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ALGEBRAIC PROPERTIES OF TRUNCATED TOEPLITZ OPERATORS

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In memory of Paul R. Halmos

(*communicated by L. Rodman*)

Abstract. Compressions of Toeplitz operators to coinvariant subspaces of H^2 are studied. Several characterizations of such operators are obtained; in particular, those of rank one are described. The paper is partly expository. Open questions are raised.

1. Introduction

Our setting is the open unit disk, $\mathcal{D}$, in the complex plane, $\mathcal{C}$. By H^2 is meant the standard Hardy space, the Hilbert space of holomorphic functions in $\mathcal{D}$ having square-summable Taylor coefficients at the origin. As usual, H^2 will be identified with its space of boundary functions, the subspace of L^2 (of normalized Lebesgue measure m on $\partial\mathcal{D}$) consisting of the functions whose Fourier coefficients with negative indices vanish.

A Toeplitz operator is the compression to H^2 of a multiplication operator on L^2. The operators of the paper's title are compressions of multiplication operators to proper invariant subspaces of the backward shift operator on H^2. While many such operators have been studied, with interesting results, there seems not to exist a treatment of them as a class, comparable, for example, to the treatment of Toeplitz operators by A. Brown and P. R. Halmos in [8], a paper which inspired much subsequent work because of the questions it raised. The present paper aims to play a role for truncated Toeplitz operators like the one the Brown–Halmos paper did for Toeplitz operators. An effort has been exerted to make the paper reasonably self-contained. Accordingly, proofs will be provided for much of the background material.

Some preparation is needed prior to precise definitions. We let P denote the orthogonal projection on L^2 with range H^2. The operator P is given explicitly as a Cauchy integral:

$$(Pf)(z) = \int \frac{f(\zeta)}{1-\bar{\zeta}z}dm(\zeta), \ |z|<1.$$

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Using the expression on the right side, we can extend P to an operator on L^1. In this incarnation, P maps L^1 into $H(\mathcal{D})$, the space of holomorphic functions in $\mathcal{D}$, and is continuous relative to the weak topology of L^1 and the topology of locally uniform convergence of $H(\mathcal{D})$.

We shall need to deal with certain unbounded Toeplitz operators. For φ in L^2, the operator T_φ, the Toeplitz operator on H^2 with symbol φ, is defined by

$$T_\varphi f = P(\varphi f).$$

It is known that T_φ is bounded if and only if φ is in L^∞, in which case $\|T_\varphi\| = \|\varphi\|_\infty$. In view of the remarks in the preceding paragraph, if φ is not in L^∞, we can, according to convenience, regard T_φ either as an unbounded operator on H^2, with domain containing H^∞, or as a transformation from H^2 to $H(\mathcal{D})$. In the latter guise T_φ is continuous relative to the weak topology of H^2 and the topology of locally uniform convergence of $H(\mathcal{D})$.

We let S denote the unilateral shift operator on H^2. Its adjoint, the backward shift, is given by

$$(S^*f)(z) = \frac{f(z) - f(0)}{z}.$$

Both S and S^* are well defined as operators on $H(\mathcal{D})$, and are continuous with respect to the topology of locally uniform convergence.

For φ in L^2 one has $S^*T_\varphi S = T_\varphi$. In fact, because of the continuity of T_φ and of $S^*T_\varphi S$ as maps from H^2 to $H(\mathcal{D})$, to establish the equality it suffices to show that $S^*T_\varphi Sf = T_\varphi f$ for all f in H^∞. For such an f, and for any g in H^2,

$$\begin{aligned} \langle S^*T_\varphi Sf, g\rangle &= \langle T_\varphi Sf, Sg\rangle = \langle \varphi Sf, Sg\rangle \\ &= \int \varphi(\zeta)\zeta f(\zeta)\overline{\zeta g(\zeta)}dm(\zeta) \\ &= \int \varphi f \bar{g} dm = \langle T_\varphi f, g\rangle, \end{aligned}$$

implying that $S^*T_\varphi Sf = T_\varphi f$, as desired.

For the remainder of the paper, u will denote a nonconstant inner function. The subspace $K_u^2 = H^2 \ominus uH^2$ is a proper nontrivial invariant subspace of S^*, the most general one by the well-known theorem of A. Beurling. The orthogonal projection on L^2 with range K_u^2 will be denoted by P_u. Like P, the projection P_u can be extended to a map from L^1 to $H(\mathcal{D})$. In fact, letting M_u and $M_{\bar{u}}$ denote the multiplication operators on L^2 induced by u and $\bar{u}$, we have $P_u = P - M_u P M_{\bar{u}}$. The operator $M_{\bar{u}}$ acts isometrically from L^1 to L^1, so $PM_{\bar{u}}$ sends L^1 into $H(\mathcal{D})$. As M_u maps $H(\mathcal{D})$ into itself, the operator $M_u P M_{\bar{u}}$, and hence also P_u, can be regarded as an operator from L^1 to $H(\mathcal{D})$, as asserted.

For φ in L^2, the truncated Toeplitz operator on K_u^2 with symbol φ is the operator A_φ defined by

$$A_\varphi f = P_u(\varphi f).$$

In the special case $u(z) = z^N$, the matrix for A_φ with respect to the monomial basis is the N-by-N Toeplitz matrix formed in the standard way from the Fourier coefficients

$\hat{\varphi}(-N+1),\ldots,\hat{\varphi}(N-1)$ ($\hat{\varphi}(0)$ on the main diagonal, $\hat{\varphi}(1)$ on the diagonal just below the main diagonal, $\hat{\varphi}(-1)$ on the diagonal just above the main diagonal, etc.).

Clearly, a truncated Toeplitz operator does not have a unique symbol; for example, $A_u = 0$. It will be shown in Section 3. that two symbols correspond to the same operator if and only if their difference lies in $uH^2 + \bar{u}\bar{H}^2$.

A truncated Toeplitz operator obviously is bounded if it has a symbol in L^∞. Does every bounded truncated Toeplitz operator possess an L^∞ symbol? Unfortunately, I have been unable to answer this basic question. It is discussed further, along with other open questions, in the exposition to follow.

Just as in the Toeplitz case, an unbounded truncated Toeplitz operator can be thought of in two ways: either as an unbounded operator on K_u^2 whose domain contains $K_u^2 \cap H^\infty$ (hereafter denoted by K_u^∞), or as an operator from K_u^2 to $H(\mathcal{D})$, continuous relative to the weak topology of K_u^2 and the topology of locally uniform convergence of $H(\mathcal{D})$. This study focuses on bounded truncated Toeplitz operators. The set of those operators will be denoted by $\mathscr{T}(K_u^2)$. It is clearly a complex vector space, and is closed under conjugation (since $A_\varphi^* = A_{\bar{\varphi}}$).

The compression of S to K_u^2 will be denoted by S_u. Its adjoint, S_u^*, is the restriction of S^* to K_u^2. The operators S_u and S_u^* are the truncated Toeplitz operators with symbols z and $\bar{z}$, respectively.

Brown and Halmos prove in [8] that a bounded operator T on H^2 is a Toeplitz operator if and only if $S^*TS = T$. There is an analogous but more involved characterization of bounded truncated Toeplitz operators, given in Section 4., which forms the basis for most of what follows. It states, roughly, that a bounded operator A on K_u^2 belongs to $\mathscr{T}(K_u^2)$ if and only if $A - S_u^*AS_u$ (or, equivalently, $A - S_uAS_u^*$) is an operator of rank at most 2 of a special kind. It will also be shown in Section 4. that $\mathscr{T}(K_u^2)$ is closed in the weak operator topology.

As shown in [8], the only compact Toeplitz operator is the zero operator. In contrast, $\mathscr{T}(K_u^2)$ contains many compact operators, in fact, many finite-rank operators. The operators in $\mathscr{T}(K_u^2)$ of rank one are determined in Section 5.. In Section 6. some additional finite-rank operators in $\mathscr{T}(K_u^2)$ are displayed. A characterization of the general such operator is an open problem.

Section 7. contains a few results for the case where K_u^2 is finite dimensional. Even the case $\dim K_u^2 = 2$ is of interest. Some open questions will be mentioned.

In Section 8. it is shown that a bounded operator on K_u^2 lies in $\mathscr{T}(K_u^2)$ if and only if it is shift invariant, in a sense to be defined. Section 9. introduces certain measures on $\partial\mathcal{D}$ that induce operators in $\mathscr{T}(K_u^2)$.

Section 10. introduces a family of rank-one perturbations of the compressed shift S_u. Among them are unitary perturbations originally analyzed by D. N. Clark [10]. Section 11. contains background on Clark's work, and Section 12. discusses the role of his unitaries in $\mathscr{T}(K_u^2)$.

Chapter 13 introduces transforms due to R. B. Crofoot which map K_u^2 onto $K_{u_w}^2$, where $u_w = \frac{u-w}{1-\bar{w}u}$ with w a point of $\mathcal{D}$. Crofoot's transforms are used in the concluding Section 14. to help determine the spectra of the operators introduced in Section 10..

The next section contains most of the needed background on the spaces K_u^2.

In addition to the notations already introduced, the following ones will be used throughout:

- The identity operator will be denoted by I. The space on which it acts will be clear from the context.
- The inner product and norm in L^2 will be denoted by $\langle \cdot,\cdot \rangle$ and $\|\cdot\|_2$.
- The usual tensor notation will be used for operators of rank one: for f and g vectors in a Hilbert space, $f\otimes g$ denotes the operator defined by $(f\otimes g)h = \langle h,g\rangle f$.

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2. Background on K_u^2

Nothing in this section is new, and the bulk of it can be found in standard sources, for example [7], [13], [16].

2.1. Kernel Functions. For λ in $\mathcal{D}$, the kernel function in H^2 for the functional of evaluation at λ will be denoted by k_λ; it is given explicitly by $k_\lambda(z) = 1/(1-\bar{\lambda}z)$, and satisfies $S^*k_\lambda = \bar{\lambda}k_\lambda$. More generally, for χ in H^2 one has $T_{\bar{\chi}}k_\lambda = \overline{\chi(\lambda)}k_\lambda$; in fact, for f in H^2,

$$\begin{aligned}\langle T_{\bar{\chi}}k_\lambda, f\rangle &= \langle k_\lambda, \chi f\rangle = \overline{\chi(\lambda)f(\lambda)}\\ &= \langle \overline{\chi(\lambda)}k_\lambda, f\rangle.\end{aligned}$$

The kernel function in K_u^2 for the functional of evaluation at λ will be denoted by k_λ^u; it equals $P_u k_\lambda$. As noted in Section 1., $P_u = P - M_u P M_{\bar{u}}$. Since $PM_{\bar{u}} \mid H^2 = T_{\bar{u}}$, we have $P_u \mid H^2 = I - T_u T_{\bar{u}}$, and accordingly $k_\lambda^u = k_\lambda - \overline{u(\lambda)}uk_\lambda$, i.e.,

$$k_\lambda^u(z) = \frac{1-\overline{u(\lambda)}u(z)}{1-\bar{\lambda}z}.$$

Note that the kernel functions k_λ^u belong to K_u^∞. Since their linear span is clearly dense in K_u^2, we see that K_u^∞ is dense in K_u^2.

2.2. Angular Derivatives. Recall that the function u is said to have an angular derivative in the sense of Carathéodory (an ADC) at the point η of $\partial\mathcal{D}$ if u has a nontangential limit $u(\eta)$ of unit modulus at η, and u' has a nontangential limit $u'(\eta)$ at η (equivalently, the difference quotient $(u(z)-u(\eta))/(z-\eta)$ has the nontangential limit $u'(\eta)$ at η). Information on this notion can be found, for example, in [16]. If η is a regular point of u on $\partial\mathcal{D}$ then u has not only an ADC at η but, obviously, an ordinary derivative there. A characterization, in terms of the zeros of u and the singular measure of its singular part, for u to have an ADC at a singularity on $\partial\mathcal{D}$ can be found in the papers [1], [2] of P. R. Ahern and D. N. Clark. Alternative characterizations can be found, for example, in [16].

The function u has an ADC at η if and only if every function in K_u^2 has a nontangential limit at η. Then the limit defines a bounded linear functional on K_u^2 (see [16], for example, for details). There is thus a corresponding kernel function k_η^u. Moreover, $k_\lambda^u \to k_\eta^u$ as λ tends to η nontangentially from $\mathcal{D}$. From the expression found earlier for k_λ^u, one obtains

$$k_\eta^u(z) = \frac{1 - \overline{u(\eta)}u}{1 - \bar{\eta}z},$$

which can be written

$$k_\eta^u(z) = \eta\overline{u(\eta)}\left(\frac{u - u(\eta)}{z - \eta}\right).$$

It is known that if η is a point of $\partial\mathcal{D}$ at which u has a nontangential limit $u(\eta)$ of unit modulus, and if the difference quotient $\frac{u-u(\eta)}{z-\eta}$ is in H^2, then u has an ADC at η (see, for example, [16]).

2.3. Conjugation. The space K_u^2, as is well known, carries a natural conjugation, an antiunitary involution C, defined by $(Cf)(\zeta) = u(\zeta)\overline{\zeta f(\zeta)}$ $(|\zeta| = 1)$. In fact, the map on L^2 defined by the same equality is clearly antiunitary and involutive, and is easily seen to map uH^2 to $\bar{H}_0^2$ and $\bar{H}_0^2$ to uH^2. It thus preserves K_u^2.

When convenient we shall write $\widetilde{f}$ for Cf. The expression

$$\widetilde{k}_\lambda^u(z) = \frac{u(z) - u(\lambda)}{z - \lambda}$$

is easily verified. In particular, $\widetilde{k}_0^u = S^*u$.

If u has an ADC at the point η of $\partial\mathcal{D}$, then one obtains the expression for $\widetilde{k}_\eta^u$ by replacing λ by η in the equality above:

$$\widetilde{k}_\eta^u = \frac{u(z) - u(\eta)}{z - \eta}.$$

Thus $\widetilde{k}_\eta^u = \bar{\eta}u(\eta)k_\eta^u$ (see Section 2.2.).

An operator A on K_u^2 is called C-symmetric if $CAC = A^*$. S. R. Garcia and M. Putinar in [12] study the notion of C-symmetry in the abstract. They give many examples, including our truncated Toeplitz operators (at least those with bounded symbols). The following result is essentially theirs.

LEMMA 2.1. [12] *The operators in $\mathscr{T}(K_u^2)$ are C-symmetric.*

Proof. Let φ be in L^2 with A_φ bounded. For f in K_u^∞ and g in K_u^2,

$$\begin{aligned}
\langle CA_\varphi Cf, g\rangle &= \langle Cg, A_\varphi Cf\rangle \\
&= \int u(\zeta)\bar{\zeta}\,\overline{g(\zeta)} \cdot \overline{\varphi(\zeta)}\,\overline{u(\zeta)}\zeta f(\zeta)dm(\zeta) \\
&= \int \bar{\varphi}f\,\bar{g}dm = \langle A_{\bar{\varphi}}f, g\rangle \\
&= \langle A_\varphi^* f, g\rangle.
\end{aligned}$$

Since K_u^∞ is dense in K_u^2, the desired conclusion follows. □

It will be shown later, in Section 5., that, for $\dim K_u^2 > 2$, not all C-symmetric operators are in $\mathscr{T}(K_u^2)$. At this point it is easy to handle the case $u(z) = z^N$, $N \geqslant 3$. For this u, the conjugation sends the monomial z^k to z^{N-k-1}. Suppose A is the operator on K_u^2 having the matrix $(a_{jk})_{j,k=0}^{N-1}$ with respect to the monomial basis: $a_{jk} = \langle Az^k, z^j\rangle$. The $(j,k)^{th}$ entry of the matrix for CAC is then equal to

$$\begin{aligned}\langle CACz^k, z^j\rangle &= \langle Cz^j, ACz^k\rangle \\ &= \langle z^{N-j-1}, Az^{N-k-1}\rangle = \bar{a}_{(N-j-1)(N-k-1)},\end{aligned}$$

whereas $\langle A^*z^k, z^j\rangle = \bar{a}_{kj}$. The condition that A be C-symmetric is thus that $a_{kj} = a_{(N-j-1)(N-k-1)}$ for all j,k, which is the condition that the matrix for A be symmetric with respect to reflection about the diagonal orthogonal to the main diagonal, clearly weaker than the Toeplitz condition. (For $N = 2$, however, the reasoning shows that all C-symmetric operators are in $\mathscr{T}(K_u^2)$.)

