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ON THE MULTIPLICITY FUNCTION OF REAL NORMAL OPERATORS

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Abstract. Properties of commuting and non-commuting operators are important in operator theory. In this paper we study the consequences of equations such as $AB = BA^*$ when A and B are real normal operators (A is real if it commutes with a conjugation operator, see [1] or [14]). It is shown that the multiplicity function of A has certain symmetry properties in this case. This generalizes to infinite dimensions some recent results concerning the eigenvalues of certain normal matrices (see [9]).

1. Introduction

For bounded normal operators A and B defined on a separable Hilbert space $\mathscr{H}$, equations like $A^*B = BA$ and $AB = BA$ imply a certain type of symmetry on the spectrum of A and B, in particular the multiplicity function of A sometimes takes only even values on certain subspaces. It is this symmetry which we investigate in this paper in the case that A and B are bounded real normal operators. (A^* is the adjoint of A, uniquely determined by the equation $\langle Af, g\rangle = \langle f, A^*g\rangle$ for $f, g \in \mathscr{H}$). An operator is real if it commutes with a conjugation operator. In a similar way one can define the notions of symmetric operator, complex symmetric operator etc. See for example [4], where complex symmetric operators are studied and many examples of conjugation operators are given.

It was shown in [9] that if A and B are real $n \times n$ normal matrices acting on n dimensional complex Hilbert space $\mathbb{C}^n$, satisfying $A^*B = BA$, then the eigenvalues of A have even multiplicity on the orthogonal complement of the subspace $\{x \in \mathbb{C}^n : Bx = B^*x\}$. As a corollary one obtains that if A has only simple eigenvalues, then $B = B^*$. In this paper it is this type of theorem we wish to investigate and to generalize (see also [7] and [8]).

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2. Preliminaries: General Spectral Theory for Normal Operators

Let Y be a compact subset of the complex plane $\mathbb{C}$, and let $\mathscr{H}$ be a separable complex Hilbert space. Given a σ-algebra $\mathscr{F}$ of Borel subsets of Y, let $E : \mathscr{F} \to B(\mathscr{H})$ be a *spectral measure*, so that for each $\Delta \in \mathscr{F}$, $E(\Delta)$ is a projection on $\mathscr{H}$ with the usual properties.

The spectral measure E is said to be of *multiplicity one* if there exists a cyclic vector $h \in \mathscr{H}$, i.e., is such that the closed linear span of $\{E(\Delta)h : \Delta \in \mathscr{F}\}$ is equal to $\mathscr{H}$.

More generally (when E may not be of multiplicity one) we let $Z(f)$ be the closed linear span of all vectors of the form $\{E(\Delta)f : \Delta \in \mathscr{F}\}$, and call $Z(f)$ the *cyclic subspace* generated by $f \in \mathscr{H}$. This is written

$$Z(f) = \overline{\text{span}}\{E(\Delta)f : \Delta \in \mathscr{F}\}.$$

For $h \in \mathscr{H}$, we define a measure μ_h on the measurable space $(Y, \mathscr{F})$ by

$$\mu_h(\Delta) = \langle E(\Delta)h, h\rangle = ||E(\Delta)h||^2.$$

This gives rise to an isometry $U : Z(h) \to L^2(Y, \mu_h)$ satisfying $UE(\Delta)U^* = E_h(\Delta)$, where E_h is the spectral measure on $L^2(Y, \mu_h)$ defined by $E_h(\Delta) = \chi_\Delta$ (where χ_D is the characteristic function of D).

The operator U is a unitary operator which is defined on simple functions by

$$U\left(\sum_{i=1}^{n} a_i E(\Delta_i)h\right) = \sum_{i=1}^{n} a_i \chi_{\Delta_i},$$

in particular, taking $n = 1$, $a_i = 1$ and $\Delta_i = Y$ gives

$$U(h) = U(E(Y)h) = \chi_Y = p_0,$$

where $p_n(z) = z^n$, for $n \geqslant 0$.

A bounded linear operator $A : \mathscr{H} \to \mathscr{H}$ is normal if $AA^* = A^*A$. In the finite dimensional case there is unitary matrix U with the property that

$$U^*AU = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. Denote the spectrum of A by $\sigma(A)$, a compact subset of the complex plane. In infinite dimensions the situation is more complex:

The Spectral Theorem for a bounded normal operator $A : \mathscr{H} \to \mathscr{H}$ on the separable Hilbert space $\mathscr{H}$ tells us that there exists a unique spectral measure E defined on the Borel subsets of the spectrum $\sigma(A)$ of A such that

(a) $A = \int_{\sigma(A)} z\, E(dz)$.
(b) $E(G) \neq 0$ for all non-empty G relatively open in $\sigma(A)$.
(c) If $T \in B(\mathscr{H})$, and $TA = AT$, then $TE(\Delta) = E(\Delta)T$ for all $\Delta \in \mathscr{F}$.

If the spectral measure E is of multiplicity one, then A is a *cyclic operator*, or is said to have *multiplicity one* (classically the term simple spectrum was used: see Akhiezer and Glazman [1], Putnam [13] or Stone [15]). One can see that a normal operator has multiplicity one if and only if there is a vector $h \in \mathscr{H}$ for which

$$\overline{\text{span}}\{A^m A^{*n} h : m, n \geqslant 0\} = \mathscr{H}.$$

If the normal operator A has multiplicity one, then A is unitarily equivalent to the operator $N_h : L^2(\sigma(A), \mu_h) \to L^2(\sigma(A), \mu_h)$, defined by $N_h f(z) = zf(z)$. In fact we have $A = U^* N_h U$ where U is the unitary operator defined previously. It readily follows that

$$U\left(\sum_{n=0}^{N}\sum_{m=0}^{K} a_{mn} A^m A^{*n} h\right) = \sum_{n=0}^{N}\sum_{m=0}^{K} a_{mn} z^m \bar{z}^n,$$

and that functions of the form $\sum_{n=0}^{N}\sum_{m=0}^{K} a_{mn} z^m \bar{z}^n$ are dense in $L^2(\sigma(A), \mu_h)$. Note that if μ is a regular Borel measure on $\sigma(A)$, with $\int_{\sigma(A)} z^m \bar{z}^n \, d\mu(z) = 0$ for all $m, n \geqslant 0$, then $\mu = 0$.

