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THE KALMAN–YAKUBOVICH–POPOV INEQUALITY FOR PASSIVE DISCRETE TIME–INVARIANT SYSTEMS

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Abstract. We consider the Kalman - Yakubovich - Popov (KYP) inequality

$$\begin{pmatrix} X - A^*XA - C^*C & -A^*XB - C^*D \\ -B^*XA - D^*C & I - B^*XB - D^*D \end{pmatrix} \geqslant 0$$

for contractive operator matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}$, where $\mathfrak{H}$, $\mathfrak{M}$, and $\mathfrak{N}$ are separable Hilbert spaces. We restrict ourselves to the positive contractive solutions X. Using the parametrization of the blocks of contractive operator matrices, the Kreĭn shorted operator, and the Möbius representation of the Schur class operator-valued function we find several equivalent forms of the KYP inequality. The properties of solutions are established and it is proved that the minimal solution of the KYP inequality satisfies the corresponding algebraic Riccati equation and can be obtained by the iterative procedure with the special choice of the initial point. In terms of the Kreĭn shorted operators the necessary condition and some sufficient conditions for uniqueness of the solution are established.

1. Introduction

The system of equations

$$\begin{cases} h_{k+1} = Ah_k + B\xi_k, \\ \sigma_k = Ch_k + D\xi_k \end{cases}, \qquad k \geqslant 0$$

describes the evolution of a *linear discrete time-invariant system* $\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\}$ with bounded linear operators A, B, C, D and separable Hilbert spaces $\mathfrak{H}$ (state space), $\mathfrak{M}$ (input space), and $\mathfrak{N}$ (output space). If the linear operator T_τ defined by the block-matrix

$$T_\tau = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}$$

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is contractive, then the corresponding system is said to be *passive*. If the operator T_τ is isometric (co-isometric, unitary) then the corresponding system is called isometric (co-isometric, conservative). Isometric and co-isometric systems have been studied by L. de Branges and J. Rovnyak [15], [16] and by T. Ando [4], conservative systems have been investigated by B. Sz.-Nagy and C. Foias [33] and M.S. Brodskiĭ [17]. Passive systems are studied by D.Z. Arov et al [5, 6, 7, 8, 10, 11, 12].

The subspaces

$$\mathfrak{H}^c_\tau := \overline{\operatorname{span}}\,\{A^n B\mathfrak{M} :\ n = 0, 1, \ldots\} \quad \text{and} \quad \mathfrak{H}^o_\tau = \overline{\operatorname{span}}\,\{A^{*n} C^*\mathfrak{N} :\ n = 0, 1, \ldots\} \tag{1.1}$$

are called the *controllable* and *observable* subspaces of the system $\tau = \left\{\begin{pmatrix} A & B \\ C & D\end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\right\}$, respectively. If $\mathfrak{H}^c_\tau = \mathfrak{H}$ ($\mathfrak{H}^o_\tau = \mathfrak{H}$) then the system τ is said to be *controllable* (*observable*), and *minimal* if τ is both controllable and observable. If $\mathfrak{H} = \text{closure}\{\mathfrak{H}^c_\tau + \mathfrak{H}^o_\tau\}$ then the system τ is said to be *simple*. Note that from (1.1) it follows that

$$(\mathfrak{H}^c_\tau)^\perp = \bigcap_{n=0}^{\infty} \ker(B^* A^{*n}),\ (\mathfrak{H}^o_\tau)^\perp = \bigcap_{n=0}^{\infty} \ker(CA^n).$$

Therefore

(1) the system τ is controllable $\iff \bigcap\limits_{n=0}^{\infty} \ker(B^* A^{*n}) = \{0\}$;

(2) the system τ is observable $\iff \bigcap\limits_{n=0}^{\infty} \ker(CA^n) = \{0\}$;

(3) the system τ is simple $\iff \left(\bigcap\limits_{n=0}^{\infty} \ker(B^* A^{*n})\right) \bigcap \left(\bigcap\limits_{n=0}^{\infty} \ker(CA^n)\right) = \{0\}$.

The function

$$\Theta_\tau(\lambda) := D + \lambda C(I_{\mathfrak{H}} - \lambda A)^{-1}B, \quad \lambda \in \mathbb{D},$$

is called the *transfer function* of the system τ.