LEMMA 2.2. *(a) For λ in $\mathcal{D}$,*

$$S_u^* k_\lambda^u = \bar{\lambda} k_\lambda^u - \overline{u(\lambda)}\widetilde{k}_0^u,\ S_u \widetilde{k}_\lambda^u = \lambda \widetilde{k}_\lambda^u - u(\lambda) k_0^u.$$

(b) For λ in $\mathcal{D}\setminus\{0\}$,

$$S_u k_\lambda^u = \frac{1}{\bar{\lambda}} k_\lambda^u - \frac{1}{\bar{\lambda}} k_0^u,\ S_u^* \widetilde{k}_\lambda^u = \frac{1}{\lambda}\widetilde{k}_\lambda^u - \frac{1}{\lambda}\widetilde{k}_0^u.$$

Proof. (a) For the first equality we have

$$\begin{aligned}S_u^* k_\lambda^u &= S^*((1-\overline{u(\lambda)}u)k_\lambda) = (1-\overline{u(\lambda)}u)S^*k_\lambda + k_\lambda(0)S^*(1-\overline{u(\lambda)}u) \\ &= \bar{\lambda}(1-\overline{u(\lambda)}u)k_\lambda - \overline{u(\lambda)}S^*u = \bar{\lambda}k_\lambda^u - \overline{u(\lambda)}\widetilde{k}_0^u,\end{aligned}$$

as desired. We obtain the second equality by applying C to the first equality:

$$\begin{aligned}S_u \widetilde{k}_\lambda^u &= CS_u^*CCk_\lambda^u = CS_u^*k_\lambda^u \\ &= C(\bar{\lambda}k_\lambda^u - \overline{u(\lambda)}\widetilde{k}_0^u) = \lambda\widetilde{k}_\lambda^u - u(\lambda)k_0^u.\end{aligned}$$

(b) For $\lambda \neq 0$ we have

$$S_u k_\lambda^u = P_u S k_\lambda^u = P_u S((1-\overline{u(\lambda)}u)k_\lambda) = P_u S k_\lambda.$$

Since

$$(Sk_\lambda)(z) = \frac{z}{1-\bar{\lambda}z} = \frac{1}{\bar{\lambda}}\left(\frac{1}{1-\bar{\lambda}z} - 1\right) = \frac{1}{\bar{\lambda}}(k_\lambda(z) - 1),$$

we get

$$S_u k_\lambda^u = \frac{1}{\bar{\lambda}}P_u(k_\lambda - 1) = \frac{1}{\bar{\lambda}}(k_\lambda^u - k_0^u),$$

which is the first equality. As in (a), the second equality is obtained from the first through an application of C. □

COROLLARY. *If u has an ADC at the point η of $\partial\mathcal{D}$, then the equalities in the lemma hold with η in place of λ.*

Proof. One obtains this by letting λ tend nontangentially to η and using continuity. □

2.4. Cyclic Vectors.

LEMMA 2.3. *The function* k_0^u *is a cyclic vector of* S_u. *The function* $\widetilde{k}_0^u$ *is a cyclic vector of* S_u^*.

Proof. The second statement follows from the first through an application of C. To prove the first statement, suppose the function f in K_u^2 is orthogonal to $S_u^n k_0^u$ for all nonnegative integers n. We have

$$0 = \langle f, S_u^n k_0^u \rangle = \langle S^{*n} f, k_0^u \rangle = (S^{*n} f)(0),$$

which says that all the Taylor coefficients of f at the origin vanish, hence that $f = 0$. □

2.5. Defect Operators.

LEMMA 2.4. $I - S_u S_u^* = k_0^u \otimes k_0^u$, $I - S_u^* S_u = \widetilde{k}_0^u \otimes \widetilde{k}_0^u$.

Proof. The second equality follows from the first through an application of C. To obtain the first equality we note that the operator S_u^* $(= S^* \mid K_u^2)$ acts isometrically on the subspace of functions in K_u^2 that vanish at 0, in other words, on the orthogonal complement of k_0^u. The self-adjoint operator $I - S_u S_u^*$ thus vanishes on that orthogonal complement, so it equals a scalar multiple of $k_0^u \otimes k_0^u$. To determine the scalar we apply Lemma 2.2(a) (with $\lambda = 0$):

$$\begin{aligned} (I - S_u S_u^*) k_0^u &= k_0^u + \overline{u(0)} S_u \widetilde{k}_0^u = (1 - |u(0)|^2) k_0^u \\ &= \langle k_0^u, k_0^u \rangle k_0^u = (k_0^u \otimes k_0^u) k_0^u. \end{aligned}$$

Thus the scalar is 1. □

2.6. Difference Quotients. For λ in $\mathcal{D}$ we define the operator Q_λ on H^2 by $Q_\lambda = S^*(I - \lambda S^*)^{-1}$. Since K_u^2 is S^*-invariant, it is also Q_λ-invariant. The operator Q_λ is the Toeplitz operator with symbol $\bar{z}/(1 - \lambda \bar{z})$ $(|z| = 1)$, so for f in H^2,

$$\begin{aligned} Q_\lambda f &= P\left(\frac{\bar{z} f}{1 - \lambda \bar{z}}\right) = P\left(\frac{f - f(\lambda)}{z - \lambda} + \frac{\bar{z} f(\lambda)}{1 - \lambda \bar{z}}\right) \\ &= \frac{f - f(\lambda)}{z - \lambda}. \end{aligned}$$

Hence, if f is in K_u^2, then the difference quotient $(f - f(\lambda))/(z - \lambda)$ is also in K_u^2. We note also that $\widetilde{k}_\lambda^u = Q_\lambda u$.

2.7. Spectrum of S_u.

LEMMA 2.5. *The spectrum of* S_u *consists of the set of singularities of* u *on* $\partial\mathcal{D}$ *together with the zero set of* u *in* $\mathcal{D}$. *The essential spectrum is the set of singularities on* $\partial\mathcal{D}$. *For* λ *in* $u^{-1}(0)$, *the operator* $S_u - \lambda I$ *is a Fredholm operator of index* 0 *with a one-dimensional kernel. The operator* S_u *has no eigenvalues on* $\partial\mathcal{D}$.

Proof. Since S_u is a contraction its spectrum is contained in $\bar{\mathcal{D}}$. If λ is a point of $\mathcal{D}$ where u is nonzero, or a regular point of u on $\partial\mathcal{D}$, one easily checks that the operator $-\frac{1}{u(\lambda)}A_{\widetilde{k}^u_\lambda}$ inverts $S_u - \lambda I$.

Suppose λ is a zero of u. Then by Lemma 2.2(a) $S_u\widetilde{k}^u_\lambda = \lambda\widetilde{k}^u_\lambda$, so λ is an eigenvalue of S_u. To verify that $S_u - \lambda I$ is a Fredholm operator we compute the product $(S_u - \lambda I)Q_\lambda$, regarding Q_λ here as an operator on K^2_u. For f in K^2_u,

$$(S_u - \lambda I)Q_\lambda f = P_u\left((z-\lambda)\left(\frac{f - f(\lambda)}{z-\lambda}\right)\right) = f - f(\lambda)k^u_0,$$

from which we conclude that $(S - \lambda I)Q_\lambda = I - (k^u_0 \otimes k^u_\lambda)$. This tells us that $S_u - \lambda I$ is right Fredholm with a range of codimension at most 1. By C-symmetry $S_u - \lambda I$ is also left Fredholm, hence Fredholm. Also by C-symmetry, the index of $S_u - \lambda I$ is 0. We know the dimension of the cokernel of $S_u - \lambda I$ is at most 1 and the dimension of the kernel is at least 1, so both are of dimension 1.

It remains to show that every singularity of u on $\partial\mathcal{D}$ is in the essential spectrum. Suppose first that the point η of $\partial\mathcal{D}$ lies in the resolvent set of u, so that $(S_u - \lambda I)^{-1}$ is defined in a neighborhood of η. We note that, for λ in $\mathcal{D}$, we have $k_\lambda = (I - \bar{\lambda}S)^{-1}k_0$, which after projection onto K^2_u gives $k^u_\lambda = (I - \bar{\lambda}S_u)^{-1}k^u_0$. In particular, then,

$$\frac{u(\lambda) - u(0)}{\lambda} = \widetilde{k}^u_0(\lambda) = \langle \widetilde{k}^u_0, (I - \bar{\lambda}S_u)^{-1}k^u_0\rangle .$$

As the function of λ on the right side extends holomorphically to a neighborhood of η, we conclude that the same is true of u, i.e., η is a regular point of u. Hence every singular point of u on $\partial\mathcal{D}$ belongs to the spectrum of S_u.

Finally, consider a singularity λ of u on $\partial\mathcal{D}$. The function $z - \lambda$ is an outer function so, by Beurling's theorem, the operator $S - \lambda I$ has a dense range. Therefore $S_u - \lambda I$ also has a dense range. This tells us that S^*_u has no eigenvalues on $\partial\mathcal{D}$. By C-symmetry, the same is true of S_u, and so $S_u - \lambda I$ is injective. But $S_u - \lambda I$, being noninvertible, is not surjective. Therefore $S_u - \lambda I$ is not Fredholm, i.e., λ is in the essential spectrum of S_u. □

Additional known properties of K^2_u will be introduced as the need arises. What stands above will suffice for the time being.

3. Condition for $A_\varphi = 0$

THEOREM 3.1. *If φ is in L^2, then $A_\varphi = 0$ if and only if φ belongs to $uH^2 + \bar{u}\bar{H}^2$.*

Proof. Suppose φ is in $uH^2 + \bar{u}\bar{H}^2$, say $\varphi = u\psi + \bar{u}\bar{\chi}$ with ψ and χ in H^2. For f in K^∞_u we have $\varphi f = \psi uf + \bar{\chi}\bar{u}f$, which is orthogonal to K^2_u because $uK^\infty_u \subset uH^\infty$ and $\bar{u}K^\infty_u \subset \bar{H}^\infty_0$. Hence $A_\varphi f = 0$ for f in K^∞_u, and so $A_\varphi = 0$.

Suppose, conversely, that $A_\varphi = 0$, and write $\varphi = \psi + \bar{\chi}$ with ψ and χ in H^2. Thus $A_\psi = -A_{\bar{\chi}}$. Now the operators $A_{\bar{\chi}}$ and S^*_u commute, being restrictions to K^2_u of the commuting operators $T_{\bar{\chi}}$ and S^*. Similarly, the operators A_ψ and S_u (the adjoints of $A_{\bar{\psi}}$ and S^*_u) commute. Hence A_ψ commutes with both S_u and S^*_u.

Then $A_\psi(I-S_uS_u^*)k_0^u=(I-S_uS_u^*)A_\psi k_0^u$, implying by Lemma 2.4 that $A_\psi k_0^u$ is a scalar multiple of k_0^u, say $A_\psi k_0^u=ck_0^u$. Thus

$$\begin{aligned}0=(A_\psi-cI)k_0^u &= P_u((\psi-c)(1-\overline{u(0)}u))\\ &= P_u(\psi-c),\end{aligned}$$

implying that $\psi-c$ is in uH^2. So we actually have $A_\psi=cI$, and accordingly $A_{\bar\chi}=-cI$. Repeating the reasoning above, we conclude that $\chi+\bar c$ is in uH^2, and hence that $\bar\chi+c$ is in $\bar u\bar H^2$. Therefore $\varphi=(\psi-c)+(\bar\chi+c)$ is in $uH^2+\bar u\bar H^2$. □

COROLLARY. *If φ is in L^2, then there is a pair of functions ψ, χ in K_u^2 such that $A_\varphi=A_{\psi+\bar\chi}$. If ψ, χ is one such pair, the most general such pair equals $\psi+ck_0^u$, $\chi-\bar ck_0^u$, with c a scalar.*

Proof. Write $\varphi=\varphi_++\varphi_-$ with φ_+ in H^2 and φ_- in $\bar H^2$. Let $\psi=P_u\varphi_+$, $\chi=P_u\bar\varphi_-$. Then $\varphi-\psi-\bar\chi$ is in $uH^2+\bar u\bar H^2$, so $A_\varphi=A_{\psi+\bar\chi}$ by the theorem.

Since $k_0^u=1-\overline{u(0)}u$ and $A_u=0$, we have $A_{k_0^u}=I$. Hence, if ψ, χ are as above and $\psi_1=\psi+ck_0^u$, $\chi_1=\chi-\bar ck_0^u$, with c a scalar, then $A_{\psi_1}=A_\psi+cI$, $A_{\bar\chi_1}=A_{\bar\chi}-cI$, so that $A_{\psi_1+\bar\chi_1}=A_\varphi$.

Finally, suppose ψ_1, χ_1 are in K_u^2 and $A_{\psi_1+\bar\chi_1}=A_\varphi$. Then $(\psi-\psi_1)+(\bar\chi-\bar\chi_1)$ is in $uH^2+\bar u\bar H^2$, by the theorem. As the projection P_u annihilates $uH^2+\bar u\bar H^2$, we have $\psi-\psi_1=-P_u(\bar\chi-\bar\chi_1)$. Since $P\bar H^2$ consists of the constant functions, the function $P_u(\bar\chi-\bar\chi_1)$ must be a scalar multiple of $P_u1=k_0^u$. Thus $\psi_1=\psi+ck_0^u$ for some scalar c, and, accordingly, $\bar\chi-\bar\chi_1-ck_0^u$ is in $uH^2+\bar u\bar H^2$. Then $\chi-\chi_1-\bar ck_0^u$ is in $uH^2+\bar u\bar H^2$. Applying P_u again, we obtain

$$\chi-\chi_1=\bar cP_uk_0^u=\bar cP_u(1-\overline{u(0)}u)=\bar cP_u1=\bar ck_0^u,$$

i.e., $\chi_1=\chi-\bar ck_0^u$. □

4. A Characterization

THEOREM 4.1. *The bounded operator A on K_u^2 belongs to $\mathscr{T}(K_u^2)$ if and only if there are functions ψ, χ in K_u^2 such that*

$$A-S_uAS_u^*=(\psi\otimes k_0^u)+(k_0^u\otimes\chi),$$

in which case $A=A_{\psi+\bar\chi}$.

The bulk of the proof will be accomplished in two lemmas.