The Hahn-Hellinger Theorem says that for a bounded normal operator A on the separable Hilbert space $\mathscr{H}$, there is a sequence of vectors (h_n), $n \in \mathbb{Z}^+$ such that

$$\mathscr{H} = Z(h_1) \oplus Z(h_2) \oplus \cdots \oplus Z(h_n) \oplus \cdots,$$

and a (essentially unique) sequence of regular Borel measures (μ_{h_i}) on $\sigma(A)$, satisfying:

$$\mu_{h_1} \gg \mu_{h_2} \gg \cdots \gg \mu_{h_n} \gg \cdots,$$

(where for two measures μ and τ, defined on the same σ-algebra, $\mu \ll \tau$ if $\tau(D) = 0$ implies $\mu(D) = 0$ and $\mu \sim \tau$ if $\mu \ll \tau$ and $\tau \ll \mu$).

These are the spectral measures of the operator A, μ_{h_1} being the *maximal spectral type* of A (which is defined only up to absolutely equivalent measures). A normal operator A is determined up to unitary equivalence by its maximal spectral type, together with a function $M : \sigma(A) \to \mathbb{Z}^+ \cup \{\infty\}$, called the *multiplicity function* of A. The maximum value of the range of M (μ_{h_1} a.e.), is called the *maximum spectral multiplicity of* A. If the maximum spectral multiplicity is 1, then A has multiplicity one. There is a decreasing sequence of Borel sets Δ_n in $\sigma(A)$ for which $\mu_{h_n} = \mu_{h_1}|\Delta_n$. Then M is defined a.e. μ_{h_1} by

$$M(z) = \begin{cases} \infty\,; & z \in \bigcap_{n=1}^{\infty} \Delta_n \\ n\,; & z \in \Delta_n \setminus \Delta_{n+1} \\ 0\,; & \text{otherwise} \end{cases}$$

Two normal operators having the same maximal spectral type and multiplicity function M are unitarily equivalent.

Given a normal operator A with spectral measure E, if $f, g \in \mathscr{H}$, we can define a complex measure $\mu_{f,g}$ on the Borel subsets of the spectrum of A by $\mu_{f,g}(\Delta) = \langle E(\Delta) f, g \rangle$. The measure $\mu_{f,g}$ is absolutely continuous with respect to both μ_f and

μ_g, i.e., $\mu_{f,g} \ll \mu_f$ and $\mu_{f,g} \ll \mu_g$. Note also that $\mu_f(\Delta) = 0$ if and only if $E(\Delta)f = 0$. In addition, $\mu_f \perp \mu_g$ implies that $\mu_{f,g} \equiv 0$, the converse being true when A has multiplicity one.

The Fourier coefficients of the measure $\mu_{f,g}$ are defined by

$$\widehat{\mu}_{f,g}(m,n) = \langle A^m f, A^n g\rangle = \int_{\sigma(A)} z^m \bar{z}^n \, d\mu_{f,g}, \quad m,n \in \mathbb{Z}^+,$$

and two such measures are equal if and only if their corresponding Fourier coefficients are equal.

For the basic ideas concerning the multiplicity theory of normal operators, see Halmos [10] or Conway [2].

3. Conjugation Operators and Real Normal Operators

In this paper we study bounded linear operators on separable Hilbert-spaces $\mathscr{H}$ for which there exists a conjugation operator (see Akhiezer and Glazman [1] and also Riesz and Nagy [14], for the notions of conjugation operator and real operator, and also Halmos [11] for the related notion of self-conjugate functional Hilbert space):

DEFINITION. A map $I : \mathscr{H} \to \mathscr{H}$ is a *conjugation operator* on the (complex) Hilbert space $\mathscr{H}$ if

1) $\langle I(f), I(g)\rangle = \overline{\langle f, g\rangle}$, for all $f, g \in \mathscr{H}$.
2) $I^2(f) = f$, for all $f \in \mathscr{H}$.

Such operators are conjugate linear: $I(\alpha f + \beta g) = \bar{\alpha} I(f) + \bar{\beta} I(g)$, for all $f, g \in \mathscr{H}$ and $\alpha, \beta \in \mathbb{C}$. Often we shall denote $I(f)$ by $\bar{f}$.

DEFINITION. A linear operator $A : \mathscr{H} \to \mathscr{H}$ is said to be *real* with respect to a given conjugation operator I, if A and I commute.

Note that any self conjugate functional Hilbert space (in the sense of Halmos [11]), with the property that $||\bar{f}|| = ||f||$ for all $f \in \mathscr{H}$ has a conjugation operator.

Our main theorem is the following result (throughout we use the convention that ∞ is even and we assume that $\mathscr{H}$ is a Hilbert space having a conjugation operator):

THEOREM 1. *Let $A, B : \mathscr{H} \to \mathscr{H}$ be bounded real normal operators. If $A^*B = BA$ where B is invertible, then in the orthogonal complement of the subspace*

$$H_0 = \{f \in \mathscr{H} : Bf = B^*f\},$$

the multiplicity function of A only takes even values.

Note that $H_0 = \{f \in \mathscr{H} : Bf = B^*f\}$, is a reducing subspace for A (i.e., $A(H_0) \subseteq H_0$ and $A(H_0^\perp) \subseteq H_0^\perp$) since if $f \in H_0$, then $BAf = A^*Bf = A^*B^*f = B^*Af$, so $Af \in H_0$. In addition, by the Fuglede-Putman Theorem, $AB = BA^*$, so $f \in H_0$ implies $A^*f \in H_0$. Clearly H_0 is a reducing subspace for B.