The result of D.Z. Arov [5] states that two minimal systems

$$\tau_1 = \left\{\begin{pmatrix} A_1 & B_1 \\ C_1 & D\end{pmatrix}; \mathfrak{H}_1, \mathfrak{M}, \mathfrak{N}\right\} \quad \text{and} \quad \tau_2 = \left\{\begin{pmatrix} A_2 & B_2 \\ C_2 & D\end{pmatrix}; \mathfrak{H}_2, \mathfrak{M}, \mathfrak{N}\right\}$$

with the same transfer function $\Theta(\lambda)$ are *pseudo-similar*, i.e., there exists a closed densely defined operator $Z : \mathfrak{H}_1 \to \mathfrak{H}_2$ such that Z is invertible, Z^{-1} is densely defined, and

$$ZA_1 f = A_2 Zf,\ C_1 f = C_2 Zf,\ f \in \operatorname{dom} Z, \text{ and } \quad ZB_1 = B_2.$$

If the system τ is passive then Θ_τ belongs to the *Schur class* $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$, i.e., $\Theta_\tau(\lambda)$ is holomorphic in the unit disk $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and its values are contractive linear operators from $\mathfrak{M}$ into $\mathfrak{N}$. It is well known [16], [33], [4], [5], [7] that every $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ can be realized as the transfer function of some passive system, which can be chosen as conservative simple (isometric controllable, co-isometric observable, passive and minimal, respectively). Moreover, two simple conservative (isometric controllable, co-isometric observable) systems τ_1 and τ_2 having the same transfer function are unitarily similar [15], [16], [17], [4], i.e., there exists a unitary operator U from $\mathfrak{H}_1$ onto $\mathfrak{H}_2$ such that $A_1 = U^{-1}A_2U,\ B_1 = U^{-1}B_2,\ C_1 = C_2U.$ In [10], [11] necessary and sufficient conditions on $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M},\mathfrak{N})$ have been established in order that all minimal passive systems having the transfer function $\Theta(\lambda)$ be unitarily similar or similar.

A system $\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\}$ is called X*-passive* with respect to the supply rate function $w(u,v) = ||u||^2_{\mathfrak{M}} - ||v||^2_{\mathfrak{N}},\ u \in \mathfrak{M},\ \ v \in \mathfrak{N}$ [31] if there exists a positive selfadjoint operator X in $\mathfrak{H}$, possibly unbounded, such that

$$A\,\mathrm{dom}\,X^{1/2} \subset \mathrm{dom}\,X^{1/2},\ \mathrm{ran}\,B \subset \mathrm{dom}\,X^{1/2}$$

and

$$\begin{gathered} ||X^{1/2}(Ax+Bu)||^2_{\mathfrak{H}} - ||X^{1/2}x||^2_{\mathfrak{H}} \leqslant ||u||^2_{\mathfrak{M}} - ||Cx+Du||^2_{\mathfrak{N}} \\ \text{for all } x \in \mathrm{dom}\,X^{1/2},\ u \in \mathfrak{M}. \end{gathered} \tag{1.2}$$

The condition (1.2) is equivalent to

$$\begin{gathered} \left\| \begin{pmatrix} X^{1/2} & 0 \\ 0 & I_{\mathfrak{M}} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} X^{1/2} & 0 \\ 0 & I_{\mathfrak{N}} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \right\|^2 \geqslant 0 \\ \text{for all} \quad x \in \mathrm{dom}\,X^{1/2},\ u \in \mathfrak{M}. \end{gathered} \tag{1.3}$$

If X is bounded then (1.3) becomes the *Kalman – Yakubovich – Popov inequality* (for short, the KYP inequality)

$$L_\tau(X) = \begin{pmatrix} X - A^*XA - C^*C & -A^*XB - C^*D \\ -B^*XA - D^*C & I - B^*XB - D^*D \end{pmatrix} \geqslant 0. \tag{1.4}$$

The classical Kalman-Yakubovich-Popov lemma states that *if* $\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\}$ *is a minimal system with finite dimensional state space* $\mathfrak{H}$ *then the set of the solutions of* (1.4) *is non-empty if and only if the transfer function* Θ_τ *belongs to the Schur class. If this is a case then the set of all solutions of (1.4) contains the minimal and maximal elements*.

For the case $\dim \mathfrak{H} = \infty$ the theory of the *generalized KYP inequality* (1.3) is developed in [8] and the following results have been established.

THEOREM 1.1. [8, Theorem 1.2, Theorem 5.1]. *Let*

$$\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\}$$

be a minimal system. Then the generalized KYP inequality (1.3) for τ *has a solution if and only if the transfer function* Θ_τ *coincides with the Schur class function in a neighborhood of the origin.*

Let $\mathscr{X}_\tau$ *be the set of all solutions* X *of (1.3) which have additional properties*

(a) $X^{1/2}\,(\mathrm{span}\,\{A^nB,\ n = 0,1,\ldots,\})$ *and* $X^{-1/2}\,(\mathrm{span}\,\{A^{*n}C^*,\ n = 0,1,\ldots,\})$ *are dense in* $\mathfrak{H}$,

