Operators
and
Matrices
Volume 1, Number 4 (2007), 527–534

WEAK SIMULTANEOUS TRIANGULARIZATION — A DETERMINANT CONDITION

CHEN DUBI

(*communicated by M. Neumann*)

Abstract. We introduce and study the notion of weak triangularization for a set of N matrices, which is a weaker version of simultaneous triangularization. We prove that for the special case of 2-by-2 matrices, weak triangularization is equivalent to simultaneous triangularization.

1. Introduction

1.1. Simultaneous triangularization

Let $\mathbb{A} = \{A_1, A_2, \dots A_N\}$ be a set of N matrices in $\mathbb{C}^{n\times n}$. We say that $\mathbb{A}$ admits a simultaneous triangularization if there exists a matrix T such that the matrix

$$\widetilde{A_j} = T^{-1} A_j T$$

is upper triangular for every $A_j \in \mathbb{A}$.

The problem of determining whether or not a set of matrices admits a simultaneous triangularization is a long studied problem. In particular, we refer to the work of Radjavi & Rosenthal [3]. One well known necessary and sufficient condition is the following (Radjavi [4]):

THEOREM 1.1. *A set of matrices $\mathbb{A} = \{A_1, A_2, \dots A_N\}$ has a simultaneous triangularization if and only if for every $1 \leqslant i, j, \ell \leqslant N$*

$$\operatorname{trace} A_i A_j A_\ell = \operatorname{trace} A_\ell A_j A_i.$$

It is also relatively easy to prove that every set of commuting matrices has a simultaneous triangularization — yet by no means is this condition a sufficient one (as every set of non commuting triangular matrices forms a counterexample). See [2] for details.

In the present study, we look at a more relaxed property than simultaneous triangularization; a property we refer to as *weak triangularization*, which will be defined later in this section. To properly motivate the definition, we first start with the following:

Mathematics subject classification (2000): 15A15.

Key words and phrases: Simultaneous triangularization, weak triangularization, characteristic polynomial, rational transfer function.

Paper OAM-01-30

THEOREM 1.2. *Let $\mathbb{A} = \{A_1, A_2, \dots A_N\}$ be a set of N matrices in $\mathbb{C}^{n\times n}$, then the following two conditions are equivalent:*

1. *$\mathbb{A}$ has a simultaneous triangularization.*
2. *There exists a set of upper triangular matrices $\widetilde{\mathbb{A}} = \{\widetilde{A}_1, \widetilde{A}_1, \dots \widetilde{A}_N\}$ and an invertible matrix T such that:*

$$(I - \sum_{j=1}^{N} z_j A_j)^{-1} = T^{-1}(I - \sum_{j=1}^{N} z_j \widetilde{A}_j)^{-1} T \tag{1.1}$$

Proof. Assume first $\mathbb{A}$ has a simultaneous triangularization. Then:

$$\begin{aligned}(I - \sum_{j=1}^{N} z_j A_j)^{-1} &= (T^{-1}T - \sum_{j=1}^{N} z_j T^{-1}\widetilde{A}_j T)^{-1} \\ &= (T^{-1}(I - \sum_{j=1}^{N} z_j \widetilde{A}_j)T)^{-1} = T^{-1}(I - \sum_{j=1}^{N} z_j \widetilde{A}_j)^{-1} T\end{aligned}$$

as stated.

Assume next equation (1.1) holds. Expanding both sides into power series (for sufficiently small z) and observing the linear terms implies that

$$A_j = T^{-1}\widetilde{A}_j T$$

which completes the proof. $\square$

At this point, two remarks are due:

1. From Theorem 1.2, the following necessary condition for simultaneous triangularization is easy to obtain:

PROPOSITION 1.3. *If $\mathbb{A}$ has a simultaneous triangularization, then the polynomial*

$$\mathrm{Det}(I - \sum_{j=1}^{N} z_j A_j)$$

is a product of linear terms.