Examples

1. Let $\mathscr{H} = \mathbb{C}^n =$ n-dimensional complex vector space. The usual complex conjugation gives a conjugation operator, and any real $n \times n$ matrix gives rise to an operator which preserves complex conjugation, so is real in the above sense. It follows from Theorem 1 that if A and B are real $n \times n$ normal matrices with $AB = BA^*$, then the eigenvalues of A occur with even multiplicity on the orthogonal complement of the subspace on which B is symmetric (the condition that B be invertible can be dropped in this case [9]). As a corollary, it follows that if all the eigenvalues of A are simple, then $B = B^*$.

2. Let $\mathscr{H} = l_2(\mathbb{Z})$, then if $f = (f_n) \in l_2$, $||f||^2 = \sum_n |f_n|^2 < \infty$. If $I(f) = (\bar{f}_n)$, then I defines a conjugation operator. The bilateral shift operator $A : l_2(\mathbb{Z}) \to l_2(\mathbb{Z})$ defined by $A(e_n) = e_{n+1}$ (where e_n is the sequence with a 1 in the n^{th} position and 0's elsewhere), is a real (unitary) operator which has multiplicity one (because it is clear that $\overline{\text{span}}\{A^n e_1 : n \in \mathbb{Z}\} = l_2(\mathbb{Z})$). Theorem 1 implies that if $A^*B = BA$ where B is real and normal, then $B = B^*$.

3. Let $(X, \mathscr{F}, \mu)$ be a separable Borel measure space. Set $\mathscr{H} = L^2(X, \mathscr{F}, \mu)$, then $I(f) = \bar{f}$ defines a conjugation operator on $\mathscr{H}$, and any operator which preserves real valued functions will be a real operator. To give a more explicit example, let $X = [0, 1]$ with $\mathscr{F}$ the Borel measurable subsets and μ equal to Lebesgue measure. Define $A : \mathscr{H} \to \mathscr{H}$ by $Af(x) = f(x + \alpha)$ where $\alpha \in [0, 1]$ (addition is taken modulo one). Then A is a unitary operator which is real, and has multiplicity one if α is irrational. Theorem 1 implies that if B is any real unitary operator with $AB = BA^*$, then $B^2 = E$, the identity operator. (This fact was known to Halmos and von Neumann as a consequence of their Discrete Spectrum Theorem).

4. Let $\mathscr{H} = L^2(S^1, \mathscr{F}, \mu)$ where S^1 is the unit circle in the complex plane, $\mathscr{F}$ is the σ-algebra of Borel subsets of the circle and μ is a finite Borel measure on S^1 which is symmetric with respect to the real axis. Define $I : \mathscr{H} \to \mathscr{H}$ by $If(z) = \bar{f}(\bar{z})$. I is a conjugation operator since μ is symmetric, and $N_\mu f(z) = zf(z)$ is real with respect to I. More generally, $Wf(z) = h(z)f(z)$ is real if $h(\bar{z}) = \bar{h}(z)$. If $WN_\mu = N_\mu^* W$ where W is real and normal, then $W = W^*$, since N_μ has multiplicity one.

5. Let $\mathscr{H}$ be the space of all functions $f : D \to \mathbb{C}$ with $\int_D |f|^2 \, d\lambda < \infty$ ($\lambda =$ planar Lebesgue measure), where D is the unit disc in the complex plane and where each f can be written in the form $f = u + iv$ where both u and v are real functions satisfying Laplace's equation. $\mathscr{H}$ is a Hilbert space when given the usual inner product. As before, $I(f) = \bar{f}$ defines a conjugation operator, and any linear operator which preserves real valued functions is real.

LEMMA 1. *Let $A : \mathscr{H} \to \mathscr{H}$ be a bounded normal operator, real with respect to a conjugation operator I.*

(1) *The spectrum of A, $\sigma(A)$ is symmetric with respect to the real axis.*
(2) *The maximal spectral type of A can be chosen to be a symmetric measure.*
(3) *A is unitarily equivalent to A^*, the adjoint of A.*

Proof. (1) It suffices to show that $\rho(A)$, the complement of $\sigma(A)$ is symmetric with respect to the real axis. If $\lambda \in \rho(A)$, then $R_\lambda = (A-\lambda E)^{-1}$ is bounded (where E is the identity operator).

We can check that $R_\lambda = IR_{\bar\lambda}I$, so that $\bar\lambda \in \rho(A)$ because $A-\bar\lambda E = I(A-\lambda E)I$.

(2) Suppose that $h \in \mathscr{H}$ realizes the maximal spectral type of A, then there exists a finite Borel measure μ_h on $\sigma(A)$ with $\langle A^m h, A^n h\rangle = \int_{\sigma(A)} z^m \bar z^n \, d\mu_h(z)$. Let us denote $I(h)$ by $\bar h$, then

$$\overline{\langle A^m h, A^n h\rangle} = \langle A^m \bar h, A^n \bar h\rangle = \int_{\sigma(A)} z^m \bar z^n \, d\mu_{\bar h}(z),$$

but also

$$\overline{\langle A^m h, A^n h\rangle} = \overline{\int_{\sigma(A)} z^m \bar z^n \, d\mu_h(z)} = \int_{\sigma(A)} \bar z^m z^n \, d_h(z) = \int_{\sigma(A)} z^m \bar z^n \, d\mu_h(\bar z).$$

Thus $\mu_{\bar h}(D) = \mu_h(\bar D)$ (for any Borel set D), and since $\mu_{\bar h} \ll \mu_h$ (because μ_h is the maximal spectral type), $\mu_h(D) = 0$ implies $\mu_h(\bar D) = 0$. Consequently, we may assume that μ_h is a symmetric measure.

(3) Suppose that A has multiplicity equal to one, then we can write $A = UN_hU^*$ for some unitary operator U and some μ_h, a symmetric measure on $\sigma(A)$ as above.