(b) span $\{A^nB,\ n=0,1,\ldots,\}$ *is a core for the operator* $X^{1/2}$.
Then the set $\mathscr{X}_\tau$ *is not empty and contains a minimal* $X_{\min}$ *and a maximal* $X_{\max}$ *elements in the sense of quadratic forms:* $X\in\mathscr{X}_\tau \ \Rightarrow\ \operatorname{dom}X_{\min}^{1/2}\supset\operatorname{dom}X^{1/2}\supset\operatorname{dom}X_{\max}^{1/2}$ *and*

$$||X_{\min}^{1/2}u||^2\leqslant ||X^{1/2}u||^2 \quad \textit{for all}\quad u\in\operatorname{dom}X^{1/2},$$
$$||X^{1/2}v||^2\leqslant ||X_{\max}^{1/2}v||^2 \quad \textit{for all}\quad v\in\operatorname{dom}X_{\max}^{1/2}.$$

Let $\Theta(\lambda)\in\mathbf{S}(\mathfrak{M},\mathfrak{N})$. A passive system

$$\dot{\tau}=\left\{\begin{pmatrix}\dot{A} & \dot{B}\\ \dot{C} & D\end{pmatrix};\dot{\mathfrak{H}},\mathfrak{M},\mathfrak{N}\right\}$$

with the transfer function $\Theta(\lambda)$ is called the *optimal* ((∗)-*optimal*) realization of $\Theta(\lambda)$ [6], [7] if for each passive system $\tau=\left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix};\mathfrak{H},\mathfrak{M},\mathfrak{N}\right\}$ with the transfer function $\Theta(\lambda)$ and for each input sequence $u_0,u_1,u_2,\ldots$ in $\mathfrak{M}$ the inequalities

$$\left\|\sum_{k=0}^n \dot{A}^k\dot{B}u_k\right\|_{\dot{\mathfrak{H}}}\leqslant\left\|\sum_{k=0}^n A^kBu_k\right\|_{\mathfrak{H}}\quad\left(\left\|\sum_{k=0}^n \dot{A}^k\dot{B}u_k\right\|_{\dot{\mathfrak{H}}}\geqslant\left\|\sum_{k=0}^n A^kBu_k\right\|_{\mathfrak{H}}\right)$$

hold for all $n=0,1,\ldots$.

Two minimal and optimal ((∗)-optimal) passive realizations of a function from the Schur class are unitarily similar [7]. In addition, the system

$$\dot{\tau}_*=\left\{\begin{pmatrix}\dot{A}_* & \dot{B}_*\\ \dot{C}_* & D\end{pmatrix};\dot{\mathfrak{H}}_*,\mathfrak{M},\mathfrak{N}\right\}$$

is (∗)-optimal minimal realization of the function $\Theta(\lambda)$ if and only if the system

$$\dot{\tau}_*^*=\left\{\begin{pmatrix}\dot{A}_*^* & \dot{B}_*^*\\ \dot{C}_*^* & D^*\end{pmatrix};\dot{\mathfrak{H}}_*,\mathfrak{N},\mathfrak{M}\right\}$$

is optimal minimal realization of the function $\Theta^*(\bar{\lambda})$ [7]. In [7] the construction of the optimal ((∗)-optimal) realization is given as the first (second) restriction of a simple conservative realization of the function Θ.

Let $\tau=\left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix};\mathfrak{H},\mathfrak{M},\mathfrak{N}\right\}$ be a minimal system. Suppose Θ_τ coincides with the Schur class function in a neighborhood of the origin. Let $X_{\min}$ and $X_{\max}$ be the minimal and maximal solutions of the KYP inequality (1.3). It is proved in [8] that the systems

$$\dot{\tau}=\left\{\begin{pmatrix}X_{\min}^{1/2}AX_{\min}^{-1/2} & X_{\min}^{1/2}B\\ CX_{\min}^{-1/2} & D\end{pmatrix};\mathfrak{H},\mathfrak{M},\mathfrak{N}\right\},\ \dot{\tau}_*=\left\{\begin{pmatrix}X_{\max}^{1/2}AX_{\max}^{-1/2} & X_{\max}^{1/2}B\\ CX_{\max}^{-1/2} & D\end{pmatrix};\mathfrak{H},\mathfrak{M},\mathfrak{N}\right\}$$

are minimal optimal and minimal (∗)-optimal realizations of Θ, respectively. Note that the contractive operators

$$\dot{T}=\begin{pmatrix}X_{\min}^{1/2}AX_{\min}^{-1/2} & X_{\min}^{1/2}B\\ CX_{\min}^{-1/2} & D\end{pmatrix}\quad\text{and}\quad\dot{T}_*=\begin{pmatrix}X_{\max}^{1/2}AX_{\max}^{-1/2} & X_{\max}^{1/2}B\\ CX_{\max}^{-1/2} & D\end{pmatrix}\qquad(1.5)$$

are defined on the dense in $\mathfrak{H}\oplus\mathfrak{M}$ linear manifolds $\operatorname{dom} X_{\min}^{-1/2}\oplus\mathfrak{M}$ and $\operatorname{dom} X_{\max}^{-1/2}\oplus\mathfrak{M}$, respectively.

