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WEYL–TITCHMARSH THEORY AND BORG–MARCHENKO–TYPE UNIQUENESS RESULTS FOR CMV OPERATORS WITH MATRIX–VALUED VERBLUNSKY COEFFICIENTS

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Dedicated with great pleasure
to Eduard Tsekanovskii
on the occasion of his 70th birthday

(*communicated by L. Rodman*)

Abstract. We prove local and global versions of Borg–Marchenko-type uniqueness theorems for half-lattice and full-lattice CMV operators (CMV for Cantero, Moral, and Velázquez [19]) with matrix-valued Verblunsky coefficients. While our half-lattice results are formulated in terms of matrix-valued Weyl–Titchmarsh functions, our full-lattice results involve the diagonal and main off-diagonal Green's matrices.

We also develop the basics of Weyl–Titchmarsh theory for CMV operators with matrix-valued Verblunsky coefficients as this is of independent interest and an essential ingredient in proving the corresponding Borg–Marchenko-type uniqueness theorems.

1. Introduction

Since Borg–Marchenko-type uniqueness theorems were first formulated in the context of scalar Schrödinger operators on half-lines, we start with a brief review of these results: Let $H_j = -\frac{d^2}{dx^2} + V_j$, $V_j \in L^1([0,R];dx)$ for all $R > 0$, V_j real-valued, $j = 1,2$, be two self-adjoint operators in $L^2([0,\infty);dx)$ which, just for simplicity, have a Dirichlet boundary condition at $x = 0$ (and possibly a self-adjoint boundary condition at infinity). Let $m_j(z)$, $z \in \mathbb{C}\backslash\mathbb{R}$, be the Weyl–Titchmarsh m-functions associated with H_j, $j = 1,2$. Then the celebrated Borg–Marchenko uniqueness theorem, in this particular context, reads as follows:

THEOREM 1.1. *Suppose*

$$m_1(z) = m_2(z), \quad z \in \mathbb{C}\backslash\mathbb{R}, \text{ then } V_1(x) = V_2(x) \text{ for a.e. } x \in [0,\infty). \tag{1.1}$$

This result was published by Marchenko [72] in 1950. Marchenko's extensive treatise on spectral theory of one-dimensional Schrödinger operators [73], repeating the proof of his uniqueness theorem, then appeared in 1952, which also marked the appear-

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ance of Borg's proof of the uniqueness theorem [14] (apparently, based on his lecture at the 11th Scandinavian Congress of Mathematicians held at Trondheim, Norway in 1949).

We emphasize that Borg and Marchenko also treat the general case of non-Dirichlet boundary conditions at $x = 0$ (in which equality of the two m-functions also identifies the two boundary conditions), moreover, Marchenko also simultaneously discussed the half-line and the finite interval case. For brevity we chose to illustrate the simplest possible case only.

To the best of our knowledge, the only alternative approaches to Theorem 1.1 are based on the Gelfand–Levitan solution [37] of the inverse spectral problem published in 1951 (see also Levitan and Gasymov [71]) and alternative variants due to M. Krein [64], [65]. For over 45 years, Theorem 1.1 stood the test of time and resisted any improvements. Finally, in 1998, Simon [89] proved the following spectacular result, a local Borg–Marchenko theorem (see part (i) below) and a significant improvement of the original Borg–Marchenko theorem (see part (ii) below):

THEOREM 1.2.
(i) *Let* $a > 0$, $0 < \varepsilon < \pi/2$ *and suppose that*

$$|m_1(z) - m_2(z)| \underset{|z|\to\infty}{=} O(e^{-2\mathrm{Im}(z^{1/2})a}) \tag{1.2}$$

along the ray $\arg(z) = \pi - \varepsilon$. *Then*

$$V_1(x) = V_2(x) \text{ for a.e. } x \in [0, a]. \tag{1.3}$$

(ii) *Let* $0 < \varepsilon < \pi/2$ *and suppose that for all* $a > 0$,

$$|m_1(z) - m_2(z)| \underset{|z|\to\infty}{=} O(e^{-2\mathrm{Im}(z^{1/2})a}) \tag{1.4}$$

along the ray $\arg(z) = \pi - \varepsilon$. *Then*

$$V_1(x) = V_2(x) \text{ for a.e. } x \in [0, \infty). \tag{1.5}$$

The ray $\arg(z) = \pi - \varepsilon$, $0 < \varepsilon < \pi/2$ chosen in Theorem 1.2 is of no particular importance. A limit taken along any non-self-intersecting curve $\mathscr{C}$ going to infinity in the sector $\arg(z) \in ((\pi/2) + \varepsilon, \pi - \varepsilon)$ is permissible. For simplicity we only discussed the Dirichlet boundary condition $u(0) = 0$ thus far. However, everything extends to the case of general boundary conditions $u'(0) + hu(0) = 0$, $h \in \mathbb{R}$. Moreover, the case of a finite interval problem on $[0, b]$, $b \in (0, \infty)$, instead of the half-line $[0, \infty)$ in Theorem 1.2 (i), with $0 < a < b$, and a self-adjoint boundary condition at $x = b$ of the type $u'(b) + h_b u(b) = 0$, $h_b \in \mathbb{R}$, can be handled as well. All of this is treated in detail in [54].

Remarkably enough, the local Borg–Marchenko theorem proven by Simon [89] was just a by-product of his new approach to inverse spectral theory for half-line Schrödinger operators. Actually, Simon's original result in [89] was obtained under a bit weaker conditions on V; the result as stated in Theorem 1.2 is taken from [54] (see also [53]). While the original proof of the local Borg–Marchenko theorem in [89] relied on the full power of a new formalism in inverse spectral theory, a short and fairly elementary proof of Theorem 1.2 was presented in [54]. Without going into further details at this point, we also mention that [54] contains the analog of the local Borg–Marchenko uniqueness result, Theorem 1.2 for Schrödinger operators on the real line. In addition, the case of half-line Jacobi operators and half-line matrix-valued Schrödinger operators was dealt with in [54].

We should also mention some work of Ramm [82], [83], who provided a proof of Theorem 1.2 (i) under the additional assumption that V_j are short-range potentials satisfying $V_j \in L^1([0,\infty);(1+|x|)dx)$, $j=1,2$. A very short proof of Theorem 1.2, close in spirit to Borg's original paper [14], was subsequently found by Bennewitz [9]. Still other proofs were presented in [60] and [61]. Various local and global uniqueness results for matrix-valued Schrödinger, Dirac-type, and Jacobi operators were considered in [21], [38], [52], [85], [86], and [87]. A local Borg–Marchenko theorem for complex-valued potentials has been proved in [16]; the case of semi-infinite Jacobi operators with complex-valued coefficients was studied in [104]. This circle of ideas has been reviewed in [48].

After this review of Borg–Marchenko-type uniqueness results for Schrödinger operators, we now turn to the principal object of our interest in this paper, the so-called CMV operators. CMV operators are a special class of unitary semi-infinite five-diagonal matrices. But for simplicity, we confine ourselves in this introduction to a discussion of CMV operators on $\mathbb{Z}$, that is, doubly infinite CMV operators. Let α be a sequence of $m\times m$ matrices, $m\in\mathbb{N}$, with entries in $\mathbb{C}$, $\alpha=\{\alpha_k\}_{k\in\mathbb{Z}}$ such that $\|\alpha_k\|_{\mathbb{C}^{m\times m}}<1$, $k\in\mathbb{Z}$. The unitary operator $\mathbb{U}$ on $\ell^2(\mathbb{Z})^m$ then can be written as a special five-diagonal doubly infinite matrix in the standard basis of $\ell^2(\mathbb{Z})^m$ as in (2.18). For the corresponding half-lattice CMV operators $\mathbb{U}_{+,k_0}$, in $\ell^2([k_0,\infty)\cap\mathbb{Z})^m$ we refer to (2.33) and (2.34).

The actual history of CMV operators (with scalar coefficients $\alpha_k\in\mathbb{C}$, $k\in\mathbb{Z}$) is quite interesting: The corresponding unitary semi-infinite five-diagonal matrices were first introduced in 1991 by Bunse–Gerstner and Elsner [17], and subsequently discussed in detail by Watkins [103] in 1993 (cf. the recent discussion in Simon [94]). They were subsequently rediscovered by Cantero, Moral, and Velázquez (CMV) in [19]. In [92, Sects. 4.5, 10.5], Simon introduced the corresponding notion of unitary doubly infinite five-diagonal matrices and coined the term "extended" CMV matrices. For simplicity, we will just speak of CMV operators whether or not they are half-lattice or full-lattice operators. We also note that in a context different from orthogonal polynomials on the unit circle, Bourget, Howland, and Joye [15] introduced a family of doubly infinite matrices with three sets of parameters which, for special choices of the parameters, reduces to two-sided CMV matrices on $\mathbb{Z}$. Moreover, it is possible to connect unitary block Jacobi matrices to the trigonometric moment problem (and hence to CMV matrices) as discussed by Berezansky and Dudkin [11], [12].

The relevance of this unitary operator $\mathbb{U}$ on $\ell^2(\mathbb{Z})^m$, more precisely, the relevance of the corresponding half-lattice CMV operator $\mathbb{U}_{+,0}$ in $\ell^2(\mathbb{N}_0)^m$ is derived from its intimate relationship with the trigonometric moment problem and hence with finite measures on the unit circle $\partial\mathbb{D}$. (Here $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.) This will be reviewed in some detail in Section 2 but we also refer to the monumental two-volume treatise by Simon [92] (see also [91] and [93]) and the exhaustive bibliography therein. For classical results on orthogonal polynomials on the unit circle we refer, for instance, to [6], [45]–[47], [62], [96]–[98], [101], [102]. More recent references relevant to the spectral theoretic content of this paper are [23], [42]–[44], [56], [57], [59], [81], and [90]. The full-lattice CMV operators $\mathbb{U}$ on $\mathbb{Z}$ are closely related to an important, and only recently intensively studied, completely integrable nonabelian version of the defocusing nonlinear Schrödinger equation (continuous in time but discrete in space), a special case of the Ablowitz–Ladik system. Relevant references in this context are, for instance, [1]–[5], [41], [49]–[51], [68], [74]–[77], [88], [100], and the literature cited therein. We emphasize that the case of matrix-valued coefficients α_k is considerably less studied than the case of scalar coefficients.

We note that our discussion of CMV operators will be undertaken in the spirit of [52], where (local and global) uniqueness theorems for full-line (resp., full-lattice) problems are formulated in terms of diagonal Green's matrices $g(z,x_0)$ and their x-derivatives $g'(z,x_0)$ at some fixed $x_0 \in \mathbb{R}$, for matrix-valued Schrödinger and Dirac-type operators on $\mathbb{R}$ and similarly for matrix-valued Jacobi operators on $\mathbb{Z}$. While we prove half-lattice and full-latice uniqueness results in our principal Section 4, we now confine ourselves in this introduction to just two typical results for CMV operators on $\mathbb{Z}$ with matrix-valued coefficients:

We use the following notation for the diagonal and for the neighboring off-diagonal entries of the Green's matrix of $\mathbb{U}$ (i.e., the discrete integral kernel of $(\mathbb{U}-zI)^{-1}$),

$$\begin{aligned} &g(z,k) = (\mathbb{U}-Iz)^{-1}(k,k), \\ &h(z,k) = \begin{cases} (\mathbb{U}-Iz)^{-1}(k-1,k), & k \text{ odd}, \\ (\mathbb{U}-Iz)^{-1}(k,k-1), & k \text{ even}, \end{cases} \quad k \in \mathbb{Z},\ z \in \mathbb{D}. \end{aligned} \tag{1.6}$$

The next uniqueness result then holds for the full-lattice CMV operator $\mathbb{U}$.

THEOREM 1.3. *Let $m \in \mathbb{N}$ and assume $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}}$ be a sequence of $m\times m$ matrices with complex entries such that $\|\alpha_k\|_{\mathbb{C}^{m\times m}} < 1$ and let $k_0 \in \mathbb{Z}$. Then any of the following two sets of data*

(*i*) *$g(z,k_0)$ and $h(z,k_0)$ for all z in a sufficiently small neighborhood of the origin under the assumption that $h(0,k_0)$ is invertible;*

(*ii*) *$g(z,k_0-1)$ and $g(z,k_0)$ for all z in a sufficiently small neighborhood of the origin and α_{k_0} under the assumption α_{k_0} is invertible;*

uniquely determine the matrix-valued Verblunsky coefficients $\{\alpha_k\}_{k\in\mathbb{Z}}$, and hence the full-lattice CMV operator $\mathbb{U}$ defined in (2.18).

In the subsequent local uniqueness result, $g^{(j)}$ and $h^{(j)}$ denote the corresponding quantities in (1.6) associated with the matrix-valued Verblunsky coefficients $\alpha^{(j)}$, $j = 1,2$.

THEOREM 1.4. *Let $m \in \mathbb{N}$ and assume $\alpha^{(\ell)} = \{\alpha_k^{(\ell)}\}_{k\in\mathbb{Z}}$ be sequences of $m\times m$ matrices with complex entries such that $\left\|\alpha_k^{(\ell)}\right\|_{\mathbb{C}^{m\times m}} < 1$, $k \in \mathbb{Z}$, $\ell = 1,2$. Moreover, assume $k_0 \in \mathbb{Z}$, $N \in \mathbb{N}$. Then for the full-lattice problems associated with $\alpha^{(1)}$ and $\alpha^{(2)}$ the following local uniqueness results hold:*

(*i*) *If either* $h^{(1)}(0,k_0)$ *or* $h^{(2)}(0,k_0)$ *is invertible and*

$$\big\|g^{(1)}(z,k_0)-g^{(2)}(z,k_0)\big\|_{\mathbb{C}^{m\times m}}+\big\|h^{(1)}(z,k_0)-h^{(2)}(z,k_0)\big\|_{\mathbb{C}^{m\times m}}\underset{z\to 0}{=}o(z^N),$$

$$\text{then } \alpha_k^{(1)}=\alpha_k^{(2)} \text{ for } k_0-N\leqslant k\leqslant k_0+N+1. \tag{1.7}$$

(*ii*) *If* $\alpha_{k_0}^{(1)}=\alpha_{k_0}^{(2)}$, $\alpha_{k_0}^{(1)}$ *is invertible, and*

$$\big\|g^{(1)}(z,k_0-1)-g^{(2)}(z,k_0-1)\big\|_{\mathbb{C}^{m\times m}}+\big\|g^{(1)}(z,k_0)-g^{(2)}(z,k_0)\big\|_{\mathbb{C}^{m\times m}}\underset{z\to 0}{=}o(z^N),$$

$$\text{then } \alpha_k^{(1)}=\alpha_k^{(2)} \text{ for } k_0-N-1\leqslant k\leqslant k_0+N+1. \tag{1.8}$$

The special case of CMV operators with scalar Verblunsky coefficients has recently been discussed in [22].

Finally, a brief description of the content of each section in this paper: In Section 2 we develop the basic Weyl–Titchmarsh theory for half-lattice CMV operators with matrix-valued Verblunsky coefficients. The analogous theory for full-line CMV operators is developed in Section 3. Weyl–Titchmarsh theory for CMV operators with matrix-valued Verblunsky coefficients is a subject of independent interest and of fundamental importance in the remainder of this paper. Section 4 contains our new Borg–Marchenko-type uniqueness results for half-lattice and full-lattice CMV operators with matrix-valued Verblunsky coefficients. Appendix A summarizes basic facts on matrix-valued Caratheodory and Schur functions relevant to this paper.

2. Weyl–Titchmarsh Theory for Half-Lattice CMV Operators with Matrix-Valued Verblunsky Coefficients

In this section we present the basics of Weyl–Titchmarsh theory for half-lattice CMV operators with matrix-valued Verblunsky coefficients. We closely follow the corresponding treatment of scalar-valued Verblunsky coefficients in [56].

We should note that while there is an extensive literature on orthogonal matrix-valued polynomials on the real line and on the unit circle, we refer, for instance, to [7], [8], [10, Ch. VII], [13], [18], [20], [24]–[35], [39], [40], [63], [66], [67], [69], [78]–[80], [84], [105]–[108], and the literature therein, the case of CMV operators with matrix-valued Verblunsky coefficients appears to be a much less explored frontier. The only references we are aware of in this context are Simon's treatise [92, Part 1, Sect. 2.13] and a recent preprint by Simon [94].

In the remainder of this paper, $\mathbb{C}^{m\times m}$ denotes the space of $m\times m$ matrices with complex-valued entries endowed with the operator norm $\|\cdot\|_{\mathbb{C}^{m\times m}}$ (we use the standard Euclidean norm in $\mathbb{C}^m$). The adjoint of an element $\gamma\in\mathbb{C}^{m\times m}$ is denoted by γ^*, and the real and imaginary parts of γ are defined as usual by $\operatorname{Re}(\gamma)=(\gamma+\gamma^*)/2$ and $\operatorname{Im}(\gamma)=(\gamma-\gamma^*)/(2i)$.

REMARK 2.1. For simplicity of exposition, we find it convenient to use the following conventions: We denote by $\mathrm{s}(\mathbb{Z})$ the vector space of all $\mathbb{C}$-valued sequences, and by $s(\mathbb{Z})^m = s(\mathbb{Z}) \otimes \mathbb{C}^m$ the vector space of all $\mathbb{C}^m$-valued sequences; that is,

$$\phi = \{\phi(k)\}_{k\in\mathbb{Z}} = \begin{pmatrix} \vdots \\ \phi(-1) \\ \phi(0) \\ \phi(1) \\ \vdots \end{pmatrix} \in s(\mathbb{Z})^m, \quad \phi(k) = \begin{pmatrix} (\phi(k))_1 \\ (\phi(k))_2 \\ \vdots \\ (\phi(k))_m \end{pmatrix} \in \mathbb{C}^m, \ k \in \mathbb{Z}. \tag{2.1}$$

Moreover, we introduce $s(\mathbb{Z})^{m\times n} = s(\mathbb{Z})^m \otimes \mathbb{C}^n$, $m,n \in \mathbb{N}$, that is, $\Phi = (\phi_1,\dots,\phi_n) \in s(\mathbb{Z})^{m\times n}$, where $\phi_j \in s(\mathbb{Z})^m$ for all $j = 1,\dots,n$.

We also note that $s(\mathbb{Z})^{m\times n} = s(\mathbb{Z}) \otimes \mathbb{C}^{m\times n}$, $m,n \in \mathbb{N}$; which is to say that the elements of $s(\mathbb{Z})^{m\times n}$ can be identified with the $\mathbb{C}^{m\times n}$-valued sequences,

$$\Phi=\{\Phi(k)\}_{k\in\mathbb{Z}}= \begin{pmatrix} \vdots \\ \Phi(-1) \\ \Phi(0) \\ \Phi(1) \\ \vdots \end{pmatrix}, \ \Phi(k)= \begin{pmatrix} (\Phi(k))_{1,1} & \dots & (\Phi(k))_{1,n} \\ \vdots & & \vdots \\ (\Phi(k))_{m,1} & \dots & (\Phi(k))_{m,n} \end{pmatrix} \in \mathbb{C}^{m\times n}, \ k \in \mathbb{Z}, \tag{2.2}$$

by setting $\Phi = (\phi_1,\dots,\phi_n)$, where

$$\phi_j = \begin{pmatrix} \vdots \\ \phi_j(-1) \\ \phi_j(0) \\ \phi_j(1) \\ \vdots \end{pmatrix} \in s(\mathbb{Z})^m, \quad \phi_j(k) = \begin{pmatrix} (\Phi(k))_{1,j} \\ \vdots \\ (\Phi(k))_{m,j} \end{pmatrix} \in \mathbb{C}^m, \ j = 1,\dots,n, \ k \in \mathbb{Z}. \tag{2.3}$$

For the elements of $s(\mathbb{Z})^{m\times n}$ we define the right-multiplication by $n\times n$ matrices with complex-valued entries by

$$\Phi C = (\phi_1,\dots,\phi_n) \begin{pmatrix} c_{1,1} & \dots & c_{1,n} \\ \vdots & & \vdots \\ c_{n,1} & \dots & c_{n,n} \end{pmatrix} = \left(\sum_{j=1}^{n} \phi_j c_{j,1}, \dots, \sum_{j=1}^{n} \phi_j c_{j,n} \right) \in s(\mathbb{Z})^{m\times n} \tag{2.4}$$

for all $\Phi \in s(\mathbb{Z})^{m\times n}$ and $C \in \mathbb{C}^{n\times n}$. In addition, for any linear transformation $\mathbb{A} : s(\mathbb{Z})^m \to s(\mathbb{Z})^m$, we define $\mathbb{A}\Phi$ for all $\Phi = (\phi_1,\dots,\phi_n) \in s(\mathbb{Z})^{m\times n}$ by

$$\mathbb{A}\Phi = (\mathbb{A}\phi_1,\dots,\mathbb{A}\phi_n) \in s(\mathbb{Z})^{m\times n}. \tag{2.5}$$

Given the above conventions, we note the subspace containment: $\ell^2(\mathbb{Z})^m = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^m \subset s(\mathbb{Z})^m$ and $\ell^2(\mathbb{Z})^{m\times n} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^{m\times n} \subset s(\mathbb{Z})^{m\times n}$. We also note that $\ell^2(\mathbb{Z})^m$ represents a Hilbert space with scalar product given by

$$(\phi,\psi)_{\ell^2(\mathbb{Z})^m} = \sum_{k=-\infty}^{\infty} \sum_{j=1}^{m} \overline{(\phi(k))_j} (\psi(k))_j, \quad \phi,\psi \in \ell^2(\mathbb{Z})^m. \tag{2.6}$$

Finally, we note that a straightforward modification of the above definitions also yields the Hilbert space $\ell^2(J)^m$ as well as the sets $\ell^2(J)^{m\times n}$, $\mathrm{s}(J)^m$, and $\mathrm{s}(J)^{m\times n}$ for any $J \subset \mathbb{Z}$.

We start by introducing our basic assumption:

HYPOTHESIS 2.2. *Let $m \in \mathbb{N}$ and assume $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}}$ is a sequence of $m\times m$ matrices with complex entries*[1] *and such that*

$$\|\alpha_k\|_{\mathbb{C}^{m\times m}} < 1, \quad k \in \mathbb{Z}. \tag{2.7}$$

Given a sequence α satisfying (2.7), we define two sequences of positive self-adjoint $m\times m$ matrices $\{\rho_k\}_{k\in\mathbb{Z}}$ and $\{\widetilde{\rho}_k\}_{k\in\mathbb{Z}}$ by

$$\rho_k = \sqrt{I_m - \alpha_k^*\alpha_k}, \quad k \in \mathbb{Z}, \tag{2.8}$$
$$\widetilde{\rho}_k = \sqrt{I_m - \alpha_k\alpha_k^*}, \quad k \in \mathbb{Z}, \tag{2.9}$$

and two sequences of $m\times m$ matrices with positive real parts, $\{a_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}^{m\times m}$ and $\{b_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}^{m\times m}$ by

$$a_k = I_m + \alpha_k, \quad k \in \mathbb{Z}, \tag{2.10}$$
$$b_k = I_m - \alpha_k, \quad k \in \mathbb{Z}. \tag{2.11}$$

Then (2.7) implies that ρ_k and $\widetilde{\rho}_k$ are invertible matrices for all $k \in \mathbb{Z}$, and using elementary power series expansions one verifies the following identities for all $k \in \mathbb{Z}$,

$$\widetilde{\rho}_k^{\pm 1}\alpha_k = \alpha_k\rho_k^{\pm 1}, \quad \alpha_k^*\widetilde{\rho}_k^{\pm 1} = \rho_k^{\pm 1}\alpha_k^*, \tag{2.12}$$
$$a_k^*\widetilde{\rho}_k^{-2}a_k = a_k\rho_k^{-2}a_k^*, \quad b_k^*\widetilde{\rho}_k^{-2}b_k = b_k\rho_k^{-2}b_k^*,$$
$$a_k^*\widetilde{\rho}_k^{-2}b_k + a_k\rho_k^{-2}b_k^* = b_k^*\widetilde{\rho}_k^{-2}a_k + b_k\rho_k^{-2}a_k^* = 2I_m. \tag{2.13}$$

According to Simon [92], we call α_k the Verblunsky coefficients in honor of Verblunsky's pioneering work in the theory of orthogonal polynomials on the unit circle [101], [102].

Next, we introduce a sequence of 2×2 block unitary matrices Θ_k with $m\times m$ matrix coefficients by

$$\Theta_k = \begin{pmatrix} -\alpha_k & \widetilde{\rho}_k \\ \rho_k & \alpha_k^* \end{pmatrix}, \quad k \in \mathbb{Z}, \tag{2.14}$$

and two unitary operators $\mathbb{V}$ and $\mathbb{W}$ on $\ell^2(\mathbb{Z})^m$ by their matrix representations in the standard basis of $\ell^2(\mathbb{Z})^m$ by

$$\mathbb{V} = \begin{pmatrix} \ddots & & & 0 \\ & \Theta_{2k-2} & & \\ & & \Theta_{2k} & \\ 0 & & & \ddots \end{pmatrix}, \quad \mathbb{W} = \begin{pmatrix} \ddots & & & 0 \\ & \Theta_{2k-1} & & \\ & & \Theta_{2k+1} & \\ 0 & & & \ddots \end{pmatrix}, \tag{2.15}$$

[1] We emphasize that $\alpha_k \in \mathbb{C}^{m\times m}$, $k \in \mathbb{Z}$, are general (not necessarily normal) matrices.

where

$$\begin{pmatrix} \mathbb{V}_{2k-1,2k-1} & \mathbb{V}_{2k-1,2k} \\ \mathbb{V}_{2k,2k-1} & \mathbb{V}_{2k,2k} \end{pmatrix} = \Theta_{2k}, \quad \begin{pmatrix} \mathbb{W}_{2k,2k} & \mathbb{W}_{2k,2k+1} \\ \mathbb{W}_{2k+1,2k} & \mathbb{W}_{2k+1,2k+1} \end{pmatrix} = \Theta_{2k+1}, \ k \in \mathbb{Z}. \tag{2.16}$$

Moreover, we introduce the unitary operator $\mathbb{U}$ on $\ell^2(\mathbb{Z})^m$ as the product of the unitary operators $\mathbb{V}$ and $\mathbb{W}$ by

$$\mathbb{U} = \mathbb{V}\mathbb{W}, \tag{2.17}$$

or in matrix form in the standard basis of $\ell^2(\mathbb{Z})^m$, by

$$\mathbb{U} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & 0 & -\alpha_0\rho_{-1} & -\alpha_0\alpha_{-1}^* & -\widetilde{\rho}_0\alpha_1 & \widetilde{\rho}_0\widetilde{\rho}_1 & & \mathbf{0} & \\ & & \rho_0\rho_{-1} & \rho_0\alpha_{-1}^* & -\alpha_0^*\alpha_1 & \alpha_0^*\widetilde{\rho}_1 & 0 & & \\ & & & 0 & -\alpha_2\rho_1 & -\alpha_2\alpha_1^* & -\widetilde{\rho}_2\alpha_3 & \widetilde{\rho}_2\widetilde{\rho}_3 & \\ & & & & \rho_2\rho_1 & \rho_2\alpha_1^* & -\alpha_2^*\alpha_3 & \alpha_2^*\widetilde{\rho}_3 & 0 \\ & \mathbf{0} & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.18}$$

Here terms of the form $-\alpha_{2k}\alpha_{2k-1}^*$ and $-\alpha_{2k}^*\alpha_{2k+1}$, $k \in \mathbb{Z}$, represent the diagonal entries $\mathbb{U}_{2k-1,2k-1}$ and $\mathbb{U}_{2k,2k}$ of the infinite matrix $\mathbb{U}$ in (2.18), respectively. We continue to call the operator $\mathbb{U}$ on $\ell^2(\mathbb{Z})^m$ the CMV operator since (2.14)–(2.18) in the context of the scalar-valued semi-infinite (i.e., half-lattice) case were obtained by Cantero, Moral, and Velázquez in [19] in 2003, but we refer to the discussion in the introduction about the involved history of these operators.