Define $B : L^2(\sigma(A), \mu_h) \to L^2(\sigma(A), \mu_h)$ by $Bf(z) = f(\bar z)$, then B is a unitary operator (because μ_h is symmetric). In addition, $BN_h = N^*_{\mu_h}B$, so N_h and N_h^* are unitarily equivalent, and thus A and A^* are unitarily equivalent.

Now suppose that A has multiplicity greater than one, then we can see that A and A^* have the same multiplicity function. A similar argument to that above shows that they are unitarily equivalent. □

4. Proof of Theorem 1

The following lemma shows that in order to prove Theorem 1 we may assume that B is a unitary operator.

LEMMA 2. *The operator B in Theorem 1 may be assumed to be unitary.*

Proof. If $A^*B = BA$, where B is invertible, then by a theorem of Putnam (see [12] or [11]), there is a unitary operator U with the property $A^*U = UA$. In fact U arises from the polar decomposition of B, $B = UP$ (uniquely), where U is unitary and P is positive. In the case that B is normal, $UP = PU$. In addition, if B is invertible then P is invertible and we can check that

$$\{f \in \mathscr{H} : Bf = B^*f\} = \{f \in \mathscr{H} : Uf = U^*f\}.$$

Now if B is a real operator, $B = IBI$, so that $UP = IUPI = IUIIPI$. It follows by the uniqueness of U and P that $U = IUI$ and $P = IPI$, so both U and P are real operators. □

Given a bounded normal operator $K : \mathscr{H} \to \mathscr{H}$ and $f \in \mathscr{H}$, the (non-closed) support of the spectral measure μ_f is a subset of $\sigma(K)$, the spectrum of K, which will be denoted by supp (μ_f). The complex plane $\mathbb{C}$ can be written as the disjoint union $\mathbb{C} = \mathbb{C}^+ \cup \mathbb{C}^- \cup \mathbb{R}$, where $\mathbb{C}^+$, $\mathbb{C}^-$ are respectively the upper and lower open half planes and $\mathbb{R}$ is the real axis.

LEMMA 3. *Let $A, B : \mathscr{H} \to \mathscr{H}$ be bounded normal operators with $A^*B = BA$. Set*

$$\mathscr{P}_+ = \{f \in \mathscr{H} : supp(\mu_f) \subseteq \mathbb{C}^+\}, \quad \mathscr{P}_- = \{f \in \mathscr{H} : supp(\mu_f) \subseteq \mathbb{C}^-\},$$

$$H_0 = \{f \in \mathscr{H} : Bf = B^*f\}, \quad H_1 = \{f \in \mathscr{H} : Bf = -B^*f\},$$

(where μ_f is a spectral measure with respect to the operator B^2).

(1) *$\mathscr{P}_+$, $\mathscr{P}_-$, H_0 and H_1 are closed subspaces of $\mathscr{H}$ which are both A and B^2 invariant and reducing.*

(2) *$\{f \in \mathscr{H} : {B^*}^2 f = B^2 f\} = \{f \in \mathscr{H} : supp(\mu_f) \subseteq \mathbb{R}\}$.*

(3) *If B is unitary operator, then*

$$\mathscr{H} = \mathscr{P}_+ \oplus \mathscr{P}_- \oplus H_0 \oplus H_1.$$

Proof. (1) Clearly all the above sets are closed subspaces, and H_0 and H_1 are B^2-invariant. H_1 is A-invariant for if $f \in H_1$, the $BAf = A^*Bf = -A^*B^*f = -B^*Af$, so $Af \in H_1$ (the proof for H_0 was given earlier). The symmetry of these definitions implies that the subspaces are also A^* and ${B^*}^2$-invariant, so they are reducing subspaces.

Also note that $A^*B = BA$ implies $AB = BA^*$ (by the Fuglede-Putnam theorem [12]), since A is normal and so $AB^2 = B^2A$, $A^*B^2 = B^2A^*$ and $A^nB = BA^{*n}$ for any $n \in \mathbb{Z}^+$.

To see that $\mathscr{P}_+$ is B^2-invariant, let $f \in \mathscr{P}_+$, then supp $(\mu_f) \subseteq \mathbb{C}^+$, and

$$\widehat{\mu}_{B^2f}(m,n) = \langle B^{2m}B^2f, B^{2n}B^2f \rangle = \langle B^{2m}f, B^{2n}{B^*}^2B^2f \rangle = \widehat{\mu}_{f,{B^*}^2B^2f}(m,n),$$

thus $\mu_{B^2f} = \mu_{f,{B^*}^2B^2f} \ll \mu_f$, so that supp $(\mu_{B^2f}) \subseteq \mathbb{C}^+$.

To show that $\mathscr{P}_+$ is A-invariant, let $f \in \mathscr{P}_+$, then for $m, n \in \mathbb{Z}^+$

$$\begin{aligned}\widehat{\mu}_{Af}(m,n) &= \langle B^{2m}Af, B^{2n}Af \rangle = \langle AB^{2m}f, B^{2n}Af \rangle \\ &= \langle B^{2m}f, A^*B^{2n}Af \rangle = \langle B^{2m}f, B^{2n}A^*Af \rangle = \widehat{\mu}_{f,A^*Af}(m,n),\end{aligned}$$

so $\mu_{Af} = \mu_{f,A^*Af}$, and in particular $\mu_{Af} \ll \mu_f$, so that $Af \in \mathscr{P}_+$.

(2) Set $H = \{f \in \mathscr{H} : B^2f = {B^*}^2f\}$, and $H_{\mathbb{R}} = \{f \in \mathscr{H} : \text{supp}(\mu_f) \subseteq \mathbb{R}\}$, then if $f \in H$ we can check that $\widehat{\mu}_f(m,n)$ is real, and this implies supp $(\mu_f) \subseteq \mathbb{R}$, or $H \subseteq H_{\mathbb{R}}$.