In this paper we consider the KYP inequality for a contractive operator $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. In this case the set of all solutions contains the identity operator. Hence the minimal solution $X_{\min}$ is a positive contraction, while the maximum one satisfies $X_{\max} \geqslant I_{\mathfrak{H}}$, so, $Z = X_{\max}^{-1}$ is also a positive contraction and is the minimal solution of the KYP inequality for the adjoint operator T^*. That's why we are interested in only contractive positive solutions X of the KYP inequality.

We will keep the following notations. The class of all continuous linear operators defined on a complex Hilbert space $\mathfrak{H}_1$ and taking values in a complex Hilbert space $\mathfrak{H}_2$ is denoted by $\mathbf{L}(\mathfrak{H}_1,\mathfrak{H}_2)$ and $\mathbf{L}(\mathfrak{H}) := \mathbf{L}(\mathfrak{H},\mathfrak{H})$. The domain, the range, and the null-space of a linear operator T are denoted by $\operatorname{dom} T$, $\operatorname{ran} T$, and $\ker T$. For a contraction $T \in \mathbf{L}(\mathfrak{H}_1,\mathfrak{H}_2)$ the nonnegative square root $D_T = (I - T^*T)^{1/2}$ is called the defect operator of T and $\mathfrak{D}_T$ stands for the closure of the range $\operatorname{ran} D_T$. It is well known that the defect operators satisfy the relation $TD_T = D_{T^*}T$ and $T\mathfrak{D}_T \subset \mathfrak{D}_{T^*}$ cf. [33]. The set of all regular points of a closed operator T is denoted by $\rho(T)$ and its spectrum by $\sigma(T)$. We denote by $I_{\mathscr{H}}$ the identity operator in a Hilbert space $\mathscr{H}$ and by $P_{\mathscr{L}}$ the orthogonal projection onto the subspace (the closed linear manifold) $\mathscr{L}$. A selfadjoint operator M is called *nonnegative* if $(Mf,f) \geqslant 0$ for all $f \in \operatorname{dom} M$ and *positive* if $(Mf,f) > 0$ for all $f \in \operatorname{dom} M \setminus \{0\}$. If M and N are bounded operators in a Hilbert space, M is a selfadjoint and nonnegative, and, in addition, $\operatorname{ran} N \subset \operatorname{ran} M^{1/2}$ then by definition

$$N^*M^{-1}N := (M^{-1/2}N)^*M^{-1/2}N,$$

where $M^{-1/2}$ is the Moore–Penrose pseudo-inverse to $M^{1/2}$. The boundedness of $M^{-1/2}N$ follows from the result of Douglas:

THEOREM 1.2. [19], [20]. *For $N, L \in \mathbf{L}(\mathfrak{H})$ the following statements are equivalent:*

- (i) $\operatorname{ran} N \subset \operatorname{ran} L$*;*
- (ii) $NN^* \leqslant \lambda LL^*$ *for some* $\lambda \geqslant 0$.
- (iii) $N = LK$ *for some* $K \in \mathbf{L}(\mathfrak{H})$, $\operatorname{ran} K \subset \overline{\operatorname{ran}}\, L^*$, $\ker K = \ker N$.

By $X_{\min}$ and $X_{\max}$ we will denote the minimal and the maximal elements of the subset $\mathscr{X}_T$ of the solutions of the generalized KYP inequality (1.3) for T defined in Theorem 1.1.

We essentially use the following tools.

(1) The parametrization of the 2×2 contractive block-operator matrix $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{K} \\ \mathfrak{N} \end{pmatrix}$ [13], [18], [30]:

$$\begin{aligned} &B = FD_D,\ C = D_{D^*}G,\\ &A = -FD^*G + D_{F^*}LD_G, \end{aligned} \tag{1.6}$$

where the operators $F \in \mathbf{L}(\mathfrak{D}_D,\mathfrak{K})$, $G \in \mathbf{L}(\mathfrak{H},\mathfrak{D}_{D^*})$ and $L \in \mathbf{L}(\mathfrak{D}_G,\mathfrak{D}_{F^*})$ are contractions.

(2) The notion of the shorted operator [22]:

$$S_{\mathscr{K}} = \max\left\{ Z \in \mathbf{L}(\mathscr{H}) : 0 \leqslant Z \leqslant S,\ \operatorname{ran} Z \subseteq \mathscr{K} \right\},$$

where S is a bounded nonnegative selfadjoint operator in the Hilbert space $\mathscr{H}$ and $\mathscr{K}$ is a subspace in $\mathscr{H}$.