LEMMA 2.3. *Let $z \in \mathbb{C}\backslash\{0\}$ and $\{U(z,k)\}_{k\in\mathbb{Z}}, \{V(z,k)\}_{k\in\mathbb{Z}}$ be two $\mathbb{C}^{m\times m}$-valued sequences. Then the following items (i)–(iii) are equivalent:*

$$(i) \quad (\mathbb{U}U(z,\cdot))(k) = zU(z,k), \quad (\mathbb{W}U(z,\cdot))(k) = zV(z,k), \quad k \in \mathbb{Z}. \tag{2.19}$$

$$(ii) \quad (\mathbb{W}U(z,\cdot))(k) = zV(z,k), \quad (\mathbb{V}V(z,\cdot))(k) = U(z,k), \quad k \in \mathbb{Z}. \tag{2.20}$$

$$(iii) \quad \begin{pmatrix} U(z,k) \\ V(z,k) \end{pmatrix} = \mathbb{T}(z,k)\begin{pmatrix} U(z,k-1) \\ V(z,k-1) \end{pmatrix}, \quad k \in \mathbb{Z}. \tag{2.21}$$

Here $\mathbb{U}$, $\mathbb{V}$, and $\mathbb{W}$ are understood in the sense of difference expressions on $\mathrm{s}(\mathbb{Z})^{m\times m}$ rather than difference operators on $\ell^2(\mathbb{Z})^m$ (cf. Remark 2.1) and the transfer matrices $\mathbb{T}(z,k)$, $z \in \mathbb{C}\backslash\{0\}$, $k \in \mathbb{Z}$, are defined by

$$\mathbb{T}(z,k) = \begin{cases} \begin{pmatrix} \widetilde{\rho}_k^{-1}\alpha_k & z\widetilde{\rho}_k^{-1} \\ z^{-1}\rho_k^{-1} & \rho_k^{-1}\alpha_k^* \end{pmatrix}, & k \text{ odd}, \\ \begin{pmatrix} \rho_k^{-1}\alpha_k^* & \rho_k^{-1} \\ \widetilde{\rho}_k^{-1} & \widetilde{\rho}_k^{-1}\alpha_k \end{pmatrix}, & k \text{ even}. \end{cases} \tag{2.22}$$

Proof. The equivalence of (2.19) and (2.20) is a consequence of (2.17) and equivalence of (2.20) and (2.21) is implied by the following computations:

Assuming k to be odd and utilizing (2.8), (2.9), and (2.12), one verifies equivalence of the following items (i)–(v):

$$(i) \quad \begin{pmatrix} U(z,k) \\ V(z,k) \end{pmatrix} = \mathbb{T}(z,k) \begin{pmatrix} U(z,k-1) \\ V(z,k-1) \end{pmatrix}. \tag{2.23}$$

$$(ii) \quad \begin{cases} \widetilde{\rho}_k U(z,k) = \alpha_k U(z,k-1) + zV(z,k-1), \\ \rho_k zV(z,k) = U(z,k-1) + \alpha_k^* zV(z,k-1). \end{cases} \tag{2.24}$$

$$(iii) \quad \begin{cases} zV(z,k-1) = -\alpha_k U(z,k-1) + \widetilde{\rho}_k U(z,k), \\ \rho_k zV(z,k) = U(z,k-1) + \alpha_k^* \big(-\alpha_k U(z,k-1) + \widetilde{\rho}_k U(z,k) \big). \end{cases} \tag{2.25}$$

$$(iv) \quad \begin{cases} zV(z,k-1) = -\alpha_k U(z,k-1) + \widetilde{\rho}_k U(z,k), \\ \rho_k zV(z,k) = \rho_k^2 U(z,k-1) + \rho_k \alpha_k^* U(z,k). \end{cases} \tag{2.26}$$

$$(v) \quad z \begin{pmatrix} V(z,k-1) \\ V(z,k) \end{pmatrix} = \Theta_k \begin{pmatrix} U(z,k-1) \\ U(z,k) \end{pmatrix}. \tag{2.27}$$

Similarly, assuming k to be even, one verifies that the items (vi)–$(viii)$ are equivalent:

$$(vi) \quad \begin{pmatrix} U(z,k) \\ V(z,k) \end{pmatrix} = \mathbb{T}(z,k) \begin{pmatrix} U(z,k-1) \\ V(z,k-1) \end{pmatrix}. \tag{2.28}$$

$$(vii) \quad \begin{cases} \widetilde{\rho}_k V(z,k) = \alpha_k V(z,k-1) + U(z,k-1), \\ \rho_k U(z,k) = V(z,k-1) + \alpha_k^* U(z,k-1). \end{cases} \tag{2.29}$$

$$(viii) \quad \begin{pmatrix} U(z,k-1) \\ U(z,k) \end{pmatrix} = \Theta_k \begin{pmatrix} V(z,k-1) \\ V(z,k) \end{pmatrix}. \tag{2.30}$$

Finally, taking into account (2.15) and (2.16), one concludes that

$$\begin{aligned} \Theta_{2k+1} \begin{pmatrix} U(z,2k) \\ U(z,2k+1) \end{pmatrix} &= z \begin{pmatrix} V(z,2k) \\ V(z,2k+1) \end{pmatrix}, \\ \Theta_{2k} \begin{pmatrix} V(z,2k-1) \\ V(z,2k) \end{pmatrix} &= z \begin{pmatrix} U(z,2k-1) \\ U(z,2k) \end{pmatrix}, \quad k \in \mathbb{Z} \end{aligned} \tag{2.31}$$

is equivalent to

$$(\mathbb{W}U(z,\cdot))(k) = zV(z,k), \quad (\mathbb{V}V(z,\cdot))(k) = U(z,k), \quad k \in \mathbb{Z}. \tag{2.32}$$

□

We note that in studying solutions of $(\mathbb{U}U(z,\cdot))(k) = zU(z,k)$ as in Lemma 2.3 (i), the purpose of the additional relation $(\mathbb{W}U(z,\cdot))(k) = zV(z,k)$ in (2.19) is to introduce a new variable V that improves our understanding of the structure of such solutions U.

If one sets $\alpha_{k_0} = I_m$ for some reference point $k_0 \in \mathbb{Z}$, then the operator $\mathbb{U}$ splits into a direct sum of two half-lattice operators $\mathbb{U}_{-,k_0-1}$ and $\mathbb{U}_{+,k_0}$ acting on $\ell^2((-\infty,k_0-1]\cap\mathbb{Z})^m$ and on $\ell^2([k_0,\infty)\cap\mathbb{Z})^m$, respectively. Explicitly, one obtains

$$\mathbb{U} = \mathbb{U}_{-,k_0-1} \oplus \mathbb{U}_{+,k_0} \text{ in } \ell^2((-\infty,k_0-1]\cap\mathbb{Z})^m \oplus \ell^2([k_0,\infty)\cap\mathbb{Z})^m. \tag{2.33}$$

(Strictly, speaking, setting $\alpha_{k_0} = I_m$ for some reference point $k_0 \in \mathbb{Z}$ contradicts our basic Hypothesis 2.2. However, as long as the exception to Hypothesis 2.2 refers to only one site, we will safely ignore this inconsistency in favor of the notational simplicity it provides by avoiding the introduction of a properly modified hypothesis on $\{\alpha_k\}_{k\in\mathbb{Z}}$.) Similarly, one obtains $\mathbb{W}_{-,k_0-1}$, $\mathbb{V}_{-,k_0-1}$ and $\mathbb{W}_{+,k_0}$, $\mathbb{V}_{+,k_0}$ such that

$$\mathbb{U}_{\pm,k_0} = \mathbb{V}_{\pm,k_0}\mathbb{W}_{\pm,k_0}. \tag{2.34}$$

LEMMA 2.4. *Let $z \in \mathbb{C}\backslash\{0\}$, $k_0 \in \mathbb{Z}$, and $\{\widehat{P}_+(z,k,k_0)\}_{k\geqslant k_0}$, $\{\widehat{R}_+(z,k,k_0)\}_{k\geqslant k_0}$ be two $\mathbb{C}^{m\times m}$-valued sequences. Then the following items (i)–(vi) are equivalent:*

$$(i)\ (\mathbb{U}_{+,k_0}\widehat{P}_+(z,\cdot,k_0))(k) = z\widehat{P}_+(z,k,k_0),$$
$$(\mathbb{W}_{+,k_0}\widehat{P}_+(z,\cdot,k_0))(k)=z\widehat{R}_+(z,k,k_0),\ k\geqslant k_0. \tag{2.35}$$
$$(ii)\ (\mathbb{W}_{+,k_0}\widehat{P}_+(z,\cdot,k_0))(k) = z\widehat{R}_+(z,k,k_0),$$
$$(\mathbb{V}_{+,k_0}\widehat{R}_+(z,\cdot,k_0))(k) = \widehat{P}_+(z,k,k_0),\ k\geqslant k_0. \tag{2.36}$$
$$(iii)\ \begin{pmatrix}\widehat{P}_+(z,k,k_0)\\ \widehat{R}_+(z,k,k_0)\end{pmatrix} = \mathbb{T}(z,k)\begin{pmatrix}\widehat{P}_+(z,k-1,k_0)\\ \widehat{R}_+(z,k-1,k_0)\end{pmatrix}, \quad k>k_0,$$
$$\textit{with initial condition } \widehat{P}_+(z,k_0,k_0) = \begin{cases} z\widehat{R}_+(z,k_0,k_0), & k_0 \textit{ odd},\\ \widehat{R}_+(z,k_0,k_0), & k_0 \textit{ even}.\end{cases} \tag{2.37}$$

Next, consider $\mathbb{C}^{m\times m}$-valued sequences $\{\widehat{P}_-(z,k,k_0)\}_{k\leqslant k_0}$, $\{\widehat{R}_-(z,k,k_0)\}_{k\leqslant k_0}$. Then the following items (iv)–(vi) are equivalent:

$$(iv)\ (\mathbb{U}_{-,k_0}\widehat{P}_-(z,\cdot,k_0))(k) = z\widehat{P}_-(z,k,k_0),$$
$$(\mathbb{W}_{-,k_0}\widehat{P}_-(z,\cdot,k_0))(k) = z\widehat{R}_-(z,k,k_0),\ k\leqslant k_0. \tag{2.38}$$
$$(v)\ (\mathbb{W}_{-,k_0}\widehat{P}_-(z,\cdot,k_0))(k) = z\widehat{R}_-(z,k,k_0),$$
$$(\mathbb{V}_{-,k_0}\widehat{R}_-(z,\cdot,k_0))(k) = \widehat{P}_-(z,k,k_0),\ k\leqslant k_0. \tag{2.39}$$
$$(vi)\ \begin{pmatrix}\widehat{P}_-(z,k-1),k_0\\ \widehat{R}_-(z,k-1,k_0)\end{pmatrix} = \mathbb{T}(z,k)^{-1}\begin{pmatrix}\widehat{P}_-(z,k,k_0)\\ \widehat{R}_-(z,k,k_0)\end{pmatrix}, \quad k\leqslant k_0,$$
$$\textit{with initial condition } \widehat{P}_-(z,k_0,k_0) = \begin{cases} -\widehat{R}_-(z,k_0,k_0), & k_0 \textit{ odd},\\ -z\widehat{R}_-(z,k_0,k_0), & k_0 \textit{ even}.\end{cases} \tag{2.40}$$

Here $\mathbb{U}_{\pm,k_0}$, $\mathbb{V}_{\pm,k_0}$, and $\mathbb{W}_{\pm,k_0}$ are understood in the sense of difference expressions on the set $\mathrm{s}(\mathbb{Z}\cap[k_0,\pm\infty))^{m\times m}$ rather than difference operators on $\ell^2(\mathbb{Z}\cap[k_0,\pm\infty))^m$ (cf. Remark 2.1).

Proof. Equivalence of (2.35) and (2.36) is a consequence of (2.34).

Next, repeating the proof of Lemma 2.3 one obtains that

$$(\mathbb{W}_{+,k_0}\widehat{P}_+(z,\cdot,k_0))(k)=z\widehat{R}_+(z,k,k_0),\ \ (\mathbb{V}_{+,k_0}\widehat{R}_+(z,\cdot,k_0))(k)=\widehat{P}_+(z,k,k_0),\ k>k_0, \tag{2.41}$$

is equivalent to

$$\begin{pmatrix} \widehat{P}_+(z,k,k_0) \\ \widehat{R}_+(z,k,k_0) \end{pmatrix} = \mathbb{T}(z,k) \begin{pmatrix} \widehat{P}_+(z,k-1,k_0) \\ \widehat{R}_+(z,k-1,k_0) \end{pmatrix}, \quad k > k_0. \tag{2.42}$$

Moreover, in the case k_0 is odd, the matrices $\mathbb{V}_{+,k_0}$ and $\mathbb{W}_{+,k_0}$ have the structure,

$$\mathbb{V}_{+,k_0} = \begin{pmatrix} \Theta_{k_0+1} & & 0 \\ & \Theta_{k_0+3} & \\ 0 & & \ddots \end{pmatrix}, \quad \mathbb{W}_{+,k_0} = \begin{pmatrix} I_m & & 0 \\ & \Theta_{k_0+2} & \\ 0 & & \ddots \end{pmatrix}, \tag{2.43}$$

and hence,

$$(\mathbb{W}_{+,k_0}\widehat{P}_+(z,\cdot,k_0))(k_0) = z\widehat{R}_+(z,k_0,k_0) \tag{2.44}$$

is equivalent to

$$\widehat{P}_+(z,k_0,k_0) = z\widehat{R}_+(z,k_0,k_0). \tag{2.45}$$

In the case k_0 is even, the matrices V_{+,k_0} and W_{+,k_0} have the structure,

$$\mathbb{V}_{+,k_0} = \begin{pmatrix} I_m & & 0 \\ & \Theta_{k_0+2} & \\ 0 & & \ddots \end{pmatrix}, \quad \mathbb{W}_{+,k_0} = \begin{pmatrix} \Theta_{k_0+1} & & 0 \\ & \Theta_{k_0+3} & \\ 0 & & \ddots \end{pmatrix}, \tag{2.46}$$

and hence,

$$(\mathbb{V}_{+,k_0}\widehat{R}_+(z,\cdot,k_0))(k_0) = \widehat{P}_+(z,k_0,k_0) \tag{2.47}$$

is equivalent to

$$\widehat{P}_+(z,k_0,k_0) = \widehat{R}_+(z,k_0,k_0). \tag{2.48}$$

Thus, one infers the equivalence of (2.36) and (2.37) from the equivalence of (2.41) and (2.42) with (2.44)–(2.45) and (2.47)–(2.48).

The results for $\{\widehat{P}_-(z,k,k_0)\}_{k\leqslant k_0}$ and $\{\widehat{R}_-(z,k,k_0)\}_{k\leqslant k_0}$ are proved analogously. $\square$

Analogous comments to those made right after the proof of Lemma 2.3 apply in the present context of Lemma 2.4.

Next, we denote by $\left(\begin{smallmatrix} P_\pm(z,k,k_0) \\ R_\pm(z,k,k_0) \end{smallmatrix}\right)_{k\in\mathbb{Z}}$ and $\left(\begin{smallmatrix} Q_\pm(z,k,k_0) \\ S_\pm(z,k,k_0) \end{smallmatrix}\right)_{k\in\mathbb{Z}}$, $z \in \mathbb{C}\backslash\{0\}$, four linearly independent solutions of (2.21) satisfying the following initial conditions:

$$\begin{pmatrix} P_+(z,k_0,k_0) \\ R_+(z,k_0,k_0) \end{pmatrix} = \begin{cases} \left(\begin{smallmatrix} zI_m \\ I_m \end{smallmatrix}\right), & k_0 \text{ odd}, \\ \left(\begin{smallmatrix} I_m \\ I_m \end{smallmatrix}\right), & k_0 \text{ even}, \end{cases} \quad \begin{pmatrix} Q_+(z,k_0,k_0) \\ S_+(z,k_0,k_0) \end{pmatrix} = \begin{cases} \left(\begin{smallmatrix} zI_m \\ -I_m \end{smallmatrix}\right), & k_0 \text{ odd}, \\ \left(\begin{smallmatrix} -I_m \\ I_m \end{smallmatrix}\right), & k_0 \text{ even}. \end{cases} \tag{2.49}$$

$$\begin{pmatrix} P_-(z,k_0,k_0) \\ R_-(z,k_0,k_0) \end{pmatrix} = \begin{cases} \left(\begin{smallmatrix} I_m \\ -I_m \end{smallmatrix}\right), & k_0 \text{ odd}, \\ \left(\begin{smallmatrix} -zI_m \\ I_m \end{smallmatrix}\right), & k_0 \text{ even}, \end{cases} \quad \begin{pmatrix} Q_-(z,k_0,k_0) \\ S_-(z,k_0,k_0) \end{pmatrix} = \begin{cases} \left(\begin{smallmatrix} I_m \\ I_m \end{smallmatrix}\right), & k_0 \text{ odd}, \\ \left(\begin{smallmatrix} zI_m \\ I_m \end{smallmatrix}\right), & k_0 \text{ even}. \end{cases} \tag{2.50}$$

Then $P_\pm(z,k,k_0)$, $Q_\pm(z,k,k_0)$, $R_\pm(z,k,k_0)$, and $S_\pm(z,k,k_0)$, $k,k_0 \in \mathbb{Z}$, are $\mathbb{C}^{m\times m}$-valued Laurent polynomials in z. In particular, one computes

k	k_0-1	k_0 odd	k_0+1
$\begin{pmatrix} P_+(z,k,k_0) \\ R_+(z,k,k_0) \end{pmatrix}$	$\begin{pmatrix} z\rho_{k_0}^{-1}(I_m-\alpha_{k_0}^*) \\ \widetilde\rho_{k_0}^{-1}(I_m-\alpha_{k_0}) \end{pmatrix}$	$\begin{pmatrix} zI_m \\ I_m \end{pmatrix}$	$\begin{pmatrix} \rho_{k_0+1}^{-1}(I_m+z\alpha_{k_0+1}^*) \\ \widetilde\rho_{k_0+1}^{-1}(zI_m+\alpha_{k_0+1}) \end{pmatrix}$
$\begin{pmatrix} Q_+(z,k,k_0) \\ S_+(z,k,k_0) \end{pmatrix}$	$\begin{pmatrix} z\rho_{k_0}^{-1}(-I_m-\alpha_{k_0}^*) \\ \widetilde\rho_{k_0}^{-1}(I_m+\alpha_{k_0}) \end{pmatrix}$	$\begin{pmatrix} zI_m \\ -I_m \end{pmatrix}$	$\begin{pmatrix} \rho_{k_0+1}^{-1}(-I_m+z\alpha_{k_0+1}^*) \\ \widetilde\rho_{k_0+1}^{-1}(zI_m-\alpha_{k_0+1}) \end{pmatrix}$
$\begin{pmatrix} P_-(z,k,k_0) \\ R_-(z,k,k_0) \end{pmatrix}$	$\begin{pmatrix} \rho_{k_0}^{-1}(-zI_m-\alpha_{k_0}^*) \\ \widetilde\rho_{k_0}^{-1}(\frac{1}{z}I_m+\alpha_{k_0}) \end{pmatrix}$	$\begin{pmatrix} I_m \\ -I_m \end{pmatrix}$	$\begin{pmatrix} \rho_{k_0+1}^{-1}(-I_m+\alpha_{k_0+1}^*) \\ \widetilde\rho_{k_0+1}^{-1}(I_m-\alpha_{k_0+1}) \end{pmatrix}$
$\begin{pmatrix} Q_-(z,k,k_0) \\ S_-(z,k,k_0) \end{pmatrix}$	$\begin{pmatrix} \rho_{k_0}^{-1}(zI_m-\alpha_{k_0}^*) \\ \widetilde\rho_{k_0}^{-1}(\frac{1}{z}I_m-\alpha_{k_0}) \end{pmatrix}$	$\begin{pmatrix} I_m \\ I_m \end{pmatrix}$	$\begin{pmatrix} \rho_{k_0+1}^{-1}(I_m+\alpha_{k_0+1}^*) \\ \widetilde\rho_{k_0+1}^{-1}(I_m+\alpha_{k_0+1}) \end{pmatrix}$

k	k_0-1	k_0 even	k_0+1
$\begin{pmatrix} P_+(z,k,k_0) \\ R_+(z,k,k_0) \end{pmatrix}$	$\begin{pmatrix} \widetilde\rho_{k_0}^{-1}(I_m-\alpha_{k_0}) \\ \rho_{k_0}^{-1}(I_m-\alpha_{k_0}^*) \end{pmatrix}$	$\begin{pmatrix} I_m \\ I_m \end{pmatrix}$	$\begin{pmatrix} \widetilde\rho_{k_0+1}^{-1}(zI_m+\alpha_{k_0+1}) \\ \rho_{k_0+1}^{-1}(\frac{1}{z}I_m+\alpha_{k_0+1}^*) \end{pmatrix}$
$\begin{pmatrix} Q_+(z,k,k_0) \\ S_+(z,k,k_0) \end{pmatrix}$	$\begin{pmatrix} \widetilde\rho_{k_0}^{-1}(I_m+\alpha_{k_0}) \\ \rho_{k_0}^{-1}(-I_m-\alpha_{k_0}^*) \end{pmatrix}$	$\begin{pmatrix} -I_m \\ I_m \end{pmatrix}$	$\begin{pmatrix} \widetilde\rho_{k_0+1}^{-1}(zI_m-\alpha_{k_0+1}) \\ \rho_{k_0+1}^{-1}(-\frac{1}{z}I_m+\alpha_{k_0+1}^*) \end{pmatrix}$
$\begin{pmatrix} P_-(z,k,k_0) \\ R_-(z,k,k_0) \end{pmatrix}$	$\begin{pmatrix} \widetilde\rho_{k_0}^{-1}(I_m+z\alpha_{k_0}) \\ \rho_{k_0}^{-1}(-zI_m-\alpha_{k_0}^*) \end{pmatrix}$	$\begin{pmatrix} -zI_m \\ I_m \end{pmatrix}$	$\begin{pmatrix} z\widetilde\rho_{k_0+1}^{-1}(I_m-\alpha_{k_0+1}) \\ \rho_{k_0+1}^{-1}(-I_m+\alpha_{k_0+1}^*) \end{pmatrix}$
$\begin{pmatrix} Q_-(z,k,k_0) \\ S_-(z,k,k_0) \end{pmatrix}$	$\begin{pmatrix} \widetilde\rho_{k_0}^{-1}(I_m-z\alpha_{k_0}) \\ \rho_{k_0}^{-1}(zI_m-\alpha_{k_0}^*) \end{pmatrix}$	$\begin{pmatrix} zI_m \\ I_m \end{pmatrix}$	$\begin{pmatrix} z\widetilde\rho_{k_0+1}^{-1}(I_m+\alpha_{k_0+1}) \\ \rho_{k_0+1}^{-1}(I_m+\alpha_{k_0+1}^*) \end{pmatrix}$

(2.51)

REMARK 2.5. Subsequently, we will have to refer to the leading-order terms of certain matrix-valued Laurent polynomials at various occasions. To put this in precise terms we now introduce the following conventions: We will refer to the terms

$$\begin{cases} z^{-(k+1)/2}, & k \text{ odd}, \\ z^{k/2}, & k \text{ even}, \end{cases} \tag{2.52}$$

as the leading-order terms of the Laurent polynomials

$$\begin{cases} z^{-1}P_+(z,k_0+k,k_0), \ R_-(z,k_0-k,k_0), & \\ z^{-1}Q_+(z,k_0+k,k_0), \ S_-(z,k_0-k,k_0), & k_0 \text{ odd}, \\ R_+(z,k_0+k,k_0), \ z^{-1}P_-(z,k_0-k,k_0), & \\ S_+(z,k_0+k,k_0), \ z^{-1}Q_-(z,k_0-k,k_0), & k_0 \text{ even}, \end{cases} \tag{2.53}$$

and similarly, we will refer to the terms

$$\begin{cases} z^{(k+1)/2}, & k \text{ odd}, \\ z^{-k/2}, & k \text{ even}, \end{cases} \tag{2.54}$$

as the leading-order term of the Laurent polynomials

$$\begin{cases} R_+(z,k_0+k,k_0),\ P_-(z,k_0-k,k_0), \\ S_+(z,k_0+k,k_0),\ Q_-(z,k_0-k,k_0), & k_0 \text{ odd}, \\ P_+(z,k_0+k,k_0),\ R_-(z,k_0-k,k_0), \\ Q_+(z,k_0+k,k_0),\ S_-(z,k_0-k,k_0), & k_0 \text{ even}. \end{cases} \tag{2.55}$$

REMARK 2.6. We note that Lemmas 2.3 and 2.4 are crucial for many of the proofs to follow. For instance, we note that the equivalence of items (i) and (iii) in Lemma 2.3 proves that for each $z \in \mathbb{C}\backslash\{0\}$, any solutions $\{U(z,k)\}_{k\in\mathbb{Z}}$ of $\mathbb{U}U(z,\cdot) = zU(z,\cdot)$ can be expressed as a linear combinations of $P_+(z,\cdot,k_0)$ and $Q_+(z,\cdot,k_0)$ (or $P_-(z,\cdot,k_0)$ and $Q_-(z,\cdot,k_0)$) with z-dependent right-multiple $\mathbb{C}^{m\times m}$-valued coefficients. This equivalence also proves that any solution of $\mathbb{U}U(z,\cdot) = zU(z,\cdot)$ is determined by the values of U and the auxiliary variable V at a site k_0. In the context of Lemma 2.4, we remark that its importance lies in the fact that it shows that in the case of half-lattice CMV operators, the analogous equations have solutions, which up to right-multiplication by z-dependent $\mathbb{C}^{m\times m}$-valued coefficients, are given by $\{P_\pm(z,k,k_0)\}_{k\in\mathbb{Z}}$ for each $z \in \mathbb{C}\backslash\{0\}$. Consequently, the corresponding solutions are determined by their value at a single site k_0.

Next, we introduce the modified matrix-valued Laurent polynomials $\widetilde{P}_\pm(z,k,k_0)$ and $\widetilde{Q}_\pm(z,k,k_0)$, $z \in \mathbb{C}\backslash\{0\}$, $k,k_0 \in \mathbb{Z}$, by

$$\widetilde{P}_+(z,k,k_0) = \begin{cases} P_+(z,k,k_0)/z, & k_0 \text{ odd}, \\ P_+(z,k,k_0), & k_0 \text{ even}, \end{cases} \quad \widetilde{P}_-(z,k,k_0) = \begin{cases} P_-(z,k,k_0), & k_0 \text{ odd}, \\ -P_-(z,k,k_0)/z, & k_0 \text{ even}, \end{cases} \tag{2.56}$$

$$\widetilde{Q}_+(z,k,k_0) = \begin{cases} Q_+(z,k,k_0)/z, & k_0 \text{ odd}, \\ Q_+(z,k,k_0), & k_0 \text{ even}, \end{cases} \quad \widetilde{Q}_-(z,k,k_0) = \begin{cases} Q_-(z,k,k_0), & k_0 \text{ odd}, \\ -Q_-(z,k,k_0)/z, & k_0 \text{ even}. \end{cases} \tag{2.57}$$

In the remainder of this paper we use the following abbreviations for subarcs A_ζ of $\partial\mathbb{D}$,

$$A_\zeta = \big\{e^{i\phi} \in \partial\mathbb{D} \,\big|\, 0 \leqslant \phi \leqslant \theta\big\}, \quad \zeta = e^{i\theta},\ \theta \in [0,2\pi). \tag{2.58}$$

The next auxiliary result is of importance in proving orthonormality of the matrix-valued Laurent polynomials $P_\pm$ and $R_\pm$.

LEMMA 2.7. *Let* $\{F_\pm(\cdot,k,k_0)\}_{k \gtreqless k_0}$ *denote two sequences of* $\mathbb{C}^{m\times m}$*-valued functions of bounded variation with* $F_\pm(1,k,k_0) = 0$ *for all* $k \gtreqless k_0$ *that satisfy*

$$(\mathbb{U}_{\pm,k_0} F_\pm(\zeta,\cdot,k_0))(k) = \int_{A_\zeta} dF_\pm(\zeta',k,k_0)\,\zeta', \quad \zeta \in \partial\mathbb{D},\ k \gtreqless k_0, \tag{2.59}$$

where $\mathbb{U}_{\pm,k_0}$ *are understood in the sense of difference expressions on* $\mathrm{s}(\mathbb{Z}\cap[k_0,\pm\infty))^{m\times m}$ *rather than difference operators on* $\ell^2(\mathbb{Z}\cap[k_0,\pm\infty))^m$ *(cf. Remark 2.1). Then,* $F_\pm(\cdot,k,k_0)$ *also satisfy*

$$F_\pm(\zeta,k,k_0) = \int_{A_\zeta} \widetilde{P}_\pm(\zeta',k,k_0)\,dF_\pm(\zeta',k_0,k_0), \quad \zeta \in \partial\mathbb{D},\ k \gtreqless k_0. \tag{2.60}$$

Proof. Let $\{G_\pm(\cdot,k,k_0)\}_{k \gtrless k_0}$ denote the two sequences of $\mathbb{C}^{m\times m}$-valued functions,

$$G_\pm(\zeta,k,k_0) = \int_{A_\zeta} \widetilde{P}_\pm(\zeta',k,k_0) dF_\pm(\zeta',k_0,k_0), \quad \zeta \in \partial\mathbb{D},\ k \gtrless k_0. \tag{2.61}$$

Then it suffices to prove that $F_\pm(\zeta,k,k_0) = G_\pm(\zeta,k,k_0)$, $\zeta \in \partial\mathbb{D}$, $k \gtrless k_0$.

First, we note that according to (2.49), (2.50), and (2.56), $\widetilde{P}_\pm(\zeta,k_0,k_0) = I_m$, and hence,

$$G_\pm(\zeta,k_0,k_0) = \int_{A_\zeta} dF_\pm(\zeta',k_0,k_0) = F_\pm(\zeta,k_0,k_0), \quad \zeta \in \partial\mathbb{D}. \tag{2.62}$$

Moreover,

$$\begin{aligned}(\mathbb{U}_{\pm,k_0} G_\pm(\zeta,\cdot,k_0))(k) &= \int_{A_\zeta} (\mathbb{U}_{\pm,k_0} \widetilde{P}_\pm(\zeta',\cdot,k_0))(k) dF_\pm(\zeta',k_0,k_0) \\ &= \int_{A_\zeta} dG_\pm(\zeta',k,k_0)\,\zeta', \quad \zeta \in \partial\mathbb{D},\ k \gtrless k_0.\end{aligned} \tag{2.63}$$

Next, defining $K_\pm(\zeta,k,k_0) = F_\pm(\zeta,k,k_0) - G_\pm(\zeta,k,k_0)$, $\zeta \in \partial\mathbb{D}$, $k \gtrless k_0$, one obtains

$$K_\pm(\zeta,k_0,k_0) = 0 \text{ and } (\mathbb{U}_{\pm,k_0} K_\pm(\zeta,\cdot,k_0))(k) = \int_{A_\zeta} dK_\pm(\zeta',k,k_0)\,\zeta',\ \zeta \in \partial\mathbb{D},\ k \gtrless k_0,$$

or equivalently,

$$K_\pm(\zeta,k_0,k_0) = 0 \text{ and } (\mathbb{U}_{\pm,k_0} K_\pm(\zeta,\cdot,k_0))(k) = (\mathbb{L}\,K_\pm(\cdot,k,k_0))(\zeta),\ \zeta \in \partial\mathbb{D},\ k \gtrless k_0, \tag{2.64}$$

where $\mathbb{L}$ denotes the boundedly invertible operator on $\mathbb{C}^{m\times m}$-valued functions K of bounded variation defined by

$$(\mathbb{L}\,K)(\zeta) = \int_{A_\zeta} dK(\zeta')\,\zeta', \quad (\mathbb{L}^{-1}K)(\zeta) = \int_{A_\zeta} dK(\zeta')\,\zeta'^{-1}. \tag{2.65}$$

Finally, since, $\mathbb{L}$ commutes with all constant $m\times m$ matrices, one can repeat the proof of Lemma 2.4 with z replaced by $\mathbb{L}$ and obtain that (2.64) has the unique solution $K_\pm(\zeta,k,k_0) = 0$, $\zeta \in \partial\mathbb{D}$, $k \gtrless k_0$, and hence, $F_\pm(\zeta,k,k_0) = G_\pm(\zeta,k,k_0)$, $\zeta \in \partial\mathbb{D}$, $k \gtrless k_0$. $\square$

Next, following [10] (see also [13]), we prove a matrix-valued version of the “orthogonality" relation for matrix-valued Laurent polynomials $P_\pm$ and $R_\pm$.