LEMMA 4.1. *For ψ, χ in K_u^2,*

$$A_{\psi+\bar\chi}-S_uA_{\psi+\bar\chi}S_u^*=(\psi\otimes k_0^u)+(k_0^u\otimes\chi).$$

Proof. Since S_u commutes with A_ψ and S_u^* commutes with $A_{\bar\chi}$, we have

$$A_{\psi+\bar\chi}-S_uA_{\psi+\bar\chi}S_u^*=A_\psi(I-S_uS_u^*)+(I-S_uS_u^*)A_{\bar\chi}.$$

Since $I - S_u S_u^* = k_0^u \otimes k_0^u$ (Lemma 2.4), the right side equals

$$(A_\psi k_0^u \otimes k_0^u) + (k_0^u \otimes A_\chi k_0^u).$$

Finally, $A_\psi k_0^u = \psi$, $A_\chi k_0^u = \chi$. □

LEMMA 4.2. *Let ψ, χ be in K_u^2. Then for f, g in K_u^∞,*

$$\langle A_{\psi+\bar\chi} f, g\rangle = \sum_{n=0}^{\infty} (\langle f, S_u^n k_0^u\rangle \langle S_u^n \psi, g\rangle + \langle f, S_u^n \chi\rangle \langle S_u^n k_0^u, g\rangle).$$

Proof. By Lemma 4.1,

$$A_{\psi+\bar\chi} - S_u A_{\psi+\bar\chi} S_u^* = (\psi \otimes k_0^u) + (k_0^u \otimes \chi).$$

Thus, for any positive integer n,

$$S_u^n A_{\psi+\bar\chi} S_u^{*n} - S_u^{n+1} A_{\psi+\bar\chi} S_u^{*(n+1)} = (S_u^n \psi \otimes S_u^n k_0^u) + (S_u^n k_0^u \otimes S_u^n \chi).$$

Adding for $n = 0, 1, \ldots, N$, we get

$$A_{\psi+\bar\chi} = \sum_{n=0}^{N} ((S_u^n \psi \otimes S_u^n k_0^u) + (S_u^n k_0^u \otimes S_u^n \chi)) + S_u^{N+1} A_{\psi+\bar\chi} S_u^{*(N+1)}.$$

Thus, for f, g in K_u^∞,

$$\langle A_{\psi+\bar\chi} f, g\rangle = \sum_{n=0}^{N} (\langle f, S_u^n k_0^u\rangle \langle S_u^n \psi, g\rangle + \langle f, S_u^n \chi\rangle \langle S_u^n k_0^u, g\rangle) + \langle A_{\psi+\bar\chi} S_u^{*(N+1)} f, S_u^{*(N+1)} g\rangle.$$

It remains to show that the last summand on the right tends to 0 as $N \to \infty$. We have

$$\langle A_{\psi+\bar\chi} S_u^{*N} f, S_u^{*N} g\rangle = \langle S_u^{*N} f, S_u^{*N} A_{\bar\psi} g\rangle + \langle S_u^{*N} A_{\bar\chi} f, S_u^{*N} g\rangle.$$

The desired conclusion now follows because $S^{*N} \to 0$ in the strong operator topology. □

Proof of Theorem 4.1. It follows by Lemma 4.1 and the corollary to Theorem 3.1 that every operator in $\mathscr{T}(K_u^2)$ satisfies the condition of the theorem. Suppose, conversely, that A is a bounded operator in K_u^2 that satisfies the condition:

$$A - S_u A S_u^* = (\psi \otimes k_0^u) + (k_0^u \otimes \chi)$$

with ψ, χ in K_u^2. The first part of the proof of Lemma 4.2 shows that, for any positive integer N,

$$A = \sum_{n=0}^{N} ((S_u^n \psi \otimes S_u^n k_0^u) + (S_u^n k_0^u \otimes S_n^n \chi)) + S_u^{N+1} A S_u^{*(N+1)}.$$

Since $S_u^{*N} \to 0$ in the strong operator topology, we conclude that

$$A = \sum_{n=0}^{\infty}((S_u^n \psi \otimes S_u^n k_0^u) + (S_u^n k_0^u \otimes S_u^n \chi)),$$

the series converging in the strong operator topology. By Lemma 4.2 we can conclude that $A = A_{\psi+\bar{\chi}}$. □

REMARK. An application of the conjugation C produces from Theorem 4.1 an alternative necessary and sufficient condition for a bounded operator A to belong to $\mathscr{T}(K_u^2)$, namely, the condition

$$A - S_u^* A S_u = (\psi \otimes \widetilde{k}_0^u) + (\widetilde{k}_0^u \otimes \chi)$$

with ψ, χ in K_u^2.

THEOREM 4.2. *$\mathscr{T}(K_u^2)$ is closed in the weak operator topology.*

Proof. Suppose the net (A_α) in $\mathscr{T}(K_u^2)$ converges weakly to the bounded operator A. By Theorem 4.1, for each index α there are functions ψ_α, χ_α in K_u^2 such that

$$A_\alpha - S_u A_\alpha S_u^* = (\psi_\alpha \otimes k_0^u) - (k_0^u \otimes \chi_\alpha). \tag{4.1}$$

Moreover, the functions χ_α can be taken to satisfy $\chi_\alpha(0) = 0$ (see the corollary to Theorem 3.1). Then we have

$$A_\alpha k_0^u - S_u A_\alpha S_u^* k_0^u = \|k_0^u\|_2^2 \psi_\alpha,$$

and it follows that the net (ψ_α) converges weakly, say to the function ψ in K_u^2. The net $(\psi_\alpha \otimes k_0^u)$ thus converges in the weak operator topology, and so by (4.1) the net $(k_0^u \otimes \chi_\alpha)$ also converges in the weak operator topology, implying that the net (χ_α) converges weakly, say to the function χ in K_u^2. Passing to the limit in (4.1), we obtain

$$A - S_u A S_u^* = (\psi \otimes k_0^u) + (k_0^u \otimes \chi),$$

and it follows by Theorem 4.1 that $A = A_{\psi+\bar{\chi}}$. □

Among the operators in $\mathscr{T}(K_u^2)$ are those in the commutant of S_u. In fact, if the bounded operator A on K_u^2 commutes with S_u, then

$$A - S_u A S_u^* = A(I - S_u S_u^*) = A k_0^u \otimes k_0^u,$$

and it follows by Theorem 4.1 that A is the truncated Toeplitz operator with symbol $A k_0^u$. It is known that such an A has an H^∞ symbol with supremum norm equal to $\|A\|$. This result, first established in [15], is a corollary of the commutant lifting theorem of B. Sz.–Nagy and C. Foiaş [17].

By the same token, a bounded operator on K_u^2 that commutes with S_u^* is in $\mathscr{T}(K_u^2)$ and has a symbol in $\bar{H}^\infty$ of supremum norm equal to the operator norm. The question whether every operator in $\mathscr{T}(K_u^2)$ has a bounded symbol concerns operators that commute with neither S_u nor S_u^*.

5. Rank-One Operators

THEOREM 5.1. *(a) For λ in $\mathcal{D}$, the operators $k_\lambda^u \otimes \widetilde{k}_\lambda^u$ and $\widetilde{k}_\lambda^u \otimes k_\lambda^u$ belong to $\mathscr{T}(K_u^2)$.*

(b) If u has an ADC at the point η of $\partial\mathcal{D}$, then the operator $k_\eta^u \otimes k_\eta^u$ belongs to $\mathscr{T}(K_u^2)$.

(c) The only rank-one operators in $\mathscr{T}(K_u^2)$ are the nonzero scalar multiples of the operators in (a) and (b).

Proof. (a) We consider first a point λ in $\mathcal{D} \setminus \{0\}$ and apply the criterion of Theorem 4.1 to the operator $k_\lambda^u \otimes \widetilde{k}_\lambda^u$. By Lemma 2.2,

$$\begin{aligned} S_u(k_\lambda^u \otimes \widetilde{k}_\lambda^u)S_u^* &= S_u k_\lambda^u \otimes S_u \widetilde{k}_\lambda^u \\ &= \left(\frac{1}{\bar\lambda}k_\lambda^u - \frac{1}{\bar\lambda}k_0^u\right) \otimes (\lambda \widetilde{k}_\lambda^u - u(\lambda)k_0^u) \\ &= (k_\lambda^u \otimes \widetilde{k}_\lambda^u) - (k_0^u \otimes \widetilde{k}_\lambda^u) - \frac{\overline{u(\lambda)}}{\bar\lambda}(k_\lambda^u \otimes k_0^u) + \frac{\overline{u(\lambda)}}{\bar\lambda}(k_0^u \otimes k_0^u). \end{aligned}$$

Thus

$$k_\lambda^u \otimes \widetilde{k}_\lambda^u - S_u(k_\lambda^u \otimes \widetilde{k}_\lambda^u)S_u^* = (k_0^u \otimes \widetilde{k}_\lambda^u) + \frac{\overline{u(\lambda)}}{\bar\lambda}((k_\lambda^u - k_0^u) \otimes k_0^u).$$

By Theorem 4.1, $k_\lambda^u \otimes \widetilde{k}_\lambda^u$ is the truncated Toeplitz operator with symbol $\frac{\overline{u(\lambda)}}{\bar\lambda}(k_\lambda^u - k_0^u) + \overline{\widetilde{k}_\lambda^u}$. The symbol can be simplified because $A_{k_0^u} = I = A_1$ and $A_{k_\lambda^u} = A_{k_\lambda}$. Replacing k_0^u by 1 and k_λ^u by k_λ in the preceding expression, we obtain the symbol (written as a function of the variable z on $\partial\mathcal{D}$)

$$\begin{aligned} \frac{\overline{u(\lambda)}}{\bar\lambda}\left(\frac{1}{1-\bar\lambda z} - 1\right) + \frac{\bar u - \overline{u(\lambda)}}{\bar z - \bar\lambda} &= \overline{u(\lambda)}\left(\frac{z}{1-\bar\lambda z}\right) + \frac{z(\bar u - \overline{u(\lambda)})}{1-\bar\lambda z} \\ &= \frac{z\bar u}{1-\bar\lambda z} = \frac{\bar u}{\bar z - \bar\lambda}. \end{aligned}$$

Conclusion. $k_\lambda^u \otimes \widetilde{k}_\lambda^u$ is the truncated Toeplitz operator with symbol $\bar u/(\bar z - \bar\lambda)$.

Conjugating the last conclusion, we find that $\widetilde{k}_\lambda^u \otimes k_\lambda^u$ is the truncated Toeplitz operator with symbol $u/(z-\lambda)$. Taking the limit as $\lambda \to 0$, we find that $k_0^u \otimes \widetilde{k}_0^u$ and $\widetilde{k}_0^u \otimes k_0^u$ are the truncated Toeplitz operators with respective symbols $\bar u/\bar z$ and u/z.

(b) Let η be a point of $\partial\mathcal{D}$ at which u has an ADC. By the corollary to Lemma 2.2, the first part of the proof of (a) can be repeated with η in place of λ to show that $k_\eta^u \otimes \widetilde{k}_\eta^u$ is the truncated Toeplitz operator with symbol $\frac{\overline{u(\eta)}}{\bar\eta}(k_\eta^u - k_0^u) + \overline{\widetilde{k}_\lambda^u}$. In Section 2.3. we observed that $\widetilde{k}_\eta^u = \bar\eta u(\eta) k_\eta^u$. Combining this with the preceding conclusion, we see that $k_\eta^u \otimes k_\eta^u$ is the truncated Toeplitz operator with symbol $k_\eta^u + \bar{k}_\eta^u - k_0^u$. As above in the proof of (a), we can replace k_0^u here by 1, obtaining the symbol $k_\eta^u + \bar{k}_\eta^u - 1$.

(c) Suppose σ, τ are nonzero functions in K_u^2 such that the operator $\sigma \otimes \tau$ belongs to $\mathscr{T}(K_u^2)$. Since $\sigma \otimes \tau$ is then C-symmetric, we have

$$\tau \otimes \sigma = C(\sigma \otimes \tau)C = \widetilde{\sigma} \otimes \widetilde{\tau}.$$

Hence τ and $\widetilde{\sigma}$ are linearly dependent, so, after a scaling, we can assume with no loss of generality that $\tau = \widetilde{\sigma}$.

By Theorem 4.1, there are functions ψ and χ in K_u^2 such that

$$(\sigma \otimes \widetilde{\sigma}) - (S_u\sigma \otimes S_u\widetilde{\sigma}) = (\psi \otimes k_0^u) + (k_0^u \otimes \chi). \tag{5.1}$$

Various cases arise.

Case 1. One of ψ, χ is the zero function.

By C-symmetry it will be enough to treat the case $\psi = 0$. Then (5.1) reduces to

$$(\sigma \otimes \widetilde{\sigma}) - (S_u\sigma \otimes S_u\widetilde{\sigma}) = k_0^u \otimes \chi. \tag{5.2}$$

For the operator on the left side of (5.2) to have rank one, either σ and $S_u\sigma$ must be linearly dependent or $\widetilde{\sigma}$ and $S_\sigma\widetilde{\sigma}$ must be linearly dependent. Suppose σ and $S_u\sigma$ are linearly dependent, say $S_u\sigma = \lambda\sigma$. By Lemma 2.5 (and its proof), $|\lambda| < 1$, $u(\lambda) = 0$, and σ is a scalar multiple of $\widetilde{k}_\lambda^u$. Thus $\sigma \otimes \widetilde{\sigma}$ is a scalar multiple of $\widetilde{k}_\lambda^u \otimes k_\lambda^u$. If $\widetilde{\sigma}$ and $S_u\widetilde{\sigma}$ are linearly dependent, $S_u\widetilde{\sigma} = \lambda\widetilde{\sigma}$, the analogous argument shows that λ is in $\mathcal{D}$ and $\sigma \otimes \widetilde{\sigma}$ is a scalar multiple of $k_\lambda^u \otimes \widetilde{k}_\lambda^u$.

For the remainder of the proof we assume that neither ψ nor χ is the zero function. By (5.1), either k_0^u is a linear combination of σ and $S_u\sigma$, or k_0^u is a linear combination of $\widetilde{\sigma}$ and $S_u\widetilde{\sigma}$. Since σ and $\widetilde{\sigma}$ are interchangeable, it will be enough to treat the case where k_0^u is a linear combination of σ and $S_u\sigma$, say $k_0^u = a\sigma + bS_u\sigma$.

Case 2. $b = 0$.

In this case σ is a scalar multiple of k_0^u, so $\sigma \otimes \widetilde{\sigma}$ is a scalar multiple of $k_0^u \otimes \widetilde{k}_0^u$.

Case 3. $a = 0$.

In this case k_0^u lies in the range of S_u. That implies $u(0) \neq 0$ (for otherwise $k_0^u = 1$ and $S_u^* k_0^u = 0$), and therefore S_u is invertible, by Lemma 2.5. By Lemma 2.4(a) we have $S_u\widetilde{k}_0^u = -u(0)k_0^u$, so $\sigma = \frac{1}{b}S_u^{-1}k_0^u$ is a scalar multiple of $\widetilde{k}_0^u$, and $\sigma \otimes \widetilde{\sigma}$ is a scalar multiple of $\widetilde{k}_0^u \otimes k_0^u$.

In all remaining cases $a \neq 0 \neq b$. Replacing σ by $a\sigma$ and letting $\lambda = -\bar{b}/\bar{a}$, we transform the equality $k_0^u = a\sigma + bS_u\sigma$ to $k_0^u = (I - \bar{\lambda}S_u)\sigma$, $\lambda \neq 0$.

Case 4. $0 < |\lambda| < 1$.

In this case the operator $I - \bar{\lambda}S_u$ is invertible, and by Lemma 2.2(b) $\widetilde{k}_\lambda^u - \bar{\lambda}S_u\widetilde{k}_\lambda^u = k_0^u$. Thus $\sigma = \widetilde{k}_\lambda^u$, and $\sigma \otimes \widetilde{\sigma} = \widetilde{k}_\lambda^u \otimes k_\lambda^u$.

Case 5. $|\lambda| > 1$, $u(1/\bar{\lambda}) \neq 0$.

In this case the operator $I - \bar{\lambda} S_u$ is invertible, by Lemma 2.5. By Lemma 2.2(a),

$$\bar{\lambda} S_u \widetilde{k}^u_{1/\bar{\lambda}} = \widetilde{k}^u_{1/\bar{\lambda}} - \bar{\lambda} u(1/\bar{\lambda}) k^u_0,$$

which tells us that $\sigma = \frac{1}{\bar{\lambda} u(1/\bar{\lambda})} \widetilde{k}^u_{1/\bar{\lambda}}$, and hence that $\sigma \otimes \widetilde{\sigma}$ is a scalar multiple of $\widetilde{k}^u_{1/\bar{\lambda}} \otimes k^u_{1/\bar{\lambda}}$.

Case 6. $|\lambda| > 1$, $u(1/\bar{\lambda}) = 0$.

In this case, by Lemma 2.5 (and its proof), the operator $I - \lambda S^*_u$ has a one-dimensional kernel spanned by the function $k_{1/\bar{\lambda}}$. If k^u_0 were in the range of $I - \bar{\lambda} S_u$ it would be orthogonal to $k_{1/\bar{\lambda}}$, which it is not because $k_{1/\bar{\lambda}}(0) = 1$. This case therefore does not arise.

Case 7. $|\lambda| = 1$.