(3) The *Möbius representation*

$$\Theta(\lambda) = \Theta(0) + D_{\Theta^*(0)} Z(\lambda)(I_{\mathfrak{D}_{\Theta(0)}} + \Theta^*(0)Z(\lambda))^{-1} D_{\Theta(0)},\ \lambda \in \mathbb{D}$$

of the Schur class operator-valued function $\Theta(\lambda)$ by means of the operator-valued parameter $Z(\lambda)$ from the Schur class $\mathbf{S}(\mathfrak{D}_{\Theta(0)}, \mathfrak{D}_{\Theta^*(0)})$.

A representation of such a kind was studied in [32], [27], [14]. In this paper using the equalities (1.6) we give a more simple and algebraic proof of the Möbius representation. In particular, we establish that if $\Theta(\lambda)$ is the transfer function of the passive system

$$\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\}$$

with entries A, B, and C given by (1.6) then the parameter $Z(\lambda)$ is the transfer function of the passive system $\nu = \left\{ \begin{pmatrix} D_{F^*}LD_G & F \\ G & 0 \end{pmatrix}; \mathfrak{H}, \mathfrak{D}_D, \mathfrak{D}_{D^*} \right\}$. Moreover, we prove that the correspondence

$$\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\} \longleftrightarrow \nu = \left\{ \begin{pmatrix} D_{F^*}LD_G & F \\ G & 0 \end{pmatrix}; \mathfrak{H}, \mathfrak{D}_D, \mathfrak{D}_{D^*} \right\}$$

preserves the properties of the system to be isometric, co-isometric, conservative, controllable, observable, simple, optimal, and $(*)$-optimal, and, in addition, preserves the set of bounded solutions of the corresponding KYP inequalities.

We show that all positive contractive solutions of (1.4) for contraction $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfy the inequality

$$(I_{\mathfrak{H}} - X)P_{\mathfrak{H}} \leqslant \left(D_T^2 + T^*(I_{\mathfrak{H}} - X)P'_{\mathfrak{H}}T\right)_{\mathfrak{H}}, \tag{1.7}$$

where $P_{\mathfrak{H}}$ ($P'_{\mathfrak{H}}$) is the orthogonal projection onto $\mathfrak{H}$ in the Hilbert space $\mathfrak{H} \oplus \mathfrak{M}$ ($\mathfrak{H} \oplus \mathfrak{N}$). We establish that the set of all positive contractive solutions of (1.7) for an observable system contains a minimal element X_0 which satisfies the Riccati equation

$$(I_{\mathfrak{H}} - X)P_{\mathfrak{H}} = \left(D_T^2 + T^*(I_{\mathfrak{H}} - X)P'_{\mathfrak{H}}T\right)_{\mathfrak{H}}$$

and can be obtained by the following iterative procedure

$$X^{(0)} = 0,\ X^{(n+1)} = I_{\mathfrak{H}} - \left(D_T^2 + T^*(I_{\mathfrak{H}} - X^{(n)})P'_{\mathfrak{H}}T\right)_{\mathfrak{H}} \upharpoonright \mathfrak{H},\ X_0 = s - \lim_{n\to\infty} X^{(n)}.$$

Moreover, $X_0 = X_{\min}$ for a minimal system. We prove that the condition

$$\left(D_T^2\right)_{\mathfrak{H}} = 0 \ \ (\iff \ \mathfrak{H} \cap \operatorname{ran} D_T = \{0\})$$

is necessary for the uniqueness of the solutions of (1.7). In the example it is shown that this condition is not sufficient. Some sufficient conditions for the uniqueness are obtained. It is proved (see Theorem 7.9) that if $T \in \mathbf{L}(\mathfrak{H}\oplus\mathfrak{M},\mathfrak{H}\oplus\mathfrak{N})$ is a contraction and if

$$\begin{cases} \left(D^2_T\right)_{\mathfrak{H}} = 0, \ \left(D^2_{T^*}\right)_{\mathfrak{H}} = 0, \\ \operatorname{ran}\left(\left(D^2_{P_{\mathfrak{N}}T}\right)_{\mathfrak{H}}\right)^{1/2} \bigcap \operatorname{ran}\left(\left(D^2_{P_{\mathfrak{M}}T^*}\right)_{\mathfrak{H}}\right)^{1/2} = \{0\} \end{cases}$$

then the passive system $\tau = \{T;\mathfrak{H},\mathfrak{M},\mathfrak{N}\}$ is minimal and $X_{\min} = X_{\max} = I_{\mathfrak{H}}$, i.e., any passive and minimal realization of the transfer function of τ is unitarily equivalent to the system τ.