Let $\Delta_k = \{\Delta_k(\ell)\}_{\ell\in\mathbb{Z}} \in s(\mathbb{Z})^{m\times m}$, $k \in \mathbb{Z}$, denote the sequences of $m\times m$ matrices defined by

$$(\Delta_k)(\ell) = \begin{cases} I_m, & \ell = k, \\ 0, & \ell \neq k, \end{cases} \quad k,\ell \in \mathbb{Z}. \tag{2.66}$$

Then using right-multiplication by $m \times m$ matrices on $\mathrm{s}(\mathbb{Z})^{m\times m}$ defined in Remark 2.1, we get the identity

$$(\Delta_k X)(\ell) = \begin{cases} X, & \ell = k, \\ 0, & \ell \neq k, \end{cases} \quad X \in \mathbb{C}^{m\times m}, \tag{2.67}$$

and hence consider Δ_k as a map $\Delta_k \colon \mathbb{C}^{m\times m} \to \mathrm{s}(\mathbb{Z})^{m\times m}$. In addition, we introduce the map $\Delta_k^* \colon \mathrm{s}(\mathbb{Z})^{m\times m} \to \mathbb{C}^{m\times m}$, $k \in \mathbb{Z}$, defined by

$$\Delta_k^* \Phi = \Phi(k), \text{ where } \Phi = \{\Phi(k)\}_{k\in\mathbb{Z}} \in \mathrm{s}(\mathbb{Z})^{m\times m}. \tag{2.68}$$

Similarly, one introduces the corresponding maps with $\mathbb{Z}$ replaced by $[k_0, \pm\infty) \cap \mathbb{Z}$, $k_0 \in \mathbb{Z}$, which, for notational brevity, we will also denote by Δ_k and Δ_k^*, respectively.

Next, we call sequences of $C^{m\times m}$-valued functions $\{\Phi_\pm(\cdot,k,k_0)\}_{k \gtrless k_0}$ orthonormal[2] on $\partial\mathbb{D}$ with respect to some $\mathbb{C}^{m\times m}$-valued measures $d\Omega_\pm(\cdot,k_0)$, defined on $\partial\mathbb{D}$, if the following identity holds for all $k, k' \gtrless k_0$,

$$\oint_{\partial\mathbb{D}} \Phi_\pm(\zeta,k,k_0)\, d\Omega_\pm(\zeta,k_0)\, \Phi_\pm(\zeta,k',k_0)^* = \delta_{k,k'} I_m. \tag{2.69}$$

We will also call the sequences of $\mathbb{C}^{m\times m}$-valued functions $\{\Phi_\pm(\cdot,k,k_0)\}_{k \gtrless k_0}$ complete with respect to the measures $d\Omega_\pm(\cdot,k_0)$ if the collections of $\mathbb{C}^m$-valued functions

$$\left\{ \phi_{\pm,j}(\cdot,k,k_0) = \begin{pmatrix} (\Phi_\pm(\cdot,k,k_0))_{1,j} \\ \vdots \\ (\Phi_\pm(\cdot,k,k_0))_{m,j} \end{pmatrix} \right\}_{j=1,\dots,m,\ k \gtrless k_0} \tag{2.70}$$

form complete systems in $L^2(\partial\mathbb{D}; d\Omega_\pm(\cdot,k_0))$.

LEMMA 2.8. *Let $k_0 \in \mathbb{Z}$. The sets of $\mathbb{C}^{m\times m}$-valued Laurent polynomials $\{P_\pm(\cdot,k,k_0)^*\}_{k \gtrless k_0}$ and $\{R_\pm(\cdot,k,k_0)^*\}_{k \gtrless k_0}$ form complete orthonormal systems on $\partial\mathbb{D}$ with respect to $\mathbb{C}^{m\times m}$-valued measures $d\Omega_\pm(\cdot,k_0)$ defined by*

$$d\Omega_\pm(\zeta,k_0) = d(\Delta_{k_0}^* E_{\mathbb{U}_{\pm,k_0}}(\zeta) \Delta_{k_0}), \quad \zeta \in \partial\mathbb{D}, \tag{2.71}$$

where $E_{\mathbb{U}_{\pm,k_0}}(\cdot)$ denotes the family of spectral projections of the half-lattice unitary operators $\mathbb{U}_{\pm,k_0}$,

$$\mathbb{U}_{\pm,k_0} = \oint_{\partial\mathbb{D}} dE_{\mathbb{U}_{\pm,k_0}}(\zeta)\, \zeta. \tag{2.72}$$

Explicitly, $P_\pm$ and $R_\pm$ satisfy,

$$\oint_{\partial\mathbb{D}} P_\pm(\zeta,k,k_0)\, d\Omega_\pm(\zeta,k_0)\, P_\pm(\zeta,k',k_0)^* = \delta_{k,k'} I_m, \quad k,k' \gtrless k_0, \tag{2.73}$$

$$\oint_{\partial\mathbb{D}} R_\pm(\zeta,k,k_0)\, d\Omega_\pm(\zeta,k_0)\, R_\pm(\zeta,k',k_0)^* = \delta_{k,k'} I_m, \quad k,k' \gtrless k_0. \tag{2.74}$$

[2]This is denoted by pseudo-orthonormality in [10, Sect. VII.2.6]

Proof. Fix an integer $k_1 \gtreqless k_0$ and let $\{F_\pm(\cdot,k,k_0)\}_{k\gtreqless k_0}$ denote two $\mathbb{C}^{m\times m}$-valued sequences of functions of bounded variation,

$$F_\pm(\zeta,k,k_0) = \Delta_k^* E_{\mathbb{U}_{\pm,k_0}}(\zeta)\Delta_{k_1}, \quad \zeta\in\partial\mathbb{D},\ k\gtreqless k_0. \tag{2.75}$$

Then,

$$\begin{aligned}(\mathbb{U}_{\pm,k_0}F_\pm(\zeta,\cdot,k_0))(k) &= (\mathbb{U}_{\pm,k_0}E_{\mathbb{U}_{\pm,k_0}}(\zeta)\Delta_{k_1})(k) = \left(\int_{A_\zeta} dE_{\mathbb{U}_{\pm,k_0}}(\zeta')\,\zeta'\Delta_{k_1}\right)(k)\\ &= \int_{A_\zeta} d\big(\Delta_k^* E_{\mathbb{U}_{\pm,k_0}}(\zeta')\Delta_{k_1}\big)\,\zeta' = \int_{A_\zeta} dF_\pm(\zeta',k,k_0)\,\zeta', \\ &\qquad \zeta\in\partial\mathbb{D},\ k\gtreqless k_0,\end{aligned} \tag{2.76}$$

and hence it follows from Lemma 2.7 that

$$F_\pm(\zeta,k,k_0) = \int_{A_\zeta} \widetilde{P}_\pm(\zeta',k,k_0)\,dF_\pm(\zeta',k_0,k_0), \quad \zeta\in\partial\mathbb{D},\ k\gtreqless k_0, \tag{2.77}$$

or equivalently,

$$\Delta_k^* E_{\mathbb{U}_{\pm,k_0}}(\zeta)\Delta_{k_1} = \int_{A_\zeta} \widetilde{P}_\pm(\zeta',k,k_0)\,d\big(\Delta_{k_0}^* E_{\mathbb{U}_{\pm,k_0}}(\zeta')\Delta_{k_1}\big), \quad \zeta\in\partial\mathbb{D},\ k\gtreqless k_0. \tag{2.78}$$

In particular, taking $k_1 = k'$ and $k_1 = k_0$, one obtains, respectively,

$$\Delta_k^* E_{\mathbb{U}_{\pm,k_0}}(\zeta)\Delta_{k'} = \int_{A_\zeta} \widetilde{P}_\pm(\zeta',k,k_0)\,d\big(\Delta_{k_0}^* E_{\mathbb{U}_{\pm,k_0}}(\zeta')\Delta_{k'}\big), \quad \zeta\in\partial\mathbb{D},\ k\gtreqless k_0, \tag{2.79}$$

and

$$\begin{aligned}\Delta_{k'}^* E_{\mathbb{U}_{\pm,k_0}}(\zeta)\Delta_{k_0} &= \int_{A_\zeta} \widetilde{P}_\pm(\zeta',k',k_0)\,d\big(\Delta_{k_0}^* E_{\mathbb{U}_{\pm,k_0}}(\zeta')\Delta_{k_0}\big)\\ &= \int_{A_\zeta} \widetilde{P}_\pm(\zeta',k',k_0)\,d\Omega_\pm(\zeta',k_0), \quad \zeta\in\partial\mathbb{D},\ k'\gtreqless k_0.\end{aligned} \tag{2.80}$$

Taking adjoints in (2.80) one also obtains

$$\Delta_{k_0}^* E_{\mathbb{U}_{\pm,k_0}}(\zeta)\Delta_{k'} = \int_{A_\zeta} d\Omega_\pm(\zeta',k_0)\,\widetilde{P}_\pm(\zeta',k',k_0)^*, \quad \zeta\in\partial\mathbb{D},\ k'\gtreqless k_0. \tag{2.81}$$

Thus, inserting (2.56) and (2.81) into (2.79) and letting $\theta\to 2\pi$, $\zeta = e^{i\theta}$, yields (2.73),

$$\begin{aligned}\delta_{k,k'}I_m &= \oint_{\partial\mathbb{D}} \widetilde{P}_\pm(\zeta,k,k_0)\,d\Omega_\pm(\zeta,k_0)\,\widetilde{P}_\pm(\zeta,k',k_0)^*\\ &= \oint_{\partial\mathbb{D}} P_\pm(\zeta,k,k_0)\,d\Omega_\pm(\zeta,k_0)\,P_\pm(\zeta,k',k_0)^*, \quad k,k'\gtreqless k_0.\end{aligned} \tag{2.82}$$

Finally, (2.74) is a consequence of (2.73) and the relation

$$R_{\pm}(z,k,k_0) = \frac{1}{z}(\mathbb{W}_{\pm,k_0}P_{\pm}(z,\cdot,k_0))(k), \quad z\in\mathbb{C}\backslash\{0\},\ k \gtreqless k_0, \tag{2.83}$$

where $\mathbb{W}_{\pm,k_0}$ are the unitary block diagonal semi-infinite matrices defined in (2.34).

To prove completeness of $\{P_{\pm}(\cdot,k,k_0)^*\}_{k\gtreqless k_0}$ and $\{R_{\pm}(\cdot,k,k_0)^*\}_{k\gtreqless k_0}$ we first note the subsequent fact that can be inferred from the definitions of $P_{\pm}$ and $R_{\pm}$ and, in particular, from (2.21), (2.22), (2.49), and (2.50),

$$\operatorname{span}\{P_{\pm}(\zeta,k,k_0)^*\}_{k\gtreqless k_0} = \operatorname{span}\{R_{\pm}(\zeta,k,k_0)^*\}_{k\gtreqless k_0} = \operatorname{span}\left\{\zeta^k I_m\right\}_{k\in\mathbb{Z}}. \tag{2.84}$$

Hence, it suffices to prove that $\left\{\zeta^k I_m\right\}_{k\in\mathbb{Z}}$ are complete with respect to $d\Omega_{\pm}(\cdot,k_0)$. Suppose $F\in L^2(\partial\mathbb{D};d\Omega_{\pm}(\cdot,k_0))$ is orthogonal to all columns of $\zeta^k I_m$ for all $k\in\mathbb{Z}$, that is,

$$\oint_{\partial\mathbb{D}} \zeta^{-k} d\Omega(\zeta,k_0)\,F(\zeta) = \begin{pmatrix} 0\\ \vdots\\ 0\end{pmatrix} \in\mathbb{C}^m, \quad k\in\mathbb{Z}. \tag{2.85}$$

Note that for a scalar complex-valued measure $d\omega$ equalities $\oint d\omega(\zeta)\,\zeta^n = 0$, $n\in\mathbb{Z}$, imply that $\oint d\mathrm{Re}(\omega(\zeta))\,\zeta^n = \oint d\mathrm{Im}(\omega(\zeta))\,\zeta^n = 0$, and hence one concludes from [36, p. 24]) that $d\omega = 0$. Applying this argument to $d(\Omega_{\pm}(\cdot,k_0)F(\cdot))_\ell$, $\ell = 1,\dots,m$, one obtains

$$d\Omega_{\pm}(\cdot,k_0)F(\cdot) = \begin{pmatrix} 0\\ \vdots\\ 0\end{pmatrix} \in\mathbb{C}^m. \tag{2.86}$$

Multiplying by $F(\cdot)^*$ on the left and integrating over the unit circle then yields

$$\|F\|^2_{L^2(\partial\mathbb{D};d\Omega_{\pm}(\cdot,k_0))} = \oint_{\partial\mathbb{D}} F(\zeta)^*\,d\Omega_{\pm}(\zeta,k_0)\,F(\zeta) = 0. \tag{2.87}$$

□

We note that $d\Omega_{\pm}(\cdot,k_0)$, $k_0\in\mathbb{Z}$, defined in (2.71) are normalized, nonnegative, nondegenerate, $\mathbb{C}^{m\times m}$-valued measures supported on infinite subsets of $\partial\mathbb{D}$, that is, for any $\mathbb{C}^{m\times m}$-valued Laurent polynomial $P(z)$ the following properties hold,

$$(i)\ \oint_{\partial\mathbb{D}} d\Omega_{\pm}(\zeta,k_0) = I_m \ \text{ and } \ \oint_{\partial\mathbb{D}} P(\zeta)\,d\Omega_{\pm}(\zeta,k_0)\,P(\zeta)^* \geqslant 0. \tag{2.88}$$

(ii) If $P(z) = z^{-n}A_{-n} + \dots + z^n A_n$ and either A_n or A_{-n} is invertible, (2.89)

$$\text{then } \oint_{\partial\mathbb{D}} P(\zeta)\,d\Omega_{\pm}(\zeta,k_0)\,P(\zeta)^* > 0. \tag{2.90}$$

$$(iii)\ \text{If } \oint_{\partial\mathbb{D}} P(\zeta)\,d\Omega_{\pm}(\zeta,k_0)\,P(\zeta)^* = 0 \text{ then } P(z) = 0. \tag{2.91}$$

The infinite support property of the spectral measure is a consequence of the fact that we have infinitely many linearly independent orthogonal Laurent polynomials

$P_{\pm}$. Property (i) follows from (2.71), and properties (ii) and (iii) are implied by the orthogonality relations (2.73), (2.74), and the fact that the matrix-valued Laurent polynomials $P_{\pm}$ and $R_{\pm}$ have invertible leading-order coefficients (cf. Remark 2.5).

COROLLARY 2.9. *Let $k_0 \in \mathbb{Z}$. Then the operators $\mathbb{U}_{\pm,k_0}$ are unitarily equivalent to the operators of multiplication by ζ on $L^2(\partial\mathbb{D}; d\Omega_{\pm}(\cdot,k_0))$. In particular,*

$$\sigma(\mathbb{U}_{\pm,k_0}) = \operatorname{supp}(d\Omega_{\pm}(\cdot,k_0)). \tag{2.92}$$

Proof. Consider the linear maps $\dot{\mathscr{U}}_{\pm}\colon \ell_0^2([k_0,\pm\infty)\cap\mathbb{Z})^m \to L^2(\partial\mathbb{D}; d\Omega_{\pm}(\cdot,k_0))$ from the space of compactly supported sequences $\ell_0^2([k_0,\pm\infty)\cap\mathbb{Z})^m$ to the set of $\mathbb{C}^m$-valued Laurent polynomials defined by

$$(\dot{\mathscr{U}}_{\pm}F)(z) = \sum_{k=k_0}^{\pm\infty} \widetilde{P}_{\pm}(1/\overline{z},k,k_0)^* F(k), \quad F \in \ell_0^2([k_0,\pm\infty)\cap\mathbb{Z})^m. \tag{2.93}$$

Using (2.73) one shows that $\widehat{F}(\zeta) = (\dot{\mathscr{U}}_{\pm}F)(\zeta)$, $F \in \ell_0^2([k_0,\pm\infty)\cap\mathbb{Z})^m$ has the property

$$\begin{aligned}
\|\widehat{F}\|^2_{L^2(\partial\mathbb{D};d\Omega_{\pm}(\cdot,k_0))} &= \oint_{\partial\mathbb{D}} \widehat{F}(\zeta)^* d\Omega_{\pm}(\zeta,k_0)\, \widehat{F}(\zeta) \qquad (2.94)\\
&= \oint_{\partial\mathbb{D}} \sum_{k=k_0}^{\pm\infty} F(k)^* \widetilde{P}_{\pm}(\zeta,k,k_0)\, d\Omega_{\pm}(\zeta,k_0) \sum_{k'=k_0}^{\pm\infty} \widetilde{P}_{\pm}(\zeta,k',k_0)^* F(k')\\
&= \sum_{k,k'=k_0}^{\pm\infty} F(k)^* \left(\oint_{\partial\mathbb{D}} \widetilde{P}_{\pm}(\zeta,k,k_0)\, d\Omega_{\pm}(\zeta,k_0)\, \widetilde{P}_{\pm}(\zeta,k',k_0)^* \right) F(k')\\
&= \sum_{k=k_0}^{\pm\infty} F(k)^* F(k) = \|F\|^2_{\ell^2([k_0,\pm\infty)\cap\mathbb{Z})^m}. \qquad (2.95)
\end{aligned}$$

Since $\ell_0^2([k_0,\pm\infty)\cap\mathbb{Z})^m$ is dense in $\ell^2([k_0,\pm\infty)\cap\mathbb{Z})^m$, $\dot{\mathscr{U}}_{\pm}$ extend to bounded linear operators $\mathscr{U}_{\pm}\colon \ell^2([k_0,\pm\infty)\cap\mathbb{Z})^m \to L^2(\partial\mathbb{D}; d\Omega_{\pm}(\cdot,k_0))$, and the identity

$$\begin{aligned}
(\mathscr{U}_{\pm}(\mathbb{U}_{\pm,k_0}F))(\zeta) &= \sum_{k=k_0}^{\pm\infty} \widetilde{P}_{\pm}(\zeta,k,k_0)^* (\mathbb{U}_{\pm,k_0}F)(k) = \sum_{k=k_0}^{\pm\infty} (\mathbb{U}^*_{\pm,k_0}\widetilde{P}_{\pm}(\zeta,\cdot,k_0))(k)^* F(k)\\
&= \sum_{k=k_0}^{\pm\infty} (\zeta^{-1}\widetilde{P}_{\pm}(\zeta,k,k_0))^* F(k) = \zeta(\mathscr{U}_{\pm}F)(\zeta), \qquad (2.96)\\
&\quad F \in \ell^2([k_0,\pm\infty)\cap\mathbb{Z})^m,
\end{aligned}$$

holds. The ranges of the operators $\mathscr{U}_{\pm}$ are all of $L^2(\partial\mathbb{D}; d\Omega_{\pm}(\cdot,k_0))$ since the sets of Laurent polynomials $\{\widetilde{P}_{\pm}(\cdot,k,k_0)^*\}_{k \gtrless k_0}$ are complete with respect to $d\Omega_{\pm}(\cdot,k_0)$, and hence $\mathscr{U}_{\pm}$ are onto. Finally, one computes the inverse operators $\mathscr{U}_{\pm}^{-1}$,

$$(\mathscr{U}_{\pm}^{-1}\widehat{F})(k) = \oint_{\partial\mathbb{D}} \widetilde{P}_{\pm}(\zeta,k,k_0)\, d\Omega_{\pm}(\zeta,k_0)\, \widehat{F}(\zeta), \quad \widehat{F} \in L^2(\partial\mathbb{D}; d\Omega_{\pm}(\cdot,k_0)), \tag{2.97}$$

which together with (2.95) implies that $\mathscr{U}_{\pm}$ are unitary. In addition, (2.96) implies that the half-lattice unitary operators $\mathbb{U}_{\pm,k_0}$ on $\ell^2([k_0,\pm\infty)\cap\mathbb{Z})^m$ are unitarily equivalent to the operators of multiplication by ζ on $L^2(\partial\mathbb{D}; d\Omega_{\pm}(\cdot,k_0))$,

$$(\mathscr{U}_{\pm}\mathbb{U}_{\pm,k_0}\mathscr{U}_{\pm}^{-1}\widehat{F})(\zeta) = \zeta\widehat{F}(\zeta), \quad \widehat{F} \in L^2(\partial\mathbb{D}; d\Omega_{\pm}(\cdot,k_0)). \tag{2.98}$$

□

COROLLARY 2.10. *Let* $k_0 \in \mathbb{Z}$.
The matrix-valued Laurent polynomials $\{P_+(\cdot,k,k_0)\}_{k\geqslant k_0}$ *can be constructed by Gram–Schmidt orthogonalizing*

$$\begin{cases} \zeta I_m,\ I_m,\ \zeta^2 I_m,\ \zeta^{-1} I_m,\ \zeta^3 I_m,\ \zeta^{-2} I_m,\ldots, & k_0 \text{ odd}, \\ I_m,\ \zeta I_m,\ \zeta^{-1} I_m,\ \zeta^2 I_m,\ \zeta^{-2} I_m,\ \zeta^2 I_m,\ldots, & k_0 \text{ even} \end{cases} \tag{2.99}$$

in the context of matrix-valued Laurent polynomials orthogonal with respect to $d\Omega_+(\cdot,k_0)$.
The matrix-valued Laurent polynomials $\{R_+(\cdot,k,k_0)\}_{k\geqslant k_0}$ *can be constructed by Gram–Schmidt orthogonalizing*

$$\begin{cases} I_m,\ \zeta I_m,\ \zeta^{-1} I_m,\ \zeta^2 I_m,\ \zeta^{-2} I_m,\ \zeta^3 I_m,\ldots, & k_0 \text{ odd}, \\ I_m,\ \zeta^{-1} I_m,\ \zeta I_m,\ \zeta^{-2} I_m,\ \zeta^2 I_m,\ \zeta^{-3} I_m,\ldots, & k_0 \text{ even} \end{cases} \tag{2.100}$$

in the context of matrix-valued Laurent polynomials orthogonal with respect to $d\Omega_+(\cdot,k_0)$.
The matrix-valued Laurent polynomials $\{P_-(\cdot,k,k_0)\}_{k\leqslant k_0}$ *can be constructed by Gram–Schmidt orthogonalizing*

$$\begin{cases} I_m,\ -\zeta I_m,\ \zeta^{-1} I_m,\ -\zeta^2 I_m,\ \zeta^{-2} I_m,\ -\zeta^3 I_m,\ldots, & k_0 \text{ odd}, \\ -\zeta I_m,\ I_m,\ -\zeta^2 I_m,\ \zeta^{-1} I_m,\ -\zeta^3 I_m,\ \zeta^{-2} I_m,\ldots, & k_0 \text{ even} \end{cases} \tag{2.101}$$

in the context of matrix-valued Laurent polynomials orthogonal with respect to $d\Omega_-(\cdot,k_0)$.
The matrix-valued Laurent polynomials $\{R_-(\cdot,k,k_0)\}_{k\leqslant k_0}$ *can be constructed by Gram–Schmidt orthogonalizing*

$$\begin{cases} -I_m,\ \zeta^{-1} I_m,\ -\zeta I_m,\ \zeta^{-2} I_m,\ -\zeta^2 I_m,\ \zeta^{-3} I_m,\ldots, & k_0 \text{ odd}, \\ I_m,\ -\zeta I_m,\ \zeta^{-1} I_m,\ -\zeta^2 I_m,\ \zeta^{-2} I_m,\ -\zeta^3 I_m,\ldots, & k_0 \text{ even} \end{cases} \tag{2.102}$$

in the context of matrix-valued Laurent polynomials orthogonal with respect to $d\Omega_-(\cdot,k_0)$.

Here the Gram–Schmidt orthogonalization procedure employs left-multiplication by constant (*i.e.,* ζ*-independent*) $m\times m$ *matrices as discussed in* [10, Sect. VII.2.8].

Proof. The result is a consequence of the definition of the Laurent polynomials $P_{\pm}$ and $R_{\pm}$ and Lemma 2.8. □

We note that the Gram–Schmidt orthogonalization process implies that all matrix-valued Laurent polynomials constructed in Corollary 2.10 have self-adjoint invertible leading-order coefficients (cf. Remark 2.5).

The next result clarifies which measures arise as spectral measures of half-lattice CMV operators and it yields the reconstruction of the matrix-valued Verblunsky coefficients from the spectral measures and the corresponding orthogonal Laurent polynomials.

THEOREM 2.11. *Let $k_0 \in \mathbb{Z}$ and $d\Omega_\pm(\cdot,k_0)$ be nonnegative finite measures on $\partial\mathbb{D}$, supported on infinite sets, and normalized by*

$$\oint_{\partial\mathbb{D}} d\Omega_\pm(\zeta,k_0) = I_m. \tag{2.103}$$

Moreover, assume that $d\Omega_\pm(\cdot,k_0)$ are nondegenerate in the sense that expressions of the form

$$\oint_{\partial\mathbb{D}} P(\zeta)d\Omega_\pm(\zeta,k_0)P(\zeta)^* \tag{2.104}$$

are invertible for all $\mathbb{C}^{m\times m}$-valued Laurent polynomials $P(z) = z^{-n}A_{-n} + \ldots + z^n A_n$ with either $A_{-n} = I_m$ or $A_n = I_m$. Then $d\Omega_\pm(\cdot,k_0)$ are necessarily the spectral measures for some half-lattice CMV operators $\mathbb{U}_{\pm,k_0}$ with coefficients $\{\alpha_k\}_{k\geqslant k_0+1}$, respectively, $\{\alpha_k\}_{k\leqslant k_0}$, defined by

$$\alpha_k = -\begin{cases} \oint_{\partial\mathbb{D}} \zeta R_+(\zeta,k-1,k_0)d\Omega_+(\zeta,k_0)P_+(\zeta,k-1,k_0)^*, & k \text{ odd}, \\ \oint_{\partial\mathbb{D}} P_+(\zeta,k-1,k_0)d\Omega_+(\zeta,k_0)R_+(\zeta,k-1,k_0)^*, & k \text{ even} \end{cases} \tag{2.105}$$

for all $k \geqslant k_0+1$, and

$$\alpha_k = -\begin{cases} \oint_{\partial\mathbb{D}} \zeta R_-(\zeta,k-1,k_0)d\Omega_-(\zeta,k_0)P_-(\zeta,k-1,k_0)^*, & k \text{ odd}, \\ \oint_{\partial\mathbb{D}} P_-(\zeta,k-1,k_0)d\Omega_-(\zeta,k_0)R_-(\zeta,k-1,k_0)^*, & k \text{ even} \end{cases} \tag{2.106}$$

for all $k \leqslant k_0$. Here the matrix-valued Laurent polynomials $\{P_\pm(\cdot,k,k_0)\}_{k\geqslant k_0}$ and $\{R_\pm(\cdot,k,k_0)\}_{k\geqslant k_0}$ denote the orthonormal Laurent polynomials constructed in Corollary 2.10.

Proof. First, using Corollary 2.10, one constructs the orthonormal polynomials $\{P_+(\cdot,k,k_0)\}_{k\geqslant k_0}$ and $\{R_+(\cdot,k,k_0)\}_{k\geqslant k_0}$.

