $\widetilde{L}(\lambda) := L(i\lambda)$ *is* $\mu(-1)^{\ell/2}$*-palindromic.*

(c) *If* $L(\lambda)$ *is* μ*-palindromic of even degree* ℓ*, and* $|\omega| = 1$ *is fixed, then the matrix polynomial* $L_\omega(\lambda) := (\omega)^{\ell/2}L(\omega\lambda)$ *is* μ*-palindromic as well.*

Proof. (a) Let $M_\alpha(\lambda) = \sum_{j=0}^{\ell+1}\lambda^j B_j$. Then

$$B_{\ell+1}^* = (\alpha A_\ell)^* = \mu\overline{\alpha}A_0 = \mu B_0,$$

and for $j = 1,2,\dots,\ell$:

$$B_j^* = (\alpha A_{j-1} + \overline{\alpha}A_j)^* = \overline{\alpha}A_{j-1}^* + \alpha A_j^* = \overline{\alpha}\mu A_{\ell-j+1}^* + \alpha\mu A_{\ell-j} = \mu B_{\ell+1-j}.$$

For part (b) write, using the μ-palindromic property of $L(\lambda)$:

$$(L(i\lambda))^* = \mu(-i\overline{\lambda})^{\ell}L((-i\overline{\lambda})^{-1}) = \mu(-1)^{\ell/2}\overline{\lambda}^{\ell}L(i\overline{\lambda}^{-1}) = \mu(-1)^{\ell/2}\overline{\lambda}^{\ell}\widetilde{L}(\overline{\lambda}^{-1}).$$

The part (c) is verified by a straightforward computation. $\square$

2. Linearizations with symmetries

DEFINITION 2. A matrix pencil $\lambda X+Y$ of size $\ell n\times\ell n$ is called a *linearization* of $L(\lambda)$ (of equation (1)) if

$$E(\lambda)(\lambda X+Y)F(\lambda)=\begin{bmatrix} L(\lambda) & 0\\ 0 & I_{n(\ell-1)}\end{bmatrix}$$

for some unimodular (i.e., having constant nonzero determinant) matrix polynomials $E(\lambda)$ and $F(\lambda)$.

There is an extensive literature on matrix polynomials and their linearizations, see for example [1], [2], [16], and in particular the monographs [7], [3] and references there. The use of Definition 2 is consistent with our hypothesis that A_ℓ is nonsingular. An important and well-known example of a linearization for $L(\lambda)$ is the pencil $\lambda A-B$, where

$$A:=\begin{bmatrix} A_1 & A_2 & \cdots & A_\ell\\ A_2 & & \iddots & 0\\ \vdots & \iddots & \iddots & \vdots\\ A_\ell & 0 & \cdots & 0\end{bmatrix},\quad B:=\begin{bmatrix} -A_0 & 0 & 0 & \cdots & 0\\ 0 & A_2 & A_3 & \cdots & A_\ell\\ 0 & A_3 & & \iddots & 0\\ \vdots & \vdots & \iddots & \iddots & \vdots\\ 0 & A_\ell & 0 & \cdots & 0\end{bmatrix}. \tag{4}$$

This pencil obviously has a useful role to play when the coefficients of $L(\lambda)$ are Hermitian. Assuming that A_ℓ is invertible, if $C=A^{-1}B$ is the *companion matrix* of $L(\lambda)$:

$$C:=\begin{bmatrix} 0 & I_n & 0 & \cdots & 0\\ \vdots & 0 & \ddots & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & I_n\\ -A_\ell^{-1}A_0 & -A_\ell^{-1}A_1 & \cdots & -A_\ell^{-1}A_{\ell-2} & -A_\ell^{-1}A_{\ell-1}\end{bmatrix}, \tag{5}$$

then the pencil $\lambda I-C$ is also a linearization.

More generally, consider the matrices

$$S_i:=S_i(L)=AC^i,\quad i=0,1,\ldots.$$

The sequence begins with $S_0=A$ and $S_1=B$ and, for $i=0,1,\ldots,\ell$, they have the form

$$S_i:=\begin{bmatrix} 0 & \cdots & 0 & -A_0 & 0 & \cdots & \cdots & 0\\ \vdots & \iddots & \iddots & \vdots & \vdots & & & \vdots\\ 0 & \iddots & & \vdots & \vdots & & & \vdots\\ -A_0 & \cdots & \cdots & -A_{i-1} & 0 & \cdots & \cdots & 0\\ 0 & \cdots & \cdots & 0 & A_{i+1} & \cdots & \cdots & A_\ell\\ \vdots & & & \vdots & \vdots & & \iddots & 0\\ \vdots & & & \vdots & \vdots & \iddots & \iddots & \vdots\\ 0 & \cdots & \cdots & 0 & A_\ell & 0 & \cdots & 0\end{bmatrix}. \tag{6}$$

In particular,

$$S_{\ell-1}=\begin{bmatrix} 0 & \cdots & 0 & -A_0 & 0 \\ \vdots & \iddots & \iddots & \vdots & \vdots \\ 0 & \iddots & & \vdots & \vdots \\ -A_0 & \cdots & \cdots & -A_{\ell-2} & 0 \\ 0 & \cdots & \cdots & 0 & A_\ell \end{bmatrix},\ S_\ell=\begin{bmatrix} 0 & \cdots & 0 & -A_0 \\ \vdots & \iddots & \iddots & -A_1 \\ 0 & \iddots & & \vdots \\ -A_0 & -A_1 & \cdots & -A_{\ell-1} \end{bmatrix}. \tag{7}$$

Note that S_i does not depend on A_i, and the formulas for the S_i make sense also when A_ℓ is singular.

It is easily seen that all S_i are nonsingular if and only if A_ℓ and A_0 are nonsingular. In this case C is nonsingular and

$$\lambda S_{k-1}-S_k=AC^{k-1}(\lambda I-C)=S_{k-1}(\lambda I-C), \quad k=1,2,\ldots,\ell.$$

This strict equivalence shows that, when *both* A_ℓ and A_0 are nonsingular, $\lambda S_{k-1}-S_k$ is a linearization of $P(\lambda)$ for each k.

Now we review some results from [13] concerning linearizations for μ-palindromic matrix polynomials. First define the $n\ell\times n\ell$ matrix

$$R_{\ell,n}:=\begin{bmatrix} 0 & & I_n \\ & \iddots & \\ I_n & & 0 \end{bmatrix}=R_{\ell,1}\otimes I_n.$$

Note that $R_{\ell,1}$ is known as the *SIP matrix* — short for Standard Involutory Permutation — of size $\ell\times\ell$: it is square with 1's on the lower left - upper right diagonal and zeros elsewhere. Define also the subset of $\mathbb{C}^\ell$,

$$\mathbb{P}:=\{c\in\mathbb{C}^\ell\ :\ c^T=c^*R_{\ell,1}\}. \tag{8}$$

Observe that $c\in\mathbb{P}$ if and only if $c_j=\overline{c}_{\ell-j+1}$ for $j=1,2,\ldots,\ell$ and $\mathbb{P}$ is a linear space over $\mathbb{R}$. With each $c\in\mathbb{P}$ define a corresponding polynomial $p_c(\lambda)=c_1+c_2\lambda+\cdots+c_\ell\lambda^{\ell-1}$.