2. Shorted operators

For every nonnegative bounded operator S in the Hilbert space $\mathscr{H}$ and every subspace $\mathscr{K} \subset \mathscr{H}$ M.G. Kreĭn [22] defined the operator $S_{\mathscr{K}}$ by the formula

$$S_{\mathscr{K}} = \max\left\{ Z \in \mathbf{L}(\mathscr{H}) : 0 \leqslant Z \leqslant S, \operatorname{ran} Z \subseteq \mathscr{K} \right\}.$$

The equivalent definition

$$(S_{\mathscr{K}}f,f) = \inf_{\varphi\in\mathscr{H}\ominus\mathscr{K}} \{(S(f+\varphi),f+\varphi)\}, \quad f \in \mathscr{H}. \tag{2.1}$$

The properties of $S_{\mathscr{K}}$, were studied in [1, 2, 3, 20, 23, 24, 25, 27]. $S_{\mathscr{K}}$ is called the *shorted operator* (see [1], [2]). It is proved in [22] that $S_{\mathscr{K}}$ takes the form

$$S_{\mathscr{K}} = S^{1/2}P_{\Omega}S^{1/2},$$

where P_Ω is the orthogonal projection in $\mathscr{H}$ onto the subspace

$$\Omega = \{f \in \overline{\operatorname{ran}}\, S : S^{1/2}f \in \mathscr{K}\} = \overline{\operatorname{ran}}\, S \ominus S^{1/2}(\mathscr{H}\ominus\mathscr{K}).$$

Hence (see [22]),

$$\operatorname{ran} S^{1/2}_{\mathscr{K}} = \operatorname{ran} S^{1/2}P_\Omega = \operatorname{ran} S^{1/2} \cap \mathscr{K}.$$

It follows that

$$S_{\mathscr{K}} = 0 \iff \operatorname{ran} S^{1/2} \cap \mathscr{K} = \{0\}.$$

The shortening operation possesses the following properties.

PROPOSITION 2.1. [2]. *Let $\mathscr{K}$ be a subspace in $\mathscr{H}$. Then*

(1) *if S_1 and S_2 are nonnegative selfadjoint operators then*

$$(S_1+S_2)_{\mathscr{K}} \geqslant (S_1)_{\mathscr{K}} + (S_2)_{\mathscr{K}};$$

(2) $S_1 \geqslant S_2 \geqslant 0 \Rightarrow (S_1)_{\mathscr{K}} \geqslant (S_2)_{\mathscr{K}}$;

(3) *if $\{S_n\}$ is a nonincreasing sequence of nonnegative bounded selfadjoint operators and $S = s - \lim\limits_{n\to\infty} S_n$ then*

$$s - \lim_{n\to\infty} (S_n)_{\mathscr{K}} = S_{\mathscr{K}}.$$

Let $\mathscr{K}^\perp = \mathscr{H} \ominus \mathscr{K}$. Then a bounded selfadjoint operator S has the block-matrix form

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^* & S_{22} \end{pmatrix} : \begin{pmatrix} \mathscr{K} \\ \mathscr{K}^\perp \end{pmatrix} \to \begin{pmatrix} \mathscr{K} \\ \mathscr{K}^\perp \end{pmatrix}.$$

It is well known (see [23]) that

the operator S is nonnegative if and only if

$$S_{22} \geqslant 0, \ \operatorname{ran} S_{12}^* \subset \operatorname{ran} S_{22}^{1/2}, \ \ S_{11} \geqslant S_{12} S_{22}^{-1} S_{12}^* \tag{2.2}$$

and the operator $S_{\mathscr{K}}$ is given by the block matrix

$$S_{\mathscr{K}} = \begin{pmatrix} S_{11} - S_{12} S_{22}^{-1} S_{12}^* & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.3}$$

If $S_{22}^{-1} \in \mathbf{L}(\mathscr{K}^\perp)$ then the right hand side of (2.3) is called the *Schur complement* of the matrix S. From (2.3) it follows that

$$S_{\mathscr{K}} = 0 \iff \operatorname{ran} S_{12}^* \subset \operatorname{ran} S_{22}^{1/2} \quad \text{and} \quad S_{11} = S_{12} S_{22}^{-1} S_{12}^*.$$

The next representation of the shorted operator is new.