Next, we will establish the recursion relation (2.37). Assume k is odd and consider the matrix-valued Laurent polynomials P and R,

$$P(\zeta) = \widetilde{\rho}_k P_+(\zeta,k,k_0) - \zeta R_+(\zeta,k-1,k_0), \tag{2.107}$$
$$R(\zeta) = \rho_k R_+(\zeta,k,k_0) - \zeta^{-1} P_+(\zeta,k-1,k_0), \tag{2.108}$$

where ρ_k, $\widetilde{\rho}_k \in \mathbb{C}^{m\times m}$ are self-adjoint invertible matrices chosen such that the leading-order terms of the Laurent polynomials $\widetilde{\rho}_k P_+(\zeta,k,k_0)$ and $\rho_k R_+(\zeta,k,k_0)$ cancel the leading-order terms of $\zeta R_+(\zeta,k-1,k_0)$ and $\zeta^{-1}P_+(\zeta,k-1,k_0)$, respectively (cf. Remark 2.5). Using Corollary 2.10 one then checks that the Laurent polynomials P and R are constant $m\times m$ matrix left-multiples of $P_+(\cdot,k-1,k_0)$ and $R_+(\cdot,k-1,k_0)$, respectively,

$$\alpha_k P_+(\zeta,k-1,k_0) = \widetilde{\rho}_k P_+(\zeta,k,k_0) - \zeta R_+(\zeta,k-1,k_0), \tag{2.109}$$
$$\widetilde{\alpha}_k R_+(\zeta,k-1,k_0) = \rho_k R_+(\zeta,k,k_0) - \zeta^{-1} P_+(\zeta,k-1,k_0), \tag{2.110}$$

with α_k, $\widetilde{\alpha}_k \in \mathbb{C}^{m\times m}$ constant $m\times m$ matrices. Moreover, using (2.109), (2.110), and Lemma 2.8 one computes,

$$\begin{aligned} I_m &= \oint_{\partial\mathbb{D}} \zeta R_+(\zeta,k-1,k_0)\, d\Omega_\pm(\zeta,k_0)\, \zeta^{-1} R_+(\zeta,k-1,k_0)^* \\ &= \oint_{\partial\mathbb{D}} \Big(\widetilde{\rho}_k P_+(\zeta,k,k_0) - \alpha_k P_+(\zeta,k-1,k_0)\Big) d\Omega_\pm(\zeta,k_0) \\ &\qquad\qquad \times \Big(\widetilde{\rho}_k P_+(\zeta,k,k_0) - \alpha_k P_+(\zeta,k-1,k_0)\Big)^* \\ &= \widetilde{\rho}_k^2 + \alpha_k \alpha_k^*, \end{aligned} \tag{2.111}$$

$$\begin{aligned} I_m &= \oint_{\partial\mathbb{D}} \zeta^{-1} P_+(\zeta,k-1,k_0)\, d\Omega_\pm(\zeta,k_0)\, \zeta P_+(\zeta,k-1,k_0)^* \\ &= \oint_{\partial\mathbb{D}} \Big(\rho_k R_+(\zeta,k,k_0) - \widetilde{\alpha}_k R_+(\zeta,k-1,k_0)\Big) d\Omega_\pm(\zeta,k_0) \\ &\qquad\qquad \times \Big(\rho_k R_+(\zeta,k,k_0) - \widetilde{\alpha}_k R_+(\zeta,k-1,k_0)\Big)^* \\ &= \rho_k^2 + \widetilde{\alpha}_k \widetilde{\alpha}_k^*, \end{aligned} \tag{2.112}$$

and

$$\begin{aligned} \alpha_k &= \oint_{\partial\mathbb{D}} \alpha_k P_+(\zeta,k-1,k_0)\, d\Omega_\pm(\zeta,k_0)\, P_+(\zeta,k-1,k_0)^* \\ &= \oint_{\partial\mathbb{D}} \Big(\widetilde{\rho}_k P_+(\zeta,k,k_0) - \zeta R_+(\zeta,k-1,k_0)\Big) d\Omega_\pm(\zeta,k_0)\, P_+(\zeta,k-1,k_0)^* \\ &= -\oint_{\partial\mathbb{D}} \zeta R_+(\zeta,k-1,k_0)\, d\Omega_\pm(\zeta,k_0)\, P_+(\zeta,k-1,k_0)^*, \end{aligned} \tag{2.113}$$

$$\begin{aligned} \widetilde{\alpha}_k &= \oint_{\partial\mathbb{D}} \widetilde{\alpha}_k R_+(\zeta,k-1,k_0)\, d\Omega_\pm(\zeta,k_0)\, R_+(\zeta,k-1,k_0)^* \\ &= \oint_{\partial\mathbb{D}} \Big(\rho_k R_+(\zeta,k,k_0) - \zeta^{-1} P_+(\zeta,k-1,k_0)\Big) d\Omega_\pm(\zeta,k_0)\, R_+(\zeta,k-1,k_0)^* \\ &= -\oint_{\partial\mathbb{D}} \zeta^{-1} P_+(\zeta,k-1,k_0)\, d\Omega_\pm(\zeta,k_0)\, R_+(\zeta,k-1,k_0)^*. \end{aligned} \tag{2.114}$$

Thus, (2.111)–(2.114) imply that $\widetilde{\alpha}_k = \alpha_k^*$, $\rho_k = \sqrt{I_m - \alpha_k^*\alpha_k}$, and $\widetilde{\rho}_k = \sqrt{I_m - \alpha_k\alpha_k^*}$, and hence (2.109) and (2.110) yield the recursion relation (2.37). A similar argument also proves the recursion relation (2.37) for the case k even.

Finally, using Lemma 2.4 one concludes that the Laurent polynomials $\{P_+(\cdot,k,k_0)\}_{k\geqslant k_0}$ form a generalized eigenvector of the operator $\mathbb{U}_{+,k_0}$ associated with the coefficients $\alpha_k, \rho_k, \widetilde{\rho}_k$ introduced above. Thus, the measure $d\Omega_+(\cdot,k_0)$ is the spectral measure of the operator $\mathbb{U}_{+,k_0}$.

Similarly one proves the result for $d\Omega_-(\cdot,k_0)$ and (2.106) for $k \leqslant k_0$. $\square$

LEMMA 2.12. *Let $z \in \mathbb{C}\backslash(\partial\mathbb{D}\cup\{0\})$ and $k_0 \in \mathbb{Z}$. Then the following identity holds,*

$$\widetilde{Q}_\pm(z,k,k_0) = \pm\oint_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z}\big(\widetilde{P}_\pm(\zeta,k,k_0) - \widetilde{P}_\pm(z,k,k_0)\big) d\Omega_\pm(\zeta,k_0), \quad k \gtrless k_0,$$

$$S_\pm(z,k,k_0) = \pm \oint_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z}\big(R_\pm(\zeta,k,k_0) - R_\pm(z,k,k_0)\big)d\Omega_\pm(\zeta,k_0), \quad k \gtrless k_0. \tag{2.115}$$

Proof. To simplify our further notation we agree to write both equalities in (2.115) as a single one,

$$\begin{pmatrix} \widetilde{Q}_\pm(z,k,k_0) \\ S_\pm(z,k,k_0) \end{pmatrix} = \pm \oint_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z}\left(\begin{pmatrix} \widetilde{P}_\pm(\zeta,k,k_0) \\ R_\pm(\zeta,k,k_0) \end{pmatrix} - \begin{pmatrix} \widetilde{P}_\pm(z,k,k_0) \\ R_\pm(z,k,k_0) \end{pmatrix}\right) d\Omega_\pm(\zeta,k_0), \ k \gtrless k_0, \tag{2.116}$$

where the integration on the right-hand side is understood componentwise, that is, an expression of the type $\oint_{\partial\mathbb{D}} \left(\begin{smallmatrix} G_1(\zeta) \\ G_2(\zeta) \end{smallmatrix}\right) d\Omega_\pm(\zeta,k_0)$ with $G_1(z)$ and $G_2(z)$ some $\mathbb{C}^{m\times m}$-valued Laurent polynomials is defined by

$$\oint_{\partial\mathbb{D}} \begin{pmatrix} G_1(\zeta) \\ G_2(\zeta) \end{pmatrix} d\Omega_\pm(\zeta,k_0) = \begin{pmatrix} \oint_{\partial\mathbb{D}} G_1(\zeta)\, d\Omega_\pm(\zeta,k_0) \\ \oint_{\partial\mathbb{D}} G_2(\zeta)\, d\Omega_\pm(\zeta,k_0) \end{pmatrix}. \tag{2.117}$$

First, we prove (2.116) for the right half-lattice Laurent polynomials and for k_0 even. In this case (2.56) and (2.57) imply that (2.116) is equivalent to

$$\begin{pmatrix} Q_+(z,k,k_0) \\ S_+(z,k,k_0) \end{pmatrix} = \oint_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z}\left(\begin{pmatrix} P_+(\zeta,k,k_0) \\ R_+(\zeta,k,k_0) \end{pmatrix} - \begin{pmatrix} P_+(z,k,k_0) \\ R_+(z,k,k_0) \end{pmatrix}\right) d\Omega_+(\zeta,k_0),$$
$$k > k_0, \ k_0 \text{ even}. \tag{2.118}$$

Let $k_0 \in \mathbb{Z}$ be even. It suffices to show that the right-hand side of (2.118), temporarily denoted by the symbol $RHS(z,k,k_0)$, satisfies

$$\mathbb{T}(z,k+1)^{-1} RHS(z,k+1,k_0) = RHS(z,k,k_0), \quad k > k_0, \tag{2.119}$$
$$\mathbb{T}(z,k_0+1)^{-1} RHS(z,k_0+1,k_0) = \begin{pmatrix} Q_+(z,k_0,k_0) \\ S_+(z,k_0,k_0) \end{pmatrix} = \begin{pmatrix} -I_m \\ I_m \end{pmatrix}. \tag{2.120}$$

One verifies these statements using the equality,

$$\begin{aligned} &\mathbb{T}(z,k+1)^{-1} RHS(z,k+1,k_0) \\ &\quad = RHS(z,k,k_0) + \oint_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z}\left(\mathbb{T}(z,k+1)^{-1} - \mathbb{T}(\zeta,k+1)^{-1}\right) \\ &\qquad \times \begin{pmatrix} P_+(\zeta,k+1,k_0) \\ R_+(\zeta,k+1,k_0) \end{pmatrix} d\Omega_+(\zeta,k_0), \quad k \in \mathbb{Z}. \end{aligned} \tag{2.121}$$

For $k > k_0$, the last term on the right-hand side of (2.121) vanishes since for k odd, $\mathbb{T}(z,k+1)$ does not depend on z, and for k even, it follows from Corollary 2.10 that $P_+(\cdot,k+1,k_0)$ and $R_+(\cdot,k+1,k_0)$ are orthogonal in $L^2(\partial\mathbb{D}; d\Omega_+(\cdot,k_0))$ to

$\operatorname{span}\{I_m, \zeta I_m\}$ and $\operatorname{span}\{I_m, \zeta^{-1} I_m\}$, respectively. Hence one computes

$$\begin{aligned}
&\oint_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z}\Big(\mathbb{T}(z,k+1)^{-1} - \mathbb{T}(\zeta,k+1)^{-1}\Big)\begin{pmatrix} P_+(\zeta,k+1,k_0) \\ R_+(\zeta,k+1,k_0)\end{pmatrix} d\Omega_+(\zeta,k_0) \\
&\quad = \oint_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z}\begin{pmatrix} 0 & (z-\zeta)\rho_{k+1}^{-1} \\ (z^{-1}-\zeta^{-1})\widetilde{\rho}_{k+1}^{-1} & 0 \end{pmatrix}\begin{pmatrix} P_+(\zeta,k+1,k_0) \\ R_+(\zeta,k+1,k_0)\end{pmatrix} d\Omega_+(\zeta,k_0) \\
&\quad = \oint_{\partial\mathbb{D}} \begin{pmatrix} 0 & -(\zeta+z)\rho_{k+1}^{-1} \\ (\zeta^{-1}+z^{-1})\widetilde{\rho}_{k+1}^{-1} & 0 \end{pmatrix}\begin{pmatrix} P_+(\zeta,k+1,k_0) \\ R_+(\zeta,k+1,k_0)\end{pmatrix} d\Omega_+(\zeta,k_0) \\
&\quad = \oint_{\partial\mathbb{D}} \begin{pmatrix} -\rho_{k+1}^{-1}(\zeta+z)R_+(\zeta,k,k_0) \\ \widetilde{\rho}_{k+1}^{-1}(\zeta^{-1}+z^{-1})P_+(\zeta,k,k_0)\end{pmatrix} d\Omega_+(\zeta,k_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned} \tag{2.122}$$

Thus, (2.119) is implied by (2.121).

For $k=k_0$ even, one obtains that $RHS(z,k_0,k_0)=0$ since by (2.49) one has the normalization $P_+(z,k_0,k_0)=R_+(z,k_0,k_0)=I_m$. Then using the fact that by Corollary 2.10, $P_+(\cdot,k_0+1,k_0)$ and $R_+(\cdot,k_0+1,k_0)$ are orthogonal to constant $m\times m$ matrices in $L^2(\partial\mathbb{D}; d\Omega_+(\cdot,k_0))$ and that by (2.51),

$$\begin{aligned}
P_+(\zeta,k_0+1,k_0) &= \widetilde{\rho}_{k_0+1}^{-1}(\zeta I_m + \alpha_{k_0+1}), \\
R_+(\zeta,k_0+1,k_0) &= \rho_{k_0+1}^{-1}(\zeta^{-1} I_m + \alpha_{k_0+1}^*),
\end{aligned} \tag{2.123}$$

one computes,

$$\begin{aligned}
&\oint_{\partial\mathbb{D}} \zeta^{-1} P_+(\zeta,k_0+1,k_0) d\Omega_+(\zeta,k_0) = \oint_{\partial\mathbb{D}} P_+(\zeta,k_0+1,k_0) d\Omega_+(\zeta,k_0)(\zeta I_m)^* \\
&\quad = \oint_{\partial\mathbb{D}} P_+(\zeta,k_0+1,k_0) d\Omega_+(\zeta,k_0) P_+(\zeta,k_0+1,k_0)^* \widetilde{\rho}_{k_0+1} = \widetilde{\rho}_{k_0+1},
\end{aligned} \tag{2.124}$$

$$\begin{aligned}
&\oint_{\partial\mathbb{D}} \zeta R_+(\zeta,k_0+1,k_0) d\Omega_+(\zeta,k_0) = \oint_{\partial\mathbb{D}} R_+(\zeta,k_0+1,k_0) d\Omega_+(\zeta,k_0)(\zeta^{-1} I_m)^* \\
&\quad = \oint_{\partial\mathbb{D}} R_+(\zeta,k_0+1,k_0) d\Omega_+(\zeta,k_0) R_+(\zeta,k_0+1,k_0)^* \rho_{k_0+1} = \rho_{k_0+1},
\end{aligned} \tag{2.125}$$

and hence,

$$\begin{aligned}
&\oint_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z}\Big(\mathbb{T}(z,k_0+1)^{-1} - \mathbb{T}(\zeta,k_0+1)^{-1}\Big)\begin{pmatrix} P_+(\zeta,k_0+1,k_0) \\ R_+(\zeta,k_0+1,k_0)\end{pmatrix} d\Omega_+(\zeta,k_0) \\
&\quad = \oint_{\partial\mathbb{D}} \begin{pmatrix} -\rho_{k+1}^{-1}(\zeta+z)R_+(\zeta,k_0+1,k_0) \\ \widetilde{\rho}_{k+1}^{-1}(\zeta^{-1}+z^{-1})P_+(\zeta,k_0+1,k_0)\end{pmatrix} d\Omega_+(\zeta,k_0) \\
&\quad = \oint_{\partial\mathbb{D}} \begin{pmatrix} -\rho_{k+1}^{-1}\zeta R_+(\zeta,k_0+1,k_0) \\ \widetilde{\rho}_{k+1}^{-1}\zeta^{-1}P_+(\zeta,k_0+1,k_0)\end{pmatrix} d\Omega_+(\zeta,k_0) = \begin{pmatrix} -\rho_{k+1}^{-1}\rho_{k+1} \\ \widetilde{\rho}_{k+1}^{-1}\widetilde{\rho}_{k+1}\end{pmatrix} = \begin{pmatrix} -I_m \\ I_m \end{pmatrix}.
\end{aligned} \tag{2.126}$$

Thus, (2.120) is a consequence of (2.121), (2.126), and the fact that $RHS(z,k_0,k_0)=0$.

Next, we prove (2.116) for the right half-lattice Laurent polynomials and k_0 odd,

$$\begin{pmatrix} S_+(z,k,k_0) \\ \widetilde{Q}_+(z,k,k_0) \end{pmatrix} = \oint_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z}\left(\begin{pmatrix} R_+(\zeta,k,k_0) \\ \widetilde{P}_+(\zeta,k,k_0) \end{pmatrix} - \begin{pmatrix} R_+(z,k,k_0) \\ \widetilde{P}_+(z,k,k_0) \end{pmatrix}\right) d\Omega_+(\zeta,k_0),$$
$$k > k_0,\ k_0 \text{ odd}. \tag{2.127}$$

Let $k_0 \in \mathbb{Z}$ be odd. We note that for $U(z,k), V(z,k) \in \mathbb{C}^{m\times m}$, $k \in \mathbb{Z}$, $z \in \mathbb{C}\backslash\{0\}$,

$$\begin{pmatrix} U(z,k) \\ V(z,k) \end{pmatrix} = \mathbb{T}(z,k) \begin{pmatrix} U(z,k-1) \\ V(z,k-1) \end{pmatrix} \tag{2.128}$$

is equivalent to

$$\begin{pmatrix} V(z,k) \\ \widetilde{U}(z,k) \end{pmatrix} = \widetilde{\mathbb{T}}(z,k) \begin{pmatrix} V(z,k-1) \\ \widetilde{U}(z,k-1) \end{pmatrix}, \tag{2.129}$$

where

$$\widetilde{U}(z,k) = z^{-1}U(z,k) \text{ and } \widetilde{\mathbb{T}}(z,k) = \begin{pmatrix} 0 & I_m \\ z^{-1}I_m & 0 \end{pmatrix} \mathbb{T}(z,k) \begin{pmatrix} 0 & zI_m \\ I_m & 0 \end{pmatrix}. \tag{2.130}$$

Thus, it suffices to show that the right-hand side of (2.127), temporarily denoted by $\widetilde{RHS}(z,k,k_0)$, satisfies

$$\widetilde{\mathbb{T}}(z,k+1)^{-1}\widetilde{RHS}(z,k+1,k_0) = \widetilde{RHS}(z,k,k_0), \quad k > k_0, \tag{2.131}$$

$$\widetilde{\mathbb{T}}(z,k_0+1)^{-1}\widetilde{RHS}(z,k_0+1,k_0) = \begin{pmatrix} S_+(z,k_0,k_0) \\ \widetilde{Q}_+(z,k_0,k_0) \end{pmatrix} = \begin{pmatrix} -I_m \\ I_m \end{pmatrix}. \tag{2.132}$$

At this point one can follow the first part of the proof replacing $\mathbb{T}$ by $\widetilde{\mathbb{T}}$, $\begin{pmatrix} P_+ \\ R_+ \end{pmatrix}$ by $\begin{pmatrix} R_+ \\ \widetilde{P}_+ \end{pmatrix}$, $\begin{pmatrix} Q_+ \\ S_+ \end{pmatrix}$ by $\begin{pmatrix} S_+ \\ \widetilde{Q}_+ \end{pmatrix}$, etc.

The result for the remaining Laurent polynomials $\widetilde{P}_-(z,k,k_0)$, $R_-(z,k,k_0)$, $\widetilde{Q}_-(z,k,k_0)$, and $S_-(z,k,k_0)$ is proved similarly. □

LEMMA 2.13. *Let $k_0 \in \mathbb{Z}$ and let $m_\pm(\cdot,k_0)$ denote the $\mathbb{C}^{m\times m}$-valued Caratheodory and anti-Caratheodory functions*

$$m_\pm(z,k_0) = \pm\Delta_{k_0}^*(\mathbb{U}_{\pm,k_0}+zI)(\mathbb{U}_{\pm,k_0}-zI)^{-1}\Delta_{k_0} \tag{2.133}$$
$$= \pm\oint_{\partial\mathbb{D}} d\Omega_\pm(\zeta,k_0)\,\frac{\zeta+z}{\zeta-z}, \quad z \in \mathbb{C}\backslash\partial\mathbb{D}, \tag{2.134}$$

with

$$\oint_{\partial\mathbb{D}} d\Omega_\pm(\zeta,k_0) = I_m. \tag{2.135}$$

Then the following relations hold,

$$Q_\pm(z,\cdot,k_0) + P_\pm(z,\cdot,k_0)m_\pm(z,k_0) \in \ell^2([k_0,\pm\infty)\cap\mathbb{Z})^{m\times m}, \quad z \in \mathbb{C}\backslash(\partial\mathbb{D}\cup\{0\}), \tag{2.136}$$

$$S_\pm(z,\cdot,k_0) + R_\pm(z,\cdot,k_0)m_\pm(z,k_0) \in \ell^2([k_0,\pm\infty)\cap\mathbb{Z})^{m\times m}, \quad z \in \mathbb{C}\backslash(\partial\mathbb{D}\cup\{0\}). \tag{2.137}$$

Proof. Equality (2.134) is implied by (2.71) and (2.72).
Next, let $\mathbb{B}_{\pm,k_0}(z)$ denote operators defined on $\ell^2([k_0,\pm\infty)\cap\mathbb{Z})^m$ by

$$\mathbb{B}_{\pm,k_0}(z) = (\mathbb{U}_{\pm,k_0} + zI)(\mathbb{U}_{\pm,k_0} - zI)^{-1}, \quad z\in\mathbb{C}\backslash\partial\mathbb{D}. \tag{2.138}$$

Since $\mathbb{U}_{\pm,k_0}$ are unitary, the operators $\mathbb{B}_{\pm,k_0}(z)$ are bounded for all $z\in\mathbb{C}\backslash\partial\mathbb{D}$, and hence one has

$$\mathbb{B}_{\pm,k_0}(z)\Delta_{k_0} = \big\{\Delta_k^*\mathbb{B}_{\pm,k_0}(z)\Delta_{k_0}\big\}_{k\in[k_0,\pm\infty)\cap\mathbb{Z}} \in \ell^2([k_0,\pm\infty)\cap\mathbb{Z})^{m\times m}. \tag{2.139}$$

Using the spectral representation for the operators $\mathbb{B}_{\pm,k_0}(z)$ and equalities (2.80), (2.115), and (2.134), one obtains

$$\begin{aligned}\Delta_k^*\mathbb{B}_{\pm,k_0}(z)\Delta_{k_0} &= \oint_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z}\widetilde{P}_\pm(\zeta,k,k_0)\,d\Omega_\pm(\zeta,k_0)\\ &= \pm\big[\widetilde{Q}_\pm(z,k,k_0) + \widetilde{P}_\pm(z,k,k_0)m_\pm(z,k_0)\big], \quad k\gtrless k_0.\end{aligned} \tag{2.140}$$

Thus, (2.136) is a consequence of (2.56), (2.57), (2.139), and (2.140).

Moreover, since $\begin{pmatrix} P_\pm(z,k,k_0) \\ R_\pm(z,k,k_0)\end{pmatrix}_{k\in\mathbb{Z}}$ and $\begin{pmatrix} Q_\pm(z,k,k_0) \\ S_\pm(z,k,k_0)\end{pmatrix}_{k\in\mathbb{Z}}$, $z\in\mathbb{C}\backslash\{0\}$, satisfy (2.21), Lemma 2.3 implies that

$$\begin{aligned}&(\mathbb{W}(Q_\pm(z,\cdot,k_0) + P_\pm(z,\cdot,k_0)m_\pm(z,k_0)))(k) = z[S_\pm(z,k,k_0) + R_\pm(z,k,k_0)m_\pm(z,k_0)],\\ &\qquad k\in\mathbb{Z},\end{aligned} \tag{2.141}$$

and hence (2.137) follows from (2.136) and (2.141). □

LEMMA 2.14. *Let $k_0\in\mathbb{Z}$. Then the relations in* (2.136) *(equivalently, those in* (2.137)*) uniquely determine the $\mathbb{C}^{m\times m}$-valued functions $m_\pm(\cdot,k_0)$ on $\mathbb{C}\backslash(\partial\mathbb{D}\cup\{0\})$.*

Proof. We will prove the lemma by contradiction. Assume that there are two $\mathbb{C}^{m\times m}$-valued functions $m_+(z,k_0)$ and $\widetilde{m}_+(z,k_0)$ satisfying (2.136) such that $m_+(z_0,k_0)\neq\widetilde{m}_+(z_0,k_0)$ for some $z_0\in\mathbb{C}\backslash(\partial\mathbb{D}\cup\{0\})$. Then there is a vector $x\in\mathbb{C}^m$ such that $(m_+(z_0,k_0)-\widetilde{m}_+(z_0,k_0))x\neq 0$ and by (2.136),

$$\begin{aligned}&p_+(z_0,\cdot,k_0) = P_+(z_0,\cdot,k_0)[m_+(z_0,k_0)-\widetilde{m}_+(z_0,k_0)]x \in \ell^2([k_0,\pm\infty)\cap\mathbb{Z})^m,\\ &\qquad z\in\mathbb{C}\backslash(\partial\mathbb{D}\cup\{0\}).\end{aligned} \tag{2.142}$$

Since $P_+(z_0,k_0,k_0)\neq 0$, the sequence of vectors $\{p_+(z,k,k_0)\}_{k\geqslant k_0}$ is not identically zero, and hence, by Lemma 2.4, $p_+(z_0,\cdot,k_0)$ is an eigenvector of the operator $\mathbb{U}_{+,k_0}$ corresponding to the eigenvalue $z_0\in\mathbb{C}\backslash\partial\mathbb{D}$. This contradicts unitarity of $\mathbb{U}_{+,k_0}$.

Similarly, one proves the result for $m_-(z,k_0)$. Moreover, one easily supplies a proof that utilizes (2.137) instead of (2.136). □

COROLLARY 2.15. *There are solutions* $\begin{pmatrix} \psi_\pm(z,k) \\ \chi_\pm(z,k) \end{pmatrix}$, $k \in \mathbb{Z}$, *of* (2.21)*, unique up to right-multiplication by constant* $m \times m$ *matrices, so that for some* (*and hence for all*) $k_1 \in \mathbb{Z}$,

$$\psi_\pm(z,\cdot),\ \chi_\pm(z,\cdot) \in \ell^2([k_1,\pm\infty)\cap\mathbb{Z})^{m\times m}, \quad z \in \mathbb{C}\backslash(\partial\mathbb{D}\cup\{0\}). \tag{2.143}$$

Proof. Since any solution of (2.21) can be expressed as a linear combination of the Laurent polynomials $\begin{pmatrix} P_\pm(z,\cdot,k_0) \\ R_\pm(z,\cdot,k_0) \end{pmatrix}$ and $\begin{pmatrix} Q_\pm(z,\cdot,k_0) \\ S_\pm(z,\cdot,k_0) \end{pmatrix}$, existence and uniqueness of the solutions $\begin{pmatrix} \psi_\pm(z,\cdot) \\ \chi_\pm(z,\cdot) \end{pmatrix}$ is implied by Lemmas 2.13 and 2.14, respectively. $\square$

For the next result we recall the definition of a_k and b_k in (2.10) and (2.11).

LEMMA 2.16. *Let* $z \in \mathbb{C}\backslash\{0\}$ *and* $k_0 \in \mathbb{Z}$. *Then the following relations hold for all* $k \in \mathbb{Z}$,

$$\begin{aligned} \begin{pmatrix} P_-(z,k,k_0-1) \\ R_-(z,k,k_0-1) \end{pmatrix} &= \begin{pmatrix} P_+(z,k,k_0) \\ R_+(z,k,k_0) \end{pmatrix} \frac{\widetilde\rho_{k_0}^{-1} b_{k_0} - \rho_{k_0}^{-1} b_{k_0}^*}{2} \\ &\quad + \begin{pmatrix} Q_+(z,k,k_0) \\ S_+(z,k,k_0) \end{pmatrix} \frac{\widetilde\rho_{k_0}^{-1} b_{k_0} + \rho_{k_0}^{-1} b_{k_0}^*}{2}, \end{aligned} \tag{2.144}$$

$$\begin{aligned} \begin{pmatrix} Q_-(z,k,k_0-1) \\ S_-(z,k,k_0-1) \end{pmatrix} &= \begin{pmatrix} P_+(z,k,k_0) \\ R_+(z,k,k_0) \end{pmatrix} \frac{\widetilde\rho_{k_0}^{-1} a_{k_0} + \rho_{k_0}^{-1} a_{k_0}^*}{2} \\ &\quad + \begin{pmatrix} Q_+(z,k,k_0) \\ S_+(z,k,k_0) \end{pmatrix} \frac{\widetilde\rho_{k_0}^{-1} a_{k_0} - \rho_{k_0}^{-1} a_{k_0}^*}{2}, \end{aligned} \tag{2.145}$$

and

$$\begin{pmatrix} P_-(z,k,k_0) \\ R_-(z,k,k_0) \end{pmatrix} = \begin{pmatrix} P_+(z,k,k_0) \\ R_+(z,k,k_0) \end{pmatrix} c(z,k_0) + \begin{pmatrix} Q_+(z,k,k_0) \\ S_+(z,k,k_0) \end{pmatrix} d(z,k_0), \tag{2.146}$$

$$\begin{pmatrix} Q_-(z,k,k_0) \\ S_-(z,k,k_0) \end{pmatrix} = \begin{pmatrix} P_+(z,k,k_0) \\ R_+(z,k,k_0) \end{pmatrix} d(z,k_0) + \begin{pmatrix} Q_+(z,k,k_0) \\ S_+(z,k,k_0) \end{pmatrix} c(z,k_0), \tag{2.147}$$

where

$$c(z,k_0) = \begin{cases} \frac{1-z}{2z}, & k_0 \text{ odd}, \\ \frac{1-z}{2}, & k_0 \text{ even} \end{cases} \quad \text{and } d(z,k_0) = \begin{cases} \frac{1+z}{2z}, & k_0 \text{ odd}, \\ \frac{1+z}{2}, & k_0 \text{ even}. \end{cases} \tag{2.148}$$

Proof. Since the left and right-hand sides of (2.144)–(2.147) satisfy the same recursion relation (2.21), it suffices to check (2.144)–(2.147) at only one point, say, at the point $k = k_0$. The latter is easily seen to be a consequence of (2.51). $\square$

THEOREM 2.17. *Let* $k_0 \in \mathbb{Z}$. *Then there exist unique* $\mathbb{C}^{m\times m}$*-valued functions* $M_\pm(\cdot,k_0)$ *such that for all* $z \in \mathbb{C}\backslash(\partial\mathbb{D}\cup\{0\})$

$$U_\pm(z,\cdot,k_0) = Q_+(z,\cdot,k_0) + P_+(z,\cdot,k_0)M_\pm(z,k_0) \in \ell^2([k_0,\pm\infty)\cap\mathbb{Z})^{m\times m}, \tag{2.149}$$

$$V_\pm(z,\cdot,k_0) = S_+(z,\cdot,k_0) + R_+(z,\cdot,k_0)M_\pm(z,k_0) \in \ell^2([k_0,\pm\infty)\cap\mathbb{Z})^{m\times m}. \tag{2.150}$$

Proof. The assertions (2.149) and (2.150) follow from Lemma 2.13, Corollary 2.15, and Lemma 2.16. □

We will call $U_\pm(z,\cdot,k_0)$ the *Weyl–Titchmarsh solutions* of $\mathbb{U}$. By Corollary 2.15, $U_\pm(z,\cdot,k_0)$ and $V_\pm(z,\cdot,k_0)$ are unique up to right-multiplication by constant $m\times m$ matrices. Similarly, we will call $m_\pm(z,k_0)$ as well as $M_\pm(z,k_0)$ the *half-lattice Weyl–Titchmarsh* m*-functions* associated with $\mathbb{U}_{\pm,k_0}$. (See also [90] for a comparison of various alternative notions of Weyl–Titchmarsh m-functions for $\mathbb{U}_{+,k_0}$ with scalar-valued Verblunsky coefficients.)