Also, for $c\in\mathbb{C}^\ell$ and any matrix polynomial $L(\lambda)$ of degree ℓ, let

$$A_c(L):=R_{\ell,n}\sum_{i=1}^{\ell}c_iS_{i-1}(L), \qquad B_c(L):=R_{\ell,n}\sum_{i=1}^{\ell}c_iS_i(L)=A_c(L)C.$$

From Section 6 of [13] we have:

LEMMA 2. *Let $L(\lambda)$ have μ-palindromic symmetry. Then:*
(a) $S_j^*=-\mu R_{\ell,n}S_{\ell-j}R_{\ell,n}$ *for* $j=0,1,\ldots,\ell.$
(b) *The linear matrix polynomial*

$$L_{c,L}(\lambda):=\lambda A_c(L)-B_c(L)=A_c(L)(\lambda I-C) \tag{9}$$

has μ-palindromic symmetry if and only if $c\in\mathbb{P}$. In this case,

$$L_{c,L}(\lambda):=\lambda A_c(L)+\mu A_c(L)^*. \tag{10}$$

THEOREM 3. *Let $L(\lambda)$ be a μ-palindromic matrix polynomial, $c \in \mathbb{P}$, and assume that $p_c(\lambda)$ is nonzero at the eigenvalues of $L(\lambda)$. Then the linear matrix polynomial of* (9), (10) *is a μ-palindromic linearization for $L(\lambda)$ (with the same μ).*

A general study of linearizations of matrix polynomials was undertaken in [16], and in [15] with emphasis on palindromic symmetry properties. The result of Theorem 3 appears (in a different context) as a particular case of Theorem 6.5 of [15]. See also Theorem 6.7 of [16], which gives a result in the spirit of Theorem 3 for nonsymmetric matrix polynomials.

Following [12], we have the following definition:

DEFINITION 3. Let $L(\lambda)$ be a μ-palindromic matrix polynomial given by (1), and assume that $\det A_\ell \neq 0$ and that in case ℓ is even, also $-1 \notin \sigma(L)$. The *primary linearization* for a μ-palindromic matrix polynomial $L(\lambda)$ is the palindromic pencil $\lambda A_c + \mu A_c^*$, where

$$A_c = \begin{cases} R_{\ell,n}(S_{k-1} + S_k) & \text{when } \ell = 2k\,, \\ R_{\ell,n} S_k & \text{when } \ell = 2k+1\,. \end{cases} \tag{11}$$

(For any ℓ this definition requires that $0 \notin \sigma(L)$ and this is guaranteed here by the assumption that $\det A_\ell \neq 0$ and the palindromic symmetry.)

EXAMPLE 1. When $\ell = 2$ we choose $p_c(\lambda) = 1+\lambda$ and the primary linearization is determined by equation (10) and

$$A_c = R_{2,n}(S_0 + S_1) = \begin{bmatrix} A_2 & A_2 \\ A_1 - A_0 & A_2 \end{bmatrix} = \begin{bmatrix} \mu A_0^* & \mu A_0^* \\ -A_0 + A_1 & \mu A_0^* \end{bmatrix},$$

provided that -1 is not in the spectrum of $L(\lambda)$.

When $\ell = 3$ we choose $p_c(\lambda) = \lambda$ and the primary linearization is determined by

$$A_c = R_{3,n} S_1 = \begin{bmatrix} 0 & A_3 & 0 \\ 0 & A_2 & A_3 \\ -A_0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mu A_0^* & 0 \\ 0 & \mu A_1^* & \mu A_0^* \\ -A_0 & 0 & 0 \end{bmatrix},$$

provided only that 0 is not an eigenvalue of $L(\lambda)$. $\square$

Because we consider only matrix polynomials $L(\lambda)$ with $\det(A_\ell) \neq 0$, any μ-palindromic matrix polynomial $L(\lambda)$ also has $\det(A_0) \neq 0$, and hence $L(\lambda)$ has no zero eigenvalue; $0 \notin \sigma(L)$. Then, using (2), it is easily verified that for both palindromic symmetries:

$$\lambda \in \sigma(L) \quad \Longleftrightarrow \quad (\overline{\lambda})^{-1} \in \sigma(L).$$

This statement implies that the spectrum of a μ-palindromic matrix polynomial is symmetric about the unit circle: either the eigenvalues satisfy $|\lambda_i| = 1$, or they occur in pairs $\lambda_j \neq \lambda_k$ where $\lambda_j \overline{\lambda_k} = 1$. This symmetry implies that the linearizations must be unitary in a suitable indefinite inner product (see Section 12.6 of [6]). Let us confirm this for the palindromic linearizations of Theorem 3. We have $L_c(\lambda) = \lambda A_c + \mu A_c^*$, and since A_c is nonsingular,

$$L_c(\lambda) = A_c(\lambda I + \mu U) \tag{12}$$

where $U = A_c^{-1}A_c^*$. But it is known that *any* matrix of the form $\pm M^{-1}M^*$ is H-unitary for some nonsingular H (see Section 4.4 of [6] for details). Indeed there is a family of indefinite inner products to choose from, namely, those generated by matrices of the form

$$H_z := \bar{z}A_c + zA_c^* \tag{13}$$

where $|z| = 1$ is such that H_z is nonsingular. Notice that H_z now depends on $c \in \mathbb{P}$ (of (8)), and also on z.

Since $\lambda I + \mu U$ is strictly equivalent to $L_c(\lambda)$ it is also a linearization (but not palindromic). Thus the canonical structures of H-unitary matrices (and not just the spectrum) are inherited by the palindromic linearization. These properties become more apparent in linearizations of the form $H(\lambda I - C)$ where H generates an indefinite scalar product on $\mathbb{C}^{\ell n}$ and C is the companion matrix of equation (5). Thus, in the next section, the palindromic symmetry of (10) is abandoned in favor of the Hermitian symmetry of H in linearizations of the form $\lambda H - HC$.

3. Unitary properties of linearizations

It will be assumed here that $L(\lambda) = \sum_{j=0}^{\ell} A_j\lambda^j$ is a μ-palindromic matrix polynomial with nonsingular A_ℓ (and hence also with nonsingular A_0). Consider the linearization in companion form, $\lambda I - C$. We seek a nonsingular Hermitian matrix H of size ℓn for which $C^*HC = H$. In this case, and in the terminology of [6], C is said to be H-unitary. This admits a deeper analysis of the spectrum, $\sigma(C)$ of C. In particular, a sign characteristic appears which is associated with the eigenvalues on the unit circle (if any). A class of matrices, H, will be determined, each of which determines an indefinite inner product on $\mathbb{C}^{\ell n}$ in which C is unitary. (This can be done using (13), but a more direct approach is instructive.)

The first proposition shows that any H in which C is H-unitary has a special structure.

PROPOSITION 4. *Let C be a block-companion matrix as in* (5). *If H is nonsingular and Hermitian and $C^*HC = H$ then H^{-1} is a block-Toeplitz matrix.*

Proof. It is easily verified that any matrix T satisfying the equation $CTC^* = T$ is block-Toeplitz. But if H is a nonsingular matrix satisfying $C^*HC = H$, then $CH^{-1}C^* = H^{-1}$. Thus, H^{-1} is block-Toeplitz. □

We mention in passing that the inverses of block-Toeplitz matrices have been extensively studied; as in [11]. In particular, these inverses are Bezoutian-like matrices; see [14], [10], and especially Gohberg-Semencul formulas and their numerous generalizations and applications (the original reference is [9]).