THEOREM 2.2. *Let X be a nonnegative selfadjoint contraction in the Hilbert space $\mathscr{H}$ and let $\mathscr{K}$ be a subspace in $\mathscr{H}$. Then holds the following equality*

$$(I_{\mathscr{H}} - X)_{\mathscr{K}} = P_{\mathscr{K}} - P_{\mathscr{K}} X^{1/2} (I_{\mathscr{H}} - X^{1/2} P_{\mathscr{K}^\perp} X^{1/2})^{-1} X^{1/2} P_{\mathscr{K}}. \tag{2.4}$$

Proof. Let us prove (2.4) for the case $||X|| < 1$. In this case the operator $I_{\mathscr{H}} - X^{1/2} P_{\mathscr{K}^\perp} X^{1/2}$ has bounded inverse and

$$P_{\mathscr{K}} X^{1/2} (I_{\mathscr{H}} - X^{1/2} P_{\mathscr{K}^\perp} X^{1/2})^{-1} X^{1/2} P_{\mathscr{K}} = P_{\mathscr{K}} (I_{\mathscr{H}} - X P_{\mathscr{K}^\perp})^{-1} X P_{\mathscr{K}}.$$

Let $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix}$ be the block-matrix representation of the operator X with respect to the decomposition $\mathscr{H} = \mathscr{K} \oplus \mathscr{K}^\perp$. Then

$$(I_{\mathscr{H}} - X P_{\mathscr{K}^\perp})^{-1} = \begin{pmatrix} I_{\mathscr{K}} & X_{12}(I_{\mathscr{K}^\perp} - X_{22})^{-1} \\ 0 & (I_{\mathscr{K}^\perp} - X_{22})^{-1} \end{pmatrix}.$$

Hence

$$P_{\mathscr{K}} - P_{\mathscr{K}} (I_{\mathscr{H}} - X P_{\mathscr{K}^\perp})^{-1} X P_{\mathscr{K}} = \begin{pmatrix} I_{\mathscr{K}} - X_{11} - X_{12}(I_{\mathscr{K}^\perp} - X_{22})^{-1} X_{12}^* & 0 \\ 0 & 0 \end{pmatrix},$$

and from (2.3) we get (2.4). Observe that (2.4) all $f \in \mathscr{H}$ is equivalent to the relation

$$||(I_{\mathscr{H}} - X^{1/2} P_{\mathscr{K}^\perp} X^{1/2})^{-1/2} X^{1/2} P_{\mathscr{K}} f||^2 = ||P_{\mathscr{K}} f||^2 - ((I_{\mathscr{H}} - X)_{\mathscr{K}} f, f).$$

The latter means that

$$\sup_{g \in \mathscr{H} \setminus \{0\}} \frac{|(X^{1/2} P_{\mathscr{K}} f, g)|^2}{((I_{\mathscr{H}} - X^{1/2} P_{\mathscr{K}^\perp} X^{1/2}) g, g)} = ||P_{\mathscr{K}} f||^2 - ((I_{\mathscr{H}} - X)_{\mathscr{K}} f, f) \quad \text{for all} \quad f \in \mathscr{H}. \tag{2.5}$$

Here we use the well known relations for a nonnegative selfadjoint operator B

$$\sup_{g\in \operatorname{dom} B\setminus\{0\}} \frac{|(h,g)|^2}{(Bg,g)} = \begin{cases} ||B^{-1/2}h||^2, & h \in \operatorname{ran} B^{1/2} \\ +\infty, & h \notin \operatorname{ran} B^{1/2} \end{cases},$$

where $B^{-1/2}$ is the Moore-Penroose pseudo-inverse.

For general case $||X|| \leqslant 1$ using αX for $0 < \alpha < 1$, letting $\alpha \uparrow 1$, and taking into account Proposition 2.1 we see that (2.5) holds also for X. □

3. Parametrization of contractive block-operator matrices

Let $\mathfrak{H}$, $\mathfrak{K}$, $\mathfrak{M}$, and $\mathfrak{N}$ be Hilbert spaces and let T be a contraction acting from $\mathfrak{H}\oplus\mathfrak{M}$ into $\mathfrak{K}\oplus\mathfrak{N}$. The following well known result gives the parametrization of the corresponding representation of T in the block-operator matrix form.

THEOREM 3.1. [13], [18], [30]. *The operator matrix*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{K} \\ \mathfrak{N} \end{pmatrix}$$

is a contraction if and only if $D \in \mathbf{L}(\mathfrak{M},\mathfrak{N})$ *is a contraction and the entries* A, B, *and* C *take the form*

$$\begin{array}{l} B = FD_D,\ C = D_{D^*}G, \\ A = -FD^*G + D_{F^*}LD_G, \end{array}$$

where the operators $F \in \mathbf{L}(\mathfrak{D}_D,\mathfrak{K})$, $G \in \mathbf{L}(\mathfrak{H},\mathfrak{D}_{D^*})$ *and* $L \in \mathbf{L}(\mathfrak{D}_G,\mathfrak{D}_{F^*})$ *are contractions. Moreover, the operators* F, G, *and* L *are uniquely determined.*