Lemma 2.13, Corollary 2.15, and Lemma 2.16 imply that for all $z \in \mathbb{C}\backslash\partial\mathbb{D}$,

$$M_+(z,k_0) = m_+(z,k_0), \tag{2.151}$$

$$M_+(0,k_0) = I_m, \tag{2.152}$$

$$M_-(z,k_0) = [(1+z)I_m + (1-z)m_-(z,k_0)][(1-z)I_m + (1+z)m_-(z,k_0)]^{-1}, \tag{2.153}$$

$$\begin{aligned} M_-(z,k_0) = {} & \big[(\widetilde{\rho}_{k_0}^{-1}a_{k_0} + \rho_{k_0}^{-1}a_{k_0}^*) + (\widetilde{\rho}_{k_0}^{-1}b_{k_0} - \rho_{k_0}^{-1}b_{k_0}^*)m_-(z,k_0-1)\big] \\ & \times \big[(\widetilde{\rho}_{k_0}^{-1}a_{k_0} - \rho_{k_0}^{-1}a_{k_0}^*) + (\widetilde{\rho}_{k_0}^{-1}b_{k_0} + \rho_{k_0}^{-1}b_{k_0}^*)m_-(z,k_0-1)\big]^{-1}, \end{aligned} \tag{2.154}$$

$$M_-(0,k_0) = (\alpha_{k_0} + I_m)(\alpha_{k_0} - I_m)^{-1}. \tag{2.155}$$

In addition, it follows from (2.134) and Theorem A.2 that $m_\pm(z,k_0)$ are $\mathbb{C}^{m\times m}$-valued Caratheodory and anti-Caratheodory functions, respectively. From (2.151) one concludes that $M_+(z,k_0)$ is also a Caratheodory function. Using (2.153) one verifies that $M_-(z,k_0)$ is analytic in $\mathbb{D}$ since the anti-Caratheodory function $m_-(\cdot,k_0)$ satisfies $\mathrm{Re}(m_-(z,k_0)) = (m_-(z,k_0) + m_-(z,k_0)^*)/2 < 0$ for $z \in \mathbb{D}$. Moreover, utilizing (2.12), (2.13), and (2.154), one computes,

$$\begin{aligned} &\mathrm{Re}(M_-(z,k_0)) = [M_-(z,k_0) + M_-(z,k_0)^*]/2 \\ &\quad = \big[(a_{k_0}^*\widetilde{\rho}_{k_0}^{-1} - a_{k_0}\rho_{k_0}^{-1}) + m_-(z,k_0-1)^*(b_{k_0}^*\widetilde{\rho}_{k_0}^{-1} + b_{k_0}\rho_{k_0}^{-1})\big]^{-1}\mathrm{Re}(m_-(z,k_0-1)) \\ &\qquad \times \big[(\widetilde{\rho}_{k_0}^{-1}a_{k_0} - \rho_{k_0}^{-1}a_{k_0}^*) + (\widetilde{\rho}_{k_0}^{-1}b_{k_0} + \rho_{k_0}^{-1}b_{k_0}^*)m_-(z,k_0-1)\big]^{-1}, \end{aligned} \tag{2.156}$$

and hence, $M_-(\cdot,k_0)$ is an anti-Caratheodory matrix.

Next, we introduce the $\mathbb{C}^{m\times m}$-valued functions $\Phi_\pm(\cdot,k)$, $k\in\mathbb{Z}$, by

$$\Phi_\pm(z,k) = [M_\pm(z,k) - I_m][M_\pm(z,k) + I_m]^{-1}, \quad z \in \mathbb{C}\backslash\partial\mathbb{D}. \tag{2.157}$$

Then (2.152) and (2.155) imply that

$$\Phi_+(0,k_0) = 0 \text{ and } \Phi_-(0,k_0)^{-1} = \alpha_{k_0}. \tag{2.158}$$

Moreover, one verifies that

$$M_\pm(z,k) = [I_m - \Phi_\pm(z,k)]^{-1}[I_m + \Phi_\pm(z,k)], \quad z \in \mathbb{C}\backslash\partial\mathbb{D}, \tag{2.159}$$

$$m_-(z,k) = [zI_m + \Phi_-(z,k)]^{-1}[zI_m - \Phi_-(z,k)], \quad z \in \mathbb{C}\backslash\partial\mathbb{D} \tag{2.160}$$

(cf. Remark 2.20). In addition, we extend these functions to the unit circle $\partial\mathbb{D}$ by taking the radial limits which exist and are finite for Lebesgue almost every $\zeta \in \partial\mathbb{D}$,

$$M_\pm(\zeta,k) = \lim_{r\uparrow 1} M_\pm(r\zeta,k), \tag{2.161}$$

$$\Phi_\pm(\zeta,k) = \lim_{r\uparrow 1} \Phi_\pm(r\zeta,k), \quad k \in \mathbb{Z}. \tag{2.162}$$

LEMMA 2.18. *Let $z \in \mathbb{C}\backslash(\partial\mathbb{D}\cup\{0\})$, $k_0, k \in \mathbb{Z}$. Then the functions $\Phi_\pm(\cdot,k)$ satisfy*

$$\begin{aligned}\Phi_\pm(z,k) &= \begin{cases} zV_\pm(z,k,k_0)U_\pm(z,k,k_0)^{-1}, & k \text{ odd}, \\ U_\pm(z,k,k_0)V_\pm(z,k,k_0)^{-1}, & k \text{ even} \end{cases} \\ &= \begin{cases} U_\pm(z,k,k_0)^{-1}V_\pm(z,k,k_0), & k \text{ odd}, \\ V_\pm(z,k,k_0)^{-1}U_\pm(z,k,k_0), & k \text{ even}, \end{cases}\end{aligned} \tag{2.163}$$

where $U_\pm(\cdot,k,k_0)$ and $V_\pm(\cdot,k,k_0)$ are the $\mathbb{C}^{m\times m}$-valued functions defined in (2.149) *and* (2.150)*, respectively.*

Proof. Using Corollary 2.15 it suffices to assume $k = k_0$. Then the statement is immediately implied by (2.49), (2.149), (2.150), and (2.157). $\square$

LEMMA 2.19. *Let $k \in \mathbb{Z}$. Then the $\mathbb{C}^{m\times m}$-valued functions $\Phi_+(\cdot,k)|_{\mathbb{D}}$ (resp., $\Phi_-(\cdot,k)|_{\mathbb{D}}$) are Schur (resp., anti-Schur) matrices. Moreover, $\Phi_\pm$ satisfy the Riccati-type equation*

$$\Phi_\pm(z,k)\widetilde{\rho}_k^{-1}\alpha_k\Phi_\pm(z,k-1) + z\Phi_\pm(z,k)\widetilde{\rho}_k^{-1} - \rho_k^{-1}\Phi_\pm(z,k-1) = z\rho_k^{-1}\alpha_k^*, \\ z \in \mathbb{C}\backslash\partial\mathbb{D},\ k \in \mathbb{Z}. \tag{2.164}$$

Proof. Lemma 2.18 and (2.157) imply that the functions $\Phi_+(\cdot,k)|_{\mathbb{D}}$ (resp., $\Phi_-(\cdot,k)|_{\mathbb{D}}$) are Schur (resp., anti-Schur) matrices.

Let k be odd. Then applying Lemma 2.18 and the recursion relation (2.21) one obtains

$$\begin{aligned}\Phi_\pm(z,k) &= zV_\pm(z,k,k_0)U_\pm(z,k,k_0)^{-1} \\ &= \rho_k^{-1}\big[U_\pm(z,k-1,k_0) + z\alpha_k^*V_\pm(z,k-1,k_0)\big] \\ &\quad \times \big[\alpha_kU_\pm(z,k-1,k_0) + zV_\pm(z,k-1,k_0)\big]^{-1}\widetilde{\rho}_k \\ &= \rho_k^{-1}\big[\Phi_\pm(z,k-1) + z\alpha_k^*\big]\big[\alpha_k\Phi_\pm(z,k-1) + zI_m\big]^{-1}\widetilde{\rho}_k.\end{aligned} \tag{2.165}$$

For k even, one similarly obtains

$$\begin{aligned}\Phi_\pm(z,k) &= U_\pm(z,k,k_0)V_\pm(z,k,k_0)^{-1}\\ &= \rho_k^{-1}\big[\alpha_k^* U_\pm(z,k-1,k_0)+V_\pm(z,k-1,k_0)\big]\\ &\quad\times\big[U_\pm(z,k-1,k_0)+\alpha_k V_\pm(z,k-1,k_0)\big]^{-1}\widetilde{\rho}_k\\ &= \rho_k^{-1}\big[z\alpha_k^*+\Phi_\pm(z,k-1)\big]\big[zI_m+\alpha_k\Phi_\pm(z,k-1)\big]^{-1}\widetilde{\rho}_k.\end{aligned} \tag{2.166}$$

□

REMARK 2.20. (i) In the special case $\alpha=\{\alpha_k\}_{k\in\mathbb{Z}}=0$, one obtains

$$M_\pm(z,k)=\pm I_m,\quad \Phi_+(z,k)=0,\quad \Phi_-(z,k)^{-1}=0,\quad z\in\mathbb{C},\ k\in\mathbb{Z}. \tag{2.167}$$

Thus, strictly speaking, one should always consider Φ_-^{-1} rather than Φ_- and hence refer to the Riccati-type equation of Φ_-^{-1},

$$\begin{gathered}z\Phi_-(z,k)^{-1}\rho_k^{-1}\alpha_k^*\Phi_-(z,k-1)^{-1}+\Phi_-(z,k)^{-1}\rho_k^{-1}-z\widetilde{\rho}_k^{-1}\Phi_-(z,k-1)^{-1}=\widetilde{\rho}_k^{-1}\alpha_k,\\ z\in\mathbb{C}\backslash\partial\mathbb{D},\ k\in\mathbb{Z},\end{gathered} \tag{2.168}$$

rather than that of Φ_-, etc. In fact, since $M_-(\cdot,k)$ is an anti-Caratheodory matrix and hence $[M_-(z,k)-I_m]$ is invertible (cf. [99, p. 137]), we should have introduced the Schur matrix

$$\Phi_-(z,k)^{-1}=[M_-(z,k)+I_m][M_-(z,k)-I_m]^{-1},\quad z\in\mathbb{C}\backslash\partial\mathbb{D}, \tag{2.169}$$

rather than the anti-Schur matrix $\Phi_-(\cdot,k)$. For simplicity of notation, we will typically avoid this complication with Φ_- and still invoke Φ_- rather than Φ_-^{-1} whenever confusions are unlikely.
(ii) We note that $\Phi_\pm(z,k)^{\pm1}$, $z\in\partial\mathbb{D}$, $k\in\mathbb{Z}$, have nontangential limits to $\partial\mathbb{D}$ Lebesgue almost everywhere. In particular, the Riccati-type equations (2.164), and (2.168) extend to $\partial\mathbb{D}$ Lebesgue almost everywhere.

The Riccati-type equation (2.164) for the Schur matrix Φ_+ implies the norm convergent expansion,

$$\Phi_+(z,k)=\sum_{j=1}^{\infty}\phi_{+,j}(k)z^j,\quad z\in\mathbb{D},\ k\in\mathbb{Z}, \tag{2.170}$$

$$\begin{aligned}\phi_{+,1}(k)&=-\alpha_{k+1}^*,\\ \phi_{+,2}(k)&=-\rho_{k+1}\alpha_{k+2}^*\widetilde{\rho}_{k+1},\\ \phi_{+,j}(k)&=\sum_{\ell=1}^{j-1}\rho_{k+1}\phi_{+,j-\ell}(k+1)\widetilde{\rho}_{k+1}^{-1}\alpha_{k+1}\phi_{+,\ell}(k)+\rho_{k+1}\phi_{+,j-1}(k+1)\widetilde{\rho}_{k+1}^{-1},\ j\geqslant 3.\end{aligned} \tag{2.171}$$

Similarly, the corresponding Riccati-type equation (2.168) for the Schur matrix Φ_-^{-1} implies the norm convergent expansion

$$\Phi_-(z,k)^{-1} = \sum_{j=0}^{\infty} \phi_{-,j}(k) z^j, \quad z \in \mathbb{D},\ k \in \mathbb{Z}, \tag{2.172}$$

$$\begin{aligned} \phi_{-,0}(k) &= \alpha_k, \\ \phi_{-,1}(k) &= \widetilde{\rho}_k \alpha_{k-1} \rho_k, \\ \phi_{-,j}(k) &= -\sum_{\ell=0}^{j-1} \phi_{-,j-1-\ell}(k-1) \rho_k^{-1} \alpha_k^* \phi_{-,\ell}(k) \rho_k + \widetilde{\rho}_k^{-1} \phi_{-,j-1}(k-1) \rho_k,\ j \geqslant 2. \end{aligned} \tag{2.173}$$

3. Weyl–Titchmarsh Theory for CMV Operators on $\mathbb{Z}$ with Matrix-valued Verblunsky Coefficients

In this section we present the basics of Weyl–Titchmarsh theory for CMV operators on the lattice $\mathbb{Z}$ with matrix-valued Verblunsky coefficients. The corresponding case of scalar-valued Verblunsky coefficients was dealt with in detail in [56].

We start by introducing the $\mathbb{C}^{m\times m}$-valued CMV Wronskian of two $\mathbb{C}^{m\times m}$-valued sequences $U_j(z,\cdot)$, $j=1,2$,

$$\begin{aligned} &W(U_1(1/\overline{z},k), U_2(z,k)) \\ &\quad = \frac{(-1)^{k+1}}{2} \Big[U_1(1/\overline{z},k)^* U_2(z,k) - (\mathbb{V}^* U_1(1/\overline{z},\,\cdot\,))(k)^* (\mathbb{V}^* U_2(z,\,\cdot\,))(k) \Big], \\ &\qquad\qquad k \in \mathbb{Z},\ z \in \mathbb{C}\backslash\{0\}, \end{aligned} \tag{3.1}$$

Next we verify that the Wronskian of any two solutions of $\mathbb{U}U(z,\cdot) = zU(z,\cdot)$ is indeed k-independent as expected:

LEMMA 3.1. *Suppose $U_j(z,\cdot)$ satisfy $\mathbb{U}U_j(z,\cdot) = zU_j(z,\cdot)$, $j=1,2$, where $\mathbb{U}$ is understood as a difference expression (rather then an operator in $\ell^2(\mathbb{Z})^{m\times m}$). Then the Wronskian in* (3.1) *is independent of $k \in \mathbb{Z}$ and the following identities hold:*

$$\begin{aligned} W(U_1(1/\overline{z},k), U_2(z,k)) &= \frac{(-1)^{k+1}}{2} \Big[U_1(1/\overline{z},k)^* U_2(z,k) - V_1(1/\overline{z},k)^* V_2(z,k) \Big] \\ &= \frac{(-1)^{k+1}}{2} \begin{pmatrix} U_1(1/\overline{z},k) \\ V_1(1/\overline{z},k) \end{pmatrix}^* \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix} \begin{pmatrix} U_2(z,k) \\ V_2(z,k) \end{pmatrix}, \\ &\qquad k \in \mathbb{Z},\ z \in \mathbb{C}\backslash\{0\}, \end{aligned} \tag{3.2}$$

where $V_j(z,\cdot) = \mathbb{V}^* U_j(z,\cdot)$, $j=1,2$, and

$$W(P_+(1/\overline{z},k,k_0), Q_+(z,k,k_0)) = I_m, \tag{3.3}$$

$$W(U_+(1/\overline{z},k,k_0), U_-(z,k,k_0)) = M_+(z,k_0) - M_-(z,k_0), \tag{3.4}$$

$$k, k_0 \in \mathbb{Z},\ z \in \mathbb{C}\backslash\{0\}.$$

Proof. First, we note that (3.2) is implied by (3.1). Next, employing Lemma 2.3, U_j and V_j, $j = 1,2$, satisfy the recursion relation

$$\begin{pmatrix} U_j(z,k) \\ V_j(z,k) \end{pmatrix} = \mathbb{T}(z,k) \begin{pmatrix} U_j(z,k-1) \\ V_j(z,k-1) \end{pmatrix}, \quad j = 1,2, \ k \in \mathbb{Z}, \ z \in \mathbb{C}\backslash\{0\}, \tag{3.5}$$

and hence

$$\begin{aligned}
W(U_1(1/\overline{z},k), U_2(z,k)) &= \frac{(-1)^{k+1}}{2} \begin{pmatrix} U_1(1/\overline{z},k) \\ V_1(1/\overline{z},k) \end{pmatrix}^* \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix} \begin{pmatrix} U_2(z,k) \\ V_2(z,k) \end{pmatrix} \\
&= \frac{(-1)^{k+1}}{2} \begin{pmatrix} U_1(1/\overline{z},k-1) \\ V_1(1/\overline{z},k-1) \end{pmatrix}^* \mathbb{T}(1/\overline{z},k)^* \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix} \mathbb{T}(z,k) \begin{pmatrix} U_2(z,k-1) \\ V_2(z,k-1) \end{pmatrix} \\
&= -\frac{(-1)^{k}}{2} \begin{pmatrix} U_1(1/\overline{z},k-1) \\ V_1(1/\overline{z},k-1) \end{pmatrix}^* \begin{pmatrix} -I_m & 0 \\ 0 & I_m \end{pmatrix} \begin{pmatrix} U_2(z,k-1) \\ V_2(z,k-1) \end{pmatrix} \qquad (3.6) \\
&= W(U_1(1/\overline{z},k-1), U_2(z,k-1)), \quad k \in \mathbb{Z}, \ z \in \mathbb{C}\backslash\{0\}.
\end{aligned}$$

Here we used the following identity which is implied by (2.12) and (2.22)

$$\mathbb{T}(1/\overline{z},k)^* \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix} \mathbb{T}(z,k) = \begin{pmatrix} -I_m & 0 \\ 0 & I_m \end{pmatrix}, \quad k \in \mathbb{Z}, \ z \in \mathbb{C}\backslash\{0\}. \tag{3.7}$$

Finally, taking $k = k_0$ and utilizing (2.49), (2.50), (2.149), (2.150), and (A.9), one obtains (3.3) and (3.4) from (3.2). $\square$

For notational simplicity we abbreviate the Wronskian of U_+ and U_- by

$$W(z,k_0) = W(U_+(1/\overline{z},k,k_0), U_-(z,k,k_0)). \tag{3.8}$$

Then, using (2.152), (2.155), and (3.4), one analytically continues $W(z,k_0)$ to $z = 0$ and obtains

$$W(z,k_0) = M_+(z,k_0) - M_-(z,k_0), \quad k \in \mathbb{Z}, \ z \in \mathbb{C}. \tag{3.9}$$

Moreover, one verifies a certain symmetry property of the Wronskian $W(z,k_0)$,

$$M_+(z,k_0)W(z,k_0)^{-1}M_-(z,k_0) = M_-(z,k_0)W(z,k_0)^{-1}M_+(z,k_0), \quad k \in \mathbb{Z}, \ z \in \mathbb{C}. \tag{3.10}$$

Next we prove an auxiliary lemma that will play a crucial role in our computation of the resolvent for the CMV operator $\mathbb{U}$.

LEMMA 3.2. *Let $k, k_0 \in \mathbb{Z}$ and $z \in \mathbb{C}\backslash\{0\}$. The the following identities hold,*

$$P_+(z,k,k_0)Q_+(1/\overline{z},k,k_0)^* + Q_+(z,k,k_0)P_+(1/\overline{z},k,k_0)^* = 2(-1)^{k+1}I_m, \tag{3.11}$$
$$R_+(z,k,k_0)S_+(1/\overline{z},k,k_0)^* + S_+(z,k,k_0)R_+(1/\overline{z},k,k_0)^* = 2(-1)^{k}I_m, \tag{3.12}$$
$$P_+(z,k,k_0)S_+(1/\overline{z},k,k_0)^* + Q_+(z,k,k_0)R_+(1/\overline{z},k,k_0)^* = 0, \tag{3.13}$$
$$R_+(z,k,k_0)Q_+(1/\overline{z},k,k_0)^* + S_+(z,k,k_0)P_+(1/\overline{z},k,k_0)^* = 0, \tag{3.14}$$

and

$$\begin{aligned}
&U_+(z,k,k_0)W(z,k_0)^{-1}U_-(1/\overline{z},k,k_0)^* - U_-(z,k,k_0)W(z,k_0)^{-1}U_+(1/\overline{z},k,k_0)^* \\
&\qquad = 2(-1)^{k+1}I_m, \qquad (3.15)\\
&V_+(z,k,k_0)W(z,k_0)^{-1}U_-(1/\overline{z},k,k_0)^* - V_-(z,k,k_0)W(z,k_0)^{-1}U_+(1/\overline{z},k,k_0)^* = 0. \qquad (3.16)
\end{aligned}$$

Proof. First, we note that for $k = k_0$ equalities (3.11)–(3.14) follow from (2.49). Then one uses an induction argument to prove (3.11)–(3.14) for $k \neq k_0$. This involves a consideration of a number of cases all of which follow the same pattern. Therefore, we limit out attention to just one of these cases. Suppose (3.11)–(3.14) hold for some $k \in \mathbb{Z}$ even. Then utilizing (2.21) together with (2.8) and (2.9), one computes

$$\begin{aligned}
&P_+(z,k+1,k_0)Q_+(1/\overline{z},k+1,k_0)^* + Q_+(z,k+1,k_0)P_+(1/\overline{z},k+1,k_0)^* \\
&\quad = \widetilde{\rho}_{k+1}^{-1}\alpha_{k+1}\big[P_+(z,k,k_0)Q_+(1/\overline{z},k,k_0)^* + Q_+(z,k,k_0)P_+(1/\overline{z},k,k_0)^*\big]\alpha_{k+1}^*\widetilde{\rho}_{k+1}^{-1} \\
&\qquad + \widetilde{\rho}_{k+1}^{-1}\big[R_+(z,k,k_0)S_+(1/\overline{z},k,k_0)^* + S_+(z,k,k_0)R_+(1/\overline{z},k,k_0)^*\big]\widetilde{\rho}_{k+1}^{-1} \\
&\qquad + z\widetilde{\rho}_{k+1}^{-1}\big[R_+(z,k,k_0)Q_+(1/\overline{z},k,k_0)^* + S_+(z,k,k_0)P_+(1/\overline{z},k,k_0)^*\big]\alpha_{k_0}^*\widetilde{\rho}_{k+1}^{-1} \\
&\qquad + \widetilde{\rho}_{k+1}^{-1}\alpha_{k_0}\big[P_+(z,k,k_0)S_+(1/\overline{z},k,k_0)^* + Q_+(z,k,k_0)R_+(1/\overline{z},k,k_0)^*\big]\widetilde{\rho}_{k+1}^{-1}z^{-1} \\
&\quad = 2(-1)^{k+1}\big[\widetilde{\rho}_{k+1}^{-1}\alpha_{k+1}\alpha_{k+1}^*\widetilde{\rho}_{k+1}^{-1} - \widetilde{\rho}_{k+1}^{-2}\big] = 2(-1)^{(k+1)+1}I_m. \qquad (3.17)
\end{aligned}$$

Similarly, one checks equalities (3.12)–(3.14) at the point $k+1$. Then inverting the matrix $\mathbb{T}(z,k)$ and utilizing (2.21) in the form,

$$\begin{pmatrix} P_-(z,k-1),k_0 \\ R_-(z,k-1,k_0)\end{pmatrix} = \mathbb{T}(z,k)^{-1}\begin{pmatrix} P_-(z,k,k_0) \\ R_-(z,k,k_0)\end{pmatrix}, \qquad (3.18)$$

one verifies (3.11)–(3.14) at the point $k-1$. Similarly, one verifies (3.11)–(3.14) at the points $k+1$ and $k-1$ under the assumption of k odd.

Next, using (2.149), (2.150), (3.9), (3.10), (3.11), and (3.14), one verifies (3.15) and (3.16) as follows:

$$\begin{aligned}
&U_+(z,k,k_0)W(z,k_0)^{-1}U_-(1/\overline{z},k,k_0)^* - U_-(z,k,k_0)W(z,k_0)^{-1}U_+(1/\overline{z},k,k_0)^* \\
&\quad = Q_+(z,k,k_0)W(z,k_0)^{-1}\big[M_+(z,k_0) - M_-(z,k_0)\big]P_+(1/\overline{z},k,k_0)^* \\
&\qquad + P_+(z,k,k_0)\big[M_+(z,k_0) - M_-(z,k_0)\big]W(z,k_0)^{-1}Q_+(1/\overline{z},k,k_0)^* \\
&\qquad + P_+(z,k,k_0)\big[M_-(z,k_0)W(z,k_0)^{-1}M_+(z,k_0) \\
&\qquad\quad - M_+(z,k_0)W(z,k_0)^{-1}M_-(z,k_0)\big]P_+(1/\overline{z},k,k_0)^* \\
&\quad = Q_+(z,k,k_0)P_+(1/\overline{z},k,k_0)^* + P_+(z,k,k_0)Q_+(1/\overline{z},k,k_0)^* = 2(-1)^{k+1}I_m, \qquad (3.19)
\end{aligned}$$

$$\begin{aligned}
&V_+(z,k,k_0)W(z,k_0)^{-1}U_-(1/\overline{z},k,k_0)^* - V_-(z,k,k_0)W(z,k_0)^{-1}U_+(1/\overline{z},k,k_0)^* \\
&\quad = S_+(z,k,k_0)W(z,k_0)^{-1}\big[M_+(z,k_0) - M_-(z,k_0)\big]P_+(1/\overline{z},k,k_0)^* \\
&\qquad + R_+(z,k,k_0)\big[M_+(z,k_0) - M_-(z,k_0)\big]W(z,k_0)^{-1}Q_+(1/\overline{z},k,k_0)^*
\end{aligned}$$

$$
\begin{aligned}
&+ R_+(z,k,k_0)\big[M_+(z,k_0)W(z,k_0)^{-1}M_-(z,k_0) \\
&\quad - M_-(z,k_0)W(z,k_0)^{-1}M_+(z,k_0)\big]P_+(1/\overline{z},k,k_0)^* \\
&= S_+(z,k,k_0)P_+(1/\overline{z},k,k_0)^* + R_+(z,k,k_0)Q_+(1/\overline{z},k,k_0)^* = 0. \qquad (3.20)
\end{aligned}
$$

□

LEMMA 3.3. *Let $z \in \mathbb{C}\backslash(\partial\mathbb{D} \cup \{0\})$ and fix $k_0 \in \mathbb{Z}$. Then the resolvent $(\mathbb{U} - zI)^{-1}$ of the unitary CMV operator $\mathbb{U}$ on $\ell^2(\mathbb{Z})^m$ is given in terms of its matrix representation in the standard basis of $\ell^2(\mathbb{Z})^m$ by*

$$
(\mathbb{U} - zI)^{-1}(k,k') = \frac{1}{2z}\begin{cases} U_-(z,k,k_0)W(z,k_0)^{-1}U_+(1/\overline{z},k',k_0)^*, \\ \qquad k < k' \text{ or } k = k' \text{ odd}, \\ U_+(z,k,k_0)W(z,k_0)^{-1}U_-(1/\overline{z},k',k_0)^*, \\ \qquad k > k' \text{ or } k = k' \text{ even}, \end{cases} \quad k,k' \in \mathbb{Z}. \tag{3.21}
$$

Moreover, since $0 \in \mathbb{C}\backslash\sigma(\mathbb{U})$, (3.21) analytically extends to $z = 0$.