Observe that

$$C^* = \begin{bmatrix} 0 & 0 & \cdots & 0 & -A_\ell A_0^{-1} \\ I & 0 & & 0 & -A_{\ell-1}A_0^{-1} \\ 0 & I & & & \vdots \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & I & -A_1 A_0^{-1} \end{bmatrix},$$

and then (as in Section II.2.4 of [6]) it is easily verified that

$$(C^*)^{-1} = \begin{bmatrix} -A_{\ell-1}A_\ell^{-1} & I & 0 & \cdots & 0 \\ -A_{\ell-2}A_\ell^{-1} & 0 & & & \vdots \\ \vdots & & & & \\ -A_1 A_\ell^{-1} & 0 & \cdots & & I \\ -A_0 A_\ell^{-1} & 0 & \cdots & \cdots & 0 \end{bmatrix}. \tag{14}$$

LEMMA 5. *The equation $XC = (C^{-1})^* X$ has linearly independent solutions $X_j = R_{\ell,n} S_j$ for $j = 0, 1, \ldots, \ell$ (with S_j defined as in (6)).*

Proof. First it is claimed that

$$(R_{\ell,n} S_0)C = (C^{-1})^*(R_{\ell,n} S_0). \tag{15}$$

But this is an easy verification using the definitions, and equations (5) and (14). Now multiply (15) on the right with C^i and use the fact that $S_i = S_0 C^i$ for each i to see that each X_i is a solution. The linear independence of the X_i follows from that of the S_i. $\square$

To find Hermitian matrices H satisfying $HC = (C^{-1})^* H$ (and so candidates for the definition of a suitable inner product), we examine linear combinations of the (generally non-Hermitian) solutions $X_0, \ldots, X_\ell$. Thus, let $z \in \mathbb{C}^{\ell+1}$ and

$$H = \sum_{i=0}^{\ell} z_i X_i = R_{\ell,n} \sum_{i=0}^{\ell} z_i S_i. \tag{16}$$

Then $H = H^*$ is equivalent to $R_{\ell,n} \sum_{i=0}^{\ell} z_i S_i = \sum_{i=0}^{\ell} \overline{z_i} S_i^* R_{\ell,n}$. But, using Lemma 13 of [12] for instance, we have $S_i^* = -\mu R_{\ell,n} S_{\ell-i} R_{\ell,n}$, so $H = H^*$ is equivalent to

$$R_{\ell,n} \sum_{i=0}^{\ell} z_i S_i = -\mu R_{\ell,n} \sum_{i=0}^{\ell} \overline{z_i} S_{\ell-i},$$

or,

$$\sum_i z_i S_i = -\mu \sum_j \overline{z_j} S_{\ell-j} = -\mu \sum_i \overline{z_{\ell-i}} S_i.$$

However, the matrices S_i are linearly independent, so H is Hermitian if and only if $-\mu z_i = \overline{z_{\ell-i}}$ for each i.

So let us define

$$\mathbb{Q}_\mu = \{z = (z_0, z_1, \dots, z_\ell) \in \mathbb{C}^{\ell+1} \ : \ -\mu z_i = \overline{z_{\ell-i}} \quad \text{for} \quad i = 0, 1, \dots, \ell\}, \tag{17}$$

and, whenever $z \in \mathbb{Q}_\mu$, define $q_z(\lambda) := \sum_{i=0}^{\ell} z_i \lambda^i$. We have proved the first part of:

THEOREM 6. *Let $L(\lambda)$ be μ-palindromic and matrices $S_0, \dots, S_\ell$ be defined as in (6). Then all matrices of the form $H = R_{\ell,n} \sum_{j=0}^{\ell} z_j S_j$ with $z \in \mathbb{Q}_\mu$, are Hermitian and satisfy the equation $HC = (C^{-1})^* H$.*

Furthermore, when $z \in \mathbb{Q}_\mu$ and $\det A_0 \neq 0$, H is nonsingular if and only if no zero of $q_z(\lambda)$ is an eigenvalue of $L(\lambda)$.

Proof. It only remains to check the last statement. Since $S_i = S_0 C^i$ and A_0 nonsingular implies S_0 nonsingular, the conclusion follows immediately from

$$H = R_{\ell,n} \sum_{i=0}^{\ell} z_i S_i = R_{\ell,n} S_0 \sum_{i=0}^{\ell} z_i C^i = R_{\ell,n} S_0 q_z(C).$$

□

EXAMPLE 2. For a palindromic matrix polynomial of even degree, say $\ell = 2k$, the matrix S_k of (6) can be used to define an appropriate inner product, H. If $\mu = +1$, an H in a relatively simple form is generated by taking $z_j = 0$ if $j \neq k$ and $z_k = i$, and it is easily verified that $z \in \mathbb{Q}_\mu$ (and $q_z(\lambda) = i\lambda^k$). Thus, if $\mu = 1$ then the matrix

$$H = iR_{\ell,n} S_k = i \begin{bmatrix} 0 & \begin{matrix} A_0^* & 0 & \cdots & 0 \\ A_1^* & A_0^* & \cdots & \vdots \\ \vdots & & & \vdots \\ A_{k-1}^* & \cdots & A_1^* & A_0^* \end{matrix} \\ \begin{matrix} -A_0 & -A_1 & \cdots & -A_{k-1} \\ 0 & -A_0 & & \\ \vdots & & & \vdots \\ 0 & \cdots & & -A_0 \end{matrix} & 0 \end{bmatrix} \tag{18}$$

is Hermitian and nonsingular and satisfies the equation $C^* H C = H$. This is the choice of inner product identified in Section 12.4 of [6]. Matrix H is obviously nonsingular since A_0 is nonsingular.

If $\mu = -1$ then the matrix $H_- = R_{\ell,n} S_k$ is nonsingular Hermitian and satisfies $C^* H_- C = H_-$. In connection with Proposition 4, it can be verified from (18) that H^{-1} and H_-^{-1} are, indeed, block Toeplitz. □

EXAMPLE 3. In contrast to Example 2, there is a $z \in \mathbb{P}$ (depending on μ) for which H has a more elegant block-Toeplitz structure:

$$H = R_{\ell,n}(S_0 - S_\ell) = \begin{bmatrix} A_\ell + A_\ell^* & A_{\ell-1}^* & \cdots & A_2^* & A_1* \\ A_{\ell-1} & A_\ell + A_\ell^* & A_{\ell-1}^* & \cdots & A_2^* \\ \vdots & & & & \vdots \\ A_2 & & & \ddots & A_{\ell-1}^* \\ A_1 & A_2 & \cdots & A_{\ell-1} & A_\ell + A_\ell^* \end{bmatrix}. \tag{19}$$

This is obtained with $q_z(\lambda) = 1 - \mu\lambda^\ell$. Clearly, H is nonsingular provided no eigenvalue of $L(\lambda)$ is either zero or one of the ℓth roots of μ. (Note that this applies whether ℓ is even or odd.) $\square$

EXAMPLE 4. For a μ-palindromic matrix polynomial of odd degree, say $\ell = 2k+1$, we may choose $q_z(\lambda) = \lambda^k - \mu\lambda^{k+1}$ and generate an H which is nonsingular as long as the spectrum of $L(\lambda)$ does not contain either of the points $0,\ \mu$. Thus, $H = R_{\ell,n}(S_k - \mu S_{k+1})$.