In this case, because the operator S_u has no eigenvalues on $\partial\mathcal{D}$ (Lemma 2.5), the function σ is uniquely determined by the condition $k^u_0 = \sigma - \bar{\lambda} S_u \sigma$. Applying the conjugation C to the last equality, we get $\widetilde{k}^u_0 = \widetilde{\sigma} - \lambda S^*_u \widetilde{\sigma}$, in other words,

$$\frac{u - u(0)}{z} = \widetilde{\sigma} - \frac{\lambda(\widetilde{\sigma} - \widetilde{\sigma}(0))}{z}.$$

Hence

$$\widetilde{\sigma} = \frac{u - \gamma}{z - \lambda},$$

where $\gamma = u(0) + \lambda \widetilde{\sigma}(0)$. That $\widetilde{\sigma}$ belongs to H^2 implies that γ must have unit modules, and must in fact be the nontangential limit of u at λ (since an H^2 function is $o\left(\frac{1}{1-|z|}\right)$ as $|z| \to 1$). Thus we can write

$$\widetilde{\sigma} = \frac{u - u(\lambda)}{z - \lambda}.$$

In particular, the function $(u - u(\lambda))/(z - \lambda)$ is in H^2. As noted in Section 2.2., we can conclude that u has an ADC at λ. So $\widetilde{\sigma} = \widetilde{k}^u_\lambda$, and $\sigma \otimes \widetilde{\sigma} = k^u_\lambda \otimes \widetilde{k}^u_\lambda$. □

COROLLARY. *If* $\dim K^2_u \geqslant 3$, *there is a* C*-symmetric rank-one operator on* K^2_u *that does not belong to* $\mathscr{T}(K^2_u)$.

Proof. Assume $\dim K^2_u \geqslant 3$. A C-symmetric rank-one operator on K^2_u has the form $f \otimes \widetilde{f}$, with f a nonzero function in K^2_u. By Theorem 5.1, we need only show we can choose f so that it is not a scalar multiple of k^u_λ or of $\widetilde{k}^u_\lambda$ for any λ.

Choose points z_1, z_2 in $\mathcal{D}$ such that $\lambda_1 = u(z_1)$ and $\lambda_2 = u(z_2)$ are not equal. Because $\dim K^2_u \geqslant 3$, the linear map $f \mapsto (f(z_1), f(z_2))$ on K^2_u has a nontrivial kernel. Thus, there is a nonzero f in K^2_u vanishing at z_1 and at z_2. If λ is in $\mathcal{D}$, or is a point of $\partial\mathcal{D}$ where u has an ADC, then f is not a scalar multiple of k^u_λ because k^u_λ has no zeros in $\mathcal{D}$. Moreover, if λ is in $\mathcal{D}$ then, because the zero set of $\widetilde{k}^u_\lambda$ is $u^{-1}(\lambda)$, and f vanishes on at least two of those zero sets, f is not a scalar multiple of $\widetilde{k}^u_\lambda$. □

We noted in Section 2. that, in the case $u = z^2$, all C-symmetric operators on K_u^2 are in $\mathscr{T}(K_u^2)$. From Theorem 7.1(a) below one can see that the same conclusion holds whenever $\dim K_u^2 = 2$.

If two commuting operators are C-symmetric then their product is C-symmetric. It thus may be worth noting that there is a self-adjoint operator in $\mathscr{T}(K_u^2)$ whose square is not in $\mathscr{T}(K_u^2)$, provided $\dim K_u^2 \geqslant 3$. We consider the rank-two self-adjoint operator $A = (k_0^u \otimes \widetilde{k}_0^u) + (\widetilde{k}_0^u \otimes k_0^u)$, which belongs to $\mathscr{T}(K_u^2)$ by Theorem 5.1, and we assume that its square also belongs to $\mathscr{T}(K_u^2)$, and that $\dim K_u^2 \neq 1$ (to eliminate a trivial case). We show that then $\dim K_u^2 = 2$.

We have

$$\begin{aligned} A^2 &= \langle k_0^u, \widetilde{k}_0^u \rangle (k_0^u \otimes \widetilde{k}_0^u) + \langle \widetilde{k}_0^u, k_0^u \rangle (\widetilde{k}_0^u \otimes k_0^u) \\ &\quad + \|\widetilde{k}_0^u\|_2^2 (k_0^u \otimes k_0^u) + \|k_0^u\|_2^2 (\widetilde{k}_0^u \otimes \widetilde{k}_0^u). \end{aligned}$$

As $k_0^u \otimes \widetilde{k}_0^u$ and $\widetilde{k}_0^u \otimes k_0^u$ are in $\mathscr{T}(K_u^2)$, and $\|k_0^u\|_2 = \|\widetilde{k}_0^u\|_2$, it follows that the operator $B = (k_0^u \otimes k_0^u) + (\widetilde{k}_0^u \otimes \widetilde{k}_0^u)$ is in $\mathscr{T}(K_u^2)$. By Theorem 4.1 the operator $B - S_u B S_u^*$ has rank one or two. By Lemma 2.2(a),

$$\begin{aligned} S_u(\widetilde{k}_0^u \otimes \widetilde{k}_0^u) S_u^* &= (S_u \widetilde{k}_0^u \otimes S_u \widetilde{k}_0^u) \\ &= (-u(0) k_0^u) \otimes (-u(0) k_0^u) = -|u(0)|^2 (k_0^u \otimes k_0^u), \end{aligned}$$

and so

$$B - S_u B S_u = (\widetilde{k}_0^u \otimes \widetilde{k}_0^u) + (1 - |u(0)|^2)(k_0^u \otimes k_0^u) - (S_u k_0^u \otimes S_u k_0^u).$$

We know from Lemma 2.3 that k_0^u is a cyclic vector for the operator S_u. Thus k_0^u and $S_u k_0^u$ are linearly independent. If $\widetilde{k}_0^u$ is not a linear combination of k_0^u and $S_u k_0^u$, then the operator on the right side of the last equality has rank three. Hence there are scalars a and b such that $\widetilde{k}_0^u = a k_0^u + b S_u k_0^u$. Applying S_u to the last equality and using Lemma 2.2(a) again, we get $-u(0) k_0^u = a S_u k_0^u + b S_u^2 k_0^u$. Since k_0^u and $S_u k_0^u$ are linearly independent, $b \neq 0$. Hence $S_u^2 k_0^u$ is a linear combination of k_0^u and $S_u k_0^u$, implying by the cyclicity of k_0^u that $\dim K_u^2 = 2$, as desired.

According to Theorem 5.1(b), if u has an ADC at the point η of $\partial\mathfrak{D}$, then the operator $k_\eta^u \otimes k_\eta^u$ belongs to $\mathscr{T}(K_u^2)$. The proof showed that $k_\eta^u \otimes k_\eta^u$ has $k_\eta^u + \bar{k}_\eta^u - 1$ as a symbol. That symbol is bounded if η is a regular point of u but not, as will be shown shortly, if η is a singular point. In the latter case the operator $k_\eta^u \otimes k_\eta^u$ provides a test case for the question whether every operator in $\mathscr{T}(K_u^2)$ has a bounded symbol.

Assume u has an ADC at the singular point η of $\partial\mathfrak{D}$. The operator $A_{k_\eta^u} + A_{\bar{k}_\eta^u}$ is then bounded, but note that the operator $A_{k_\eta^u}$ is unbounded, for if it were bounded it would invert the operator $I - \bar{\eta} S_u$, which according to Lemma 2.5 is not invertible. In particular, then, k_η^u is not bounded, and in fact $k_\eta^u + uH^2$ contains no bounded functions.

To see that $k_\eta^u + \bar{k}_\eta^u$ is unbounded takes a bit more work. First note that, for z in $\partial\mathcal{D}$,

$$\begin{aligned} k_\eta^u + \bar{k}_\eta^u &= 2\,\mathrm{Re}\left(\frac{1}{1-\bar{\eta}z}\right) - 2\,\mathrm{Re}\left(\frac{\overline{u(\eta)}u}{1-\bar{\eta}z}\right) \\ &= \frac{1}{1-\bar{\eta}z} + \frac{1}{1-\eta\bar{z}} - 2\,\mathrm{Re}\left(\frac{\overline{u(\eta)}u}{1-\bar{\eta}z}\right) \\ &= \frac{2\,\mathrm{Re}(1-\bar{\eta}z)}{|1-\bar{\eta}z|^2} - 2\,\mathrm{Re}\left(\frac{\overline{u(\eta)}u}{1-\bar{\eta}z}\right) \\ &= 1 - 2\,\mathrm{Re}\left(\frac{\overline{u(\eta)}u}{1-\bar{\eta}z}\right) \end{aligned}$$

(since $|1-c|^2 = 2\,\mathrm{Re}(1-c)$ for $|c| = 1$). Our task, therefore, is to show that $\mathrm{Re}(\overline{u(\eta)}u/(1-\bar{\eta}z))$ is unbounded. A lemma is needed.

LEMMA 5.1. *Suppose α is an open subarc of $\partial\mathcal{D}$ on which the values of u lie almost everywhere in the proper closed subarc β. Then every point of α is a regular point of u.*

Taking the lemma temporarily for granted, fix ε in $\left(0, \frac{\pi}{2}\right)$, and let $\alpha = \{e^{i\theta} : |\theta - \arg\eta| < \varepsilon\}$. Then on $\alpha \setminus \{\eta\}$ the argument of $1-\bar{\eta}z$ lies in the union of the intervals $\left(\frac{\pi}{2}-\frac{\varepsilon}{2}, \frac{\pi}{2}+\frac{\varepsilon}{2}\right)$, $\left(-\frac{\pi}{2}-\frac{\varepsilon}{2}, -\frac{\pi}{2}+\frac{\varepsilon}{2}\right)$. By the lemma, there is a subset of α of positive measure on which the argument of $\overline{u(\eta)}u$ lies in the union of the same two intervals, and hence on which the argument of $\overline{u(\eta)}u/(1-\bar{\eta}z)$ lies in the union of the intervals $(-\varepsilon, \varepsilon)$, $(\pi-\varepsilon, \pi+\varepsilon)$. Thus, on a subset of α of positive measure we have

$$\left|\mathrm{Re}\,\frac{\overline{u(\eta)}u}{1-\bar{\eta}z}\right| \geqslant \cos\varepsilon \left|\frac{\overline{u(\eta)}u}{1-\bar{\eta}z}\right| \geqslant \frac{\cos\varepsilon}{|1-e^{i\varepsilon}|},$$

which establishes the unboundedness of $\mathrm{Re}(\overline{u(\eta)}u/(1-\bar{\eta}z))$.

In the case at hand, if the operator $k_\eta^u \otimes k_\eta^u$ has a bounded symbol then there is a function in $uH^2 + \bar{u}\bar{H}^2$ whose difference with $\mathrm{Re}(\overline{u(\eta)}u/(1-\bar{\eta}z))$ is bounded, in which case such a function can be taken to be real valued. Whether $k_\eta^u \otimes k_\eta^u$ has a bounded symbol boils down to the question: *If η is a singular point of u on $\partial\mathcal{D}$ at which u has an ADC, does there exist a function h in H^2 such that*

$$\mathrm{Re}\left(\frac{\overline{u(\eta)}u}{1-\bar{\eta}z} - uh\right)$$

is bounded?

Proof of Lemma 5.1. After composing u from the left with a conformal automorphism of $\mathcal{D}$, we can assume β has arc length less than $\pi/2$, say. For z in $\mathcal{D}$ we let

P_z be the corresponding Poisson kernel: $P_z(\zeta) = \frac{1-|z|^2}{|\zeta-z|^2}(|\zeta| = 1)$. Thus

$$u(z) = \int P_z(\zeta)u(\zeta)dm(\zeta).$$

Pick any point $\zeta_\#$ in β, and let $u^\#$ be the function on $\partial\mathcal{D}$ that equals u on α and equals $\zeta_\#$ off α. Then

$$u(z) = \int P_z(\zeta)u^\#(\zeta)dm(\zeta) + \int P_z(\zeta)(u(\zeta) - u^\#(\zeta))dm(\zeta).$$

The first summand on the right side lies in the convex hull of β, which does not contain 0, and the second summand tends to 0 as z tends to α, uniformly on closed subarcs of α. It follows that u is bounded away from 0 near any point of α, which is known to imply that such a point is a regular point of u (see for example [13, p. 63]). □

6. Finite-Rank Operators

The function $\lambda \mapsto k_\lambda^u$ on $\mathcal{D}$ is conjugate holomorphic. For j a natural number, the derivative $d^j k_\lambda^u/d\bar{\lambda}^j$ is the kernel function for the functional on K_u^2 of evaluation of the j^{th} derivative at λ:

$$f^{(j)}(\lambda) = \left\langle f, \frac{d^j k_\lambda^u}{d\bar{\lambda}^j} \right\rangle.$$

The image of $d^j k_\lambda^u/d\bar{\lambda}^j$ under the conjugation C is $d^j\widetilde{k}_\lambda^u/d\lambda^j$.

THEOREM 6.1. *For n a natural number and λ in $\mathcal{D}$, the operators*

$$\sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{d^j\widetilde{k}_\lambda^u}{d\lambda^j} \otimes \frac{d^{n-j-1}k_\lambda^u}{d\bar{\lambda}^{n-j-1}} \right) \tag{6.1}$$

$$\sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{d^j k_\lambda^u}{d\bar{\lambda}^j} \otimes \frac{d^{n-j-1}\widetilde{k}_\lambda^u}{d\lambda^{n-j-1}} \right) \tag{6.2}$$

are in $\mathscr{T}(K_u^2)$, with respective symbols $(n-1)!u/(z-\lambda)^n$ and $(n-1)!\bar{u}/(\bar{z}-\bar{\lambda})^n$.

Proof. The adjoint operation transforms (6.1) into (6.2), so it will suffice to deal with (6.1). The case $n = 1$ was obtained in Section 5.: letting $\varphi_\lambda = u/(z-\lambda)$, we have

$$A_{\varphi_\lambda} = \widetilde{k}_\lambda^u \otimes k_\lambda^u. \tag{6.3}$$

To obtain the desired conclusion one simply applies $d^{n-1}/d\lambda^{n-1}$ to both sides of (6.3), using the Leibniz formula for the derivative of a bilinear expression on the right side.

□

COROLLARY. *If r is a rational function without poles on $\partial\mathcal{D}$, then the operators A_{ru} and $A_{r\bar{u}}$ have finite rank.*

Proof. It will suffice to consider A_{ru}, because $A_{r\bar{u}} = A_{r^*u}^*$, where $r^*(z) = \overline{r(1/\bar{z})}$. In case r has only one pole, Theorem 6.1 implies A_{ru} has finite rank, from which the general case obviously follows. □

There is a boundary version of Theorem 6.1 which requires some preliminaries. It was noted in Section 2.2. that u has an ADC at the point η of $\partial\mathbb{D}$ if and only if every function in K_u^2 has a nontangential limit at η. For n a natural number, one says that u has an ADC of order n at η if each function in K_u^2 and its derivatives up to order $n-1$ have nontangential limits at η. If that happens, then u and its first n derivatives have nontangential limits at η. Moreover, for $j=0,\ldots,n-1$, the functions $d^j k_\lambda^u/d\bar{\lambda}^j$ and $d^j \widetilde{k}_\lambda^u/d\lambda^j$ converge in norm as λ tends nontangentially to η. The limit functions are denoted by $d^j k_\eta^u/d\bar{\eta}^j$ and $d^j \widetilde{k}_\eta^u/d\eta^j$. See [1], [16] for more information.

THEOREM 6.2. *Let u have an ADC of order n at the point η of $\partial\mathbb{D}$. Then the operator*

$$\sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{d^j \widetilde{k}_\eta^u}{d\eta^j} \otimes \frac{d^{n-j-1} k_\eta^u}{d\bar{\eta}^j} \right) \tag{6.4}$$

and its adjoint belong to $\mathscr{T}(K_u^2)$.

Proof. In view of the preceding remarks, as λ tends nontangentially to η the operator (6.1) converges in norm to the operator (6.4). Because $\mathscr{T}(K_u^2)$ is closed in operator norm, the desired conclusion follows. □

The following questions remain open.

What is the most general finite-rank operator in $\mathscr{T}(K_u^2)$? Is every such operator a finite linear combination of the operators in Theorems 6.1 and 6.2?

What is the general rank-two self-adjoint operator in $\mathscr{T}(K_u^2)$?

7. Finite-Dimensional Case

THEOREM 7.1. *Let K_u^2 have finite dimension N.*
(a) The dimension of $\mathscr{T}(K_u^2)$ is $2N-1$.
(b) If $\lambda_1,\ldots,\lambda_{2N-1}$ are distinct points of $\mathbb{D}$, then the operators $k_{\lambda_j}^u \otimes \widetilde{k}_{\lambda_j}^u$, $j=1,\ldots,2N-1$, form a basis for $\mathscr{T}(K_u^2)$.