In particular, one obtains for any $z \in \mathbb{C}\backslash\partial\mathbb{D}$,

$$
\begin{aligned}
&(\mathbb{U} - zI)^{-1}(k,k) \\
&\quad = \frac{1}{2z}\begin{cases} [I_m + M_-(z,k)]W(z,k)^{-1}[I_m - M_+(z,k)], & k \text{ odd}, \\ [I_m - M_+(z,k)]W(z,k)^{-1}[I_m + M_-(z,k)], & k \text{ even}, \end{cases} \qquad (3.22) \\
&(\mathbb{U} - zI)^{-1}(k-1,k-1) \\
&\quad = \frac{1}{2z}\begin{cases} \rho_k^{-1}[a_k^* - b_k^*M_+(z,k)]W(z,k)^{-1}[a_k + M_-(z,k)b_k]\rho_k^{-1}, & k \text{ odd}, \\ \widetilde{\rho}_k^{-1}[a_k + b_kM_-(z,k)]W(z,k)^{-1}[a_k^* - M_+(z,k)b_k^*]\widetilde{\rho}_k^{-1}, & k \text{ even}, \end{cases} \qquad (3.23) \\
&(\mathbb{U} - zI)^{-1}(k-1,k) \\
&\quad = \frac{-1}{2z}\begin{cases} \rho_k^{-1}[a_k^* - b_k^*M_-(z,k)]W(z,k)^{-1}[I_m - M_+(z,k)], & k \text{ odd}, \\ \widetilde{\rho}_k^{-1}[a_k + b_kM_-(z,k)]W(z,k)^{-1}[I_m + M_+(z,k)], & k \text{ even}, \end{cases} \qquad (3.24) \\
&(\mathbb{U} - zI)^{-1}(k,k-1) \\
&\quad = \frac{-1}{2z}\begin{cases} [I_m + M_+(z,k)]W(z,k)^{-1}[a_k + M_-(z,k)b_k]\rho_k^{-1}, & k \text{ odd}, \\ [I_m - M_+(z,k)]W(z,k)^{-1}[a_k^* - M_-(z,k)b_k^*]\widetilde{\rho}_k^{-1}, & k \text{ even}. \end{cases} \qquad (3.25)
\end{aligned}
$$

Proof. Let

$$
G(z,k,k',k_0) = \begin{cases} U_-(z,k,k_0)W(z,k_0)^{-1}U_+(1/\overline{z},k',k_0)^*, & k < k' \text{ or } k = k' \text{ odd}, \\ U_+(z,k,k_0)W(z,k_0)^{-1}U_-(1/\overline{z},k',k_0)^*, & k > k' \text{ or } k = k' \text{ even}, \end{cases} \quad k,k' \in \mathbb{Z}. \tag{3.26}
$$

Then (3.21) is equivalent to

$$
(\mathbb{U} - zI)G(z,\cdot,k',k_0) = 2z\Delta_{k'}, \quad k',k_0 \in \mathbb{Z}. \tag{3.27}
$$

First, assume k' to be odd. Then,

$$((\mathbb{U}-zI)G(z,\cdot,k',k_0))(\ell)=((\mathbb{V}\mathbb{W}-zI)G(z,\cdot,k',k_0))(\ell)=0, \quad \ell\in\mathbb{Z}\backslash\{k',k'+1\}, \tag{3.28}$$

and by (3.15), (3.16),

$$\begin{aligned}
&\begin{pmatrix} ((\mathbb{U}-zI)G(z,\cdot,k',k_0))(k') \\ ((\mathbb{U}-zI)G(z,\cdot,k',k_0))(k'+1)\end{pmatrix} = \begin{pmatrix} ((\mathbb{V}\mathbb{W}-zI)G(z,\cdot,k',k_0))(k') \\ ((\mathbb{V}\mathbb{W}-zI)G(z,\cdot,k',k_0))(k'+1)\end{pmatrix} \\
&\quad = \Theta_{k'+1}\begin{pmatrix} zV_-(z,k',k_0)W(z,k_0)^{-1}U_+(1/\overline{z},k',k_0)^* \\ zV_+(z,k'+1,k_0)W(z,k_0)^{-1}U_-(1/\overline{z},k',k_0)^*\end{pmatrix} - z\begin{pmatrix} G(z,k',k',k_0) \\ G(z,k'+1,k',k_0)\end{pmatrix} \\
&\quad = z\Theta_{k'+1}\begin{pmatrix} V_+(z,k',k_0)W(z,k_0)^{-1}U_-(1/\overline{z},k',k_0)^* \\ V_+(z,k'+1,k_0)W(z,k_0)^{-1}U_-(1/\overline{z},k',k_0)^*\end{pmatrix} - z\begin{pmatrix} G(z,k',k',k_0) \\ G(z,k'+1,k',k_0)\end{pmatrix} \\
&\quad = z\begin{pmatrix} U_+(z,k',k_0)W(z,k_0)^{-1}U_-(1/\overline{z},k',k_0)^* \\ U_+(z,k'+1,k_0)W(z,k_0)^{-1}U_-(1/\overline{z},k',k_0)^*\end{pmatrix} - z\begin{pmatrix} G(z,k',k',k_0) \\ G(z,k'+1,k',k_0)\end{pmatrix} \\
&\quad = z\begin{pmatrix} 2(-1)^{k'+1}I_m \\ 0\end{pmatrix} = \begin{pmatrix} 2zI_m \\ 0\end{pmatrix}.
\end{aligned} \tag{3.29}$$

Thus, for k' odd, (3.27) is a consequence of (3.28) and (3.29).

Next, assume k' to be even. Then,

$$((\mathbb{U}-zI)G(z,\cdot,k',k_0))(\ell)=((\mathbb{V}\mathbb{W}-zI)G(z,\cdot,k',k_0))(\ell)=0, \quad \ell\in\mathbb{Z}\backslash\{k'-1,k'\}, \tag{3.30}$$

and by (3.15), (3.16),

$$\begin{aligned}
&\begin{pmatrix} ((\mathbb{U}-zI)G(z,\cdot,k',k_0))(k'-1) \\ ((\mathbb{U}-zI)G(z,\cdot,k',k_0))(k')\end{pmatrix} = \begin{pmatrix} ((\mathbb{V}\mathbb{W}-zI)G(z,\cdot,k',k_0))(k'-1) \\ ((\mathbb{V}\mathbb{W}-zI)G(z,\cdot,k',k_0))(k')\end{pmatrix} \\
&\quad = \Theta_{k'}\begin{pmatrix} zV_-(z,k'-1,k_0)W(z,k_0)^{-1}U_+(1/\overline{z},k',k_0)^* \\ zV_+(z,k',k_0)W(z,k_0)^{-1}U_-(1/\overline{z},k',k_0)^*\end{pmatrix} - z\begin{pmatrix} G(z,k'-1,k',k_0) \\ G(z,k',k',k_0)\end{pmatrix} \\
&\quad = z\Theta_{k'}\begin{pmatrix} V_-(z,k'-1,k_0)W(z,k_0)^{-1}U_+(1/\overline{z},k',k_0)^* \\ V_-(z,k',k_0)W(z,k_0)^{-1}U_+(1/\overline{z},k',k_0)^*\end{pmatrix} - z\begin{pmatrix} G(z,k'-1,k',k_0) \\ G(z,k',k',k_0)\end{pmatrix} \\
&\quad = z\begin{pmatrix} U_-(z,k'-1,k_0)W(z,k_0)^{-1}U_+(1/\overline{z},k',k_0)^* \\ U_-(z,k',k_0)W(z,k_0)^{-1}U_+(1/\overline{z},k',k_0)^*\end{pmatrix} - z\begin{pmatrix} G(z,k'-1,k',k_0) \\ G(z,k',k',k_0)\end{pmatrix} \\
&\quad = z\begin{pmatrix} 0 \\ 2(-1)^{k'}I_m\end{pmatrix} = \begin{pmatrix} 0 \\ 2zI_m\end{pmatrix}.
\end{aligned} \tag{3.31}$$

Thus, for k' even, (3.27) follows from (3.30) and (3.31), and hence one obtains (3.21).

Finally, using (2.51) and (2.149) one verifies the identities

$$U_\pm(z,k,k) = \begin{cases} z[I_m + M_\pm(z,k)], & k \text{ odd}, \\ -I_m + M_\pm(z,k), & k \text{ even}, \end{cases} \tag{3.32}$$

$$U_\pm(z,k-1,k) = \begin{cases} -z\rho_k^{-1}[a_k^* - b_k^* M_\pm(z,k)], & k \text{ odd}, \\ \widetilde{\rho}_k^{-1}[a_k + b_k M_\pm(z,k)], & k \text{ even}. \end{cases} \tag{3.33}$$

Inserting (3.32) and (3.33) into (3.21) and utilizing the fact that (anti-)Caratheodory matrices satisfy $M_{\pm}(1/\overline{z},k)^* = -M_{\pm}(z,k)$, $k \in \mathbb{Z}$, $z \in \mathbb{C}$, one obtains (3.22)–(3.25). □

Next, we briefly turn to Weyl–Titchmarsh theory for CMV operators with matrix-valued Verblunsky coefficients on $\mathbb{Z}$. We denote by $d\Omega(\cdot,k)$, $k \in \mathbb{Z}$, the $2m \times 2m$ matrix-valued measure,

$$\begin{aligned} d\Omega(\zeta,k) &= d\begin{pmatrix} \Omega_{0,0}(\zeta,k) & \Omega_{0,1}(\zeta,k) \\ \Omega_{1,0}(\zeta,k) & \Omega_{1,1}(\zeta,k) \end{pmatrix} \\ &= d\begin{pmatrix} \Delta_{k-1}^* E_{\mathbb{U}}(\zeta)\Delta_{k-1} & \Delta_{k-1}^* E_{\mathbb{U}}(\zeta)\Delta_k \\ \Delta_k^* E_{\mathbb{U}}(\zeta)\Delta_{k-1} & \Delta_k^* E_{\mathbb{U}}(\zeta)\Delta_k \end{pmatrix}, \quad \zeta \in \partial\mathbb{D}, \end{aligned} \tag{3.34}$$

where $E_{\mathbb{U}}(\cdot)$ denotes the family of spectral projections of the unitary CMV operator $\mathbb{U}$ on $\ell^2(\mathbb{Z})^m$,

$$\mathbb{U} = \oint_{\partial\mathbb{D}} dE_{\mathbb{U}}(\zeta)\,\zeta. \tag{3.35}$$

We also introduce the $2m \times 2m$ matrix-valued function $\mathscr{M}(\cdot,k)$, $k \in \mathbb{Z}$, by

$$\begin{aligned} \mathscr{M}(z,k) &= \begin{pmatrix} M_{0,0}(z,k) & M_{0,1}(z,k) \\ M_{1,0}(z,k) & M_{1,1}(z,k) \end{pmatrix} \\ &= \begin{pmatrix} \Delta_{k-1}^*(\mathbb{U}+zI)(\mathbb{U}-zI)^{-1}\Delta_{k-1} & \Delta_{k-1}^*(\mathbb{U}+zI)(\mathbb{U}-zI)^{-1}\Delta_k \\ \Delta_k^*(\mathbb{U}+zI)(\mathbb{U}-zI)^{-1}\Delta_{k-1} & \Delta_k^*(\mathbb{U}+zI)(\mathbb{U}-zI)^{-1}\Delta_k \end{pmatrix} \\ &= \oint_{\partial\mathbb{D}} d\Omega(\zeta,k)\,\frac{\zeta+z}{\zeta-z}, \quad z \in \mathbb{C}\backslash\partial\mathbb{D}. \end{aligned} \tag{3.36}$$

We note that

$$M_{0,0}(\cdot,k+1) = M_{1,1}(\cdot,k), \quad k \in \mathbb{Z} \tag{3.37}$$

and

$$\begin{aligned} M_{1,1}(z,k) &= \Delta_k^*(\mathbb{U}+zI)(\mathbb{U}-zI)^{-1}\Delta_k \\ &= \oint_{\partial\mathbb{D}} d\Omega_{1,1}(\zeta,k)\,\frac{\zeta+z}{\zeta-z}, \quad z \in \mathbb{C}\backslash\partial\mathbb{D},\ k \in \mathbb{Z}, \end{aligned} \tag{3.38}$$

where

$$d\Omega_{1,1}(\zeta,k) = d\Delta_k^* E_{\mathbb{U}}(\zeta)\Delta_k, \quad \zeta \in \partial\mathbb{D}. \tag{3.39}$$

Thus, $M_{0,0}|_{\mathbb{D}}$ and $M_{1,1}|_{\mathbb{D}}$ are $m \times m$ Caratheodory matrices. Moreover, by (3.38) one infers that

$$M_{1,1}(0,k) = I_m, \quad k \in \mathbb{Z}. \tag{3.40}$$

LEMMA 3.4. *Let $z \in \mathbb{C}\backslash\partial\mathbb{D}$. Then the functions $M_{\ell,\ell'}(\cdot,k)$, $\ell,\ell' = 0,1$, and $M_\pm(\cdot,k)$, $k \in \mathbb{Z}$, satisfy the following relations*

$$M_{0,0}(z,k)=I_m+\begin{cases} \rho_k^{-1}[a_k^*-b_k^*M_+(z,k)]W(z,k)^{-1}[a_k+M_-(z,k)b_k]\rho_k^{-1}, & k \text{ odd}, \\ \widetilde{\rho}_k^{-1}[a_k+b_kM_-(z,k)]W(z,k)^{-1}[a_k^*-M_+(z,k)b_k^*]\widetilde{\rho}_k^{-1}, & k \text{ even}, \end{cases} \tag{3.41}$$

$$M_{1,1}(z,k) = I_m + \begin{cases} [I_m + M_-(z,k)]W(z,k)^{-1}[I_m - M_+(z,k)], & k \text{ odd}, \\ [I_m - M_+(z,k)]W(z,k)^{-1}[I_m + M_-(z,k)], & k \text{ even}, \end{cases} \tag{3.42}$$

$$M_{0,1}(z,k) = -\begin{cases} \rho_k^{-1}[a_k^* - b_k^*M_-(z,k)]W(z,k)^{-1}[I_m - M_+(z,k)], & k \text{ odd}, \\ \widetilde{\rho}_k^{-1}[a_k + b_kM_-(z,k)]W(z,k)^{-1}[I_m + M_+(z,k)], & k \text{ even}, \end{cases} \tag{3.43}$$

$$M_{1,0}(z,k) = -\begin{cases} [I_m + M_+(z,k)]W(z,k)^{-1}[a_k + M_-(z,k)b_k]\rho_k^{-1}, & k \text{ odd}, \\ [I_m - M_+(z,k)]W(z,k)^{-1}[a_k^* - M_-(z,k)b_k^*]\widetilde{\rho}_k^{-1}, & k \text{ even}, \end{cases} \tag{3.44}$$

where $a_k = I_m + \alpha_k$ and $b_k = I_m - \alpha_k$, $k \in \mathbb{Z}$.

Proof. The result is a consequence of Lemma 3.3 since by (3.36) one has

$$\begin{aligned} M_{\ell,\ell'}(z,k) &= \Delta_{k-1+\ell}^*(I + 2z(\mathbb{U} - zI)^{-1})\Delta_{k-1+\ell'} \\ &= I_m\delta_{\ell,\ell'} + (\mathbb{U} - zI)^{-1}(k-1+\ell, k-1+\ell'). \end{aligned} \tag{3.45}$$

□

Finally, introducing the $m \times m$ matrix-valued functions $\Phi_{1,1}(\cdot,k)$, $k \in \mathbb{Z}$, by

$$\begin{aligned} \Phi_{1,1}(z,k) &= [M_{1,1}(z,k) - I_m][M_{1,1}(z,k) + I_m]^{-1} \\ &= I_m - 2[M_{1,1}(z,k) + I_m]^{-1}, \quad z \in \mathbb{C}\backslash\partial\mathbb{D}, \end{aligned} \tag{3.46}$$

then,

$$\begin{aligned} M_{1,1}(z,k) &= [I_m + \Phi_{1,1}(z,k)][I_m - \Phi_{1,1}(z,k)]^{-1} \\ &= 2[I_m - \Phi_{1,1}(z,k)]^{-1} - I_m, \quad z \in \mathbb{C}\backslash\partial\mathbb{D}. \end{aligned} \tag{3.47}$$

LEMMA 3.5. *The $\mathbb{C}^{m\times m}$-valued function $\Phi_{1,1}|_{\mathbb{D}}$ is a Schur matrix and $\Phi_{1,1}$ is related to $\Phi_\pm$ by*

$$\Phi_{1,1}(z,k) = \begin{cases} \Phi_-(z,k)^{-1}\Phi_+(z,k), & k \text{ odd}, \\ \Phi_+(z,k)\Phi_-(z,k)^{-1}, & k \text{ even}, \end{cases} \quad z \in \mathbb{C}\backslash\partial\mathbb{D},\ k \in \mathbb{Z}. \tag{3.48}$$

Proof. Suppose k is odd. Then(3.9), (3.10), and (3.42) imply that

$$\begin{aligned} M_{1,1}(z,k) + I_m &= [I_m + M_-(z,k)]W(z,k)^{-1}[I_m - M_+(z,k)] \\ &\quad + [M_+(z,k) - M_-(z,k)]W(z,k)^{-1} + W(z,k)^{-1}[M_+(z,k) - M_-(z,k)] \\ &= [I_m + M_+(z,k)]W(z,k)^{-1}[I_m - M_-(z,k)]. \end{aligned} \tag{3.49}$$

Using (2.157), (3.9), (3.46), and (3.49), one computes

$$\begin{aligned}\Phi_{1,1}(z,k) &= I_m - 2[I_m - M_-(z,k)]^{-1}W(z,k)[I_m + M_+(z,k)]^{-1} \\ &= [I_m - M_-(z,k)]^{-1}\big[[I_m - M_-(z,k)][I_m + M_+(z,k)] - 2W(z,k)\big] \\ &\quad \times [I_m + M_+(z,k)]^{-1} \\ &= [I_m - M_-(z,k)]^{-1}[I_m + M_-(z,k)][I_m - M_+(z,k)][I_m + M_+(z,k)]^{-1} \\ &= \Phi_-(z,k)^{-1}\Phi_+(z,k), \quad z \in \mathbb{C}\backslash\partial\mathbb{D},\ k \in \mathbb{Z}.\end{aligned} \tag{3.50}$$

The result for k even is proved similarly. □

Next we introduce a sequence of $\mathbb{C}^{m\times 2m}$-valued Laurent polynomials $\{P(z,k,k_0)\}_{k\in\mathbb{Z}}$ by

$$\begin{aligned}P(z,k,k_0) &= \big(P_0(z,k,k_0), P_1(z,k,k_0)\big) \\ &= \begin{cases} \big(P_+(z,k,k_0), Q_+(z,k,k_0)\big)\begin{pmatrix} \frac{1}{2z}\rho_{k_0} & \frac{1}{2z}a^*_{k_0} \\ -\frac{1}{2z}\rho_{k_0} & \frac{1}{2z}b^*_{k_0}\end{pmatrix}, & k_0 \text{ odd}, \\ \big(P_+(z,k,k_0), Q_+(z,k,k_0)\big)\begin{pmatrix} \frac{1}{2}\widetilde{\rho}_{k_0} & \frac{1}{2}a_{k_0} \\ \frac{1}{2}\widetilde{\rho}_{k_0} & -\frac{1}{2}b_{k_0}\end{pmatrix}, & k_0 \text{ even}.\end{cases}\end{aligned} \tag{3.51}$$

Then it is easy to see that $P_j(z,\cdot,k_0)$, $j=0,1$ are linear combinations of $P_+(z,\cdot,k_0)$ and $Q_+(z,\cdot,k_0)$, and hence satisfy $\mathbb{U}P_j(z,\cdot,k_0) = zP_j(z,\cdot,k_0)$, $j=0,1$. Moreover, (2.51) and (3.51) imply that

$$\begin{aligned}P(z,k_0-1,k_0) &= (P_0(z,k_0-1,k_0), P_1(z,k_0-1,k_0)) = (I_m, 0), \\ P(z,k_0,k_0) &= (P_0(z,k_0,k_0), P_1(z,k_0,k_0)) = (0, I_m),\end{aligned} \tag{3.52}$$

and hence any solution $U(z,\cdot)$ of $\mathbb{U}U(z,\cdot) = zU(z,\cdot)$ can be expressed as

$$U(z,\cdot) = P_0(z,\cdot,k_0)U(z,k_0-1) + P_1(z,\cdot,k_0)U(z,k_0). \tag{3.53}$$

Our next goal is to show that the Laurent polynomials $\{P(z,k,k_0)^*\}_{k\in\mathbb{Z}}$ form complete orthonormal system in $L^2(\partial\mathbb{D}; d\Omega(\cdot,k_0))$. To do that we first prove an auxiliary result analogous to Lemma 2.7.

LEMMA 3.6. *Suppose $\{F(\cdot,k)\}_{k\in\mathbb{Z}}$ is a sequence of $\mathbb{C}^{m\times m}$-valued functions of bounded variation with $F(1,k)=0$ for all $k\in\mathbb{Z}$ that satisfies*

$$(\mathbb{U}F(\zeta,\cdot))(k) = \int_{A_\zeta} dF(\zeta',k)\,\zeta', \quad \zeta\in\partial\mathbb{D},\ k\in\mathbb{Z}, \tag{3.54}$$

where $\mathbb{U}$ are understood in the sense of difference expressions rather than difference operators on $\ell^2(\mathbb{Z})^m$. Then, $F(\cdot,k)$ also satisfies

$$\begin{aligned}F(\zeta,k) = \int_{A_\zeta} P_0(\zeta',k,k_0)\,dF(\zeta',k_0-1) + \int_{A_\zeta} P_1(\zeta',k,k_0)\,dF(\zeta',k_0), \\ \zeta\in\partial\mathbb{D},\ k,k_0\in\mathbb{Z}.\end{aligned} \tag{3.55}$$

Proof. Let $\{G(\cdot,k,k_0)\}_{k\in\mathbb{Z}}$ denote the sequence of $\mathbb{C}^{m\times m}$-valued functions,

$$G(\zeta,k,k_0)=\int_{A_\zeta} P_0(\zeta',k,k_0)\,dF(\zeta',k_0-1)+\int_{A_\zeta} P_1(\zeta',k,k_0)\,dF(\zeta',k_0), \\ \zeta\in\partial\mathbb{D},\ k,k_0\in\mathbb{Z}. \tag{3.56}$$

Then it suffices to prove that $F(\zeta,k)=G(\zeta,k,k_0)$, $\zeta\in\partial\mathbb{D}$, $k,k_0\in\mathbb{Z}$.

First, we note that (3.52) and (3.56) imply that

$$\begin{aligned} G(\zeta,k_0-1,k_0)&=\int_{A_\zeta} dF(\zeta',k_0-1)=F(\zeta,k_0-1),\\ G(\zeta,k_0,k_0)&=\int_{A_\zeta} dF(\zeta',k_0)=F(\zeta,k_0),\quad \zeta\in\partial\mathbb{D},\ k_0\in\mathbb{Z}, \end{aligned}\tag{3.57}$$

and

$$\begin{aligned} (\mathbb{U}G(\zeta,\cdot,k_0))(k)&=\int_{A_\zeta} (\mathbb{U}P_0(\zeta',\cdot,k_0))(k)\,dF(\zeta',k_0-1)\\ &\quad+\int_{A_\zeta} (\mathbb{U}P_1(\zeta',\cdot,k_0))(k)\,dF(\zeta',k_0)\\ &=\int_{A_\zeta} dG(\zeta',k,k_0)\,\zeta',\quad \zeta\in\partial\mathbb{D},\ k,k_0\in\mathbb{Z}. \end{aligned}\tag{3.58}$$

Next, defining $K(\zeta,k,k_0)=F(\zeta,k)-G(\zeta,k,k_0)$, $\zeta\in\partial\mathbb{D}$, $k,k_0\in\mathbb{Z}$, one obtains

$$\begin{aligned} &K(\zeta,k_0-1,k_0)=K(\zeta,k_0,k_0)=0,\\ &(\mathbb{U}K(\zeta,\cdot,k_0))(k)=\int_{A_\zeta} dK(\zeta',k,k_0)\,\zeta',\quad \zeta\in\partial\mathbb{D},\ k,k_0\in\mathbb{Z}, \end{aligned}$$

or equivalently,

$$\begin{aligned} &K(\zeta,k_0-1,k_0)=K(\zeta,k_0,k_0)=0,\\ &(\mathbb{U}K(\zeta,\cdot,k_0))(k)=(\mathbb{L}\,K(\cdot,k,k_0))(\zeta),\quad \zeta\in\partial\mathbb{D},\ k,k_0\in\mathbb{Z}, \end{aligned}\tag{3.59}$$

where $\mathbb{L}$ denotes the boundedly invertible operator on $\mathbb{C}^{m\times m}$-valued functions K of bounded variation defined by

$$(\mathbb{L}\,K)(\zeta)=\int_{A_\zeta} dK(\zeta')\,\zeta',\quad (\mathbb{L}^{-1}K)(\zeta)=\int_{A_\zeta} dK(\zeta')\,{\zeta'}^{-1}. \tag{3.60}$$

Finally, since, $\mathbb{L}$ commutes with all constant $m\times m$ matrices, one can repeat the proof of Lemma 2.3 with z replaced by $\mathbb{L}$ and using (3.53) obtain that (3.59) has the unique solution $K(\zeta,k,k_0)=0$, $\zeta\in\partial\mathbb{D}$, $k,k_0\in\mathbb{Z}$, and hence, $F(\zeta,k)=G(\zeta,k,k_0)$, $\zeta\in\partial\mathbb{D}$, $k,k_0\in\mathbb{Z}$. $\square$

LEMMA 3.7. *Let $k_0 \in \mathbb{Z}$. Then the set of $\mathbb{C}^{2m\times m}$-valued Laurent polynomials $\{P(\cdot,k,k_0)^*\}_{k\in\mathbb{Z}}$ forms a complete orthonormal system on $\partial\mathbb{D}$ with respect to $\mathbb{C}^{2m\times 2m}$-valued measure $d\Omega(\cdot,k_0)$. Explicitly, $P(\cdot,k,k_0)$, $k\in\mathbb{Z}$, satisfy,*

$$\oint_{\partial\mathbb{D}} P(\zeta,k,k_0)\,d\Omega(\zeta,k_0)\,P(\zeta,k',k_0)^* = \delta_{k,k'}I_m, \quad k,k'\in\mathbb{Z} \tag{3.61}$$

and the collection of $\mathbb{C}^{2m}$-valued Laurent polynomials

$$\left\{\begin{pmatrix} (P(\cdot,k,k_0))_{1,j} \\ \vdots \\ (P(\cdot,k,k_0))_{2m,j}\end{pmatrix}\right\}_{j=1,\dots,2m,\,k\in\mathbb{Z}} \tag{3.62}$$

form complete systems in $L^2(\partial\mathbb{D};d\Omega(\cdot,k_0))$.

Proof. Fix an integer $k'\in\mathbb{Z}$ and let $\{F(\cdot,k,k')\}_{k\in\mathbb{Z}}$ denote the $\mathbb{C}^{m\times m}$-valued sequences of functions of bounded variation defined by

$$F(\zeta,k,k') = \Delta_k^* E_{\mathbb{U}}(\zeta)\Delta_{k'}, \quad \zeta\in\partial\mathbb{D},\ k\in\mathbb{Z}. \tag{3.63}$$

Then,

$$\begin{aligned}(\mathbb{U}F(\zeta,\cdot,k'))(k) &= (\mathbb{U}E_{\mathbb{U}}(\zeta)\Delta_{k'})(k) = \left(\int_{A_\zeta} dE_{\mathbb{U}}(\zeta')\,\zeta'\Delta_{k'}\right)(k) \\ &= \int_{A_\zeta} d\big(\Delta_k^*E_{\mathbb{U}}(\zeta')\Delta_{k'}\big)\,\zeta' = \int_{A_\zeta} dF(\zeta',k,k')\,\zeta', \quad \zeta\in\partial\mathbb{D},\ k\in\mathbb{Z},\end{aligned} \tag{3.64}$$

and hence (3.55) in Lemma 3.6 implies that

$$\begin{aligned} dF(\zeta,k,k') = P_0(\zeta,k,k_0)\,dF(\zeta,k_0-1,k') + P_1(\zeta,k,k_0)\,dF(\zeta,k_0,k'),& \\ \zeta\in\partial\mathbb{D},\ k\in\mathbb{Z},&\end{aligned} \tag{3.65}$$

or equivalently,

$$\begin{aligned} dF(\zeta,k',k)=dF(\zeta,k',k)^*=dF(\zeta,k',k_0-1)P_0(\zeta,k,k_0)^*+dF(\zeta,k',k_0)P_1(\zeta,k,k_0)^*,& \\ \zeta\in\partial\mathbb{D},\ k\in\mathbb{Z}.&\end{aligned} \tag{3.66}$$

In particular, taking $k'=k_0-1$ and $k'=k_0$, one obtains from (3.66),

$$\begin{aligned} dF(\zeta,k_0-1,k) &= dF(\zeta,k_0-1,k_0-1)\,P_0(\zeta,k,k_0)^* \\ &\quad + dF(\zeta,k_0-1,k_0)\,P_1(\zeta,k,k_0)^*, \\ dF(\zeta,k_0,k) &= dF(\zeta,k_0,k_0-1)\,P_0(\zeta,k,k_0)^* + dF(\zeta,k_0,k_0)\,P_1(\zeta,k,k_0)^*, \\ &\qquad \zeta\in\partial\mathbb{D},\ k\in\mathbb{Z}.\end{aligned} \tag{3.67}$$

Next, setting $k=k'$ in (3.67) and plugging it into (3.65), one obtains

$$\begin{aligned} dF(\zeta,k,k') = \sum_{\ell,\ell'=0}^{1} P_\ell(\zeta,k,k_0)\,dF(\zeta,k_0-1+\ell,k_0-1+\ell')\,P_{\ell'}(\zeta,k',k_0)^*,& \\ \zeta\in\partial\mathbb{D},\ k,k'\in\mathbb{Z}.&\end{aligned} \tag{3.68}$$

Integrating (3.68) over the unit circle $\partial\mathbb{D}$ and observing that by (3.34) and (3.63) $dF(\zeta,k_0-1+\ell,k_0-1+\ell')=d\Omega_{\ell,\ell'}(z,k_0)$, $\ell,\ell'=0,1$, one obtains

$$\delta_{k,k'}I_m=\oint_{\partial\mathbb{D}}\sum_{\ell,\ell'=0}^{1}P_\ell(\zeta,k,k_0)\,d\Omega_{\ell,\ell'}(\zeta,k_0)\,P_{\ell'}(\zeta,k',k_0)^*,\quad \zeta\in\partial\mathbb{D},\ k,k'\in\mathbb{Z}, \tag{3.69}$$

which is equivalent to (3.61).