The polynomial $\mathrm{Det}(I - \sum_{j=1}^{N} z_j A_j)$ will play a key role in our further analysis. Hereon, we will refer to $\mathrm{Det}(I - \sum_{j=1}^{N} z_j A_j)$ as the *characteristic polynomial* of $\mathbb{A}$ and denote

$$\mathrm{Det}(I - \sum_{j=1}^{N} z_j A_j) = P_{\mathbb{A}}(z).$$

2. Since both statements in Theorem 1.2 are equivalent, both statement can be chosen to define simultaneous triangularization.

This last statement is the motivation for the next definition, which is the key notion in our study:

DEFINITION 1.4. Let $\mathbb{A} = \{A_1, A_2, \dots A_N\}$ be a tuple of N matrices in $\mathbb{C}^{n\times n}$. We say that $\mathbb{A}$ has a *weak triangularization* if there exists a tuple of upper triangular matrices $\widetilde{\mathbb{A}} = \{\widetilde{A}_1, \widetilde{A}_2, \dots, \widetilde{A}_N\}$ ($A_k \in \mathbb{C}^{n_1 \times n_1}$), and $T \in \mathbb{C}^{n_1 \times n}, T^\ell \in \mathbb{C}^{n \times n_1}$ such that:

$$(I - \sum_{j=1}^N z_j A_j)^{-1} = T^\ell (I - \sum_{j=1}^N z_j \widetilde{A}_j)^{-1} T, \quad \text{and} \quad T^\ell T = I_n.$$

The main objective in this study is to prove the following theorem

THEOREM 1.5. *Let $\mathbb{A} = \{A_1, A_2, \dots A_N\}$ be a tuple of N matrices in $\mathbb{C}^{n \times n}$, then $\mathbb{A}$ has a weak triangularization if and only if the characteristic polynomial $P_{\mathbb{A}}(z)$ is a product of linear terms.*

We remark that following the arguments of Theorem 1.2, if $\mathbb{A} = \{A_1, A_2, \dots A_N\}$ has a weak triangularization, then there for $1 \leqslant j \leqslant N$ we have that

$$A_j = T^\ell \widetilde{A}_j T. \tag{1.2}$$

However, the existence of an upper triangular set $\widetilde{\mathbb{A}} = \{\widetilde{A}_1, \widetilde{A}_2, \dots, \widetilde{A}_N\}$ and a left invertible matrix T satisfying equality (1.2) do not, in general, imply weak triangularization.

The paper is arranged as follows: hereon, Section 1. is devoted to provide the proper background for the proof of Theorem 1.5, which will be presented in Section 2.. In Section 3. we show that for the special case of a tuple of N 2-by-2 matrices, weak triangularization is equivalent to simultaneous triangularization.

Throughout the study, we will use the following two notations : for a $\mathbb{C}^{p \times n}$ matrix valued function $F(z)$, we denote by $\mathcal{S}(F)$ the linear span of the columns of $F(z)$. In other words, we denote:

$$\mathcal{S}(F) = \{F(z)\xi \mid \xi \in \mathbb{C}^n\}.$$

For a matrix valued function $F(z) : \mathbb{C}^N \to \mathbb{C}^{n \times n}$, we denote by $M_F(z)$ the algebraic adjoint of $F(z)$.

1.2. Cramer's rule

Let $P(z)$ be a matrix polynomial with values in $\mathbb{C}^{n \times n}$. It is well known that $P(z_0)$ is invertible if and only if $\operatorname{Det} P(z_0) \neq 0$, and so, roots of the (scalar) polynomial $\operatorname{Det} P(z)$ are exactly the singular point of $P^{-1}(z)$ .By Cramer's rule we can write:

$$P^{-1}(z) = \frac{1}{\operatorname{Det} P(z)} M_P(z).$$

The following proposition will prove useful later, as we turn to prove Theorem 1.5:

PROPOSITION 1.6. *Assume $\operatorname{Det} P(z)$ is of the form $\operatorname{Det} P(z) = p_1^\ell(z) p_2(z)$ and p_1 is not a factor of p_2. Then $M_P(z)$ is not of the form $M_P(z) = p_1^\ell(z) Q(z)$ (notice, here $p_1(z)$ is scalar valued, whereas $Q(z)$ is matrix valued of appropriate dimensions).*