Proof. (a) Every operator in $\mathscr{T}(K_u^2)$ can be written as the sum of an operator with a symbol in H^∞ and an operator with a symbol in $\bar{H}^\infty$. The subspace of operators with symbols in H^∞ has dimension equal to the dimension of H^∞/uH^∞, which is N. The subspace of operators with symbols in $\bar{H}^\infty$ (consisting of the adjoints of the operators in the preceding subspace) also has dimension N. The intersection of the two subspaces consists of the scalar multiples of the identity operator; its dimension is 1. Hence $\dim \mathscr{T}(K_u^2) = N+N-1$.

(b) We first show that if $\lambda_1,\ldots,\lambda_N$ are distinct points of $\mathbb{D}$ then the kernel functions $k_{\lambda_j}^u$, $j=1,\ldots,N$, are linearly independent. In fact, because K_u^2 has dimension N, the inner function u is a Blaschke product of order N, say with zeros $z_1,\ldots,z_N$. The general function in K_u^2 is of the form p/q, where $q(z)=\Pi_{n=1}^N(1-\bar{z}_n z)$, and p is a polynomial of degree at most $N-1$. Hence, given distinct points $\lambda_1,\ldots,\lambda_N$ in $\mathbb{D}$, and given any complex numbers $w_1,\ldots,w_N$, there is a function f in K_u^2 such that $f(\lambda_j) = w_j$, $j = 1, \dots, N$, implying the asserted linear independence of k_{λ_j}, $j = 1, \dots, N$.

Now let $\lambda_1, \dots, \lambda_{2N-1}$ be distinct points of $\mathcal{D}$. By (a), it will suffices to show that the operators $k^u_{\lambda_j} \otimes \widetilde{k}^u_{\lambda_j}$, $j = 1, \dots, 2N-1$, are linearly independent. Let $a_1, \dots, a_{2N-1}$ be scalars such that

$$\sum_{j=1}^{2N-1} a_j (k^u_{\lambda_j} \otimes \widetilde{k}^u_{\lambda_j}) = 0.$$

It will suffice to show that $a_1 = 0$. The linear independence of $k^u_{\lambda_j}$, $j = 1, \dots, N$, implies the linear independence of $\widetilde{k}^u_{\lambda_j}$, $j = 1, \dots, N$, so there is a function g in K^2_u such that $\langle g, \widetilde{k}^u_{\lambda_1} \rangle = 1$ and $\langle g, \widetilde{k}^u_{\lambda_j} \rangle = 0$ for $j = 2, \dots, N$. Then

$$\begin{aligned} 0 &= \sum_{j=1}^{2N-1} a_j (k^u_{\lambda_j} \otimes \widetilde{k}^u_{\lambda_j}) g \\ &= a_1 k^u_{\lambda_1} + \sum_{j=N+1}^{2N-1} a_j \langle g, \widetilde{k}^u_{\lambda_j} \rangle \, k^u_{\lambda_j}. \end{aligned}$$

Since $k^u_{\lambda_1}, k^u_{\lambda_{N+1}}, \dots, k^u_{\lambda_{2N-1}}$ are linearly independent, it follows that $a_1 = 0$. $\square$

The question whether every operator in $\mathscr{T}(K^2_u)$ has a bounded symbol has a trivial affirmative answer if $\dim \mathscr{T}(K^2_u)$ is finite. However, there are quantitative versions of the question that are nontrivial, and to my knowledge unsettled, in the finite-dimensional case. For example:

Let K^2_u be finite dimensional. What is the least upper bound of

$$\inf\{\|\varphi\|_\infty : A_\varphi = A\}$$

as A ranges over the operators in $\mathscr{T}(K^2_u)$ of unit norm? What is the supremum of that least upper bound as K^2_u ranges over all spaces of the same dimension?

The case $u(z) = z^N$ is interesting. Let A be an operator in $\mathscr{T}(K^2_u)$ for this u, and let $(a_{j-k})_{j,k=0}^{N-1}$ be the matrix for A with respect to the monomial basis. We have $A = A_\varphi$ precisely when $\hat{\varphi}(n) = a_n$ for $n = -N+1, \dots, N-1$. The first of the two preceding questions for this situation can be formulated thus:

Among all sequences $a_{-N+1}, \dots, a_{N-1}$ of complex numbers such that the matrix $(a_{j-k})_{j,k=0}^{N-1}$ has norm 1 what is the supremum of $\inf\{\|\varphi\|_\infty : \hat{\varphi}(n) = a_n,\ n = -N+1, \dots, N-1\}$*?*

The analogous question in which the matrix for A is restricted to be lower triangular (or upper triangular) is settled by the solution of the classical Carathéodory interpolation problem (see, for example, [15]): If the matrix $(a_{j-k})_{j,k=0}^{\infty}$ is lower triangular and of unit norm, and if the function φ in H^∞ satisfies $\varphi(z) = a_0 + a_1 z + \cdots + a_{N-1} z^{N-1} + O(z^N)$, then $\|\varphi\|_\infty \geqslant 1$, and equality holds for a unique φ, a Blaschke product of order less than N. As will be seen shortly, the same bound does not hold in the general case.

If our sequence $a_{-N+1}, \ldots, a_{N-1}$ satisfies $a_{-n} = \bar{a}_n$ then the corresponding operator A is self-adjoint. If $A = A_\varphi$ in this case, then also $A = A_{\operatorname{Re}\varphi}$. So, to find the infimum of interest, we need only consider real φ such that $A = A_\varphi$. In that case we can write $\varphi = h + \bar{h}$ with h in H^2, and with $\frac{a_0}{2} + a_1 z + \cdots + a_{N-1} z^{N-1}$ as its Taylor polynomial of order $N-1$ at the origin. We obtain the following questions:

Given a self-adjoint Toeplitz matrix $(a_{j-k})_{j,k=0}^{N-1}$, what is the infimum of $2\|\operatorname{Re} h\|_\infty$ over all functions h in H^2 with power series of the form $\frac{a_0}{2} + a_1 z + \cdots + a_{N-1} z^{N-1} + O(z^N)$? What is the supremum of that infimum under the condition that the matrix $(a_{j-k})_{j,k=0}^{N-1}$ have unit norm?

The next theorem answers these questions for the case $N = 2$. The method of proof seems not to generalize easily to larger N. We note that the norm of the two-by-two Toeplitz matrix $\begin{pmatrix} a & \bar{b} \\ b & a \end{pmatrix}$ $(a \in \mathcal{R})$ equals $|a| + |b|$.

THEOREM 7.2. *Let a be a real number and b a nonzero complex number.*

(a) *If the function h in H^2 satisfies $h(0) = a/2$, $h'(0) = b$, and if $c = 2\|Reh\|_\infty < \infty$, then $\frac{2c}{\pi} \cos \frac{\pi a}{2c} \geqslant |b|$. Equality holds for a unique function h.*

(b) *The supremum of $\frac{c}{|a|+|b|}$ over all a, b, c satisfying $\frac{2c}{\pi} \cos \frac{\pi a}{2c} = |b|$ equals $\frac{\pi}{2}$ and is attained only for $a = 0$.*

Proof. (a) The extremal function h will be determined in the course of the proof. Let h be as described. The strategy is to compose h with a conformal map of the strip $|\operatorname{Re} z| < \frac{c}{2}$ onto $\mathcal{D}$ and then to apply Schwarz's lemma.

Define the function h_0 in $\mathcal{D}$ by

$$h_0(z) = -\frac{ic}{\pi} \operatorname{Log} \left(\frac{1+z}{1-z} \right);$$

here Log denotes the principal branch of log. The function h_0 is a conformal map of $\mathcal{D}$ onto the strip $|\operatorname{Re} z| < \frac{c}{2}$. A short calculation shows that the inverse h_0^{-1} is given by $h_0^{-1}(z) = i \tan \frac{\pi z}{2c}$. Let $g = h_0^{-1} \circ h$, a self-map of $\mathcal{D}$. Letting $\alpha = g(0)$, $\beta = g'(0)$, we have $\alpha = i \tan \frac{\pi a}{4c}$ and $\beta = \frac{\pi i b}{2c} \sec^2 \frac{\pi a}{4c}$. The function $\frac{g-\alpha}{1-\bar{\alpha} g}$ maps $\mathcal{D}$ into $\mathcal{D}$ and vanishes at 0. An application of Schwarz's lemma gives $|g'(0)| \leqslant 1 - |\alpha|^2$, i.e.,

$$\frac{\pi|b|}{2c} \sec^2 \frac{\pi a}{4c} \leqslant 1 - \tan^2 \frac{\pi a}{4c}.$$

Using the identity $(1 - \tan^2\theta)/\sec^2\theta = \cos 2\theta$, we can rewrite the preceding inequality as

$$\frac{2c}{\pi} \cos \frac{\pi a}{2c} \geqslant |b|, \tag{7.1}$$

the desired inequality. In particular, $c \geqslant \frac{\pi|b|}{2}$.

The conditions $h(0) = \frac{a}{2}$, $h'(0) = b \neq 0$ imply that $c > |a|$, giving $-\frac{\pi}{2} < \frac{\pi a}{2c} < \frac{\pi}{2}$. The inequality $c \geqslant \frac{\pi|b|}{2}$ mentioned above can become an equality only for $a = 0$.

If (7.1) becomes an equality then, by the condition for equality in Schwarz's lemma, $\frac{g(z)-\alpha}{1-\bar{\alpha} g(z)} = \omega z$ for a unimodular constant ω. One can determine ω by computing the derivative of $\frac{g-\alpha}{1-\bar{\alpha}z}$ at 0. After a short calculation one finds that $\omega = ib/|b|$. Since $g(z) = \frac{\omega z+\alpha}{1+\bar{\alpha}\omega z}$, the extremal function in case (7.1) reduces to an equality is the function

$$h(z) = h_0\left(\frac{\omega z+\alpha}{1+\bar{\alpha}\omega z}\right),$$

a conformal map of $\mathcal{D}$ onto the strip $|\operatorname{Re} z| < \frac{c}{2}$.

(b) One easily checks that the function $x\cos\frac{1}{x}$ is strictly increasing on the interval $\left[\frac{2}{\pi},\infty\right)$. It follows that for each a and b there is a unique c for which (7.1) becomes an equality, When $a = 0$, (7.1) becomes an equality for $c = \pi|b|/2$, in which case $\frac{c}{|a|+|b|} = \frac{\pi}{2}$. It remains to show that $\frac{c}{|a|+|b|} < \frac{\pi}{2}$ in other cases where (7.1) is an equality.

The general case can be reduced to the case $a = 1$. Namely, if a,b,c make (7.1) an equality and ρ is a nonzero real number, then $\rho a, \rho b, |\rho|c$ also make (7.1) an equality. It remains to show that $\frac{c}{1+b} < \frac{\pi}{2}$ if $c > 1$, $b > 0$, and $\frac{2c}{\pi}\cos\frac{\pi}{2c} = b$. With the preceding conditions we have

$$\frac{1+b}{c} = \frac{1}{c} + \frac{2}{\pi}\cos\frac{\pi}{2c}.$$

Denoting by $f(c)$ the function on the right side, we have

$$f'(c) = -\frac{1}{c^2} + \frac{1}{c^2}\sin\frac{\pi}{2} = -\frac{1}{c^2}\left(1-\sin\frac{\pi}{2c}\right),$$

which is negative for $c > 1$. Thus f decreases on $(1,\infty)$. Also $f(c) \to \frac{2}{\pi}$ as $c \to \infty$. Hence $f(c) > \frac{2}{\pi}$ on $(1,\infty)$, and $\frac{1+b}{c} > \frac{2}{\pi}$, the desired conclusion. $\square$

COROLLARY. *Let $u(z) = z^2$. Let A be the operator in $\mathscr{T}(K_u^2)$ whose matrix with respect to the monomial basis is $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. If $A = A_\varphi$ with φ real, then $\|\varphi\|_\infty \geqslant \frac{\pi}{2}$. Equality holds for the unique function φ_0 given by*

$$\varphi_0(e^{i\theta}) = \operatorname{Arg}\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right) = \begin{cases} \pi/2, & 0<\theta<\pi \\ -\pi/2, & -\pi<\theta<0. \end{cases} \tag{7.2}$$

Proof. The corollary is the special case of the theorem where $a = 0$ and $b = -i$. Part (a) of the theorem gives, with c as defined in the theorem, $c \geqslant \frac{\pi}{2}$. By the way c is defined, this implies $\|\varphi\|_\infty \geqslant \frac{\pi}{2}$ if φ is real and $A = A_\varphi$. By part (a) and its proof, there is a unique φ_0 such that $\|\varphi_0\|_\infty = \frac{\pi}{2}$ and $A = A_{\varphi_0}$, given by

$$\varphi_0(z) = \operatorname{Re}\left(i\operatorname{Log}\left(\frac{1+z}{1-z}\right)\right),$$

from which (7.2) follows. (In the present case, α and ω from the proof of (a) are given by $\alpha = 0$, $\omega = 1$. Equality is achieved in (7.1) when $h = h_0$.) $\square$

8. Shift Invariance

Given a bounded operator A on K_u^2, we let Q_A denote the associated quadratic form on K_u^2:

$$Q_A(f) = \langle Af, f \rangle .$$

We shall say that A is shift invariant if $Q_A(f) = Q_A(Sf)$ whenever f and Sf are both in K_u^2. If this happens then, by the polarization identity, we also have $\langle Af, g\rangle = \langle ASf, Sg\rangle$ whenever f, g, Sf, Sg are in K_u^2.

We remark that if f is in K_u^2, then Sf is in K_u^2 if and only if f is orthogonal to $\widetilde{k}_0^u$. This follows from the equality $I - S_u^* S_u = \widetilde{k}_0^u \otimes \widetilde{k}_0^u$ (Lemma 2.4), according to which $\|f\| = \|S_u f\|$ $(= \|P_u Sf\|)$ if and only if f is orthogonal to $\widetilde{k}_0^u$.

THEOREM 8.1. *A bounded operator on K_u^2 belongs to $\mathscr{T}(K_u^2)$ if and only if it is shift invariant.*

Proof. Suppose the operator $A = A_{\psi+\bar{\chi}}$ is in $\mathscr{T}(K_u^2)$, where ψ and χ are in H^2. To prove A is shift invariant, it will suffice to show that $Q_A(Sf) = Q_A(f)$ for all f in $(\widetilde{k}_0^u)^\perp \cap K_u^\infty$. (This is so because K_u^∞ is dense in K_u^2, and $\widetilde{k}_0^u$ is itself in K_u^∞, implying that $(\widetilde{k}_0^u)^\perp \cap K_u^\infty$ is dense in $(\widetilde{k}_0^u)^\perp$.)

Let f be in $(\widetilde{k}_0^u)^\perp \cap K_u^\infty$. Because S_u commutes with A_ψ and with A_χ, we get

$$\begin{aligned} Q_A(Sf) &= \langle A_\psi S_u f, S_u f\rangle + \langle S_u f, A_\chi S_u f\rangle \\ &= \langle S_u A_\psi f, S_u f\rangle + \langle S_u f, S_u A_\chi f\rangle \\ &= \langle A_\psi f, S_u^* S_u f\rangle + \langle S_u^* S_u f, A_\chi f\rangle . \end{aligned}$$

Because $S_u f = Sf$ we have $S_u^* S_u f = f$, so the right side reduces to $\langle A_{\psi+\bar{\chi}} f, f\rangle = Q_A(f)$, which establishes the shift invariance of A.

For the other direction, suppose the bounded operator A on K_u^2 is shift invariant. We shall prove A is in $\mathscr{T}(K_u^2)$ by showing that A satisfies the criterion in the remark following the proof of Theorem 4.1.

Let $B = A - S_u^* A S_u$. The shift invariance of A implies that Q_B annihilates $(\widetilde{k}_0^u)^\perp$: if the function f in K_u^2 is orthogonal to $\widetilde{k}_0^u$ then

$$\begin{aligned} Q_B(f) &= \langle Af, f\rangle - \langle S_u^* A S_u f, f\rangle \\ &= \langle Af, f\rangle - \langle ASf, Sf\rangle = 0. \end{aligned}$$

By the polarization identity, then, $\langle Bf, g\rangle = 0$ whenever f and g are in $(\widetilde{k}_0^u)^\perp$; in other words, the compression of B to $(\widetilde{k}_0^u)^\perp$ is the zero operator.