We illustrate with the case $\ell = 7$ ($k = 3$). The lower triangle of H is displayed and the matrix can be completed by Hermitian symmetry:

$$H = R_{7,n}(S_3 - \mu S_4) = \begin{bmatrix} 0 & & & & & & \\ 0 & 0 & & & & & \\ 0 & 0 & 0 & & & & \\ \mu A_0 & \mu A_1 & \mu A_2 & \mu A_3 + A_4 & & & \\ -A_0 & \mu A_0 - A_1 & \mu A_1 - A_2 & \mu A_2 & 0 & & \\ 0 & -A_0 & \mu A_0 - A_1 & \mu A_1 & 0 & 0 & \\ 0 & 0 & -A_0 & \mu A_0 & 0 & 0 & 0 \end{bmatrix}.$$

$\square$

In the sequel the choice of H will be that suggested by Examples 2 and 4. Thus, when $\ell = 2k$ we choose

$$H = \begin{cases} iR_{\ell,n}S_k & \text{when } \mu = +1, \\ R_{\ell,n}S_k & \text{when } \mu = -1. \end{cases} \tag{20}$$

and when $\ell = 2k+1$ it is assumed that $\mu \notin \sigma(L)$ and

$$H = R_{\ell,n}(S_k - \mu S_{k+1}). \tag{21}$$

4. Standard triples for μ-palindromic polynomials

We now take advantage of the well-developed theory of standard triples for matrix polynomials (see [7], for example). Recall that a triple of matrices (X,T,Y) is said to be a *standard triple* of the $n \times n$ matrix polynomial $L(\lambda) = \sum_{j=0}^{\ell} A_j\lambda^j$ with $\det A_\ell \neq 0$ if X is $n \times n\ell$, T is $n\ell \times n\ell$, Y is $n\ell \times n$, and the equality

$$L(\lambda)^{-1} = X(\lambda I - T)^{-1}Y$$

holds for every complex λ for which $L(\lambda)$ is a nonsingular matrix. We will use several properties of standard triples that are found, for instance, in [7]:

(a) A standard triple is determined uniquely by $L(\lambda)$ up to similarity: If (X_1,T_1,Y_1) and (X_2,T_2,Y_2) are two standard triples for the same $L(\lambda)$, then there exists an invertible matrix S such that

$$X_1 = X_2S, \quad T_1 = S^{-1}T_2S, \quad Y_1 = S^{-1}Y_2. \tag{22}$$

Moreover, the invertible matrix S satisfying (22) is unique.

(b) If (X,T,Y) is a standard triple for $L(\lambda)$, then Y is the unique solution of

$$\begin{bmatrix} X \\ XT \\ \vdots \\ XT^{\ell-1} \end{bmatrix} Y = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ A_\ell^{-1} \end{bmatrix}.$$

(c) One example of a standard triple is given by

$$X = \begin{bmatrix} I & 0 & \dots & 0 \end{bmatrix}, \quad T = C, \quad Y = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ A_\ell^{-1} \end{bmatrix}, \tag{23}$$

where C is the companion matrix (5) of $L(\lambda)$.

For Hermitian matrix polynomials there are standard triples which reflect the symmetry of the polynomial; see Theorem 12.2.2 of [6], for example. Analogous properties for palindromic polynomials are contained in the next two theorems.

THEOREM 7. *Let $\ell = 2k$, $L(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ with A_ℓ and A_0 nonsingular, and let H be given by* (20). *Then the following are equivalent:*

(i) *$L(\lambda)$ is μ-palindromic,*

(ii) *$C^*H = HC^{-1}$,*

(iii) *If (X,T,Y) is a standard triple for $L(\lambda)$ then so is*

$$(iY^*(T^*)^{k-1},\ (T^{-1})^*,\ i(T^*)^{k-1}X^*) \quad \textit{when } \mu = +1, \tag{24}$$

$$(Y^*(T^*)^{k-1},\ (T^{-1})^*,\ (T^*)^{k-1}X^*) \quad \textit{when } \mu = -1. \tag{25}$$

Proof. The implication $(i) \Rightarrow (ii)$ is just Theorem 6 above with a particular choice of $q_z(\lambda) \in Q_\mu$.

For the implication $(ii) \Rightarrow (iii)$ suppose first that $\mu = +1$. It is enough to consider the case when (X,T,Y) is given by (23) (with $\ell = 2k$). Since $H = H^*$ we also have $H = -i(C^*)^k A^* R_{\ell,n}$. Now consider

$$iY^*(C^*)^{k-1}H = \begin{bmatrix} 0 & \dots & 0 & A_0^{-1} \end{bmatrix} (C^*)^{2k-1}A^*R_{\ell,n} = \begin{bmatrix} 0 & \dots & 0 & A_0^{-1} \end{bmatrix} (S_{\ell-1})^*R_{\ell,n}.$$

Use equation (7) to obtain

$$iY^*(C^*)^{k-1}H = \begin{bmatrix} 0 & \dots & 0 & I \end{bmatrix} R_{\ell,n} = X. \tag{26}$$

Since (X,C,Y) is a standard triple, so is the similar triple

$$(XH^{-1},\, HCH^{-1},\, HY) = (XH^{-1},\, (C^{-1})^*,\, HY).$$

But it follows from (26) that $XH^{-1} = iY^*(C^*)^{k-1}$. So there is a standard triple

$$\left(iY^*(C^*)^{k-1},\, (C^{-1})^*,\, HY\right).$$

It only remains to show that $HY = i(C^*)^{k-1}X^*$. From (26), $X^* = -iHC^{k-1}Y$, and assumption (ii) means that $HC^{k-1} = (C^*)^{-(k-1)}H$. Thus $X^* = -i(C^*)^{-(k-1)}HY$ and, finally, $HY = i(C^*)^{k-1}X^*$, as required.

To show that $(ii) \Rightarrow (iii)$ when $\mu = -1$ it is only necessary to make adjustments to the above argument for the case $\mu = +1$ noting the change in H and showing first that $Y^*(C^*)^{k-1} = X$.

Now consider the implication $(iii) \Rightarrow (i)$. Applying a similarity transformation to the triple of (24) we see that $\left(Y^*,\, (T^{-1})^*,\, -\mu(T^*)^{2k-2}X^*\right)$ is a standard triple. So the resolvent form for $L(\lambda)$ is

$$L(\lambda)^{-1} = X(\lambda I - T)^{-1}Y = -\mu Y^*(\lambda I - (T^{-1})^*)^{-1}(T^*)^{\ell-2}X^*, \tag{27}$$

and

$$\begin{aligned}\left(\left(L(\bar{\lambda})\right)^{-1}\right)^* &= -\mu XT^{\ell-2}(\lambda I - T^{-1})^{-1}Y = -\mu XT^{\ell-1}(\lambda T - I)^{-1}Y \\ &= \mu XT^{\ell-1}(I - \lambda T)^{-1}Y.\end{aligned} \tag{28}$$

We also have

$$\lambda^{-\ell}L(\lambda^{-1})^{-1} = \lambda^{-\ell}X(\lambda^{-1}I - T)^{-1}Y = \lambda^{-\ell+1}X(I + \lambda T + \lambda^2T^2 + \cdots)Y$$

for $|\lambda| \neq 0$, $\lambda \notin \sigma(L)$. But the orthogonality relations $XT^rY = 0$ hold for $r = 0, 1, \ldots, \ell-2$ (see equation (1.3) of [7]), so

$$\lambda^{-\ell}L(\lambda^{-1})^{-1} = XT^{\ell-1}(I + \lambda T + \lambda^2T^2 + \cdots)Y = XT^{\ell-1}(I - \lambda T)^{-1}Y. \tag{29}$$

Comparing with (28) we find that $\mu\left(\left(L(\bar{\lambda})\right)^{-1}\right)^* = \lambda^{-\ell}L(\lambda^{-1})^{-1}$. But (see equation (2)) this is equivalent to the statement that $L(\lambda)$ is μ-palindromic. $\square$

Now we consider the case when the polynomial $L(\lambda)$ has odd degree, $\ell = 2k+1$. For the statement of the next theorem, as well as for use later on in the paper, we introduce the function

$$\phi(z) = z^k(1 - \mu z)^{-1}. \tag{30}$$

Note that ϕ depends on μ, and $\phi(C)$ and $\phi((C^{-1})^*)$ are well-defined (provided $\mu \notin \sigma(C)$).