Proof. Assume, to arrive at a contradiction, that $M_P(z)$ is of the form $M_P(z) = p_1^\ell(z) Q(z)$. By Cramer's rule we have that

$$P^{-1}(z) = \frac{1}{\operatorname{Det} P(z)} M_P(z) = \frac{1}{p_2(z)} Q(z).$$

Since $p_1(z)$ is not a factor of $p_2(z)$, there exists z_1 such that $p_1(z_1) = 0$ and $p_2(z_1) \neq 0$. Since $p_2(z_1) \neq 0$, the matrix $\frac{1}{p_2(z_1)}Q(z_1)$ is well defined, and hence $P(z_1)$ is invertible. This serves as a contradiction, since $\operatorname{Det} P(z_1) = 0$. $\square$

In a more simple manner, proposition 1.6 states that none of the irreducible factors of $\operatorname{Det} P(z)$ are "lost" when evaluating the quotient $\frac{1}{\operatorname{Det} P(z)} M_P(z)$.

1.3. Triangular realization of N variable matrix valued function

When thinking of realization of a rational matrix valued function in N complex variables $R(z)$, several possibilities come to mined. One possible realization is as a transfer function of Fornasini-Marchesini linear system, given by

$$R(z) = D + C(I - \sum_{j=1}^{N} z_j A_j)^{-1} \sum_{j=1}^{N} z_j B_j$$

(see [6]). Such realizations, as shown in [1], turn to be a general form of realization; that is, every rational matrix valued function $R(z)$, analytic at the origin, can be presented as a transfer function of a Fornasini-Marchesini system. In [5], the problem of determining whether or not a transfer function R (analytic at the origin) can be realized with upper triangular state operators $\{A_1, A_2, \ldots, A_N\}$ was addressed, and a necessary and sufficient criterion was given through the following theorem:

THEOREM 1.7. *A rational matrix valued function $R(z)$ can be realized with upper triangular state operators $\{A_1, A_2, \ldots, A_N\}$ if and only if the complement of the points of analyticity of R is a finite union of hyperplanes.*

2. Proof of Theorem 1.5

Proof of main theorem. Assume first that the characteristic polynomial $P_{\mathbb{A}}(z)$ is a product of linear terms. Using Cramer's formula for the inverse, this implies that the set of singular points of the rational function

$$R(z) = (I - \sum_{j=1}^{N} z_j A_j)^{-1}$$

is a finite union of hyperplanes (which are exactly the roots of $P_{\mathbb{A}}(z)$). By Theorem 1.7 $R(z)$ has the realization

$$R(z) = I + C_1(I - \sum_{j=1}^{N} A_{1,j} z_j)^{-1} \sum_{j=1} z_j B_{1,j}$$

here

$$C_1 \in \mathbb{C}^{n \times n_1}, \quad A_{1,j} \in \mathbb{C}^{n_1 \times n_1}, \qquad B_{1,j} \in \mathbb{C}^{n_1 \times n}$$

and $A_{1,j}$ is upper triangular for $1 \leqslant j \leqslant N$.

We denote

$$R_1(z) = \left[\, C_1(I - \textstyle\sum_{j=1}^N A_{1,j}z_j)^{-1}, \quad (I - \sum_{j=1}^N z_jA_j)^{-1} \,\right].$$

Clearly, $\mathcal{S}(R(z)) \subseteq \mathcal{S}(R_1(z))$. We define:

$$\widetilde{A}_j = \begin{bmatrix} A_{1,j} & B_{1,j} \\ 0_{n\times n_1} & 0_{n\times n} \end{bmatrix}$$

and proceed with the following:

$$\begin{aligned} R_1(z) - R_1(0) &= \left[\, C_1(I - \textstyle\sum_{j=1}^N A_{1,j}z_j)^{-1}, \quad (I - \sum_{j=1}^N z_jA_j)^{-1} \,\right] - [\, C_1, \quad I \,] \\ &= \left[\, C_1(I - \textstyle\sum_{j=1}^N A_{1,j}z_j)^{-1} - C_1, \quad C_1(I - \sum_{j=1}^N z_jA_j)^{-1} \sum_{j=1}^N B_{1,j} \,\right] \\ &= C_1(I - \sum_{j=1}^N z_jA_j)^{-1} \left[\, I - (I - \textstyle\sum_{j=1}^N A_{1,j}), \quad \sum_{j=1}^N B_{1,j} \,\right] \\ &= \left[\, C_1(I - \textstyle\sum_{j=1}^N A_{1,j}z_j)^{-1}, \quad (I - \sum_{j=1}^N z_jA_j)^{-1} \,\right] \sum_{j=1}^N z_j \begin{bmatrix} A_{1,j} & B_{1,j} \\ 0_{n\times n_1} & 0_{n\times n} \end{bmatrix} \\ &= R_1(z) \sum_{j=1}^N z_j\widetilde{A}_j. \end{aligned}$$

From the last we have that

$$R_1(z) = R_1(0)(I - \sum_{j=1}^N z_j\widetilde{A}_j)^{-1}.$$

Since $\mathcal{S}(R(z)) \subseteq \mathcal{S}(R_1(z))$, there exist a matrix T such that

$$(I - \sum_{j=1}^N z_jA_j)^{-1} = R_1(z)T = R_1(0)(I - \sum_{j=1}^N z_j\widetilde{A}_j)^{-1}T. \tag{2.3}$$

Taking $z = 0$ on both sides of (2.3) we have that $R_1(0)T = I$. Upon setting $R(0) = T^\ell$, we have

$$(I - \sum_{j=1}^N z_jA_j)^{-1} = T^\ell(I - \sum_{j=1}^N z_j\widetilde{A}_j)^{-1}T$$

as stated.

Next, assume that $\mathbb{A}$ has a weak triangularization. We denote:

$$D(z) = I - \sum_{j=1}^N z_jA_j, \quad \widetilde{D}(z) = I - \sum_{j=1}^N z_j\widetilde{A}_j.$$

Using Cramer's rule, we can the write:

$$D_1(z) = \frac{1}{\operatorname{Det} D(z)} M_D(z) = T^\ell \left(\frac{1}{\operatorname{Det} \widetilde{D}(z)} M_{\widetilde{D}} \right) T = \frac{1}{\operatorname{Det} \widetilde{D}(z)} T^\ell M_{\widetilde{D}} T$$

or

$$\frac{\operatorname{Det} \widetilde{D}}{\operatorname{Det} D(z)} M_D(z) = T^\ell M_{\widetilde{D}} T.$$

Since $T^{\ell}M_{\widetilde{D}}T$ is a polynomial matrix, so is $\frac{\mathrm{Det}\,\widetilde{D}}{\mathrm{Det}\,D(z)}M_D(z)$, and so all factors of $\mathrm{Det}\,D(z)$ are factors of $\mathrm{Det}\,\widetilde{D}(z)M_D(z)$ (with the same multiplicity). By Proposition 1.6, this implies that all factors of $\mathrm{Det}\,D(z)$ are factors of $\mathrm{Det}\,\widetilde{D}(z)$. Since $\widetilde{A}$ is upper triangular, $\mathrm{Det}\,\widetilde{D}(z)$ is a product of linear terms - and thus $\mathrm{Det}\,D(z)$ is a product of linear factors. □

3. The special case $N = 2$

In the final section of this study, we prove the following theorem, which states the interesting fact that for $n = 2$, weak triangularization is equivalent to simultaneous triangularization.

THEOREM 3.1. *A set* $\mathbb{A} = \{A_1, A_2, \ldots, A_N\}$ *of* $\mathbb{C}^{2\times 2}$ *matrices has a simultaneous triangularization if and only if the characteristic polynomial*

$$P_{\mathbb{A}}(z) = \mathrm{Det}(I - \sum_{j=1}^{N} z_j A_j)$$

is a product of linear terms.