The orthogonal projection in K_u^2 with range $(\widetilde{k}_0^u)^\perp$ equals $I - c(\widetilde{k}_0^u \otimes \widetilde{k}_0^u)$, where $c = \|\widetilde{k}_0^u\|_2^{-2}$. We have, then,

$$0 = (I - c(\widetilde{k}_0^u \otimes \widetilde{k}_0^u))B(I - c(\widetilde{k}_0^u \otimes \widetilde{k}_0^u)),$$

implying that

$$A - S_u^* A S_u = B = c(\widetilde{k}_0^u \otimes B^* \widetilde{k}_0^u) + c(B\widetilde{k}_0^u \otimes \widetilde{k}_0^u) - c^2 \langle B\widetilde{k}_0^u, \widetilde{k}_0^u\rangle\, (\widetilde{k}_0^u \otimes \widetilde{k}_0^u).$$

By Theorem 4.1 and the remark following its proof, A is in $\mathscr{T}(K_u^2)$. □

9. u-Compatible Measures

We let $K_u^{\infty+}$ denote the space of functions in K_u^2 that are continuous on $\bar{\mathbb{D}}$. A remarkable theorem of A. B. Aleksandrov states that $K_u^{\infty+}$ is dense in K_u^2; a clearly presented proof can be found in [9, Section 8.5]. Of course, if $\dim K_u^2$ is finite then $K_u^2 = K_u^{\infty+}$. This is a trivial case as far as the present section goes, so we shall assume for the remainder of the section that K_u^2 is infinite dimensional.

The finite positive Borel measure μ on $\partial\mathbb{D}$ will be called u-compatible if $K_u^{\infty+}$ is, relative to the K_u^2 norm, boundedly embedded in $L^2(\mu)$. Thus, μ is u-compatible if there is a positive constant c such that $\int |f|^2 d\mu \leqslant c\|f\|_2^2$ for all f in $K_u^{\infty+}$. If this happens then also $\left|\int f\bar{g}d\mu\right| \leqslant c\|f\|_2\|g\|_2$ for all f and g in $K_u^{\infty+}$, so, by the Aleksandrov density theorem, the sesquilinear functional

$$(f,g) \mapsto \int f\bar{g}d\mu$$

on $K_u^{\infty+} \times K_u^{\infty+}$ extends by continuity to a bounded sesquilinear functional on $K_u^2 \times K_u^2$. The operator on K_u^2 that induces the extended sesquilinear functional will be denoted by A_μ. It is a positive operator on K_u^2, of norm at most c, satisfying

$$\langle A_\mu f, g\rangle = \int f\bar{g}d\mu$$

for all f and g in $K_u^{\infty+}$.

THEOREM 9.1. *If the measure μ is u-compatible, then the operator A_μ lies in $\mathscr{T}(K_u^2)$.*

A lemma is needed.

LEMMA 9.1. *$K_u^{\infty+} \cap (\widetilde{k}_0^u)^\perp$ is dense in $(\widetilde{k}_0^u)^\perp$.*

Proof. By Aleksandrov's theorem there is a function h in $K_u^{\infty+}$ such that $\langle h, \widetilde{k}_0^u\rangle = 1$. Let g belong to $(\widetilde{k}_0^u)^\perp$. By Aleksandrov's theorem, again, there is a sequence $(f_n)_1^\infty$ in $K_u^{\infty+}$ converging in norm to g. Then $\langle f_n, \widetilde{k}_0^u\rangle \to 0$. Thus, the functions $f_n - \langle f_n, \widetilde{k}_0^u\rangle h$ lie in $K_u^{\infty+} \cap (\widetilde{k}_0^u)^\perp$ and converge to g as $n \to \infty$. □

Proof of Theorem 9.1. By Theorem 8.1, it will suffice to show that the quadratic form Q_{A_μ} is shift invariant. By Lemma 9.1, it will suffice for this to show that $\int |Sf|^2 d\mu = \int |f|^2 d\mu$ for all f in $K_u^{\infty+} \cap (\widetilde{k}_0^u)^\perp$, which is obviously true because $|Sf| = |f|$ on $\partial\mathbb{D}$. □

A complex Borel measure ν on $\partial\mathbb{D}$ will be called u-compatible if its total variation $|\nu|$ is u-compatible. In that case, just as in the positive case, there is a corresponding operator A_ν in $\mathscr{T}(K_u^2)$, defined initially by $\langle A_\nu f, g\rangle = \int f\bar{g}d\nu$ $(f, g \in K_u^{\infty+})$. There are some open questions:

Is every operator in $\mathscr{T}(K_u^2)$ equal to A_ν for a u-compatible measure ν?

An affirmative answer would be implied by an affirmative answer to the question whether every operator in $\mathscr{T}(K_u^2)$ has a bounded symbol.

Is every positive operator in $\mathscr{T}(K_u^2)$ equal to A_μ for a positive u-compatible measure μ?

Can one characterize the positive u-compatible measures?

Aleksandrov [5] has answered a special case of the last question by characterizing the positive measures μ on $\partial\mathcal{D}$ such that K_u^2 embeds isometrically in $L^2(\mu)$; these are the positive u-compatible measures μ such that $A_\mu = I$. Aleksandrov's result: K_u^2 embeds isometrically in $L^2(\mu)$ if and only if the Poisson integral of μ equals $\operatorname{Re}\left(\frac{1+uh}{1-uh}\right)$ with h a function in the unit ball of H^∞. The next theorem answers another (and simpler) case of the last question.

THEOREM 9.2. *For η a point of $\partial\mathcal{D}$, the point mass δ_η is u-compatible if and only if u has an ADC at η.*

Proof. One sees immediately that if u has an ADC at η then δ_η is u-compatible, and $A_{\delta_\eta} = k_\eta^u \otimes k_\eta^u$.

For the other direction, assume δ_η is u-compatible. One could rely here on a theorem of Aleksandrov [4] which implies that, if the positive measure μ is u-compatible, then the functions in K_u^2 have nontangential limits almost everywhere with respect to μ. In the case $\mu = \delta_\eta$, this tells us that the functions in K_u^2 have nontangential limits at η, which (as noted in Section 2.2.) implies u has an ADC at η. An alternative approach will be presented.

Since δ_η is u-compatible, the linear functional $f \mapsto f(\eta)$ extends from $K_u^{\infty+}$ to a bounded linear functional on K_u^2. Let h be the function in K_u^2 that induces the functional. For f and g in $K_u^{\infty+}$ we have

$$\begin{aligned} \langle A_{\delta_\eta} f, g\rangle &= f(\eta)\overline{g(\eta)} \\ &= \langle (h\otimes h)f, g\rangle, \end{aligned}$$

from which one concludes (by Aleksandrov's density theorem) that $A_{\delta_\eta} = h \otimes h$.

By Theorem 5.1, the rank-one operators in $\mathscr{T}(K_u^2)$ are the scalar multiples of the operators $k_\lambda^u \otimes \widetilde{k}_\lambda^u$ and $\widetilde{k}_\lambda^u \otimes k_\lambda^u$ with λ in $\mathcal{D}$, and the scalar multiples of the operators $k_{\eta'}^u \otimes k_{\eta'}^u$, with η' a point of $\partial\mathcal{D}$ at which u has an ADC. A simple argument shows that, for λ in $\mathcal{D}$, the functions k_λ^u and $\widetilde{k}_\lambda^u$ are linearly independent except when u is a particular Blaschke factor, a case excluded at the beginning of the section by the assumption that K_u^2 is infinite dimensional. It must be, then, that h is a scalar multiple of $k_{\eta'}^u$ for some η' on $\partial\mathcal{D}$ at which u has an ADC. It remains to prove that $\eta' = \eta$. By Lemma 9.2 below, if $\eta' \neq \eta$ then there is a function f in $K_u^{\infty+}$ such that $f(\eta') \neq 0 = f(\eta)$. Since $f(\eta) = 0$ we have $\langle A_{\delta_\eta} f, f\rangle = |f(\eta)|^2 = 0$, implying that $A_{\delta_\eta} f = 0$, but since $f(\eta') \neq 0$ we have $A_{\delta_\eta} f = \text{const.}(k_{\eta'}^u \otimes k_{\eta'}^u)f = \text{const.} f(\eta'))k_{\eta'}^u \neq 0$, a contradiction. □

COROLLARY. *If the positive u-compatible measure μ has a point mass at the point η of $\partial\mathcal{D}$, then u has an ADC at η.*

Proof. Under the hypotheses δ_η is u-compatible, and Theorem 9.2 applies. □

LEMMA 9.2. *If η and η' are distinct points of $\partial\mathbb{D}$, then there is a function f in $K_u^{\infty+}$ such that $f(\eta') \neq 0 = f(\eta)$.*

Proof. Choose a linearly independent pair f_1, f_2 of functions in $K_u^{\infty+}$. If the pairs $(f_1(\eta), f_1(\eta'))$, $(f_2(\eta), f_2(\eta'))$ are linearly independent, then a linear combination of f_1 and f_2 has the desired property. In the contrary case there is a nontrivial linear combination f_3 of f_1 and f_2 such that $f_3(\eta) = f_3(\eta') = 0$. Applying a suitable power of S^* to f_3, we obtain a function f_4 in $K_u^{\infty+}$ such that $f_4(\eta) = f_4(\eta') = 0$ and $f_4(0) \neq 0$. The function

$$f_5 = S^*f_4 - \left(\frac{(S^*f_4)(0)}{f_4(0)}\right) f_4$$

is then in $K_u^{\infty+}$ and satisfies $f_5(0) = 0$, $f_5(\eta) = -\bar{\eta} f_4(0)$, $f_5(\eta') = -\bar{\eta}' f_4(0)$, and the function $f_6 = S^*f_5$ is in $K_u^{\infty+}$ and satisfies $f_6(\eta) = -\bar{\eta}^2 f_4(0)$, $f_6(\eta') = -\bar{\eta}'^2 f_4(0)$. A linear combination of f_5 and f_6 has the desired property. □

10. Modified Compressed Shifts

The compressed shift S_u agrees with S on the subspace $(\widetilde{k}_0^u)^\perp$ of K_u^2, and it maps $\widetilde{k}_0^u$ to $-u(0)k_0^u$ (see Lemma 2.2(a)). For c a complex number, we define the operator $S_{u,c}$ by $S_{u,c} = S_u + c(k_0^u \otimes \widetilde{k}_0^u)$. Thus $S_{u,c}$, being the sum of two operators in $\mathscr{T}(K_u^2)$, is itself in $\mathscr{T}(K_u^2)$. It agrees with S on $(\widetilde{k}_0^u)^\perp$ and maps $\widetilde{k}_0^u$ to $(c(1-|u(0)|^2) - u(0))k_0^u$ (since $\|\widetilde{k}_0^u\|_2^2 = \|k_0^u\|_2^2 = 1 - |u(0)|^2$).

The next theorem extends the criterion from Theorem 4.1 for an operator to belong to $\mathscr{T}(K_u^2)$.

THEOREM 10.1. *Let c be a complex number. The bounded operator A on K_u^2 belongs to $\mathscr{T}(K_u^2)$ if and only if there are functions ψ and χ in K_u^2 such that*

$$A - S_{u,c}AS_{u,c}^* = (\psi \otimes k_0^u) + (k_0^u \otimes \chi).$$

Proof. A calculation gives

$$\begin{aligned} A - S_uAS_u^* \;&=\; A - S_{u,c}AS_{u,c}^* \\ &\quad + c(k_0^u \otimes S_uA^*\widetilde{k}_0^u) + \bar{c}(S_uAk_0^u\widetilde{k}_0^u \otimes k_0^u) \\ &\quad + |c|^2\langle A\widetilde{k}_0^u, \widetilde{k}_0^u\rangle\,(k_0^u \otimes k_0^u), \end{aligned}$$

showing that the criterion in this theorem and the one in Theorem 4.1 imply each other. □

COROLLARY. *If the bounded operator A on K_u^2 commutes with $S_{u,c}$ for some c, then A belongs to $\mathscr{T}(K_u^2)$.*

Proof. A calculation, using the equality $S_u\widetilde{k}_0^u = -u(0)k_0^u$, gives

$$I - S_{u,c}S_{u,c}^* = (|1 + c\overline{u(0)}|^2 - |c|^2)(k_0^u \otimes k_0^u). \tag{10.1}$$

Hence, if $AS_{u,c} = S_{u,c}A$, then

$$\begin{aligned} A - S_{u,c}AS_{u,c}^* &= A(I - S_{u,c}S_{u,c}^*) \\ &= (|1+\bar{c}u(0)|^2 - |c|^2)(Ak_0^u \otimes k_0^u), \end{aligned}$$

and it follows by the theorem that A is in $\mathscr{T}(K_u^2)$. □

Recall from above that the operator $S_{u,c}$ maps $\widetilde{k}_0^u$ to $(c(1-|u(0|^2) - u(0))k_0^u$. The operator $S_{u,c}$ agrees with S on $(\widetilde{k}_0^u)^\perp$ and maps $(\widetilde{k}_0^u)^\perp$ isometrically onto $(k_0^u)^\perp$. If c is such that $c(1-|u(0)|^2) - u(0)$ has unit modulus, then $S_{u,c}$ is an isometry of K_u^2 onto itself, in other words, a unitary operator. For $|\alpha| = 1$, we let c_α denote the value of c such that $c(1-|u(0)|^2) - u(0) = \alpha$, namely $c_\alpha = (\alpha + u(0))/(1-|u(0)|^2)$, and we let U_α denote the unitary operator S_{u,c_α}. The operators U_α were first studied by D. N. Clark [10] who, in particular, found concrete spectral representations for them, and noted that they are the only rank-one unitary perturbations of S_u. The following section is devoted to background material on Clark's unitary perturbations.

By equality (10.1) from the preceding proof, the numbers c_α satisfy $|c_\alpha| = |1 + c_\alpha\overline{u(0)}|$, which one can easily check directly; in fact, $1 + c_\alpha\overline{u(0)} = \alpha\bar{c}_\alpha$.

11. Clark's Unitary Perturbations

This section contains background on the operators U_α, including Clark's spectral representation and mention of subsequent results of Aleksandrov and A. G. Poltoratski. Some proofs are included.

11.1. Clark Measures. For α on $\partial\mathfrak{D}$ we let μ_α be the measure on $\partial\mathfrak{D}$ whose Poisson integral is the real part of the function $\frac{\alpha+u}{\alpha-u}$. Thus, we have the Herglotz integral representation

$$\frac{\alpha + u(z)}{\alpha - u(z)} = \int \frac{\zeta + z}{\zeta - z} d\mu_\alpha(\zeta) + i\,\mathrm{Im}\,\frac{\alpha + u(0)}{\alpha - u(0)} \quad (|z| < 1). \tag{11.1}$$

The function $\mathrm{Re}\,\frac{\alpha+u}{\alpha-u} = \frac{1-|u|^2}{|\alpha-u|^2}$ has the nontangential limit 0 almost everywhere (with respect to m) on $\partial\mathfrak{D}$, implying that μ_α is a singular measure. It also has nontangential limit ∞ almost everywhere with respect to μ_α, implying that u has nontangential limit α almost everywhere with respect to μ_α. The measures μ_α are thus mutually singular.

Aleksandrov [3] proved that the measures μ_α form a disintegration of Lebesgue measure: one has

$$m = \int \mu_\alpha dm(\alpha)$$

in the sense that

$$\int f\,dm = \int \left(\int f\,d\mu_\alpha\right) dm(\alpha) \tag{11.2}$$

for every continuous function f on $\partial\mathfrak{D}$. While the proof of this is not difficult, Aleksandrov also established the deeper result that (11.2) extends to L^1 functions: if f is an integrable Borel function on $\partial\mathfrak{D}$ then f is in $L^1(\mu_\alpha)$ for almost every α, and (11.2) holds. Full details can be found in [9, pp. 212–217].