To prove completeness of $\{P(\cdot,k,k_0)^*\}_{k\in\mathbb{Z}}$ we first note the fact,

$$\begin{aligned}\operatorname{span}\{P(z,k,k_0)^*\}_{k\in\mathbb{Z}}&=\operatorname{span}\left\{\begin{pmatrix}z^kI_m\\ z^{k-1}I_m\end{pmatrix},\begin{pmatrix}z^{k-1}I_m\\ z^kI_m\end{pmatrix},\begin{pmatrix}I_m\\ 0\end{pmatrix},\begin{pmatrix}0\\ I_m\end{pmatrix}\right\}_{k\in\mathbb{Z}}\\ &=\operatorname{span}\left\{\begin{pmatrix}z^kI_m\\ 0\end{pmatrix},\begin{pmatrix}0\\ z^kI_m\end{pmatrix}\right\}_{k\in\mathbb{Z}}.\end{aligned} \tag{3.70}$$

This is a consequence of investigating the leading-order coefficients of $P_+(z,k,k_0)$ and $Q_+(z,k,k_0)$ (cf. Remark 2.5) and (3.51)). Thus, it suffices to prove that $\left\{\begin{pmatrix}\zeta^kI_m\\ 0\end{pmatrix},\begin{pmatrix}0\\ \zeta^kI_m\end{pmatrix}\right\}_{k\in\mathbb{Z}}$ is a complete system with respect to $d\Omega(\cdot,k_0)$.

Let $F=\begin{pmatrix}F_0\\ F_1\end{pmatrix}\in L^2(\partial\mathbb{D};d\Omega(\cdot,k_0))$ and suppose F is orthogonal to all columns of $\begin{pmatrix}\zeta^kI_m\\ 0\end{pmatrix}$ and $\begin{pmatrix}0\\ \zeta^kI_m\end{pmatrix}$ for all $k\in\mathbb{Z}$, that is,

$$\begin{aligned}\oint_{\partial\mathbb{D}}\begin{pmatrix}\zeta^kI_m\\ 0\end{pmatrix}^*d\Omega(\zeta,k_0)\,F(\zeta)&=\oint_{\partial\mathbb{D}}\zeta^{-k}\,[d\Omega_{0,0}(\zeta,k_0)\,F_0(\zeta)+d\Omega_{0,1}(\zeta,k_0)\,F_1(\zeta)]\\ &=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\in\mathbb{C}^{2m}\end{aligned} \tag{3.71}$$

and

$$\begin{aligned}\oint_{\partial\mathbb{D}}\begin{pmatrix}0\\ \zeta^kI_m\end{pmatrix}^*d\Omega(\zeta,k_0)\,F(\zeta)&=\oint_{\partial\mathbb{D}}\zeta^{-k}\,[d\Omega_{1,0}(\zeta,k_0)\,F_0(\zeta)+d\Omega_{1,1}(\zeta,k_0)\,F_1(\zeta)]\\ &=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\in\mathbb{C}^{2m}\end{aligned} \tag{3.72}$$

for all $k\in\mathbb{Z}$. Note that for a scalar complex-valued measure $d\omega$ equalities $\oint d\omega(\zeta)\,\zeta^n=0$, $n\in\mathbb{Z}$, imply $\oint d\mathrm{Re}(\omega(\zeta))\,\zeta^n=\oint d\mathrm{Im}(\omega(\zeta))\,\zeta^n=0$, and hence [36, p. 24]) implies that $d\omega=0$. Applying this argument to $d(\Omega_{0,0}F_0+\Omega_{0,1}F_1)_\ell$ and $d(\Omega_{1,0}F_0+\Omega_{1,1}F_1)_\ell$, $\ell=1,\dots,2m$, one obtains

$$d\Omega_{0,0}F_0+d\Omega_{0,1}F_1=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\in\mathbb{C}^{2m}, \tag{3.73}$$

$$d\Omega_{1,0}F_0+d\Omega_{1,1}F_1=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\in\mathbb{C}^{2m}. \tag{3.74}$$

Multiplying (3.73) by F_0^* on the left and (3.74) by F_1^* on the left and adding the results then yields

$$\|F\|^2_{L^2(\partial\mathbb{D};d\Omega(\cdot,k_0))} = \oint_{\partial\mathbb{D}} F(\zeta)^*\, d\Omega(\zeta,k_0)\, F(\zeta) = 0. \tag{3.75}$$

□

COROLLARY 3.8. *The full-lattice CMV operator* $\mathbb{U}$ *is unitarily equivalent to the operator of multiplication by* ζ *on* $L^2(\partial\mathbb{D};d\Omega(\cdot,k_0))$ *for any* $k_0\in\mathbb{Z}$. *In particular,*

$$\sigma(\mathbb{U}) = \operatorname{supp}(d\Omega(\cdot,k_0)), \quad k_0\in\mathbb{Z}. \tag{3.76}$$

Proof. Consider the linear map $\dot{\mathscr{U}}\colon \ell^2_0(\mathbb{Z})^m \to L^2(\partial\mathbb{D};d\Omega(\cdot,k_0))$ from the space of compactly supported sequences $\ell^2_0(\mathbb{Z})^m$ to the set of $\mathbb{C}^{2m}$-valued Laurent polynomials defined by

$$(\dot{\mathscr{U}}F)(z) = \sum_{k=-\infty}^{\infty} P(1/\overline{z},k,k_0)^* F(k), \quad F\in\ell^2_0(\mathbb{Z})^m. \tag{3.77}$$

Using (3.61) one shows that $\widehat{F}(\zeta) = (\dot{\mathscr{U}}F)(\zeta)$, $F\in\ell^2_0(\mathbb{Z})^m$ has the property

$$\begin{aligned}
\|\widehat{F}\|^2_{L^2(\partial\mathbb{D};d\Omega(\cdot,k_0))} &= \oint_{\partial\mathbb{D}} \widehat{F}(\zeta)^* d\Omega(\zeta,k_0)\, \widehat{F}(\zeta) && (3.78)\\
&= \oint_{\partial\mathbb{D}} \sum_{k=-\infty}^{\infty} F(k)^* P(\zeta,k,k_0)\, d\Omega_\pm(\zeta,k_0) \sum_{k'=-\infty}^{\infty} P_\pm(\zeta,k',k_0)^* F(k') \\
&= \sum_{k,k'=-\infty}^{\infty} F(k)^* \left(\oint_{\partial\mathbb{D}} P(\zeta,k,k_0)\, d\Omega(\zeta,k_0)\, P(\zeta,k',k_0)^*\right) F(k') \\
&= \sum_{k=-\infty}^{\infty} F(k)^* F(k) = \|F\|^2_{\ell^2(\mathbb{Z})^m}. && (3.79)
\end{aligned}$$

Since $\ell^2_0(\mathbb{Z})^m$ is dense in $\ell^2(\mathbb{Z})^m$, $\dot{\mathscr{U}}$ extends by continuity to a bounded linear operator $\mathscr{U}\colon \ell^2(\mathbb{Z})^m \to L^2(\partial\mathbb{D};d\Omega(\cdot,k_0))$, and the identity

$$\begin{aligned}
(\mathscr{U}(\mathbb{U}F))(\zeta) &= \sum_{k=-\infty}^{\infty} P(\zeta,k,k_0)^*(\mathbb{U}F)(k) = \sum_{k=-\infty}^{\infty} (\mathbb{U}^* P(\zeta,\cdot,k_0))(k)^* F(k) \\
&= \sum_{k=-\infty}^{\infty} (\zeta^{-1}P(\zeta,k,k_0))^* F(k) = \zeta(\mathscr{U}F)(\zeta), \quad F\in\ell^2(\mathbb{Z})^m,
\end{aligned} \tag{3.80}$$

holds. The range of the operator $\mathscr{U}$ is all of $L^2(\partial\mathbb{D};d\Omega(\cdot,k_0))$ since the $\mathbb{C}^{2m\times m}$-valued Laurent polynomials $\{P(\cdot,k,k_0)^*\}_{k\in\mathbb{Z}}$ are complete with respect to $d\Omega(\cdot,k_0)$. Hence the inverse operator $\mathscr{U}^{-1}$ exists on $L^2(\partial\mathbb{D};d\Omega(\cdot,k_0))$ and is given by

$$(\mathscr{U}^{-1}\widehat{F})(k) = \oint_{\partial\mathbb{D}} P(\zeta,k,k_0)\, d\Omega(\zeta,k_0)\, \widehat{F}(\zeta), \quad \widehat{F}\in L^2(\partial\mathbb{D};d\Omega(\cdot,k_0)), \tag{3.81}$$

which together with (3.79) implies that $\mathscr{U}$ is unitary. In addition, (3.80) shows that the full-lattice unitary operator $\mathbb{U}$ on $\ell^2(\mathbb{Z})^m$ is unitarily equivalent to the operators of multiplication by ζ on $L^2(\partial\mathbb{D}; d\Omega(\cdot,k_0))$,

$$(\mathscr{U}\,\mathbb{U}\,\mathscr{U}^{-1}\widehat{F})(\zeta) = \zeta\widehat{F}(\zeta), \quad \widehat{F} \in L^2(\partial\mathbb{D}; d\Omega(\cdot,k_0)). \tag{3.82}$$

□

4. Borg–Marchenko-type Uniqueness Results for CMV Operators with Matrix-valued Verblunsky Coefficients

In this section we prove (local and global) Borg–Marchenko-type uniqueness results for CMV operators with matrix-valued Verblunsky coefficients on half-lattices and on the full lattice $\mathbb{Z}$. We freely use the notation established in Sections 2, 3, and Appendix A.

We start with uniqueness results on half-lattices.

THEOREM 4.1. *Assume Hypothesis 2.2 and let $k_0 \in \mathbb{Z}$, $N \in \mathbb{N}$. Then for the right half-lattice problem the following sets of data $(i)-(v)$ are equivalent:*

$$(i)\ \big\{\alpha_{k_0+k}\big\}_{k=1}^{N}. \tag{4.1}$$

$$(ii)\ \bigg\{\oint_{\partial\mathbb{D}} d\Omega_+(\zeta,k_0)\,\zeta^k\bigg\}_{k=1}^{N}. \tag{4.2}$$

(iii) $\big\{m_{+,k}(k_0)\big\}_{k=1}^{N}$, *where $m_{+,k}(k_0)$, $k \geqslant 0$, are the Taylor coefficients of $m_+(z,k_0)$ at $z=0$, that is,*

$$m_+(z,k_0) = \sum\nolimits_{k=0}^{\infty} m_{+,k}(k_0) z^k,\ z\in\mathbb{D}. \tag{4.3}$$

(iv) $\big\{M_{+,k}(k_0)\big\}_{k=1}^{N}$, *where $M_{+,k}(k_0)$, $k \geqslant 0$, are the Taylor coefficients of $M_+(z,k_0)$ at $z=0$, that is,*

$$M_+(z,k_0) = \sum\nolimits_{k=0}^{\infty} M_{+,k}(k_0) z^k,\ z\in\mathbb{D}. \tag{4.4}$$

(v) $\big\{\phi_{+,k}(k_0)\big\}_{k=1}^{N}$, *where $\phi_{+,k}(k_0)$, $k \geqslant 0$, are the Taylor coefficients of $\Phi_+(z,k_0)$ at $z=0$, that is,*

$$\Phi_+(z,k_0) = \sum\nolimits_{k=0}^{\infty} \phi_{+,k}(k_0) z^k,\ z\in\mathbb{D}. \tag{4.5}$$

Similarly, for the left half-lattice problem, the following sets of data $(vi)-(x)$ are equivalent:

$$(vi)\ \big\{\alpha_{k_0-k}\big\}_{k=0}^{N-1}. \tag{4.6}$$

$$(vii)\ \bigg\{\oint_{\partial\mathbb{D}} d\Omega_-(\zeta,k_0)\,\zeta^k\bigg\}_{k=1}^{N}. \tag{4.7}$$

(viii) $\big\{m_{-,k}(k_0)\big\}_{k=1}^{N}$, *where $m_{-,k}(k_0)$, $k \geqslant 0$, are the Taylor coefficients of $m_-(z,k_0)$ at $z=0$, that is,*

$$m_-(z,k_0) = \sum\nolimits_{k=0}^{\infty} m_{-,k}(k_0) z^k. \tag{4.8}$$

(ix) $\big\{M_{-,k}(k_0)\big\}_{k=0}^{N-1}$, *where $M_{-,k}(k_0)$, $k \geqslant 0$, are the Taylor coefficients of $M_-(z,k_0)$ at $z=0$, that is,*

$$M_-(z,k_0) = \sum\nolimits_{k=0}^{\infty} M_{-,k}(k_0) z^k. \tag{4.9}$$

(x) $\{\phi_{-,k}(k_0)\}_{k=0}^{N-1}$, *where* $\phi_{-,k}(k_0)$, $k \geqslant 0$, *are the Taylor coefficients of* $\Phi_-(z,k_0)^{-1}$ *at* $z=0$, *that is,* $\Phi_-(z,k_0)^{-1} = \sum_{k=0}^{\infty} \phi_{-,k}(k_0) z^k$. (4.10)

Proof. $(i) \Rightarrow (ii)$ and $(vi) \Rightarrow (vii)$: First, utilizing relations (2.37) and (2.40) with the initial conditions (2.49) and (2.50), one constructs $\{P_\pm(z,k_0 \pm k,k_0)\}_{k=1}^N$ and $\{R_\pm(z,k_0 \pm k,k_0)\}_{k=1}^N$. We note that the Laurent polynomials

$$\begin{cases} z^{-1}P_+(z,k_0+k,k_0),\ R_-(z,k_0-k,k_0), & k_0 \text{ odd},\\ R_+(z,k_0+k,k_0),\ z^{-1}P_-(z,k_0-k,k_0), & k_0 \text{ even}, \end{cases} \tag{4.11}$$

are linear combinations of

$$\begin{cases} I_m, z^{-1}I_m, zI_m, z^{-2}I_m, z^2I_m, \dots, z^{(k-1)/2}I_m, z^{-(k+1)/2}I_m, & k \text{ odd},\\ I_m, z^{-1}I_m, zI_m, z^{-2}I_m, z^2I_m, \dots, z^{-k/2}I_m, z^{k/2}I_m, & k \text{ even}, \end{cases} \tag{4.12}$$

and

$$\begin{cases} R_+(z,k_0+k,k_0),\ P_-(z,k_0-k,k_0), & k_0 \text{ odd},\\ P_+(z,k_0+k,k_0),\ R_-(z,k_0-k,k_0), & k_0 \text{ even}, \end{cases} \tag{4.13}$$

are linear combinations of

$$\begin{cases} I_m, zI_m, z^{-1}I_m, z^2I_m, z^{-2}I_m, \dots, z^{-(k-1)/2}I_m, z^{(k+1)/2}I_m, & k \text{ odd},\\ I_m, zI_m, z^{-1}I_m, z^2I_m, z^{-2}I_m, \dots, z^{k/2}I_m, z^{-k/2}I_m, & k \text{ even}. \end{cases} \tag{4.14}$$

Moreover, the last elements of the sequences in (4.12) and (4.14) represent the leading-order terms of the Laurent polynomials in (4.11) and (4.13), respectively, and the corresponding leading-order coefficients are invertible $m \times m$ matrices (cf. Remark 2.5).

Next, assume k_0 and k to be odd. Then utilizing (4.13) and (4.14) one finds $m \times m$ matrices $C_{\pm,j}$ and $D_{\pm,j}$, $0 \leqslant j \leqslant k$, such that

$$\begin{aligned} z^{-(k-1)/2}I_m &= \sum_{j=0}^{k} C_{+,j}\, R_+(z,k_0+j,k_0),\\ z^{(k+1)/2}I_m &= \sum_{j=0}^{k} D_{+,j}\, R_+(z,k_0+j,k_0), \end{aligned} \tag{4.15}$$

$$\begin{aligned} z^{-(k-1)/2}I_m &= \sum_{j=0}^{k} C_{-,j}\, P_-(z,k_0-j,k_0),\\ z^{(k+1)/2}I_m &= \sum_{j=0}^{k} D_{-,j}\, P_-(z,k_0-j,k_0), \end{aligned} \tag{4.16}$$

and, using (2.73) and (2.74), computes

$$\oint_{\partial\mathbb{D}} d\Omega_\pm(\zeta,k_0)\,\zeta^k = \oint_{\partial\mathbb{D}} \big(\zeta^{(k+1)/2}I_m\big)\, d\Omega_\pm(\zeta,k_0)\, \big(\zeta^{-(k-1)/2}I_m\big)^* = \sum_{j=0}^{k} D_{\pm,j}\, C_{\pm,j}^*. \tag{4.17}$$

The other cases of k_0 and k follow similarly.

$(ii) \Rightarrow (i)$ and $(vii) \Rightarrow (vi)$: Since $d\Omega_\pm(\cdot,k_0)$ are nonnegative normalized measures, one has

$$\oint_{\partial\mathbb{D}} d\Omega_\pm(\zeta,k_0)\,\zeta^{-k} = \bigg(\oint_{\partial\mathbb{D}} d\Omega_\pm(\zeta,k_0)\,\zeta^{k}\bigg)^* \quad \text{and} \quad \oint_{\partial\mathbb{D}} d\Omega_\pm(\zeta,k_0) = I_m, \tag{4.18}$$

that is, the knowledge of positive moments imply the knowledge of negative ones. Applying Corollary 2.10 one constructs the matrix-valued orthonormal Laurent polynomials $\{P_\pm(\zeta,k_0\pm k,k_0)\}_{k=1}^N$ and $\{R_\pm(\zeta,k_0\pm k,k_0)\}_{k=1}^N$. Subsequently applying Theorem 2.11, in particular, formulas (2.105) and (2.106), one obtains the coefficients (i) and (vi).

$(ii) \Leftrightarrow (iii)$ and $(vii) \Leftrightarrow (viii)$: These follow from (2.134) and (4.18),

$$m_\pm(z,k_0) = \pm\oint_{\partial\mathbb{D}} d\Omega_\pm(\zeta,k_0)\,\frac{\zeta+z}{\zeta-z} = \pm I_m \pm 2\sum_{k=1}^\infty z^k \bigg(\oint_{\partial\mathbb{D}} d\Omega_\pm(\zeta,k_0)\,\zeta^k\bigg)^*, \quad z\in\mathbb{D}. \tag{4.19}$$

$(iii) \Leftrightarrow (iv)$: This is implied by (2.151).

$(iv) \Leftrightarrow (v)$: This is a consequence of (2.157) and (2.159), together with the facts: For $|z|$ sufficiently small, $\|M_+(z,k_0)-I_m\|_{\mathbb{C}^{m\times m}} < 1$ by (2.152), and $\|\Phi_+(z,k_0)\|_{\mathbb{C}^{m\times m}} < 1$ by (2.158). Hence,

$$\begin{aligned} M_+(z,k_0) &= [I_m-\Phi_+(z,k_0)]^{-1}[I_m+\Phi_+(z,k_0)] \\ &\underset{z\to 0}{=} [I_m+\Phi_+(z,k_0)]\sum_{k=0}^\infty \Phi_+(z,k_0)^k, \end{aligned} \tag{4.20}$$

$$\begin{aligned} \Phi_+(z,k_0) &= \big[2^{-1}[M_+(z,k_0)-I_m]\big]\big[I_m+2^{-1}[M_+(z,k_0)-I_m]\big]^{-1} \\ &\underset{z\to 0}{=} -\sum_{k=1}^\infty 2^{-k}[I_m-M_+(z,k_0)]^k. \end{aligned} \tag{4.21}$$

$(ix) \Leftrightarrow (x)$: This is implied by (2.155), (2.157), (2.159), and the fact that, for $|z|$ sufficiently small, $\|\Phi_-(z,k_0)^{-1}\|_{\mathbb{C}^{m\times m}} < 1$ by (2.7) and (2.158). Hence,

$$\begin{aligned} M_-(z,k_0) &= [\Phi_-(z,k_0)^{-1}-I_m]^{-1}[\Phi_-(z,k_0)^{-1}+I_m] \\ &\underset{z\to 0}{=} -[\Phi_-(z,k_0)^{-1}+I_m]\sum_{k=0}^\infty \Phi_-(z,k_0)^{-k}, \end{aligned} \tag{4.22}$$

$$\begin{aligned} \Phi_-(z,k_0)^{-1} &= [M_-(z,k_0)+I_m][M_-(z,k_0)-M_-(0,k_0)+M_-(0,k_0)-I_m]^{-1} \\ &= \big[[M_-(z,k_0)+I_m][M_-(0,k_0)-I_m]^{-1}\big] \\ &\quad\times \big[[M_-(z,k_0)-M_-(0,k_0)][M_-(0,k_0)-I_m]^{-1}+I_m\big]^{-1} \\ &\underset{z\to 0}{=} \big[[M_-(z,k_0)+I_m][M_-(0,k_0)-I_m]^{-1}\big] \\ &\quad\times \sum_{k=0}^\infty \big[[M_-(z,k_0)-M_-(0,k_0)][I_m-M_-(0,k_0)]^{-1}\big]^k. \end{aligned} \tag{4.23}$$

$(viii) \Leftrightarrow (x)$: This follows because (2.135), (2.160), and the fact that $\|\Phi_-(z,k_0)^{-1}\|_{\mathbb{C}^{m\times m}} \leqslant 1$, $z\in\mathbb{D}$, together imply that

$$\begin{aligned} m_-(z,k_0) &= [z\Phi_-(z,k_0)^{-1}+I_m]^{-1}[z\Phi_-(z,k_0)^{-1}-I_m] \\ &\underset{z\to 0}{=} [z\Phi_-(z,k_0)^{-1}-I_m]\sum_{k=0}^{\infty}\big[-z\Phi_-(z,k_0)^{-1}\big]^k, \end{aligned} \tag{4.24}$$

$$\begin{aligned} z\Phi_-(z,k_0)^{-1} &= [I_m+m_-(z,k_0)][I_m-m_-(z,k_0)]^{-1} \\ &= 2^{-1}[I_m+m_-(z,k_0)]\big[I_m-2^{-1}[I_m+m_-(z,k_0)]\big]^{-1} \\ &\underset{z\to 0}{=} \sum_{k=1}^{\infty} 2^{-k}[I_m+m_-(z,k_0)]^k. \quad \square \end{aligned} \tag{4.25}$$

$\square$

Next, we restate Theorem 4.1:

THEOREM 4.2. *Assume Hypothesis 2.2 for two sequences* $\alpha^{(1)}$, $\alpha^{(2)}$ *and let* $k_0\in\mathbb{Z}$, $N\in\mathbb{N}$. *Then for the right half-lattice problems associated with* $\alpha^{(1)}$ *and* $\alpha^{(2)}$ *the following items* (i)–(iv) *are equivalent:*

$$(i) \quad \alpha_k^{(1)} = \alpha_k^{(2)}, \quad k_0+1 \leqslant k \leqslant k_0+N. \tag{4.26}$$

$$(ii) \quad m_+^{(1)}(z,k_0) - m_+^{(2)}(z,k_0) \underset{z\to 0}{=} o(z^N). \tag{4.27}$$

$$(iii) \quad M_+^{(1)}(z,k_0) - M_+^{(2)}(z,k_0) \underset{z\to 0}{=} o(z^N). \tag{4.28}$$

$$(iv) \quad \Phi_+^{(1)}(z,k_0) - \Phi_+^{(2)}(z,k_0) \underset{z\to 0}{=} o(z^N). \tag{4.29}$$

Similarly, for the left half-lattice problems associated with $\alpha^{(1)}$ *and* $\alpha^{(2)}$, *the following items* (v)–$(viii)$ *are equivalent:*

$$(v) \quad \alpha_k^{(1)} = \alpha_k^{(2)}, \quad k_0-N+1 \leqslant k \leqslant k_0. \tag{4.30}$$

$$(vi) \quad m_-^{(1)}(z,k_0) - m_-^{(2)}(z,k_0) \underset{z\to 0}{=} o(z^N). \tag{4.31}$$

$$(vii) \quad M_-^{(1)}(z,k_0) - M_-^{(2)}(z,k_0) \underset{z\to 0}{=} o(z^{N-1}). \tag{4.32}$$

$$(viii) \quad \Phi_-^{(1)}(z,k_0)^{-1} - \Phi_-^{(2)}(z,k_0)^{-1} \underset{z\to 0}{=} o(z^{N-1}). \tag{4.33}$$

Proof. This is an immediate consequence of Theorem 4.1. $\square$

Finally, we turn to CMV operators on $\mathbb{Z}$ and start with two auxiliary results that play a role in the proofs of analogous Borg–Marchenko-type uniqueness results for CMV operators on $\mathbb{Z}$.

LEMMA 4.3. *Let A, B, C, D denote some $m \times m$ matrices. Suppose that $A \neq 0$, B is invertible, and A, B, C, D satisfy*

$$\big[2\sqrt{\|A\|\,\|D\|} + \|C\|\,\big]\,\|B^{-1}\| < 1. \tag{4.34}$$

Then the matrix-valued Riccati-type equation

$$XAX + BX + XC + D = 0, \quad \|X\| < \frac{1 - \|C\|\,\|B^{-1}\|}{2\,\|A\|\,\|B^{-1}\|}, \tag{4.35}$$

has a unique solution $X \in \mathbb{C}^{m\times m}$ given by

$$X = \lim_{n\to\infty} X_n \ \text{ with } \ \|X\| \leqslant \frac{1 - \|C\|\,\|B^{-1}\|}{2\|A\|\,\|B^{-1}\|} - \sqrt{\left(\frac{1 - \|C\|\,\|B^{-1}\|}{2\|A\|\,\|B^{-1}\|}\right)^2 - \frac{\|D\|}{\|A\|}}, \tag{4.36}$$

where

$$X_0 = 0, \quad X_n = F(X_{n-1}),\ n \in \mathbb{N}, \ \text{ and } F(X) = -B^{-1}XAX - B^{-1}XC - B^{-1}D. \tag{4.37}$$

A similar result also holds if $A \neq 0$, C is invertible, and A, B, C, D satisfying

$$\big[2\sqrt{\|A\|\,\|D\|} + \|B\|\,\big]\,\|C^{-1}\| < 1. \tag{4.38}$$

In this case, the matrix-valued Riccati-type equation

$$XAX + BX + XC + D = 0, \quad \|X\| < \frac{1 - \|B\|\,\|C^{-1}\|}{2\,\|A\|\,\|C^{-1}\|}, \tag{4.39}$$

has a unique solution $X \in \mathbb{C}^{m\times m}$ given by

$$X = \lim_{n\to\infty} X_n \ \text{ with } \ \|X\| \leqslant \frac{1 - \|B\|\,\|C^{-1}\|}{2\|A\|\,\|C^{-1}\|} - \sqrt{\left(\frac{1 - \|B\|\,\|C^{-1}\|}{2\|A\|\,\|C^{-1}\|}\right)^2 - \frac{\|D\|}{\|A\|}}, \tag{4.40}$$

where

$$X_0 = 0, \quad X_n = G(X_{n-1}),\ n \in \mathbb{N}, \ \text{ and } G(X) = -XAXC^{-1} - BXC^{-1} - DC^{-1}. \tag{4.41}$$

Proof. Since B is invertible, the equation for X in (4.35) is equivalent to $F(X) = X$. Therefore, it suffices to show that $F(\cdot)$ is a strict contraction on some closed ball of radius λ centered at the origin, $B_\lambda = \{X \in \mathbb{C}^{m\times m} \mid \|X\| \leqslant \lambda\}$, and that $F(\cdot)$ preserves B_λ, that is $\|F(X)\| \leqslant \lambda$ whenever $\|X\| \leqslant \lambda$.

First, we check that for any $\lambda < \frac{1-\|C\|\|B^{-1}\|}{2\|A\|\|B^{-1}\|}$, the map $F(\cdot)$ is a strict contraction on B_λ. Let $X, Y \in B_\lambda$, then

$$\begin{aligned} \|F(X) - F(Y)\| &\leqslant \big[\,\|A\|\,\|B^{-1}\|\,\|X\| + \|A\|\,\|B^{-1}\|\,\|Y\| + \|C\|\,\|B^{-1}\|\big]\,\|X - Y\| \\ &\leqslant \big[2\lambda\,\|A\|\,\|B^{-1}\| + \|C\|\,\|B^{-1}\|\big]\,\|X - Y\|, \\ &\quad 2\lambda\,\|A\|\,\|B^{-1}\| + \|C\|\,\|B^{-1}\| < 1. \end{aligned} \tag{4.42}$$

Next, we check that $F(\cdot)$ preserves B_λ for any λ satisfying

$$\frac{1-\|C\|\,\|B^{-1}\|}{2\|A\|\,\|B^{-1}\|}-\sqrt{\left(\frac{1-\|C\|\,\|B^{-1}\|}{2\|A\|\,\|B^{-1}\|}\right)^2-\frac{\|D\|}{\|A\|}}\leqslant\lambda<\frac{1-\|C\|\,\|B^{-1}\|}{2\|A\|\,\|B^{-1}\|}. \tag{4.43}$$

Let $X\in B_\lambda$, then by (4.43)

$$\|F(X)\|\leqslant\|A\|\,\|B^{-1}\|\lambda^2+\|C\|\,\|B^{-1}\|\lambda+\|D\|\,\|B^{-1}\|\leqslant\lambda. \tag{4.44}$$

Thus, Banach's contraction mapping principle implies that $F(\cdot)$ has a unique fixed point X for which (4.35) and (4.36) hold.