Proof. Following the remark in Section 1., one direction is trivial. Assume $\mathrm{Det}(I - \sum_{j=1}^{N} z_j A_j)$ is a product of linear terms. Let us denote

$$A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}.$$

We adopt the following notations: we denote

$$a = (a_1, a_2, \ldots a_N), \quad b = (b_1, b_2, \ldots b_N), \quad c = (c_1, c_2, \ldots c_N), \quad d = (d_1, d_2, \ldots d_N),$$

and for $v = (v_1, v_2, \ldots v_N)$, $u = (u_1, u_2, \ldots u_N) \in \mathbb{C}^N$ we denote

$$\langle v, u \rangle = \sum_{j=1}^{N} v_j u_j.$$

In the following, we assume $b \neq (0, 0, \ldots, 0)$ and $c \neq (0, 0, \ldots, 0)$, for if one of the two is indeed the zero vector, the statement is trivial.

Through direct calculation we have that the characteristic polynomial $P_{\mathbb{A}}(z)$ is given by:

$$\mathrm{Det}\,D(z) = (1 - \langle z, a \rangle)(1 - \langle z, d \rangle) - \langle z, b \rangle \langle z, c \rangle.$$

And so, the characteristic polynomial is a product of linear terms if and only if there exists $v, w \in \mathbb{C}^N$ such that:

$$(1 - \langle z, a \rangle)(1 - \langle z, d \rangle) - \langle z, b \rangle \langle z, c \rangle = (1 - \langle z, v \rangle)(1 - \langle z, w \rangle).$$

Comparing the linear and quadratic terms on both sides we obtain that

$$\langle z, a + d \rangle = \langle z, v + w \rangle \tag{3.4}$$

and

$$\langle z, v \rangle \langle z, w \rangle = \langle z, a \rangle \langle z, d \rangle - \langle z, b \rangle \langle z, c \rangle \tag{3.5}$$

We proceed in several steps:

Step 1:

$$\mathrm{Det}\begin{bmatrix} \langle z, v-d\rangle & \langle z, b\rangle \\ \langle z, c\rangle & \langle z, v-a\rangle \end{bmatrix} = 0$$

Proof of Step 1. We compute

$$\begin{aligned} \mathrm{Det}\begin{bmatrix} \langle z, v-d\rangle & \langle z, b\rangle \\ \langle z, c\rangle & \langle z, v-a\rangle \end{bmatrix} &= \langle z, v-d\rangle \langle z, v-a\rangle - \langle z, b\rangle \langle z, c\rangle \\ &= \langle z, v\rangle^2 - \langle z, v\rangle \langle z, a\rangle - \langle z, v\rangle \langle z, d\rangle \\ &\quad + \langle z, d\rangle \langle z, a\rangle - \langle z, b\rangle \langle z, c\rangle \\ &= \langle z, v\rangle^2 - \langle z, v\rangle \langle z, a\rangle - \langle z, v\rangle \langle z, d\rangle + \langle z, v\rangle \langle z, w\rangle \\ &= \langle z, v\rangle \left(\langle z, v\rangle - \langle z, a\rangle - \langle z, d\rangle + \langle z, w\rangle\right) \\ &= \langle z, v\rangle \left(\langle z, v+w\rangle - \langle z, a+d\rangle\right) = 0 \end{aligned}$$

as stated.

Step 2: There exists λ such that $v-d=\lambda b$ <u>or</u> $v-d=\lambda c$.

Proof of Step 2. By Step 1, we have that

$$\langle z, v-d\rangle \langle z, v-a\rangle = \langle z, c\rangle \langle z, b\rangle \tag{3.6}$$

Let us denote

$$V_1 = (v-d)^{\perp}, \quad V_2 = (v-a)^{\perp}, \quad V_3 = (b)^{\perp}, \quad V_4 = (c)^{\perp}$$

Since $V_1 \cup V_2$ and $V_3 \cup V_4$ are the set of roots of the left and right hand side of (3.6) respectively, it is clear that

$$V_1 \subseteq V_3 \cup V_4$$

PROPOSITION 3.2. $V_1 = V_3$ *or* $V_1 = V_4$

Proof. We first notice, since all spaces are finite dimensional of dimension $N-1$, equality is equivalent to inclusion. And so, to arrive at a contradiction, we assume that $V_1 \nsubseteq V_3$ and $V_1 \nsubseteq V_4$. Since $V_1 \subseteq V_3 \cup V_4$, there exists w_1, w_2 such that

$$w_1, w_2 \in V_1, \quad w_1 \in V_3, \quad w_1 \notin V_4, \quad w_2 \notin V_3, \quad w_2 \in V_4 \tag{3.7}$$