11.2. Clark's Transforms. A key step in Clark's analysis of the operators U_α is the recognition that, for each α, there is a natural isometry, involving Cauchy integrals, mapping $L^2(\mu_\alpha)$ onto K_u^2. The Cauchy integral $\mathscr{K}\nu$ of the finite complex Borel measure ν on $\partial\mathcal{D}$ is defined by

$$(\mathscr{K}\nu)(z) = \int \frac{1}{1-\bar{\zeta}z} d\nu(\zeta).$$

The integral on the right is well defined for z off the support of ν, but here our concern is $|z| < 1$. We shall also be concerned with the case where ν is singular. In that case $\mathscr{K}\nu$ is zero only if $\nu = 0$. In fact, the n^{th} Taylor coefficient of $\mathscr{K}\nu$ at the origin is $\int \bar{\zeta}^n d\nu(\zeta)$. If this vanishes for all n, and ν is singular, the F. and M. Riesz theorem implies that $\nu = 0$.

For $|\alpha| < 1$ we define the linear map $V_\alpha : L^2(\mu_\alpha) \to H(\mathcal{D})$ by $V_\alpha q = (1 - \bar{\alpha}u)\mathscr{K}(q\mu_\alpha)$.

THEOREM A (Clark [10]). *The map V_α is an isometry of $L^2(\mu_\alpha)$ onto K_u^2.*

Proof. For λ and z in $\mathcal{D}$ we reexpress the inner product

$$\langle k_\lambda, k_z\rangle_{L^2(\mu_\alpha)} = \int \frac{1}{(1-\bar{\lambda}\zeta)(1-z\bar{\zeta})} d\mu_\alpha(\zeta). \tag{11.3}$$

The integrand in the integral on the right can be rewritten as

$$\frac{1}{2(1-\bar{\lambda}z)}\left[\frac{\bar{\zeta}+\bar{\lambda}}{\bar{\zeta}-\bar{\lambda}} + \frac{\zeta+z}{\zeta-z}\right],$$

so that, by (11.1),

$$\langle k_\lambda, k_z\rangle_{L^2(\mu_\alpha)} = \frac{1}{2(1-\bar{\lambda}z)}\left[\frac{\bar{\alpha}+\overline{u(\lambda)}}{\bar{\alpha}-\overline{u(\lambda)}} + \frac{\alpha+u(z)}{\alpha-u(z)}\right].$$

After a bit of algebra one finds that

$$\langle k_\lambda, k_z\rangle_{L^2(\mu_\alpha)} = (1-\alpha\overline{u(\lambda)})^{-1}(1-\bar{\alpha}u(z))^{-1}k_\lambda^u(z). \tag{11.4}$$

Looking back at (11.3), we see (11.4) tells us first that $V_\alpha k_\lambda = (1-\alpha\overline{u(\lambda)})^{-1}k_\lambda^u$, and second that $\langle k_\lambda, k_z\rangle_{L^2(\mu_\alpha)} = \langle V_\alpha k_\lambda, V_\alpha k_z\rangle$. Hence V_α maps the linear span of the functions k_λ in $L^2(\mu_\alpha)$ isometrically onto the linear span of the functions k_λ^u in K_u^2. A limit argument then shows V_α is an isometry from the closure of the first span onto the closure of the second span. The latter closure is clearly K_u^2. The former one is $L^2(\mu_\alpha)$, for a function q in $L^2(\mu_\alpha)$ orthogonal to it satisfies $\mathscr{K}(q\mu_\alpha) = 0$, implying (as remarked above) that $q = 0$. □

It should be noted that Clark's original work has been vastly generalized, for example in [6], and the treatment here has benefited from this subsequent work, although the essentials are already in [10]. Instead of V_α, Clark worked with V_α^{-1}, which, by a remarkable theorem of Poltoratski [14], is a standard boundary map. Poltoratski's theorem: *Each function f in K_u^2 has a nontangential limit μ_α-almost everywhere on $\partial\mathbb{D}$ coinciding with $V_\alpha^{-1}f$.*

Because u has the nontangential limit α almost everywhere with respect to μ_α, Poltoratski's theorem is easy to verify for the functions k_λ^u and $\widetilde{k}_\lambda^u$. From this alone one can see that V_α^{-1} transforms the conjugation C on K_u^2 into the conjugation $q(\zeta) \mapsto \alpha\overline{\zeta q(\zeta)}$ on $L^2(\mu_\alpha)$.

11.3. Transform of S_u. We let Z_α denote the canonical multiplication operator on $L^2(\mu_\alpha)$, the operator of multiplication by the coordinate function.

LEMMA 11.1. $V_\alpha^{-1}S_uV_\alpha = Z_\alpha - (1-\alpha\overline{u(0)})(1\otimes Z_\alpha^*1)$.

Proof. We shall establish the corresponding expression for $V_\alpha^{-1}S_u^*V_\alpha$, from which the desired one follows immediately. Working with S_u^* is convenient because $S_u^* = S^* \mid K_u^2$, and S^* interacts with the Cauchy integral in a simple way:

$$(S^*\mathscr{K}\nu)(z) = \int \frac{\bar{\zeta}}{1-\bar{\zeta}z}d\nu(\zeta). \tag{11.5}$$

We shall need the expressions for $V_\alpha^{-1}k_0^u$ and $V_\alpha^{-1}\widetilde{k}_0^u$. Setting $\lambda = 0$ in the expression for $V_\alpha k_\lambda$ found earlier, we see that $V_\alpha^{-1}k_0^u$ is the constant function $(1-\alpha\overline{u(0)})$. Applying a conjugation, we get $V_\alpha^{-1}\widetilde{k}_0^u = \alpha(1-\bar{\alpha}u(0))Z_\alpha^*1$.

Let q be a function in $L^2(\mu_\alpha)$. Using (11.5) we get

$$\begin{aligned} S_u^*V_\alpha q &= S^*((1-\bar{\alpha}u)\mathscr{K}(q\mu_\alpha)) \\ &= (1-\bar{\alpha}u)S^*\mathscr{K}(q\mu_\alpha) + (\mathscr{K}(q\mu_\alpha))(0)S^*(1-\bar{\alpha}u) \\ &= V_\alpha Z_\alpha^* q - \bar{\alpha}\langle q, 1\rangle_{L^2(\mu_\alpha)}\widetilde{k}_0^u. \end{aligned}$$

Applying V_α^{-1} to this equality and using the expression above for $V_\alpha^{-1}\widetilde{k}_0^u$, we obtain the desired equality. ☐

11.4. Clark's Spectral Representation. From the expressions for $V_\alpha^{-1}k_0^u$ and $V_\alpha^{-1}\widetilde{k}_0^u$ found in the preceding subsection, we have

$$\begin{aligned} V_\alpha^{-1}(k_0^u\otimes\widetilde{k}_0^u)V_\alpha &= (V_\alpha^{-1}k_0^u\otimes V_\alpha^{-1}\widetilde{k}_0^u) \\ &= \bar{\alpha}(1-\alpha\overline{u(0)})^2(1\otimes Z_\alpha^*1). \end{aligned}$$

This together with Lemma 11.1 enables us to determine how the operator $S_{u,c}$ is transformed under V_α^{-1}:

$$V_\alpha^{-1}S_{u,c}V_\alpha = Z_\alpha + (1-\alpha\overline{u(0)})(c(\bar{\alpha}-\overline{u(0)})-1)(1\otimes Z_\alpha^*1).$$

Thus, for the particular value $c = (\bar{\alpha}-\overline{u(0)})^{-1}$ we have $V_\alpha^{-1}S_{u,c}V_\alpha = Z_\alpha$, so $S_{u,c}$, being unitary, must equal U_β for some β on $\partial\mathbb{D}$. We denote this β by β_α. To find the expression for β_α in terms of α we note that, for $c = (\bar{\alpha}-\overline{u(0)})^{-1}$, we have

$$\begin{aligned} \beta_\alpha k_0^u &= S_{u,c}\widetilde{k}_0^u = (c(1-|u(0)|^2)-u(0))k_0^u \\ &= \left(\frac{1-|u(0)|^2}{\bar{\alpha}-\overline{u(0)}} - u(0)\right)k_0^u = \left(\frac{1-\bar{\alpha}u(0)}{\bar{\alpha}-\overline{u(0)}}\right)k_0^u. \end{aligned}$$

Hence

$$\beta_\alpha = \frac{1-\bar{\alpha}u(0)}{\bar{\alpha}-\overline{u(0)}} = \frac{\alpha - u(0)}{1-\overline{u(0)}\alpha}.$$

THEOREM B (Clark [10]). $V_\alpha^{-1} U_{\beta_\alpha} V_\alpha = Z_\alpha$.

12. Clark's Unitary Perturbations – The Sequel

Let α be a point of $\partial\mathfrak{D}$, and let $\beta_\alpha = \frac{\alpha-u(0)}{1-\overline{u(0)}\alpha}$, as in Clark's Theorem B, which says $V_\alpha^{-1} U_{\beta_\alpha} V_\alpha = Z_\alpha$. By the corollary to Theorem 10.1, the commutant of U_{β_α} is contained in $\mathscr{T}(K_u^2)$. That commutant is the transform under V_α of the commutant of Z_α, which consists of the multiplication operators on $L^2(\mu_\alpha)$ induced by the functions in $L^\infty(\mu_\alpha)$. Thus $\mathscr{T}(K_u^2)$ contains the von Neumann algebra $\{q(U_{\beta_\alpha}) : q \in L^\infty(\mu_\alpha)\}$.

The measure μ_α is u-compatible; in fact, it is one of the measures μ, characterized by Aleksandrov, with the property that $A_\mu = I$ (see Section 9.). So, if q is in $L^\infty(\mu_\alpha)$ then $q\mu_\alpha$ is also u-compatible. For f and g in K_u^2, thanks to Poltoratski's theorem, we can write

$$\begin{aligned} \langle\, q(U_{\beta_\alpha})f, g\rangle &= \langle\, qf, g\rangle_{L^2(\mu_\alpha)} = \int f\bar{g}q\,d\mu_\alpha \\ &= \langle A_{q\mu_\alpha} f, g\rangle. \end{aligned}$$

Thus $q(U_{\beta_\alpha}) = A_{q\mu_\alpha}$. The map $q \mapsto A_{q\mu_\alpha}$ is a $*$-isomorphism of $L^\infty(\mu_\alpha)$ onto the commutant of U_{β_α}.

In a certain very weak sense, the Clark unitary operators generate $\mathscr{T}(K_u^2)$. Let A_φ be an operator in $\mathscr{T}(K_u^2)$, where $\varphi = \psi + \bar{\chi}$ with ψ, χ in K_u^2. By Poltoratski's theorem, for α in $\partial\mathfrak{D}$, the functions ψ and χ extend via boundary limits to functions in $L^2(\mu_\alpha)$, coinciding respectively with $V_\alpha^{-1}\psi$ and $V_\alpha^{-1}\chi$. The function φ thus similarly extends. We can form the (possibly unbounded) operator $\varphi(Z_\alpha)$, which transforms under V_α to $\varphi(U_{\beta_\alpha})$. For f and g in $K_u^{\infty+}$ we use Aleksandrov's disintegration theorem to obtain

$$\begin{aligned} \langle A_\varphi f, g\rangle &= \int \varphi f\bar{g}\,dm = \int\left[\int \varphi f\bar{g}\,d\mu_\alpha\right] dm(\alpha) \\ &= \int \langle\, \varphi(Z_\alpha f, g\rangle_{L^2(\mu_\alpha)} dm(\alpha). \end{aligned}$$

Since $\langle\, \varphi(Z_\alpha)f, g\rangle_{L^2(\mu_\alpha)} = \langle\, \varphi(U_{\beta_\alpha})f, g\rangle$, we get

$$\langle A_\varphi f, g\rangle = \int \langle\, \varphi(U_{\beta_\alpha})f, g\rangle\, dm(\alpha). \tag{12.1}$$

Thus,

$$A_\varphi = \int \varphi(U_{\beta_\alpha})\,dm(\alpha) \tag{12.2}$$

in the sense that (12.1) holds for all f and g in $K_u^{\infty+}$.

To illustrate the last result we consider the case $\varphi = k_\eta^u + \bar{k}_\eta^u$, where η is a singular point of u on $\partial\mathfrak{D}$ at which u has an ADC. For this φ we have $A_\varphi = I + (k_\eta^u \otimes k_\eta^u)$

(see the proof of Theorem 5.1(b)). It will be shown that the operator $\varphi(U_{\beta_\alpha})$ is unbounded except for two values of α, indicating that life is more complicated than the neat equality (12.2) suggests.

It is known that, because u has an ADC at η, the measure $\mu_{u(\eta)}$ has a mass point at η of magnitude $1/|u'(\eta)|$, while $\mu_\alpha(\{\eta\}) = 0$ for $\alpha \neq u(\eta)$ (see for example [16, pp. 51–52]. We first show that φ belongs to $L^\infty(\mu_\alpha)$ for $\alpha = u(\eta)$ and for $\alpha = -u(\eta)$.

The value of k_η^u at η is $|\varphi'(\eta)|$. Hence $\varphi(\eta) = 2|\varphi'(\eta)|$. However, if $u(\zeta) = u(\eta)$ and $\zeta \neq \eta$, then

$$k_\eta^u(\zeta) = \frac{1 - \overline{u(\eta)}u(\zeta)}{1 - \bar{\eta}\zeta} = \frac{1 - \overline{u(\eta)}u(\eta)}{1 - \bar{\eta}\zeta} = 0,$$

and so $\varphi(\zeta) = 0$. As a function in $L^\infty(\mu_{u(\eta)})$, then, φ equals $2|\varphi'(\eta)|$ times the characteristic function of the singleton $\{\eta\}$. Since $\mu_{u(\eta)}$ has a mass point at η of magnitude $1/|\varphi'(\eta)|$, we have $\varphi\mu_{u(\eta)} = 2\delta_\eta$. As $A_{\delta_\eta} = k_\eta^u \otimes k_\eta^u$, we see that $\varphi(U_{\beta_{u(\eta)}}) = 2(k_\eta^u \otimes k_\eta^u)$.

If $u(\zeta) = -u(\eta)$ then

$$\varphi(\zeta) = 2\,\mathrm{Re}\left(\frac{1 - \overline{u(\eta)}u(\zeta)}{1 - \bar{\eta}\zeta}\right) = 4\,\mathrm{Re}\,\frac{1}{1 - \bar{\eta}\zeta} = 2.$$

Hence, as a function in $L^2(\mu_{-u(\eta)})$, φ is the constant function 2, and we have $\varphi(U_{\beta_{-u(\eta)}}) = 2I$. (Note, incidentally, that A_φ is the average of $\varphi(U_{\beta_{u(\eta)}})$ and $\varphi(U_{\beta_{-u(\eta)}})$.)

Finally, assume α is different for $u(\eta)$ and from $-u(\eta)$. We show that φ is not in $L^\infty(\mu_\alpha)$, and hence that $\varphi(U_{\beta_\alpha})$ is unbounded. If $u(\zeta) = \alpha$ then

$$\varphi(\zeta) = \frac{1 - \overline{u(\eta)}\alpha}{1 - \bar{\eta}\zeta} + \frac{1 - u(\eta)\bar{\alpha}}{1 - \eta\bar{\zeta}} = 1 - \frac{\overline{u(\eta)}\alpha}{1 - \bar{\eta}\zeta} - \frac{u(\eta)\bar{\alpha}}{1 - \eta\bar{\zeta}},$$

so it will suffice to show that the function

$$\varphi_1(\zeta) = \frac{\overline{u(\eta)}\alpha}{1 - \bar{\eta}\zeta} + \frac{u(\eta)\bar{\alpha}}{1 - \eta\bar{\zeta}}$$

is not in $L^\infty(\mu_\alpha)$. A bit of algebra gives

$$\varphi_1(\zeta) = \frac{\mathrm{Re}(u(\eta)\bar{\alpha}(1 - \bar{\eta}\zeta))}{\mathrm{Re}(1 - \bar{\eta}\zeta)}.$$

We write $\zeta = e^{i\theta}$, $\eta = e^{i\theta_0}$, $u(\eta)\bar{\alpha} = e^{it_0}$. Since $\alpha \neq \pm u(\eta)$, we can assume $0 < |t_0| < \pi$. Then, by an easy calculation,

$$1 - \bar{\eta}z = -2i\exp\left(i\left(\frac{\theta - \theta_0}{2}\right)\right)\sin\left(\frac{\theta - \theta_0}{2}\right),$$

giving

$$\operatorname{Re}(1-\bar{\eta}\zeta) = 2\sin^2\left(\frac{\theta-\theta_0}{2}\right).$$

Similarly,

$$u(\eta)\bar{\alpha}(1-\bar{\eta}\zeta) = -2i\exp\left(i\left(\frac{\theta-\theta_0}{2}+t_0\right)\right)\sin\left(\frac{\theta-\theta_0}{2}\right),$$

giving

$$\operatorname{Re}(u(\eta)\bar{\alpha}(1-\bar{\eta}\zeta)) = 2\sin\left(\frac{\theta-\theta_0}{2}+t_0\right)\sin\left(\frac{\theta-\theta_0}{2}\right).$$

It follows that

$$\varphi_1(\zeta) = \sin\left(\frac{\theta-\theta_0}{2}+t_0\right)\Big/\sin\left(\frac{\theta-\theta_0}{2}\right).$$

As $\theta \to \theta_0$, i.e., as $\zeta \to \eta$, the denominator in the fraction on the right tends to 0, and the numerator tends to $\sin t_0$, which is not 0., The fraction thus tends to ∞. Since η is a singularity of u it lies in the support of the measure μ_α. The desired conclusion, that φ_1 is not in $L^\infty(\mu_\alpha)$, follows.