THEOREM 8. *Let* $L(\lambda) = \sum_{j=0}^{2k+1} \lambda^j A_j$ *be* μ*-palindromic with* A_{2k+1} *and* A_0 *nonsingular and assume* $\mu \notin \sigma(C)$. *As in* (21), *let*

$$H = R_{\ell,n}(S_k - \mu S_{k+1}) = R_{\ell,n}AC^k(I - \mu C) = H^*. \tag{31}$$

Then the following are equivalent:

(*i*) $L(\lambda)$ *is* μ *-palindromic,*
(*ii*) $C^*H = HC^{-1}$,
(*iii*) *If* (X,T,Y) *is a standard triple for* $L(\lambda)$ *then so is*

$$\left(Y^*\phi(T^*),\ (T^{-1})^*,\ (\phi((T^{-1})^*))^{-1}X^*\right).$$

Proof. The implication $(i) \Rightarrow (ii)$ is an application of Theorem 6 with a particular choice of $q_z(\lambda)$. As in the preceding proof, for the implication $(ii) \Rightarrow (iii)$, it is enough to consider the standard triple (23) (with A_{2k} replaced by A_{2k+1}).

Since $H = H^* = R_{\ell,n}AC^k(I-\mu C)$ we also have $H = (C^*)^k(I-\mu C^*)A^*R_{\ell,n}$. Now consider

$$\begin{aligned} Y^*\phi(C^*)H &= \left[\,0 \ \ldots \ 0 \ A_0^{-1}\,\right](C^*)^k(I-\mu C^*)^{-1}(C^*)^k(I-\mu C^*)A^*R_{\ell,n} \\ &= \left[\,0 \ \ldots \ 0 \ A_0^{-1}\,\right](C^*)^{2k}A^*R_{\ell,n} \\ &= \left[\,0 \ \ldots \ 0 \ A_0^{-1}\,\right](S_{\ell-1})^*R_{\ell,n} \\ &= X. \end{aligned} \tag{32}$$

Since (X,C,Y) is a standard triple, so is the similar triple

$$(XH^{-1},\ HCH^{-1},\ HY) = (XH^{-1},\ (C^{-1})^*,\ HY).$$

But it follows from (32) that $XH^{-1} = Y^*\phi(C^*)$. So there is a standard triple

$$(Y^*\phi(C^*),\ (C^{-1})^*,\ HY),$$

and it only remains to verify that $HY = (\phi((C^{-1})^*)^{-1})X^*$. But (32) gives $X^* = H\phi(C)Y$ and since $HC^j = (C^*)^{-j}H$, it follows that $H\phi(C) = \phi((C^{-1})^*)H$ and $X^* = \phi((C^{-1})^*)HY$. Thus, $HY = \phi((C^{-1})^*)^{-1}X^*$, as required.

It remains to prove that $(iii) \Rightarrow (i)$. First, a little manipulation shows that

$$\phi((T^{-1})^*)^{-1} = -\mu(T^*)^{\ell-2}\{\phi(T^*)\}^{-1}. \tag{33}$$

Substitute this in the triple of (iii) and apply an obvious similarity to find that

$$(Y^*,\ (T^{-1})^*, -\mu(T^*)^{\ell-2}X^*)$$

is a standard triple. Hence, for the resolvent form:

$$L(\lambda)^{-1} = X(\lambda I - T)^{-1}Y = -\mu Y^*(\lambda I - (T^{-1})^*)^{-1}(T^*)^{\ell-2}X^*.$$

But this has the same form as (27) (for the even degree case). Furthermore, (29) has the same form whether ℓ is even or odd. Consequently $(iii) \Rightarrow (i)$ is proved just as in Theorem 7. $\square$

5. Unitary triples

In the case of Hermitian matrix polynomials there are standard triples of the form (X, J, PX^*) which reveal the Jordan structure of the spectrum and the intimate relationship between left and right eigenvectors. Our objective here is the derivation of corresponding canonical structures for palindromic polynomials, $L(\lambda)$. Theorems 7 and 8 suggest that there is a natural choice of scalar product in which the companion matrix, C, is unitary. It is the (indefinite) scalar product defined on $\mathbb{C}^n$ by the matrices H of (20) when $\ell = 2k$, and of (31) when $\ell = 2k+1$. Thus, $C^*HC = H$ in both cases. With this understanding, the ideas of this section apply whether ℓ is odd or even.

Let J be a Jordan form for C and write J in block diagonal form:

$$J = J_1 \oplus J_2 \oplus J_3, \tag{34}$$

where the eigenvalues of J_1 and J_2 are all those of J inside, and on the unit circle, respectively. Then J_3 has the same structure as J_1, but the eigenvalues are the images of those of J_1 under the transformation $\alpha \mapsto \overline{\alpha^{-1}}$ (in particular, they are all outside the unit circle).

Let

$$P_{\varepsilon,J} = \begin{bmatrix} 0 & 0 & P_1 \\ 0 & P_2 & 0 \\ P_1 & 0 & 0 \end{bmatrix}, \tag{35}$$

where P_1 is a block diagonal matrix formed by SIP matrices on the main diagonal (one SIP matrix for each Jordan block in J_1 with the same size as the Jordan block), and P_2 is a block diagonal matrix formed by *signed* SIP matrices on the main diagonal (one SIP matrix for each Jordan block, and of the same size as the Jordan block, of J_2). Notice that $P_{\varepsilon,J}^2 = I$ and $P_{\varepsilon,J}^* = P_{\varepsilon,J}$.

Let ω be chosen so that $|\omega| = 1$ and $I + \omega C$ is invertible, and let K_0 be obtained from J by replacing every eigenvalue λ_0 by the linear fractional transform $\frac{i(1-\omega\lambda_0)}{1+\omega\lambda_0}$. Then consider the matrix

$$K = \omega^{-1}(I + iK_0)(I - iK_0)^{-1}. \tag{36}$$

The matrix K has the following properties:

1. If J is written in the standard form: $J = \operatorname{diag}(J^{(i)})_{i=1}^q$ where, for $i = 1, 2, \ldots, q$, $J^{(i)}$ is a $k^{(i)} \times k^{(i)}$ Jordan block with eigenvalue $\lambda^{(i)}$, then
$$K = \operatorname{diag}(K^{(i)})_{i=1}^q,$$
where $K^{(i)}$ is an upper triangular matrix of size $k^{(i)} \times k^{(i)}$ with $\lambda^{(i)}$ on the main diagonal.
2. Each matrix $K^{(i)}$ is similar to $J^{(i)}$ (so that K has the same eigenvalues and partial multiplicities as J).
3. K and J have the same lattice of invariant subspaces.