For the converse, suppose that $f \in H_{\mathbb{R}}$ and let $g \in \mathscr{H}$. We can write $g = g_0 + g_1$, where supp $(\mu_{g_0}) \subseteq \mathbb{R}$ and $g_1 \perp g_0$. It can be seen that $\langle B^2(f + g_0), f + g_0 \rangle$ is real and this can be used to show that $\langle B^2f, g_0 \rangle = \langle f, B^2g_0 \rangle$ (see Riesz and Nagy [14], p. 229). It follows that

$$\langle B^2f, g \rangle = \langle B^2f, g_0 \rangle = \langle f, B^2g_0 \rangle = \langle f, B^2g \rangle,$$

and this implies that $B^2 f = {B^*}^2 f$.

(3) In the case where B is unitary, $H = \{f \in \mathscr{H} : B^4 f = f\} = H_{\mathbb{R}}$, and it suffices to show that $H_{\mathbb{R}} = H_0 \oplus H_1$, where now $H_0 = \{f \in \mathscr{H} : B^2 f = f\}$, $H_1 = \{f \in \mathscr{H} : B^2 f = -f\}$ (this is because $\mathscr{H} = \mathscr{P}_+ \oplus \mathscr{P}_- \oplus H_{\mathbb{R}}$, using (2)). This follows from the fact that any $f \in H$ can be written as

$$f = g_0 + g_1, \quad \text{where} \quad g_0 = 1/2(f + B^2 f), \quad g_1 = 1/2(f - B^2 f),$$

$g_0 \in H_0$ and $g_1 \in H_1$. In addition, we can see that $H_0 \perp H_1$. □

To show that a normal operator A has multiplicity greater than one, it is enough to show there exists $f, g \in \mathscr{H}$ with $Z(f) \perp Z(g)$ and $\mu_f \sim \mu_g$. Our method of proof to show that the multiplicity function of A takes only even values on a certain subspace, is to show that for $f \in \mathscr{P}_+$, $Z(f) \subseteq \mathscr{P}_+$, $Z(B\bar{f}) \subseteq \mathscr{P}_-$ (so $Z(f) \perp Z(B\bar{f})$), with $\mu_f \sim \mu_{B\bar{f}}$. If this is the case, A can be thought of as being unitarily equivalent to an operator V having a decomposition of the form

$$V = N_h \oplus N_h \oplus \text{other terms.}$$

Finally, we remark that if A has multiplicity one, the commutant $\{A\}'$ of A is Abelian.

In the following proposition it is not assumed that B is invertible.

PROPOSITION 1. *Let $A, B : \mathscr{H} \to \mathscr{H}$ be bounded real normal operators. If $A^* B = BA$ then A cannot be a cyclic operator when restricted to either of the subspaces H_1 or $\mathscr{P}_+ \oplus \mathscr{P}_-$, when they are non-trivial.*

The proof is split into two lemmas:

LEMMA 4. *If A and B are as in Proposition 1 above, then A has multiplicity greater than one on the subspace $\mathscr{P}_+ \oplus \mathscr{P}_-$.*

Proof. Initially we suppose that μ_f denotes the spectral measure of f with respect to B^2, we saw in Lemma 3 that if $f \in \mathscr{P}_+$, then $Af \in \mathscr{P}_+$, so that $f \in \mathscr{P}_+$ implies that $Z(f) \subseteq \mathscr{P}_+$ (the cyclic subspace $Z(f)$ with respect to A).

In addition, we can see that if $f \in \mathscr{P}_+$, then $\bar{f} \in \mathscr{P}_-$. This is because

$$\int_{\sigma(B^2)} z^m \bar{z}^n \, d\mu_{\bar{f}}(z) = \widehat{\mu}_{\bar{f}}(m,n) = \langle B^{2m}\bar{f}, B^{2n}\bar{f}\rangle = \overline{\langle B^{2m} f, B^{2n} f\rangle}$$

$$= \langle B^{2n} f, B^{2m} f\rangle = \int_{\sigma(B^2)} \bar{z}^m z^n \, d\mu_f(z) = \int_{\sigma(B^2)} z^m \bar{z}^n \, d\mu_f(\bar{z}),$$

i.e., $\mu_{\bar{f}}(D) = \mu_f(\bar{D})$ for all Borel sets $D \subseteq \sigma(B^2)$, so if $\operatorname{supp}(\mu_f) \subseteq \mathbb{C}^+$, then $\operatorname{supp}(\mu_{\bar{f}}) \subseteq \mathbb{C}^-$.

In a similar way we can see that for $f \in \mathscr{P}_+$, $\mu_{B\bar{f}} = \mu_{\bar{f}, B^* Bf}$ (where we are assuming $Bf \neq 0$), so that $\mu_{B\bar{f}} \ll \mu_{\bar{f}}$ (with respect to B^2), and in particular $B\bar{f} \in \mathscr{P}_-$, and generally $Z(B\bar{f}) \subseteq \mathscr{P}_-$ (with respect to A).

We now claim that $\mu_{B\bar{f}} \ll \mu_f$ (with respect to A), for each $f \in \mathscr{P}_+$, and this, together with $Z(f) \perp Z(B\bar{f})$ is enough to show that A is not of multiplicity one on

$\mathscr{P}_+ \oplus \mathscr{P}_-$. In the following our spectral measures are with respect to A:

$$\begin{aligned}\widehat{\mu}_{B\bar{f}}(m,n) &= \langle A^m B\bar{f}, A^n B\bar{f}\rangle = \langle A^m B\bar{f}, BA^{*n}\bar{f}\rangle = \langle B^* A^m B\bar{f}, A^{*n}\bar{f}\rangle \\ &= \langle A^{*m} B^* B\bar{f}, A^{*n}\bar{f}\rangle = \langle B^* B\bar{f}, A^m A^{*n}\bar{f}\rangle = \langle A^m A^{*n} f, B^* Bf\rangle \\ &= \langle A^m f, A^n B^* Bf\rangle = \widehat{\mu}_{f,B^*Bf}(m,n),\end{aligned}$$

i.e., $\mu_{B\bar{f}} = \mu_{f,B^*Bf} \ll \mu_f$ and $\mu_{B\bar{f}} \ll \mu_{B^*Bf}$ (with respect to A). Notice that in addition we have

$$\mu_{B\bar{f}} \ll \mu_{BB^*f} \ll \mu_{B^*\bar{f}}$$

and by similar reasoning $\mu_{B^*\bar{f}} \ll \mu_{B\bar{f}}$, so $\mu_{B^*\bar{f}} \sim \mu_{B\bar{f}}$. □