The second part of the Lemma is proved similarly. □

COROLLARY 4.4. *Let A_j, B_j, C_j, D_j, $j=1,2$, denote some $m\times m$ matrices. Suppose that either B_1 and B_2 are invertible and*

$$0<\|A_j\|\,,\,\|B_j^{-1}\|\leqslant a,\quad \|C_j\|\,,\,\|D_j\|\leqslant b,\quad j=1,2, \tag{4.45}$$

or C_1 and C_2 are invertible and

$$0<\|A_j\|\,,\,\|C_j^{-1}\|\leqslant a,\quad \|B_j\|\,,\,\|D_j\|\leqslant b,\quad j=1,2, \tag{4.46}$$

for some $a,b>0$ satisfying $2ab(1+2a^2)\leqslant 1$. Then there exist unique solutions X_j, $j=1,2$, of the matrix-valued Riccati-type equations

$$X_jA_jX_j+B_jX_j+X_jC_j+D_j=0,\quad \|X_j\|<\frac{1-ab}{2a^2},\quad j=1,2, \tag{4.47}$$

and the following estimate holds

$$\|X_1-X_2\|\leqslant\lambda(a,b)\big[\,\|A_1-A_2\|+\|B_1-B_2\|+\|C_1-C_2\|+\|D_1-D_2\|\,\big], \tag{4.48}$$

where $\lambda(a,b)$ is given by

$$\lambda(a,b)=\frac{\max\left\{a,\,\frac{2a^2b}{1-ab},\,a^2b+\frac{2a^3b^2}{1-ab}+\frac{4a^5b^2}{(1-ab)^2},\,\frac{4a^3b^2}{(1-ab)^2}\right\}}{(1-ab)-\frac{4a^3b}{1-ab}}>0. \tag{4.49}$$

Proof. Suppose B_j, $j=1,2$, are invertible and note that $b\leqslant 1/(2a(1+2a^2))$ implies

$$\begin{aligned}\left(2\sqrt{\|A_j\|\,\|D_j\|}+\|C_j\|\right)\|B_j^{-1}\|&\leqslant(2\sqrt{ab}+b)a\\ &\leqslant\frac{2a\sqrt{2+4a^2}+1}{2(1+2a^2)}<\frac{2a(2a+\frac{1}{2a})+1}{2(1+2a^2)}=1\end{aligned} \tag{4.50}$$

and

$$\begin{aligned}
&\frac{1-\|C_j\|\,\|B_j^{-1}\|}{2\|A_j\|\,\|B_j^{-1}\|}-\sqrt{\left(\frac{1-\|C_j\|\,\|B_j^{-1}\|}{2\|A_j\|\,\|B_j^{-1}\|}\right)^2-\frac{\|D_j\|}{\|A_j\|}}\\
&\qquad\leqslant \frac{2ab}{1-ab}<\frac{1-ab}{2a^2}\leqslant\frac{1-\|C_j\|\,\|B_j^{-1}\|}{2\,\|A_j\|\,\|B_j^{-1}\|}. \qquad (4.51)
\end{aligned}$$

Then Lemma 4.3 implies that the matrix-valued Riccati-type equations in (4.47) have unique solutions X_j satisfying $\|X_j\| \leqslant \frac{2ab}{1-ab}$, $j=1,2$ and $X_j=F_j(X_j)$, where $F_j(X)=-B_j{}^{-1}XA_jX-B_j^{-1}XC_j-B_j^{-1}D_j$, $j=1,2$. Hence, one computes

$$\begin{aligned}
\|X_1-X_2\| &= \|F_1(X_1)-F_2(X_2)\|\\
&\leqslant \big\|B_1^{-1}X_1A_1X_1-B_2^{-1}X_2A_2X_2\big\|+\big\|B_1^{-1}X_1C_1-B_2^{-1}X_2C_2\big\|\\
&\quad+\big\|B_1^{-1}D_1-B_2^{-1}D_2\big\|\\
&\leqslant \big[\,\|A_1\|\,\|B_2^{-1}\|\,\|X_1\|+\|A_2\|\,\|B_2^{-1}\|\,\|X_2\|+\|B_2^{-1}\|\,\|C_1\|\,\big]\,\|X_1-X_2\|\\
&\quad+\|B_2^{-1}\|\,\|X_1\|\,\|X_2\|\,\|A_1-A_2\|+\|B_2^{-1}\|\,\|X_2\|\,\|C_1-C_2\|\\
&\quad+\|B_2^{-1}\|\,\|D_1-D_2\|\\
&\quad+\big[\,\|A_1\|\,\|X_1\|^2+\|C_1\|\,\|X_1\|+\|D_1\|\,\big]\,\|B_1^{-1}\|\,\|B_2^{-1}\|\,\|B_1-B_2\|\\
&\leqslant \left(\frac{4a^3b}{1-ab}+ab\right)\|X_1-X_2\|+\left(\frac{4a^5b^2}{(1-ab)^2}+\frac{2a^3b^2}{1-ab}+a^2b\right)\|B_1-B_2\|\\
&\quad+\frac{4a^3b^2}{(1-ab)^2}\,\|A_1-A_2\|+\frac{2a^2b}{1-ab}\,\|C_1-C_2\|+a\,\|D_1-D_2\|. \qquad (4.52)
\end{aligned}$$

Finally, utilizing $b\leqslant 1/(2a(1+2a^2))$, one verifies that

$$\begin{aligned}
1-\left(\frac{4a^3b}{1-ab}+ab\right)&=1-\frac{4a^3b+ab(1-ab)}{1-ab}\\
&>1-\frac{ab(1+4a^2)}{1-ab}\geqslant 1-\frac{1+4a^2}{2(1+2a^2)-1}=0, \qquad (4.53)
\end{aligned}$$

and hence (4.48) and (4.49) follow from (4.52), and (4.53).

The case of C_j being invertible, $j=1,2$, is proved analogously. □

Given these preliminaries, we introduce the following notation for the diagonal and for the neighboring off-diagonal entries of the Green's matrix of $\mathbb{U}$ (i.e., the discrete integral kernel of $(\mathbb{U}-zI)^{-1}$),

$$g(z,k)=(\mathbb{U}-Iz)^{-1}(k,k), \qquad (4.54)$$

$$h(z,k)=\begin{cases}(\mathbb{U}-Iz)^{-1}(k-1,k), & k \text{ odd},\\ (\mathbb{U}-Iz)^{-1}(k,k-1), & k \text{ even},\end{cases} \qquad k\in\mathbb{Z},\ z\in\mathbb{D}. \qquad (4.55)$$

Then the subsequent uniqueness results hold for the full-lattice CMV operator $\mathbb{U}$:

THEOREM 4.5. *Assume Hypothesis 2.2 and let* $k_0 \in \mathbb{Z}$. *Then any of the following two sets of data*

(*i*) $g(z,k_0)$ *and* $h(z,k_0)$ *for all* z *in some open* (*nonempty*) *neighborhood of the origin under the assumption that* $h(0,k_0)$ *is invertible;*

(*ii*) $g(z,k_0-1)$ *and* $g(z,k_0)$ *for all* z *in some open* (*nonempty*) *neighborhood of the origin and* α_{k_0} *under the assumption* α_{k_0} *is invertible;*

uniquely determine the matrix-valued Verblunsky coefficients $\{\alpha_k\}_{k\in\mathbb{Z}}$, *and hence the full-lattice CMV operator* $\mathbb{U}$.

Proof. Case (*i*). First, we note that (2.18) implies that

$$g(0,k_0) = (\mathbb{U}^{-1})_{k_0,k_0} = (\mathbb{U}^*)_{k_0,k_0} = (\mathbb{U}_{k_0,k_0})^* = \begin{cases} -\alpha_{k_0}\alpha_{k_0+1}^*, & k_0 \text{ odd}, \\ -\alpha_{k_0+1}^*\alpha_{k_0}, & k_0 \text{ even}, \end{cases} \tag{4.56}$$

$$h(0,k_0) = \begin{cases} (\mathbb{U}^{-1})_{k_0-1,k_0} = (\mathbb{U}_{k_0,k_0-1})^* = -\rho_{k_0}\alpha_{k_0+1}^*, & k_0 \text{ odd}, \\ (\mathbb{U}^{-1})_{k_0,k_0-1} = (\mathbb{U}_{k_0-1,k_0})^* = -\alpha_{k_0+1}^*\widetilde{\rho}_{k_0}, & k_0 \text{ even}. \end{cases} \tag{4.57}$$

Since $h(0,k_0)$ is invertible, one can solve the above equalities for ρ_{k_0} and α_{k_0},

$$g(0,k_0)h(0,k_0)^{-1} = \alpha_{k_0}\rho_{k_0}^{-1}, \quad k_0 \text{ odd}, \tag{4.58}$$

$$h(0,k_0)^{-1}g(0,k_0) = \widetilde{\rho}_{k_0}^{-1}\alpha_{k_0} = \alpha_{k_0}\rho_{k_0}^{-1}, \quad k_0 \text{ even}, \tag{4.59}$$

implying

$$\rho_{k_0} = \begin{cases} \big[I_m + [g(0,k_0)h(0,k_0)^{-1}]^*[g(0,k_0)h(0,k_0)^{-1}]\big]^{-1/2}, & k_0 \text{ odd}, \\ \big[I_m + [h(0,k_0)^{-1}g(0,k_0)]^*[h(0,k_0)^{-1}g(0,k_0)]\big]^{-1/2}, & k_0 \text{ even}, \end{cases} \tag{4.60}$$

and hence,

$$\alpha_{k_0} = \begin{cases} g(0,k_0)h(0,k_0)^{-1}\rho_{k_0}, & k_0 \text{ odd}, \\ h(0,k_0)^{-1}g(0,k_0)\rho_{k_0}, & k_0 \text{ even}. \end{cases} \tag{4.61}$$

Using (2.10) and (2.11), one also obtains $a_{k_0} = I_m + \alpha_{k_0}$ and $b_{k_0} = I_m - \alpha_{k_0}$.

Next, utilizing (3.22), (3.24), and (3.25), one computes,

$$\begin{aligned} g(z,k_0)h(z,k_0)^{-1} &= -[I_m + M_-(z,k_0)][a_{k_0}^* - b_{k_0}^* M_-(z,k_0)]^{-1}\rho_{k_0}, \quad k_0 \text{ odd}, \\ h(z,k_0)^{-1}g(z,k_0) &= -\widetilde{\rho}_{k_0}[a_{k_0}^* - b_{k_0}^* M_-(z,k_0)]^{-1}[I_m + M_-(z,k_0)], \quad k_0 \text{ even}. \end{aligned} \tag{4.62}$$

Solving for $M_-(z,k_0)$, one then obtains

$$M_-(z,k_0) = \begin{cases} 2g(z,k_0)[b_{k_0}^* g(z,k_0) - \rho_{k_0}h(z,k_0)]^{-1} - I_m, & k_0 \text{ odd}, \\ 2[g(z,k_0)b_{k_0}^* - h(z,k_0)\widetilde{\rho}_{k_0}]^{-1}g(z,k_0) - I_m, & k_0 \text{ even}. \end{cases} \tag{4.63}$$

The right-hand side of the above formula is well-defined for sufficiently small $|z|$ since $b_{k_0}^* g(z,k_0) - \rho_{k_0}h(z,k_0)$ for k_0 odd and $g(z,k_0)b_{k_0}^* - h(z,k_0)\widetilde{\rho}_{k_0}$ for k_0 even are $\mathbb{C}^{m\times m}$-valued analytic functions having invertible values at the origin,

$$\begin{aligned} b_{k_0}^* g(0,k_0) - \rho_{k_0}h(0,k_0) &= (\alpha_{k_0} - I_m)\rho_{k_0}^{-1}h(0,k_0), \quad k_0 \text{ odd}, \\ g(0,k_0)b_{k_0}^* - h(0,k_0)\widetilde{\rho}_{k_0} &= h(0,k_0)\widetilde{\rho}_{k_0}^{-1}(\alpha_{k_0} - I_m), \quad k_0 \text{ even}. \end{aligned} \tag{4.64}$$

Next, having $M_-(z,k_0)$ for sufficiently small $|z|$, one solves the equation

$$h(z,k_0) = -\frac{1}{2z}\begin{cases} \rho_{k_0}^{-1}[a_{k_0}^* - b_{k_0}^* M_-(z,k_0)][M_+(z,k_0) - M_-(z,k_0)]^{-1}[I_m - M_+(z,k_0)], \\ \qquad k_0 \text{ odd}, \\ [I_m - M_+(z,k_0)][M_+(z,k_0) - M_-(z,k_0)]^{-1}[a_{k_0}^* - M_-(z,k_0) b_{k_0}^*]\widetilde{\rho}_{k_0}^{-1}, \\ \qquad k_0 \text{ even}, \end{cases} \tag{4.65}$$

for $M_+(z,k_0)$ and obtains,

$$M_+(z,k_0) = \begin{cases} 2[I_m + zg(z,k_0)]\big[I_m + z[b_{k_0}^* g(z,k_0) - \rho_{k_0} h(z,k_0)]\big]^{-1} - I_m, & k \text{ odd}, \\ 2\big[I_m + z[g(z,k_0) b_{k_0}^* - h(z,k_0)\widetilde{\rho}_{k_0}]\big]^{-1}[I_m + zg(z,k_0)] - I_m, & k_0 \text{ even}. \end{cases} \tag{4.66}$$

The right-hand side of (4.66) is well-defined for sufficiently small $|z|$ since both $I_m + z(b_{k_0}^* g(z,k_0) - \rho_{k_0} h(z,k_0))$ and $I_m + z(g(z,k_0) b_{k_0}^* - h(z,k_0)\widetilde{\rho}_{k_0})$ are $\mathbb{C}^{m\times m}$-valued analytic functions having invertible values at the origin.

Finally, Theorem 4.1 (parts (i), (iv) and (vi), (ix)) implies that $M_\pm(z,k_0)$ for z in some small neighborhood of the origin uniquely determine Verblunsky coefficients $\{\alpha_k\}_{k\in\mathbb{Z}}$.

Case (ii). Suppose k_0 is odd. Then (2.157), (3.22), (3.23), and

$$\begin{aligned} 2[I_m + zg(z,k_0)] &= [I_m + M_-(z,k_0)]W(z,k_0)^{-1}[I_m - M_+(z,k_0)] \\ &\quad + [M_+(z,k_0) - M_-(z,k_0)]W(z,k_0)^{-1} \\ &\quad + W(z,k_0)^{-1}[M_+(z,k_0) - M_-(z,k_0)] \\ &= [I_m + M_+(z,k_0)]W(z,k_0)^{-1}[I_m - M_-(z,k_0)] \end{aligned} \tag{4.67}$$

imply the identity,

$$\begin{aligned} &z\rho_{k_0} g(z,k_0-1)\rho_{k_0} \\ &\quad = \frac{1}{2}\big[(I_m+\alpha_{k_0}^*)-(I_m-\alpha_{k_0}^*)M_+(z,k_0)\big]W(z,k_0)^{-1}\big[(I_m+\alpha_{k_0})+M_-(z,k_0)(I_m-\alpha_{k_0})\big] \\ &\quad = \frac{1}{2}\big[[I_m - M_+(z,k_0)] + \alpha_{k_0}^*[I_m + M_+(z,k_0)]\big]W(z,k_0)^{-1} \\ &\qquad \times \big[[I_m + M_-(z,k_0)] + [I_m - M_-(z,k_0)]\alpha_{k_0}\big] \\ &\quad = \frac{1}{2}[-\Phi_+(z,k_0)+\alpha_{k_0}^*][I_m+M_+(z,k_0)]W(z,k_0)^{-1}[I_m-M_-(z,k_0)][-\Phi_-(z,k_0)^{-1}+\alpha_{k_0}] \\ &\quad = [\alpha_{k_0}^* - \Phi_+(z,k_0)][I_m + zg(z,k_0)][\alpha_{k_0} - \Phi_-(z,k_0)^{-1}]. \end{aligned} \tag{4.68}$$

Moreover, (4.67) also implies

$$
\begin{aligned}
&zg(z,k_0)[I_m+zg(z,k_0)]^{-1} = [I_m+zg(z,k_0)]^{-1}zg(z,k_0) = I_m - [I_m+zg(z,k_0)]^{-1} \\
&\quad = [I_m - M_-(z,k_0)]^{-1}\big[[I_m - M_-(z,k_0)][I_m + M_+(z,k_0)] - 2W(z,k_0)\big][I_m + M_+(z,k_0)]^{-1} \\
&\quad = [I_m - M_-(z,k_0)]^{-1}\big[I_m + M_-(z,k_0) - M_+(z,k_0) - M_-(z,k_0)M_+(z,k_0)\big] \\
&\qquad \times [I_m + M_+(z,k_0)]^{-1} \\
&\quad = [I_m - M_-(z,k_0)]^{-1}[I_m + M_-(z,k_0)][I_m - M_+(z,k_0)][I_m + M_+(z,k_0)]^{-1} \\
&\quad = \Phi_-(z,k_0)^{-1}\Phi_+(z,k_0).
\end{aligned} \tag{4.69}
$$

Introducing the $\mathbb{C}^{m\times m}$-valued analytic functions $A(z,k_0)$ and $B(z,k_0)$ by

$$
A(z,k_0) = I_m + zg(z,k_0) \text{ and } B(z,k_0) = z\rho_{k_0}g(z,k_0-1)\rho_{k_0} - \alpha_{k_0}^* A(z,k_0)\alpha_{k_0}, \tag{4.70}
$$

one rewrites (4.68) as

$$
\begin{aligned}
\Phi_+(z,k_0)A(z,k_0)\alpha_{k_0} + B(z,k_0) - \Phi_+(z,k_0)A(z,k_0)\Phi_-(z,k_0)^{-1}& \\
+ \alpha_{k_0}^* A(z,k_0)\Phi_-(z,k_0)^{-1} = 0.&
\end{aligned} \tag{4.71}
$$

Multiplying both sides by $\Phi_+(z,k_0)$ on the right and utilizing (4.69) then yields the Riccati-type equation for $\Phi_+(z,k_0)$,

$$
\begin{aligned}
\Phi_+(z,k_0)A(z,k_0)\alpha_{k_0}\Phi_+(z,k_0) + B(z,k_0)\Phi_+(z,k_0) - \Phi_+(z,k_0)zg(z,k_0)& \\
+ \alpha_{k_0}^* zg(z,k_0) = 0.&
\end{aligned} \tag{4.72}
$$

Since by (2.152) and (2.157) $\Phi_+(0,k_0) = 0$ and by (4.70)

$$
zg(z,k_0) \underset{z\to 0}{\longrightarrow} 0, \quad A(z,k_0) \underset{z\to 0}{\longrightarrow} I_m, \quad B(z,k_0) \underset{z\to 0}{\longrightarrow} \alpha_{k_0}^*\alpha_{k_0}, \tag{4.73}
$$

Lemma 4.3 implies that equation (4.72) uniquely determines the analytic function $\Phi_+(z,k_0)$ for $|z|$ sufficiently small.

Having $\Phi_+(z,k_0)$, one obtains $\Phi_-(z,k_0)^{-1}$ from (4.68) for $|z|$ sufficiently small,

$$
\Phi_-(z,k_0)^{-1} = \alpha_{k_0} - [I_m + zg(z,k_0)]^{-1}[\alpha_{k_0}^* - \Phi_+(z,k_0)]^{-1}z\rho_{k_0}g(z,k_0-1)\rho_{k_0}. \tag{4.74}
$$

The right-hand side of (4.74) is well-defined since $I_m + zg(z,k_0)$ and $\alpha_{k_0}^* - \Phi_+(z,k_0)$ are $\mathbb{C}^{m\times m}$-valued analytic functions invertible at the origin.

Finally, Theorem 4.1 (parts (i), (v) and (vi), (x)) implies that $\Phi_\pm(z,k_0)^{\pm 1}$ for $|z|$ sufficiently small uniquely determine the Verblunsky coefficients $\{\alpha_k\}_{k\in\mathbb{Z}}$.

The case of k_0 even is proved similarly. □

In the subsequent result, $g^{(j)}$ and $h^{(j)}$ denote the corresponding quantities (4.54) and (4.55) associated with the Verblunsky coefficients $\alpha^{(j)}$, $j = 1,2$.

THEOREM 4.6. *Assume Hypothesis 2.2 for two sequences $\alpha^{(1)}$, $\alpha^{(2)}$ and let $k_0 \in \mathbb{Z}$, $N \in \mathbb{N}$. Then for the full-lattice problems associated with $\alpha^{(1)}$ and $\alpha^{(2)}$ the following local uniqueness results hold:*

(i) *If either $h^{(1)}(0,k_0)$ or $h^{(2)}(0,k_0)$ is invertible and*

$$\big\|g^{(1)}(z,k_0) - g^{(2)}(z,k_0)\big\|_{\mathbb{C}^{m\times m}} + \big\|h^{(1)}(z,k_0) - h^{(2)}(z,k_0)\big\|_{\mathbb{C}^{m\times m}} \underset{z\to 0}{=} o(z^N),$$

$$\text{then } \alpha_k^{(1)} = \alpha_k^{(2)} \text{ for } k_0 - N \leqslant k \leqslant k_0 + N + 1. \tag{4.75}$$

(ii) *If $\alpha_{k_0}^{(1)} = \alpha_{k_0}^{(2)}$, $\alpha_{k_0}^{(1)}$ is invertible, and*

$$\big\|g^{(1)}(z,k_0-1)-g^{(2)}(z,k_0-1)\big\|_{\mathbb{C}^{m\times m}}+\big\|g^{(1)}(z,k_0)-g^{(2)}(z,k_0)\big\|_{\mathbb{C}^{m\times m}} \underset{z\to 0}{=} o(z^N),$$

$$\text{then } \alpha_k^{(1)} = \alpha_k^{(2)} \text{ for } k_0 - N - 1 \leqslant k \leqslant k_0 + N + 1. \tag{4.76}$$

Proof. Case (i). The result is implied by Theorem 4.2 (parts (i), (iii) and (v), (vii)) upon verifying that (4.63), (4.66), and (4.75) imply

$$\begin{aligned}
&\big\|M_+^{(1)}(z,k_0) - M_+^{(2)}(z,k_0)\big\|_{\mathbb{C}^{m\times m}} \underset{z\to 0}{=} o(z^{N+1}),\\
&\big\|M_-^{(1)}(z,k_0) - M_-^{(2)}(z,k_0)\big\|_{\mathbb{C}^{m\times m}} \underset{z\to 0}{=} o(z^{N}).
\end{aligned} \tag{4.77}$$

Case (ii). The result is a consequence of Theorem 4.2 (parts (i), (iv) and (v), $(viii)$) upon verifying that Corollary 4.4, (4.70), (4.72), (4.74), and (4.76) imply

$$\Big\|\Phi_+^{(1)}(z,k_0)-\Phi_+^{(2)}(z,k_0)\Big\|_{\mathbb{C}^{m\times m}} + \Big\|\Phi_-^{(1)}(z,k_0)^{-1}-\Phi_-^{(2)}(z,k_0)^{-1}\Big\|_{\mathbb{C}^{m\times m}} \underset{z\to 0}{=} o(z^{N+1}). \tag{4.78}$$

□

Appendix A. Basic Facts on Caratheodory and Schur Functions

In this appendix we summarize a few basic properties of matrix-valued Caratheodory and Schur functions used throughout this manuscript. (For the analogous case of matrix-valued Herglotz functions we refer to [55] and the extensive list of references therein.)

We denote by $\mathbb{D}$ and $\partial\mathbb{D}$ the open unit disk and the counterclockwise oriented unit circle in the complex plane $\mathbb{C}$,

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}, \quad \partial\mathbb{D} = \{\zeta \in \mathbb{C} \mid |\zeta| = 1\}. \tag{A.1}$$

Moreover, we denote as usual $\mathrm{Re}(A) = (A + A^*)/2$ and $\mathrm{Im}(A) = (A - A^*)/(2i)$ for square matrices A with complex-valued entries.

DEFINITION A.1. *Let $m \in \mathbb{N}$ and $F_\pm$, Φ_+, and Φ_-^{-1} be $m \times m$ matrix-valued analytic functions in $\mathbb{D}$.*
(i) *F_+ is called a Caratheodory matrix if $\mathrm{Re}(F_+(z)) \geqslant 0$ for all $z \in \mathbb{D}$ and F_- is called an anti-Caratheodory matrix if $-F_-$ is a Caratheodory matrix.*
(ii) *Φ_+ is called a Schur matrix if $\|\Phi_+(z)\|_{\mathbb{C}^{m\times m}} \leqslant 1$, for all $z \in \mathbb{D}$. Φ_- is called an anti-Schur matrix if Φ_-^{-1} is a Schur matrix.*

THEOREM A.2. *Let F be an $m\times m$ Caratheodory matrix, $m\in\mathbb{N}$. Then F admits the Herglotz representation*

$$F(z)=iC+\oint_{\partial\mathbb{D}}d\Omega(\zeta)\,\frac{\zeta+z}{\zeta-z},\quad z\in\mathbb{D}, \tag{A.2}$$

$$C=\mathrm{Im}(F(0)),\quad \oint_{\partial\mathbb{D}}d\Omega(\zeta)=\mathrm{Re}(F(0)), \tag{A.3}$$

where $d\Omega$ denotes a nonnegative $m\times m$ matrix-valued measure on $\partial\mathbb{D}$. The measure $d\Omega$ can be reconstructed from F by the formula

$$\Omega\big(\mathrm{Arc}\big(\big(e^{i\theta_1},e^{i\theta_2}\big]\big)\big)=\lim_{\delta\downarrow 0}\lim_{r\uparrow 1}\frac{1}{2\pi}\oint_{\theta_1+\delta}^{\theta_2+\delta}d\theta\,\mathrm{Re}\big(F\big(r\zeta\big)\big), \tag{A.4}$$

where

$$\mathrm{Arc}\big(\big(e^{i\theta_1},e^{i\theta_2}\big]\big)=\big\{\zeta\in\partial\mathbb{D}\,|\,\theta_1<\theta\leqslant\theta_2\big\},\quad \theta_1\in[0,2\pi),\ \theta_1<\theta_2\leqslant\theta_1+2\pi. \tag{A.5}$$

Conversely, the right-hand side of equation (A.2) *with $C=C^*$ and $d\Omega$ a finite nonnegative $m\times m$ matrix-valued measure on $\partial\mathbb{D}$ defines a Caratheodory matrix.*

We note that additive nonnegative $m\times m$ matrices on the right-hand side of (A.2) can be absorbed into the measure $d\Omega$ since

$$\oint_{\partial\mathbb{D}}d\mu_0(\zeta)\,\frac{\zeta+z}{\zeta-z}=1,\quad z\in\mathbb{D}, \tag{A.6}$$

where

$$d\mu_0(\zeta)=\frac{d\theta}{2\pi},\quad \zeta=e^{i\theta},\ \theta\in[0,2\pi) \tag{A.7}$$

denotes the normalized Lebesgue measure on the unit circle $\partial\mathbb{D}$.

Given a Caratheodory (resp., anti-Caratheodory) matrix F_+ (resp. F_-) defined in $\mathbb{D}$ as in (A.2), one extends $F_\pm$ to all of $\mathbb{C}\backslash\partial\mathbb{D}$ by

$$F_\pm(z)=iC_\pm\pm\oint_{\partial\mathbb{D}}d\Omega_\pm(\zeta)\,\frac{\zeta+z}{\zeta-z},\quad z\in\mathbb{C}\backslash\partial\mathbb{D},\ \ C_\pm=C_\pm^*. \tag{A.8}$$

In particular,

$$F_\pm(z)=-F_\pm(1/\overline{z})^*,\quad z\in\mathbb{C}\backslash\overline{\mathbb{D}}. \tag{A.9}$$

Of course, this continuation of $F_\pm|_{\mathbb{D}}$ to $\mathbb{C}\backslash\overline{\mathbb{D}}$, in general, is not an analytic continuation of $F_\pm|_{\mathbb{D}}$.

Next, given the functions $F_\pm$ defined in $\mathbb{C}\backslash\partial\mathbb{D}$ as in (A.8), we introduce the functions $\Phi_\pm$ by

$$\Phi_\pm(z)=[F_\pm(z)-I_m][F_\pm(z)+I_m]^{-1},\quad z\in\mathbb{C}\backslash\partial\mathbb{D}. \tag{A.10}$$

We recall (cf., e.g., [99, p. 167]) that if $\pm\mathrm{Re}(F_\pm)\geqslant 0$, then $[F_\pm\pm I_m]$ is invertible. In particular, $\Phi_+|_{\mathbb{D}}$ and $[\Phi_-]^{-1}|_{\mathbb{D}}$ are Schur matrices (resp., $\Phi_-|_{\mathbb{D}}$ is an anti-Schur matrix). Moreover,

$$F_\pm(z)=[I_m-\Phi_\pm(z)]^{-1}[I_m+\Phi_\pm(z)],\quad z\in\mathbb{C}\backslash\partial\mathbb{D}. \tag{A.11}$$

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