Since V_1 is a linear space, $w_1 + w_2 \in V_1 \subseteq V_3 \cup V_4$, implying either $w_1 \in V_4$ or $w_2 \in V_3$, contradicting (3.7). $\square$

By Proposition 3.2, either $(v-d)^{\perp} = (b)^{\perp}$ or $(v-d)^{\perp} = (c)^{\perp}$ and so there exists λ such that $v-d=\lambda b$ or $v-d=\lambda c$.

Step 3: The matrices $A_1, A_2, \ldots A_N$ have a common eigenvector.

Proof of Step 3. We divide the proof into two possible cases:

CASE 1: $v-d=\lambda b$: In such case, we clearly have $\langle z, v-d\rangle = \lambda \langle z, b\rangle$. Next, since $\mathrm{Det}\begin{bmatrix} \langle z, v-d\rangle & \langle z, b\rangle \\ \langle z, c\rangle & \langle z, v-a\rangle \end{bmatrix} = 0$, it is clear that $\lambda \langle z, v-a\rangle = \langle z, c\rangle$, yielding

$$v_i - d_i = \lambda b_i, \quad \lambda(v_i - a_i) = c_i.$$

Upon setting $\xi_0 = \begin{pmatrix} 1 \\ -\lambda \end{pmatrix}$ we have:

$$A_i\xi_0 = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \begin{pmatrix} 1 \\ -\lambda \end{pmatrix} = \begin{pmatrix} a_i - \lambda b_i \\ c_i - \lambda d_i \end{pmatrix} = \begin{pmatrix} a_i - v_i + d_i \\ \lambda(v_i - a_i - d_i) \end{pmatrix} = \begin{pmatrix} 1 \\ -\lambda \end{pmatrix} (a_i - v_i + d_i)$$
$$= w_i\xi_0$$

as stated.

CASE 2: $v - d = \lambda c$: Once again, by using the determinant equivalence shown in Step 1, it is easy to verify that $\lambda(v_i - a_i) = b_i$ for every $1 \leqslant i \leqslant N$. Repeating the computation done in the first case, now with $\xi_0 = \begin{pmatrix} -\lambda \\ 1 \end{pmatrix}$, is readably seen that

$$A_j\xi_0 = v_i\xi_0$$

which completes the proof of Step 3.

Step 4: The tuple $\mathbb{A}$ admits a simultaneous triangularization.

Proof of Step 4. To finalize the theorem, we notice that if ξ_1 is any linearly independent vector to ξ_0, the matrix representation of linear transformation $v \mapsto A_jv$ with respect to the ordered bases $\mathcal{B} = \{\xi_0, \xi_1\}$ is upper triangular, which completes the proof. $\square$

REFERENCES

[1] D. ALPAY, C. DUBI: *A realization theorem for functions of several complex variables*, System and control letters, 49 (2003), 225-229.

[2] I. GOHBERG, P. LANCASTER AND L. RODMAN: *Invariant subspaces of matrices with applications*, Canadian Mathematical Society Series of Monographs and Advanced Texts, (1986).

[3] H. RADJAVI AND P. ROSENTHAL: *Simultaneous triangularization*, Springer-Verlag New-York, (2000).

[4] H. RADJAVI: *A trace condition equivalent to simultaneous triangularizability*; Can. J. Math., Vol. 38, No.2 (1986), 476-386.

[5] C. DUBI: *Triangular realization of rational functions of N complex variables*, Multidim. Sys. Sign. Process.,to appear.

[6] E. FORNASINI AND G. MARCHESINI: *Doubly-indexed dynamical systems: state-space models and structural properties*, Math. Systems Theory, 12 (1978/79), 59–72.

(Received March 29, 2007)

Chen Dubi
Department of Mathematics
Shammon College of engineering
84 Jabotinsky St.
Ashdod
Israel
e-mail: chend@sce.ac.il