LEMMA 5. *The normal operator A restricted to the invariant subspace H_1 does not have multiplicity one.*

Proof. We saw that H_1 is both A and B^2 reducing. Let $f \in H_1$ with $Bf \neq 0$, then $B^*f = -Bf$ and we see that for all $m,n \in \mathbb{Z}^+$

$$\begin{aligned}\langle A^m f, A^n B\bar{f}\rangle &= \langle A^m f, -A^n B^*\bar{f}\rangle = \langle A^m f, -B^* A^{*n}\bar{f}\rangle = -\langle BA^m f, A^{*n}\bar{f}\rangle \\ &= -\langle A^{*m} Bf, A^{*n}\bar{f}\rangle = -\overline{\langle A^{*n}\bar{f}, A^{*m} Bf\rangle} = -\langle A^{*n} f, A^{*m} B\bar{f}\rangle \\ &= -\langle A^m f, A^n B\bar{f}\rangle,\end{aligned}$$

or $\langle A^m f, A^n B\bar{f}\rangle = 0$ for all $m,n \in \mathbb{Z}^+$, and this implies $Z(f) \perp Z(B\bar{f})$.

Consider the spectral measures μ_f and $\mu_{B\bar{f}}$ of f and $B\bar{f}$ (with respect to A). The same argument that we used previously shows that $\mu_{B\bar{f}} = \mu_{f,B^*Bf} \ll \mu_f$.

Again we have shown that $Z(f) \perp Z(B\bar{f})$, with $\mu_{B\bar{f}} \ll \mu_f$, and this is enough to show that A does not have multiplicity one on H_1. □

Completion of the Proof of Theorem 1

We have shown that A is non-cyclic on $\mathscr{P}_+ \oplus \mathscr{P}_- \oplus H_1$ and that this subspace is equal to $H_0^\perp$ (using Lemma 3 when B is invertible).

To prove that the multiplicity function only takes even values on this subspace, we now use the fact that we may assume that B is a unitary operator. Suppose that $f \in \mathscr{P}_+$, then we showed previously that $B\bar{f} \in \mathscr{P}_-$, and $Z(f) \perp Z(B\bar{f})$ (cyclic subspaces with respect to A) and that $\mu_{B\bar{f}} \ll \mu_f$ and

$$\mu_{BB^*f} \ll \mu_{B^*\bar{f}} \quad \text{and} \quad \mu_{B^*\bar{f}} \sim \mu_{B\bar{f}}.$$

Since B is unitary, $BB^*f = f$ and it follows that $\mu_f \sim \mu_{B\bar{f}}$, so that A has even multiplicity on $Z(f) \oplus Z(B\bar{f})$. The following argument shows the even multiplicity on H_1, and a similar argument shows even multiplicity on $\mathscr{P}_+ \oplus \mathscr{P}_-$:

Let $Z(f_1)$ be a maximal cyclic subspace in H_1, then $Z(B\bar{f})$ is a maximal cyclic subspace also in H_1, with $\mu_{f_1} \sim \mu_{B\bar{f}_1}$ and $Z(f_1) \perp Z(B\bar{f}_1)$, giving even multiplicity on the direct sum. Continue in this way by taking a maximal cyclic subspace $Z(f_2)$

in $H_1 \ominus (Z(f_1) \oplus Z(B\bar{f_1}))$, and form the cyclic subspace $Z(B\bar{f}_2)$, which is clearly orthogonal to $Z(f_2) \oplus Z(B\bar{f}_1)$, so we need only prove that $Z(f_1) \perp Z(B\bar{f}_2)$. Now

$$\begin{aligned}\langle A^m f_1, A^n B\bar{f}_2\rangle &= \langle A^m f_1, -A^n B^* \bar{f}_2\rangle = \langle A^m f_1, -B^* A^{*n} \bar{f}_2\rangle = -\langle BA^m f_1, A^{*n}\bar{f}_2\rangle \\ &= -\langle A^{*m} B f_1, A^{*n}\bar{f}_2\rangle = -\overline{\langle A^{*n}\bar{f}_2, A^{*m} B f_1\rangle} = -\langle A^{*n} f_2, A^{*m} B\bar{f}_1\rangle \\ &= -\langle A^m f_2, A^n B\bar{f}_1\rangle = 0,\end{aligned}$$

because $Z(f_2) \perp Z(B\bar{f}_1)$. Now continue to find a sequence $f_n \in H_1$ with

$$H_1 = \bigoplus_{n=1}^{\infty} Z(f_n) \oplus \bigoplus_{n=1}^{\infty} Z(B\bar{f_n}),$$

and we see that A has an even multiplicity function on $H_0^{\perp}$. □

COROLLARY 1. *Let $A, B : \mathscr{H} \to \mathscr{H}$ be bounded real normal operators. Suppose that A has multiplicity one. If $A^*B = BA$ then $B = B^*$.*

Proof. Since A has multiplicity one, $\mathscr{P}_+ \oplus \mathscr{P}_- = \{0\}$, so $\mathscr{H} = \{f \in \mathscr{H} : B^2 f = B^{*2} f\}$.

Let $f \in \mathscr{H}$, then if $g = (B - B^*)f$, we see that $g \in H_1$. But Lemma 2 implies that $H_1 = \{0\}$, so $Bf = B^*f$ and the result follows. □

5. Even Multiplicity for A^2

Suppose that A is a unitary operator with the property that $AB = B^*A$ for some bounded operator B with $B \neq B^*$. If A^2 has multiplicity one, then $\{A^2\}'$ (the commutant of A^2), is an Abelian sub-algebra with $A, B \in \{A^2\}'$ so that $AB = BA = B^*A$, contradicting $B^* \neq B$. We conclude that A^2 cannot have multiplicity one. This is a result we make more precise for the case of real operators on spaces with conjugation operators. In the following theorem, condition (a) holds if $AB = B^*A$ and B is normal, or A is unitary.