THEOREM 9. *If C is H-unitary, then there is a nonsingular matrix S reducing H and C simultaneously to the forms:*

$$(S^*)^{-1}HS^{-1} = P_{\varepsilon,J}, \qquad SCS^{-1} = K. \tag{37}$$

Proof. It is easily verified that $T := i(I-\omega C)(I+\omega C)^{-1}$ is H-selfadjoint. The canonical forms for selfadjoint matrices in an indefinite inner product (Theorem 5.1.1 of [6]) implies that there is an invertible S such that

$$H = S^*\widehat{P}_{\varepsilon,Z}S, \qquad T = S^{-1}ZS \tag{38}$$

where Z is a Jordan form of T, and where $\widehat{P}_{\varepsilon,Z}$ is constructed analogously to (35), taking advantage of the symmetry of the eigenvalues of Z relative to the real axis; thus, rather than using partition (34), one uses the partition

$$Z = Z_1 \oplus Z_2 \oplus Z_3,$$

where the eigenvalues of Z_1 have positive imaginary parts, the eigenvalues of Z_2 are real, and Z_3 is obtained from Z_1 by replacing each eigenvalue with its complex conjugate. But, with K_0 as defined above (and noting property 2 above), it follows that $Z = K_0$, and we obtain that in fact $\widehat{P}_{\varepsilon,Z} = P_{\varepsilon,J}$. Now

$$C = \omega^{-1}(I+iT)(I-iT)^{-1} = S^{-1}\left(\omega^{-1}(I+iZ)(I-iZ)^{-1}\right)S = S^{-1}KS.$$

□

Observe that, since $HC = (C^{-1})^*H$, (37) implies

$$\begin{aligned} P_{\varepsilon,Z}K &= ((S^{-1})^*HS^{-1})(SCS^{-1}) = (S^{-1})^*HCS^{-1} = (S^{-1})^*((C^{-1})^*H)S^{-1} \\ &= ((S^{-1})^*)(C^{-1})^*S^*)((S^{-1})^*HS^{-1}) = (K^{-1})^*P_{\varepsilon,Z}. \end{aligned}$$

Thus, $K^*P_{\varepsilon,Z}K = P_{\varepsilon,Z}$, i.e. K is $P_{\varepsilon,Z}$-unitary.

DEFINITION 4. A standard triple (X, T, Y) for a μ-palindromic polynomial $L(\lambda)$ is said to be a *unitary triple* if $T = K$, where K is defined by (36), and

$$X = \begin{cases} iY^*(K^*)^{k-1}P_{\varepsilon,J} & \text{when } \ell = 2k \text{ and } \mu = +1\,, \\ -Y^*(K^*)^{k-1}P_{\varepsilon,J} & \text{when } \ell = 2k \text{ and } \mu = -1\,, \\ \mu Y^*\phi(K^*)P_{\varepsilon,J} & \text{when } \ell = 2k+1\,. \end{cases} \tag{39}$$

Here, the function ϕ is defined by (30).

The following theorem implies, in particular, that unitary triples do exist:

THEOREM 10. *Let $L(\lambda)$ be a μ-palindromic matrix polynomial with nonsingular leading coefficient. Assume that a standard triple (X, T, Y) for $L(\lambda)$ is given (using* (23)*) in the form*

$$X = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} S^{-1}, \qquad T = SCS^{-1}, \qquad Y = S\begin{bmatrix} 0 \\ \vdots \\ 0 \\ A_\ell^{-1} \end{bmatrix}, \tag{40}$$

where S is a matrix reducing H (of (20) *or* (31)*) and C to canonical form, as in* (37)*. In the case of odd degree ℓ, assume in addition that $\mu \notin \sigma(C)$. Then (X, T, Y) is a unitary triple for $L(\lambda)$.*

Equation (40) says that (X,T,Y) is similar to the triple of (23) and is therefore a standard triple. It is, of course, the special choice of transforming matrix S which ensures that (X,T,Y) is a *unitary* triple.

Proof. First consider the case that ℓ is even: $\ell = 2k$. It follows immediately from (37) that $T = K$ and, from the definition of a unitary triple it remains to show that $X = iY^*(K^*)^{k-1}P_{\varepsilon,J}$ and $X = -Y^*(K^*)^{k-1}P_{\varepsilon,J}$ according as $\mu = +1$ and $\mu = -1$, respectively.

Using (40),

$$Y^*(K^*)^{k-1}P_{\varepsilon,J} = \begin{bmatrix} 0 & \dots & 0 & (A_\ell^{-1})^* \end{bmatrix} S^*(K^*)^{k-1}P_{\varepsilon,J}. \tag{41}$$

Now (37) implies $K^* = (S^{-1})^*C^*S^*$ and hence $S^*(K^*)^{k-1} = (C^*)^{k-1}S^*$. But $P_{\varepsilon,J} = (S^{-1})^*HS^{-1}$, so $S^*(K^*)^{k-1} = (C^*)^{k-1}HS^{-1}P_{\varepsilon,J}$ and, since $C^*H = HC^{-1}$,

$$S^*(K^*)^{k-1}P_{\varepsilon,J} = (C^*)^{k-1}HS^{-1} = HC^{-(k-1)}S^{-1}.$$

Using this in (41),

$$Y^*(K^*)^{k-1}P_{\varepsilon,J} = \begin{bmatrix} 0 & \dots & 0 & \mu A_0^{-1} \end{bmatrix} HC^{-(k-1)}S^{-1}. \tag{42}$$

Use the definition of H for the case $\mu = +1$ to obtain,

$$\begin{aligned} Y^*(K^*)^{k-1}P_{\varepsilon,J} &= i\begin{bmatrix} 0 & \dots & 0 & A_0^{-1} \end{bmatrix} R_{\ell,n}S_kC^{-(k-1)}S^{-1} \\ &= i\begin{bmatrix} 0 & \dots & 0 & A_0^{-1} \end{bmatrix} R_{\ell,n}S_1S^{-1} \\ &= -i\begin{bmatrix} I & 0 & \dots & 0 \end{bmatrix} S^{-1} \quad \text{(using definitions of } R_{\ell,n},\ S_1) \\ &= -iX. \end{aligned}$$

Thus, $iY^*(K^*)^{k-1}P_{\varepsilon,J} = X$ and $(X,\ T,\ Y)$ is indeed a unitary triple. The case $\mu = -1$ requires a simple modification of H at the last step and, in the same way, leads to $-Y^*(K^*)^{k-1}P_{\varepsilon,J} = X$. This concludes the argument for even ℓ.

Now consider the case of ℓ odd: $\ell = 2k+1$. We have to show that $Y^*\phi(K^*)P_{\varepsilon,J} = \mu X$. The preceding line of argument can be followed (with $\phi(z)$ in place z^{k-1}) up to and including equation (42), which now takes the form

$$Y^*\phi(K^*)P_{\varepsilon,J} = \begin{bmatrix} 0 & \dots & 0 & \mu A_0^{-1} \end{bmatrix} H\phi(C^{-1})S^{-1}.$$

Substituting $H = R_{\ell,n}(S_k - \mu S_{k+1})$, we obtain

$$\begin{aligned} Y^*\phi(K^*)P_{\varepsilon,J} &= \begin{bmatrix} 0 & \dots & 0 & \mu A_0^{-1} \end{bmatrix} R_{\ell,n}AC^k(I-\mu C)\{-\mu C^{-(k-1)}(I-\mu C)^{-1}\}S^{-1} \\ &= -\begin{bmatrix} 0 & \dots & 0 & A_0^{-1} \end{bmatrix} R_{\ell,n}ACS^{-1} \\ &= -\begin{bmatrix} 0 & \dots & 0 & A_0^{-1} \end{bmatrix} R_{\ell,n}S_1S^{-1} \\ &= \begin{bmatrix} I & 0 & \dots & 0 \end{bmatrix} S^{-1} = X. \end{aligned}$$

Thus, whether ℓ is even or odd, $(X,\ T,\ Y)$ of (40) forms a unitary triple. $\square$

The next result shows that the nonsingular matrices commuting with the canonical matrix K generate a class of unitary triples by similarity.