Question: *Is $\mathscr{T}(K_u^2)$ generated by the Clark unitaries in a stronger sense than the one described above?*

13. Crofoot's Transforms

For w in $\mathfrak{D}$ we let u_w denote the inner function $\frac{u-w}{1-\bar{w}u}$. We define the linear map $J_w : K_u^2 \to H^2$ by $J_w f = \sqrt{1-|w|^2} f/(1-\bar{w}u)$. The following theorem is a special case of a result of R. B. Crofoot [11].

THEOREM 13.1. *J_w is an isometry of K_u^2 onto $K_{u_w}^2$.*

Proof. We show first that $J_w K_u^2 \subset K_{u_w}^2$. Let f be in K_u^2 and g in H^2. We need to show that $\langle J_w f, u_w g\rangle = 0$, which amounts to showing that the Toeplitz operator with symbol $\bar{u}_w/(1-\bar{w}u)$ annihilates K_u^2. Since, on $\partial\mathfrak{D}$,

$$\frac{\bar{u}_w}{1-\bar{w}u} = \frac{1}{u_w(1-\bar{w}u)} = \frac{1}{u-w} = \frac{\bar{u}}{1-w\bar{u}},$$

the Toeplitz operator in question equals $T_{1/(1-w\bar{u})}T_{\bar{u}}$. Since $T_{\bar{u}}$ annihilates K_u^2, the desired conclusion follows.

By the same token, since $u = \frac{u_w+w}{1+\bar{w}u_w}$, the map $g \mapsto \sqrt{1-|w|^2}g/(1+\bar{w}u_w)$ on $K_{u_w}^2$ has range in K_u^2. But that map inverts J_w:

$$(1-\bar{w}u)(1+\bar{w}u_w) = (1-\bar{w}u)\left(1+\bar{w}\left(\frac{u-w}{1-\bar{w}u}\right)\right) = 1-|w|^2.$$

Hence $J_w K_u^2 = K_{u_w}^2$.

To prove J_w is an isometry, let f and g belong to K_u^2. We have

$$\begin{aligned}\langle J_w f, J_w g\rangle &= (1-|w|^2)\left\langle \frac{f}{1-\bar{w}u}, \frac{g}{1-\bar{w}u}\right\rangle \\ &= (1-|w|^2)\sum_{j,k=0}^{\infty} \bar{w}^j w^k \langle u^j f, u^k g\rangle .\end{aligned}$$

If $j<k$ then $\langle u^j f, u^k g\rangle = \langle \bar{u}^{k-j} f, g\rangle = 0$ since $\bar{u}f$ is orthogonal to H^2. Similarly, the inner product is 0 if $j>k$, while if $j=k$ it equals $\langle f, g\rangle$. Hence

$$\langle J_w f, J_w g\rangle = (1-|w|^2)\sum_{j=0}^{\infty}|w|^{2j}\langle f, g\rangle = \langle f, g\rangle .$$

$\square$

THEOREM 13.2. $J_w \mathscr{T}(K_u^2) J_w^{-1} = \mathscr{T}(K_{u_w}^2)$.

In other words, the Crofoot transform of a truncated Toeplitz operator is a truncated Toeplitz operator. A few lemmas are needed

LEMMA 13.1. *The transformation J_w intertwines the conjugation on K_u^2 with the conjugation on $K_{u_w}^2$.*

Proof. For f in K_u^2 we have (on $\partial\mathbb{D}$)

$$\begin{aligned}J_w \widetilde{f} &= \sqrt{1-|w|^2}\, u\bar{z}\bar{f}/(1-\bar{w}u) \\ &= \sqrt{1-|w|^2}\left(\frac{u_w u\bar{z}\bar{f}}{u-w}\right) = \sqrt{1-|w|^2}\left(u_w\bar{z}\left(\frac{\bar{f}}{1-w\bar{u}}\right)\right) \\ &= (J_w f)^{\sim}.\end{aligned}$$

$\square$

LEMMA 13.2. *For λ in $\mathbb{D}$,*

$$J_w^{-1}k_\lambda^{u_w} = \frac{\sqrt{1-|w|^2}}{1-w\overline{u(\lambda)}}k_\lambda^u,\ J_w^{-1}\widetilde{k}_\lambda^u = \frac{\sqrt{1-|w|^2}}{1-\bar{w}u(\lambda)}\widetilde{k}_\lambda^{u_w}.$$

Proof. Because of Lemma 13.1, it will suffice to establish the first equality. For f in K_u^2,

$$\begin{aligned}\langle f, k_\lambda^u\rangle &= f(\lambda) = \frac{1-\bar{w}u(\lambda)}{\sqrt{1-|w|^2}}(J_w f)(\lambda) = \frac{1-\bar{w}u(\lambda)}{\sqrt{1-|w|^2}}\langle J_w f, k_\lambda^{u_w}\rangle \\ &= \left\langle f, \frac{1-w\overline{u(\lambda)}}{\sqrt{1-|w|^2}}J_w^{-1}k_\lambda^{u_w}\right\rangle .\end{aligned}$$

$\square$

LEMMA 13.3.

$$\begin{aligned} J_w^{-1}S_{u_w}^*J_w &= S_u^* + \frac{\bar{w}}{1-\bar{w}u(0)}(\widetilde{k}_0^u \otimes k_0^u) \\ J_w^{-1}S_{u_w}J_w &= S_u + \frac{w}{1-w\overline{u(0)}}(k_0^u \otimes \widetilde{k}_0^u). \end{aligned}$$

Proof. The two equalities are adjoints of each other, so it will suffice to establish the first one. We first need to determine $S^*\left(\frac{1}{1-\bar{w}u}\right)$. We have

$$\begin{aligned} S^*\left(\frac{1}{1-\bar{w}u}\right) &= S^*\left(\frac{1}{1-\bar{w}u}+1\right) = S^*\left(\frac{\bar{w}u}{1-\bar{w}u}\right) \\ &= \bar{w}uS^*\left(\frac{1}{1-\bar{w}u}\right) + \frac{\bar{w}}{1-\bar{w}u(0)}S^*u. \end{aligned}$$

Consequently

$$S^*\left(\frac{1}{1-\bar{w}u}\right) = \left(\frac{\bar{w}}{1-\bar{w}u(0)}\right)\frac{\widetilde{k}_0^u}{1-\bar{w}u}.$$

Using this we get, for f in K_u^2,

$$\begin{aligned} S_w^*J_wf &= \sqrt{1-|w|^2}S^*\left(\frac{f}{1-\bar{w}u}\right) \\ &= \frac{\sqrt{1-|w|^2}}{1-\bar{w}u}S^*f + \sqrt{1-|w|^2}f(0)S^*\left(\frac{1}{1-\bar{w}u}\right) \\ &= J_wS_u^*f + \sqrt{1-|w|^2}f(0)\left(\frac{\bar{w}}{1-\bar{w}u(0)}\right)\frac{\widetilde{k}_0^u}{1-\bar{w}u} \\ &= J_wS_u^*f + \frac{\bar{w}f(0)}{1-\bar{w}u(0)}J_w\widetilde{k}_0^u \\ &= J_w\left(S_u^* + \frac{\bar{w}}{1-\bar{w}u(0)}(\widetilde{k}_0^u \otimes k_0^u)\right)f. \end{aligned}$$

□

Proof of Theorem 13.2. Let the operator B belong to $\mathscr{T}(K_{u_w}^2)$, and let $A = J_w^{-1}BJ_w$. We shall show that A is in $\mathscr{T}(K_u^2)$, which will give the theorem, by verifying the criterion in Theorem 10.1.

Since B is in $\mathscr{T}(K_{u_w}^2)$, there are functions ψ and χ in $K_{u_w}^2$ such that

$$B - S_{u_w}BS_{u_w}^* = (\psi \otimes k_0^{u_w}) + (k_0^{u_w} \otimes \chi).$$

By Lemma 13.3, $J_w^{-1}S_{u_w}J_w = S_{u,c}$ where $c = \frac{w}{1-w\overline{u(0)}}$. Therefore

$$A - S_{u,c}AS_{u,c}^* = (J_w^{-1}\psi \otimes J_w^{-1}k_0^{u_w}) + (J_w^{-1}k_0^{u_w} \otimes J_w^{-1}\chi).$$

By Lemma 13.2 the right side equals

$$\frac{\sqrt{1-|w|^2}}{1-w\overline{u(0)}}(J_w^{-1}\psi\otimes k_0^u)+\frac{\sqrt{1-|w|^2}}{1-\bar{w}u(0)}(k_0^u\otimes J_w^{-1}\chi).$$

By Theorem 10.1, A is in $\mathscr{T}(K_u^2)$. □

14. More on the Operators $S_{u,c}$

In this section we determine the spectra of the operators $S_{u,c}$. It is convenient to adopt an alternative notation, suggested by Lemma 13.3. For w a complex number different from $1/\overline{u(0)}$ we define the operator $A_{u,w}$ in $\mathscr{T}(K_u^2)$ by

$$A_{u,w}=S_u+\frac{w}{1-\overline{u(0)}w}(k_0^u\otimes\widetilde{k}_0^u).$$

Thus $A_{u,w}=S_{u,c}$ with $c=w/(1-\overline{u(0)}w)$. When $|w|=1$ one has $A_{u,w}=U_\beta$ with $\beta=\frac{w-u(0)}{1-\overline{u(0)}w}$ (a Clark unitary).

The spectrum of S_u is given in Lemma 2.5. The essential spectrum is the set of singularities of u on $\partial\mathcal{D}$. The only other points in the spectrum are the zeros of u in $\mathcal{D}$, if there are any, each of which is an eigenvalue of unit multiplicity. The operators $A_{u,w}$, being compact perturbations of S_u, have the same essential spectrum. If λ is a spectral point of $A_{u,w}$ not in the essential spectrum, then $A_{u,w}-\lambda I$ is a Fredholm operator, of index 0 by C-symmetry, so λ must be an eigenvalue.

We consider first the case $|w|<1$. In this case Lemma 13.3 tells us that $A_{u,w}$ is unitarily equivalent under Crofoot's transformation J_w to S_{u_w}. By Lemma 2.5, the eigenvalues of $S_{u,w}$ are the zeros of u_w, i.e., the points λ in $\mathcal{D}$ where $u(\lambda)=w$, each an eigenvalue unit multiplicity. Note that, by the corollary to Theorem 10.1, the commutant of $A_{u,w}$ belongs to $\mathscr{T}(K_u^2)$. That commutant is mapped by Crofoot's transformation to the commutant of S_{u_w}, which by the commutant lifting theorem equals $\{h(S_{u_w}):h\in H^\infty\}$, and is isometrically isomorphic to H^∞/u_wH^∞.

When $|w|=1$ we have, as noted above, $A_{u,w}=U_\beta$ with $\beta=\frac{w-u(0)}{1-\overline{u(0)}w}$. By Clark's Theorem B, U_β is unitarily equivalent under Clark's transformation V_w to the multiplication operator Z_w on $L^2(\mu_w)$. The spectrum of $A_{u,w}$ is thus the support of μ_w, which consists of the set of singularities of u on $\partial\mathcal{D}$ together with the set of regular points of u on $\partial\mathcal{D}$ that belong to supp μ_w. From (11.1) one sees that a point of the latter kind is a regular point of u on $\partial\mathcal{D}$ where u takes the value w, hence (as pointed out in Section 12.) a mass point of μ_w (of magnitude $1/|u'(w)|$). The point is therefore an eigenvalue of $A_{u,w}$ of unit multiplicity.

The following lemma reduces the case $|w|>1$ to the case $|w|<1$.

LEMMA 14.1. *If $|w|<1$ and $w\neq u(0)$, then $A^*_{u,1/\bar{w}}=A^{-1}_{u,w}$.*

Proof. Let w be as described. We know from the discussion above that 0 is not in the spectrum of $A_{u,w}$, i.e., $A_{u,w}$ is invertible. Let c be any complex number.

We compute $A_{u,w}S_{u,c}^*$ and show we get the identity when $c = \frac{1}{\bar{w}-\overline{u(0)}}$, i.e., when $S_{u,c} = A_{u,1/\bar{w}}$. We have

$$\begin{aligned} A_{u,w}S_{u,c}^* &= \left(S_u + \frac{w}{1-\overline{u(0)}w}(k_0^u \otimes \widetilde{k}_0^u)\right)(S_u^* + \bar{c}(\widetilde{k}_0^u \otimes k_0^u)) \\ &= I - \left[1 + \bar{c}u(0) + \frac{w\overline{u(0)}}{1-\overline{u(0)}w} - \frac{\bar{c}w(1-|u(0)|^2)}{1-\overline{u(0)}w}\right](k_0^u \otimes k_0^u). \end{aligned}$$

After some algebra the expression in square brackets on the right side reduces to

$$\frac{1}{1-\overline{u(0)}w}(1-\bar{c}(w-u(0))),$$

so we get $A_{u,w}S_{u,c}^* = I$ when $c = 1/(\bar{w}-\overline{u(0)})$. □

Now suppose $|w| > 1$, $w \neq 1/\overline{u(0)}$. By Lemma 14.1, $A_{u,w}^*$ is the inverse of $A_{u,1/\bar{w}}$. We know that the spectral points of $A_{u,1/\bar{w}}$ not in the essential spectrum are the points λ such that $u(\lambda) = 1/\bar{w}$. The corresponding points in the spectrum of the adjoint of the inverse of $A_{u,1/\bar{w}}$ are thus the points λ such that $u(1/\bar{\lambda}) = 1/\bar{w}$. But since u is an inner function it satisfies $u(1/\bar{z}) = 1/\overline{u(z)}$. We conclude that the points in the spectrum of $A_{u,w}$ not in the essential spectrum are the points λ such that $u(\lambda) = w$. As in the previous cases, each such point is an eigenvalue of unit multiplicity.

The following theorem summarizes the results on the spectrum of $A_{u,w}$.

THEOREM 14.1. *Let w be a complex number different from $1/\overline{u(0)}$. The essential spectrum of $A_{u,w}$ is the set of singular points of u on $\partial\mathcal{D}$. A regular point λ of u lies in the spectrum if and only if $u(\lambda) = w$, in which case λ is an eigenvalue of unit multiplicity.*

The preceding theorem does not apply in one case, the case $S_{u,c}$ with $c = -1/\overline{u(0)}$. This is the limiting case of $A_{u,w}$ where $u(0) \neq 0$, and $w \to \infty$. A calculation like that in the proof of Lemma 14.1 (but simpler) shows that $S_u - \frac{1}{\overline{u(0)}}(k_0^u \otimes \widetilde{k}_0^u)$ is the inverse of S_u^*. Its spectral points outside the essential spectrum are thus the points λ such that $u(1/\bar{\lambda}) = 0$, in other words, the poles to u.

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