THEOREM 2. *Suppose that $A, B : \mathscr{H} \to \mathscr{H}$ are bounded real linear operators. If A is normal and $B \neq B^*$ and either (a) or (b) below holds:*

(a) $AB = B^*A$ and $BA = AB^*$

(b) $AB = -BA$,

then A^2 is not of multiplicity one. If A is invertible, then A^2 has an even multiplicity function on the orthogonal complement of the subspace

$$H_0 = \{f \in \mathscr{H} : Bf = B^*f\}.$$

Proof. (a) Let $K = B - B^*$, then $K^* = B^* - B = -K$, so is a skew symmetric operator (and hence normal) whose spectrum is contained in the imaginary axis. Since $AB = B^*A$ and $AB^* = BA$ we deduce that $AK = K^*A$.

Now $H_0 = \{f \in \mathscr{H} : Kf = 0\}$ is the subspace corresponding to the eigenvalue 0 of K, so is certainly K–invariant. H_0 is A invariant (and reducing), since if $f \in H_0$, then $KAf = -K^*Af = -AKf = 0$, so $Af \in H_0$. In the following, $\mathrm{supp}(\mu_f)$ denotes the support of the spectral measure μ_f with respect to K. In this case $\mathrm{supp}(\mu_f)$ is a (not necessarily closed) subset of the imaginary axis I.

Write $I = I^+ \cup I^- \cup \{0\}$, the disjoint union of the upper and lower imaginary axes. Let

$$\mathscr{P}_+ = \{f \in \mathscr{H} : \mathrm{supp}(\mu_f) \subseteq I^+\}, \quad \mathscr{P}_- = \{f \in \mathscr{H} : \mathrm{supp}(\mu_f) \subseteq I^-\}.$$

(In each case the spectral measure is with respect to K).

$\mathscr{P}_+$ and $\mathscr{P}_-$ are K–invariant for if $f \in \mathscr{P}_+$, $\mu_{Kf} \ll \mu_f$ (with respect to K), since $\widehat{\mu}_{Kf}(n) = \langle K^n(Kf), Kf\rangle = \langle K^n f, K^*Kf\rangle$, so that $\mu_{Kf} = \mu_{f,K^*Kf} \ll \mu_f$, and then $Kf \in \mathscr{P}_+$.

Since K is a normal operator, $\mathscr{P}_+$ and $\mathscr{P}_-$ are orthogonal subspaces and

$$\mathscr{H} = H_0 \oplus \mathscr{P}_+ \oplus \mathscr{P}_-.$$

Let $f \in \mathscr{P}_+$, then $\mathrm{supp}(\mu_f) \subseteq I^+$ and we claim that $Af \in \mathscr{P}_-$. Let $n \in \mathbb{Z}^+$ and consider

$$\begin{aligned}\widehat{\mu}_{Af}(n) &= \langle K^n Af, Af\rangle = \langle AK^{*n}f, Af\rangle = \langle (K^n)^* f, A^*Af\rangle \\ &= \langle f, K^n A^*Af\rangle = \overline{\langle K^n A^*Af, f\rangle} = \langle K^n A^*A\bar{f}, \bar{f}\rangle = \widehat{\mu}_{A^*A\bar{f},\bar{f}}(n).\end{aligned}$$

We have shown that $\mu_{Af} \ll \mu_{\bar{f}}$, and if we show that $\bar{f} \in \mathscr{P}_-$, it will follow that $Af \in \mathscr{P}_-$. To see this last statement, note that

$$\int z^n\, d\mu_{\bar{f}}(z) = \langle K^n\bar{f}, \bar{f}\rangle = \overline{\langle K^n f, f\rangle} = \overline{\int z^n\, d\mu_f(z)} = \int \bar{z}^n\, d\mu_f(z) = \int z^n\, d\mu_f(\bar{z}),$$

and this implies that $\mu_{\bar{f}}(D) = \mu_f(\bar{D})$ for all Borel sets D contained in $\sigma(K)$.

We have shown that $\mathscr{P}_+$ and $\mathscr{P}_-$ are invariant with respect to both K and A^2, since if $f \in \mathscr{P}_+$ then $Af \in \mathscr{P}_-$ so that $A^2 f \in \mathscr{P}_+$.

Furthermore

$$\langle A^{2m}(Af), A^{2n}(Af)\rangle = \langle A^{2m}f, A^{2n}(A^*Af)\rangle, \quad m,n \in \mathbb{Z}^+.$$

This shows that $Z(Af) \perp Z(f)$ with $\mu_{Af} = \mu_{f,A^*Af} \ll \mu_f$ (spectral measure with respect to A^2). This is enough to show that A^2 does not have multiplicity one on $\mathscr{P}_+ \oplus \mathscr{P}_-$. In the case that A is invertible, setting $A_1 = A^2|\mathscr{P}_+$ (the restriction) and $A_2 = A^2|\mathscr{P}_-$, we have that $AA_1 = A_2A$, so that A_1 and A_2 are unitarily equivalent (A can be replaced by a unitary operator, by Putnam's Theorem), so that A^2 has even multiplicity on $\mathscr{P}_+ \oplus \mathscr{P}_- = H_0^\perp$, and the result follows.

(b) If instead $AB = -BA$, set $K = B - B^*$ as before, then we can check that $AK = K^*A$, so we are in the situation of (a). □

COROLLARY 2. *Let $A, B : \mathscr{H} \to \mathscr{H}$ be bounded real linear operators. Suppose that A is normal and A^2 has multiplicity one.*

(*a*) *If $AB = B^*A$ and $BA = AB^*$ then $B = B^*$.*

(*b*) *If $AB = -BA$, then $B = B^*$.*

6. Commuting Operators

Suppose now that our operators commute. Theorem 3 gives information about the multiplicity function of a self–adjoint (Hermitian) operator. The corresponding spectral measures are now measures on the real line and the multiplicity function is defined on the real line.