THEOREM 11. *Let $L(\lambda)$ be a μ-palindromic polynomial with a unitary triple (X, K, Y), and let V be a nonsingular matrix which commutes with K (so that $K = V^{-1}KV$) and is $P_{\varepsilon,J}$-unitary (so that $V^*P_{\varepsilon,J}V = P_{\varepsilon,J}$). Then the standard triple*

$$(XV,\ K = V^{-1}KV,\ V^{-1}Y)$$

is also unitary.

Proof. Let $\widehat{X} = XV$ and $\widehat{Y} = V^{-1}Y$, and first let the polynomial have even degree $\ell = 2k$, and assume $\mu = 1$. We have only to prove that $\widehat{X} = i\widehat{Y}^*(K^*)^{k-1}P_{\varepsilon,J}$.

Since (X, K, Y) is unitary, $X = iY^*(K^*)^{k-1}P_{\varepsilon,J}$. Then

$$\widehat{X}V^{-1} = iY^*(K^*)^{k-1} = i\widehat{Y}^*V^*(K^*)^{k-1}P_{\varepsilon,J} = i\widehat{Y}^*(K^*)^{k-1}V^*P_{\varepsilon,J},$$

where the first and the second equalities follow in view of $Y = V\widehat{Y}$ and of $K^*V^* = V^*K^*$, respectively. Thus $\widehat{X} = i\widehat{Y}^*(K^*)^{k-1}(V^*P_{\varepsilon,J}V)$ and, since V is $P_{\varepsilon,J}$-unitary, $\widehat{X} = i\widehat{Y}^*(K^*)^{k-1}P_{\varepsilon,J}$, as required.

Clearly, this argument is easily adapted to show that the same result holds when $\mu = -1$. Furthermore, for polynomials of odd degree, a similar argument applies with the function $i(K^*)^{k-1}$ replaced by $\phi(K^*)$. $\square$

The results of Theorem 9 imply that the linearization $L_p(\lambda) := H\lambda - HC$ of the μ-palindromic matrix polynomial $L(\lambda)$ is *congruent* to the matrix pencil

$$P_{\varepsilon,J}\lambda - P_{\varepsilon,J}K = (S^*)^{-1}L_p(\lambda)S^{-1}.$$

We also know that for a standard triple (X, J, Y) of $L(\lambda)$ with J in the Jordan form, we have $Q^{-1}CQ = J$, where

$$Q = \begin{bmatrix} X \\ XJ \\ \vdots \\ XJ^{\ell-1} \end{bmatrix}.$$

We conclude this section with remarks on the relationship between the matrix S and the matrix Q of eigen/principal vectors of C.

Let $J = \Lambda + N \equiv \operatorname{diag}(J^{(i)})_{i=1}^q$ be the Jordan matrix of C, where Λ is the diagonal matrix of the eigenvalues and N is nilpotent. As before we assume that $I_{n\ell} + \omega C$ is nonsingular for some $\omega \in \mathbb{C}$ with $|\omega| = 1$, and it will be useful to define the functions ψ and its inverse ψ^{-1} by

$$\psi(\lambda) = i\frac{(1-\omega\lambda)}{(1+\omega\lambda)}, \qquad \psi^{-1}(\lambda) = \frac{1}{\omega}\frac{(1+i\lambda)}{(1-i\lambda)}.$$

Then, by definition (see equation (36)), $K_0 = \psi(\Lambda) + N$ and $K = \psi^{-1}(K_0)$. Now consider the upper triangular matrix

$$\psi(J) = i(I_{n\ell} - \omega J)(I_{n\ell} + \omega J)^{-1}.$$

There exist a nonsingular matrix X_ψ such that

$$(X_\psi)^{-1}\psi(J)X_\psi = J_\psi := \psi(\Lambda) + N = K_0 \tag{43}$$

is the Jordan matrix of $\psi(J)$. Note that the nilpotent matrix N is the same as for J. For the matrix K we now have

$$K = \psi^{-1}(K_0) = \psi^{-1}(J_\psi) = \psi^{-1}((X_\psi)^{-1}\psi(J)X_\psi) = (X_\psi)^{-1}JX_\psi.$$

Thus, K and J are similar. From (37) we have $SCS^{-1} = K$ and so $(X_\psi S)C(X_\psi S)^{-1} = J$. We have established:

PROPOSITION 12. *With X_ψ and S defined as in* (43) *and* (37)*, we have that $Q := (X_\psi S)^{-1}$ is a matrix of eigen/principal vectors of C.*

It is interesting that X_ψ has an explicit form. To derive this form recall that, for $i = 1,...,q$, the matrix $J^{(i)}$ is of size $k^{(i)} \times k^{(i)}$, and define $k_m := \sum_{i=1}^{m} k^{(i)}$. In particular, we have $k_1 = k^{(1)}$ and $k_q = n\ell$. Let us abbreviate $V_i := \psi(J) - \psi(\lambda_i)I_{n\ell}$ and denote the k-th column of $I_{n\ell}$ by e_k. Then

$$\begin{aligned} X_\psi = [V_1^{k^{(1)}-1}e_{k_1}, \cdots, V_1e_{k_1}, e_{k_1}, V_2^{k^{(2)}-1}e_{k_2}, \cdots, V_2e_{k_2}, e_{k_2}, \cdots \\ \cdots, V_q^{k^{(q)}-1}e_{k_q}, \cdots, V_qe_{k_q}, e_{k_q}]\Omega, \end{aligned}$$

where Ω is a nonsingular diagonal matrix to be used for an appropriate normalization. So if we solve the eigenvalue problem $CQ = QJ$ then $QX_\psi = S^{-1}$. Note, that in the semisimple case the Jordan matrix J is diagonal and hence $K = J$ and $X_\psi = I_{n\ell}$ which implies $S^{-1} = Q$.

6. Perturbation theory for difference equations

In this section we apply the results of Section 5 to study stable boundedness (see the definition below) properties of the following difference equation:

$$A_0x_i + A_1x_{i+1} + \cdots + A_\ell x_{i+\ell} = 0, \quad i = 0, 1, \ldots, \tag{44}$$

where $\{x_i\}_{i=0}^{\infty}$ is a sequence of vectors in $\mathbb{C}^n$ to be determined, and $A_0, \ldots, A_\ell$ are given matrices in $\mathbb{C}^{n\times n}$. We study this equation under the hypotheses that the polynomial $L(\lambda) = \sum_{j=0}^{\ell} A_j\lambda^j$ is μ-palindromic, where $\mu = 1$ or $\mu = -1$, and the leading coefficient A_ℓ is nonsingular. These hypotheses will apply throughout this section. In addition, in case ℓ is odd, we assume that $\mu \notin \sigma(L)$, to ensure full applicability of Theorem 8 (see Example 4).