THEOREM 3. *Suppose that $A, B : \mathscr{H} \to \mathscr{H}$ are bounded real linear operators. If A is self–adjoint with $AB = BA$, then A has an even multiplicity function on the orthogonal complement of the subspace*

$$H_0 = \{f \in \mathscr{H} : Bf = B^* f\}.$$

Proof. We may assume that B is normal since if we set $K = B - B^*$, then $AK = KA$ and $\{f \in \mathscr{H} : Bf = B^* f\} = \{f \in \mathscr{H} : Kf = K^* f\}$.

As before write

$$\mathscr{P}_+ = \{f \in L^2(X, \mu) : \operatorname{supp}(\mu_f) \subseteq \mathbb{C}^+\}, \quad \mathscr{P}_- = \{f \in \mathscr{H} : \operatorname{supp}(\mu_f) \subseteq \mathbb{C}^-\},$$

but now the spectral measures are with respect to B.

Clearly $H_0 = \{f \in \mathscr{H} : \operatorname{supp}(\mu_f) \subseteq \mathbb{R}\}$ and it can be seen that $\mathscr{H} = H_0 \oplus \mathscr{P}_+ \oplus \mathscr{P}_-$, and so $H_0^\perp = \mathscr{P}_+ \oplus \mathscr{P}_-$.

Suppose that $f \in \mathscr{P}_+$, then for all $m, n \in \mathbb{Z}^+$

$$\begin{aligned}\int_{S^1} z^m \bar{z}^n \, d\mu_{\bar{f}}(z) &= \langle B^m \bar{f}, B^n \bar{f} \rangle = \overline{\langle B^m f, B^n f \rangle} \\ &= \overline{\int_{S^1} z^m \bar{z}^n \, d\mu_f(z)} = \int_{S^1} \bar{z}^m z^n \, d\mu_f(z) = \int_{S^1} z^m \bar{z}^n \, d\mu_f(\bar{z}).\end{aligned}$$

It follows that $\mu_{\bar{f}}(D) = \mu_f(\bar{D})$ for all Borel sets D, and this implies that $\bar{f} \in \mathscr{P}_-$.

Both $\mathscr{P}_+$ and $\mathscr{P}_-$ are A and B–invariant. B–invariance is clear, to see the A–invariance, note that

$$\widehat{\mu}_{Af}(m,n) = \langle B^m Af, B^n Af \rangle = \langle AB^m f, B^n Af \rangle = \langle B^m f, B^n A^* Af \rangle = \widehat{\mu}_{f,A^2 f}(m,n),$$

so that $\mu_{Af} = \mu_{f,A^2 f} \ll \mu_f$. In particular, if $f \in \mathscr{P}_+$, then $A^n f \in \mathscr{P}_+$ for all $n \geqslant 0$.

Finally we show that $\mu_f = \mu_{\bar{f}}$ (spectral measures now with respect to A). This is because, for $n \geqslant 0$

$$\widehat{\mu}_f(n) = \langle A^n f, f \rangle = \langle f, (A^n)^* f \rangle = \langle f, A^n f \rangle = \overline{\langle A^n f, f \rangle} = \langle A^n \bar{f}, \bar{f} \rangle = \widehat{\mu}_{\bar{f}}(n).$$

If $f \in \mathscr{P}_+$, it is clear that $\langle A^n f, \bar{f} \rangle = 0$ for all $n \in \mathbb{Z}^+$, or $Z(f) \perp Z(\bar{f})$ (since $\mathscr{P}_+$ and $\mathscr{P}_-$ are orthogonal), and also $\mu_f = \mu_{\bar{f}}$.

To show that A has even multiplicity on $\mathscr{P}_+ \oplus \mathscr{P}_-$, let $Z(f_1)$ be a maximal cyclic subspace in $\mathscr{P}_+$ with corresponding subspace $Z(\bar{f}_1)$ in $\mathscr{P}_-$. Now take $Z(f_2)$ maximal in $\mathscr{P}_+ \ominus Z(f_1)$. Clearly $Z(\bar{f}_2) \perp Z(f_2)$ and $Z(\bar{f}_2) \perp Z(f_1)$, so we need only check that $Z(\bar{f}_2) \perp Z(\bar{f}_1)$. But $\langle A^n \bar{f}_1, \bar{f}_2 \rangle = \overline{\langle A^n f_1, f_2 \rangle} = 0$, $n \geqslant 0$, so this is clear. Now continuing in this way, the result follows.

7. Final Remarks

1. Motivation for the results of this paper also came from the study of joinings and intertwinings involving (ergodic) invertible measure preserving transformations (automorphisms) on Borel probability spaces. Given a Borel probability space $(X,\mathscr{F},\mu)$, an automorphism of this space is an invertible transformation $T : X \to X$ for which $T^{-1}A \in \mathscr{F}$ and $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathscr{F}$. Recall that for automorphisms S and T on the probability space $(X,\mathscr{F},\mu)$, λ is a joining of S and T if λ is a probability measure on the product space which is invariant under $S\times T$ and each projection on X is equal to μ. Joinings can be realized as intertwining operators $J : L^2(X,\mu) \to L^2(X,\mu)$ with the property that $J\widehat{S} = \widehat{T}J$ (where $\widehat{T} : L^2(X,\mu) \to L^2(X,\mu)$ is the unitary operator induced by T and defined by $\widehat{T}f(x) = f(Tx)$), and where $J1 = 1$, $Jf \geqslant 0$ when $f \geqslant 0$ (see [6]). It was shown in [8] that if $\widehat{T}$ has multiplicity one and $J\widehat{T} = \widehat{T}^{-1}J$, then $J = J^*$.

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