Difference equations of this kind have been studied in the context of spectral theory of matrix polynomials [8], [3], also matrix polynomials with symmetries [5], [6], and see [17] for a comprehensive treatment of first degree equations (44) with various symmetries.

We say that equation (44) is *bounded*, if every solution sequence is bounded. A criterion for boundedness will be given in the next theorem in terms of the following concept:

DEFINITION 5. A matrix polynomial $L(\lambda)$ is said to have *simple structure with respect to the unit circle* if the spectrum $\sigma(L)$ lies on the unit circle and all elementary divisors of $L(\lambda)$ are linear.

This condition implies that, for every $\lambda_0 \in \sigma(L)$, the multiplicity of λ_0 as a zero of $\det(L(\lambda))$ coincides with the dimension of the kernel of $L(\lambda_0)$.

THEOREM 13. *Let the matrix polynomial $L(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ have μ-palindromic symmetry with A_ℓ nonsingular. Then equation* (44) *is bounded if and only if $L(\lambda)$ has simple structure with respect to the unit circle.*

This theorem was proved in [6] (Theorem 13.7.3) for the case of even degree ℓ and $\mu = 1$. In the present more general context the proof is essentially the same, but uses the fact that the companion matrix of a μ-palindromic matrix polynomial is H-unitary for a suitable nonsingular Hermitian matrix H (Theorems 7 and 8 above). The details are omitted.

We are also interested in the case when equation (44) is *stably bounded*, i.e., all solutions of (44) are bounded and, in addition, all solutions of every system

$$\widetilde{A}_0 y_i + \widetilde{A}_1 y_{i+1} + \cdots + \widetilde{A}_\ell y_{i+\ell} = 0, \quad i = 0, 1, \ldots,$$

with

$$\widetilde{A}_j^* = \mu \widetilde{A}_{\ell-j}, \qquad j = 0, 1, \ldots, \ell,$$

are bounded provided $\|\widetilde{A}_j - A_j\|$ is small enough (for $j = 0, 1, \ldots, \ell$). To characterize stably bounded equations, it will be convenient to treat the cases of even and odd degrees separately.

THEOREM 14. *Let $L(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ be a μ-palindromic matrix polynomials of even degree $\ell = 2k$ with A_ℓ nonsingular. Then the following statements are equivalent:*

(a) *Equation* (44) *is stably bounded.*

(b) *For every eigenvalue λ_0 of the companion matrix C, the quadratic form*

$$(x, i^{(1+\mu)/2} R_{\ell,n} S_k x), \quad x \in \operatorname{Ker}(C - \lambda_0 I), \tag{45}$$

is either positive definite or negative definite.

(c) *the spectrum of $L(\lambda)$ lies on the unit circle, and the quadratic form*

$$\left(x, i^{(1+\mu)/2} \lambda_0 \left(\lambda^{-\ell/2} L(\lambda)\right)^{(1)} (\lambda_0) x\right), \quad x \in \operatorname{Ker} L(\lambda_0) \tag{46}$$

is either positive definite or negative definite, for every $\lambda_0 \in \sigma(L)$.

Proof. For the case $\mu = 1$, Theorem 14 was established in [4], see also [5] and, in particular, [6, Theorem 13.8.1].

Assume now that $\mu = -1$. Since C is $R_{\ell,n} S_k$-unitary (Theorem 7), the implication (b) $\Longrightarrow$ (a) follows from the general perturbation theory of unitary matrices with respect to an indefinite inner product (see [5], [6]).

Next, for each $x \in \operatorname{Ker} L(\lambda_0)$, write

$$\widehat{x} = \begin{bmatrix} x \\ \lambda_0 x \\ \vdots \\ \lambda_0^{\ell-1} x \end{bmatrix}.$$

Then $\widehat{x} \in \operatorname{Ker}(C - \lambda_0 I)$ and, conversely, every vector in $\operatorname{Ker}(C - \lambda_0 I)$ has the form $\widehat{x}$ for some $x \in \operatorname{Ker} L(\lambda_0)$. A computation shows that if, in addition, $|\lambda_0| = 1$, then

$$\left(x, \lambda_0 \left(\lambda^{-\ell/2} L(\lambda)\right)^{(1)} (\lambda_0) y\right) = (\widehat{x}, R_{\ell,n} S_k \widehat{y}), \quad x, y \in \operatorname{Ker} L(\lambda_0) \tag{47}$$

(cf. formula (12.6.31) in [6]). Now, the spectral theory of unitary matrices with respect to indefinite inner products (see, e.g., [6, Section 4.3]) shows that, in view of the definiteness of the quadratic form (45), all eigenvalues of C, or equivalently of $L(\lambda)$, must be unimodular. Thus, using (47), we find that (b) and (c) are equivalent.

It remains to prove that $(a) \implies (c)$ in the case $\mu = -1$. We follow the line of proof used in [6, Theorem 13.8.1] which, in turn, goes back to [4]. Let $\omega \in \mathbb{C}$, $|\omega| = 1$, and define the matrix polynomials

$$L_\omega(\lambda) := \omega^{\ell/2} L(\omega\lambda)$$

and

$$R(\lambda) := (1 - i\lambda)^\ell L_\omega \left(\frac{1 + i\lambda}{1 - i\lambda}\right).$$

Select ω so that the leading coefficient of $R(\lambda)$ is invertible. A straightforward computation, using the palindromic property of $L(\lambda)$ (with $\mu = -1$) shows that $R(\lambda)^* = -R(\lambda)$. Now repeat the arguments in the proof of [6, Theorem 13.8.1] using the matrix polynomial $iR(\lambda)$ (with Hermitian coefficients) instead of $R(\lambda)$. $\square$

We are not aware of a direct analogue to Theorem 14 for palindromic matrix polynomials of odd degree. However, a sufficient condition for (44) to be stably bounded can be given:

PROPOSITION 15. *Let $L(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ be a μ-palindromic matrix polynomials of odd degree $\ell = 2k + 1$ and with nonsingular A_ℓ. Assume that for every eigenvalue λ_0, the quadratic form*

$$(x, R_{\ell,n}(S_k - \mu S_{k+1})x), \quad x \in \operatorname{Ker}(C - \lambda_0 I), \tag{48}$$

is either positive definite or negative definite. Then equation (44) is stably bounded.

There is a proof of Proposition 15 similar to that of Theorem 14 in which Theorem 8 is used, together with the general perturbation theory of matrices which are unitary with respect to an indefinite inner product.

We do not know whether or not the converse statement of Proposition 15 holds. In other words, whether the definiteness of the quadratic forms (48) is also necessary for stable boundedness.

We note also that one can use Proposition $1(a)$ to obtain from Theorem 14 a sufficient condition for stable boundedness of difference equations (44) corresponding to palindromic matrix polynomials of odd degree $2k+1$. Indeed, if the difference equation corresponding to a palindromic matrix polynomial of the form (3) is stably bounded, then the stable boundedness holds also for the difference equation corresponding to the palindromic matrix polynomial $L(\lambda)$ (but the converse is generally false). Thus, the statement (c) in Theorem 14 applied to the polynomial $M_\alpha(\lambda)$, for some unimodular α, gives a sufficient condition for the stable boundedness of the difference equation associated with $L(\lambda)$. It should be noted however that this condition is rather restrictive.

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