Next we derive expressions for the shorted operators $(D_T^2)_{\mathfrak{H}}$, $(D^2_{P_{\mathfrak{N}}T})_{\mathfrak{H}}$, $(D^2_{T^*})_{\mathfrak{K}}$, and $(D^2_{P_{\mathfrak{M}}T^*})_{\mathfrak{K}}$ for a contraction T given by the block matrix form

$$T = \begin{pmatrix} -FD^*G + D_{F^*}LD_G & FD_D \\ D_{D^*}G & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{K} \\ \mathfrak{N} \end{pmatrix}.$$

Let $f \in \mathfrak{H}$, $h \in \mathfrak{M}$, $\varphi \in \mathfrak{D}_D$, $\psi \in \mathfrak{D}_{F^*}$. Using the relations $F^*D_{F^*} = D_F F^*$, $D^*D_{D^*} = D_D D^*$ one can see that

$$\begin{aligned} &||\varphi||^2 + ||\psi||^2 - ||F\varphi + D_{F^*}\psi||^2 = ||D_F\varphi - F^*\psi||^2, \\ &||h||^2 + ||f||^2 - ||Dh + D_{D^*}Gf||^2 = ||D_Gf||^2 + ||D_Dh - D^*Gh||^2. \end{aligned}$$

Putting

$$\varphi = D_Dh - D^*Gf,\ \psi = LD_Gf,$$

we get

$$\begin{aligned} &\left\|\begin{pmatrix} f \\ h \end{pmatrix}\right\|^2 - \left\|T\begin{pmatrix} f \\ h \end{pmatrix}\right\|^2 = ||f||^2 + ||h||^2 - ||F(D_Dh - D^*Gf) + D_{F^*}LD_Gf||^2 \\ &-||D_{D^*}Gf + Dh||^2 = ||D_Gf||^2 + ||D_Dh - D^*Gh||^2 - ||F(D_Dh - D^*Gf) + D_{F^*}LD_Gf||^2 \\ &= ||D_Gf||^2 - ||LD_Gf||^2 + ||D_F(D_Dh - D^*Gh) - F^*LD_Gf||^2 \\ &= ||D_LD_Gf||^2 + ||D_F(D_Dh - D^*Gh) - F^*LD_Gf||^2. \end{aligned}$$

Thus,

$$\left\|D_T\begin{pmatrix} f\\ h\end{pmatrix}\right\|^2=||D_F\left(D_Dh-D^*Gf\right)-F^*LD_Gf||^2+||D_LD_Gf||^2. \tag{3.1}$$

Similarly

$$\left\|D_{T^*}\begin{pmatrix} g\\ \varphi\end{pmatrix}\right\|^2=||D_{G^*}\left(D_{D^*}\varphi-DF^*g\right)-GL^*D_{F^*}g||^2+||D_{L^*}D_{F^*}g||^2,\; g\in\mathfrak{K},\varphi\in\mathfrak{N}, \tag{3.2}$$

$$\left\|D_{P_{\mathfrak{N}}T}\begin{pmatrix} f\\ h\end{pmatrix}\right\|^2=||D_Dh-D^*Gf||^2+||D_Gf||^2,\; f\in\mathfrak{H},h\in\mathfrak{M}, \tag{3.3}$$

$$\left\|D_{P_{\mathfrak{M}}T^*}\begin{pmatrix} g\\ \varphi\end{pmatrix}\right\|^2=||D_{D^*}\varphi-DF^*g||^2+||D_{F^*}g||^2,\; g\in\mathfrak{K},\varphi\in\mathfrak{N}. \tag{3.4}$$

Since $\operatorname{ran}F^*\subset\mathfrak{D}_D,\ \operatorname{ran}L\subset\mathfrak{D}_{F^*},\ \operatorname{ran}G\subset D_{D^*}$, $D^*\mathfrak{D}_{D^*}\subset\mathfrak{D}_D$, and $F^*\mathfrak{D}_{F^*}\subset\mathfrak{D}_F$, for given $f\in\mathfrak{H}$ there exists a sequence $\{h_n\}_{n=1}^{\infty}\subset\mathfrak{D}_D$ such that

$$\lim_{n\to\infty}D_FD_Dh_n=D_FD^*Gf+F^*LD_Gf.$$

Using similar arguments from (3.1)– (3.4) we get

$$\inf_{h\in\mathfrak{M}}\left\{\left\|D_T\begin{pmatrix} f\\ h\end{pmatrix}\right\|^2\right\}=||D_LD_Gf||^2,\ \inf_{h\in\mathfrak{M}}\left\{\left\|D_{P_{\mathfrak{N}}T}\begin{pmatrix} f\\ h\end{pmatrix}\right\|^2\right\}=||D_Gf||^2,$$
$$\inf_{\varphi\in\mathfrak{N}}\left\{\left\|D_{T^*}\begin{pmatrix} g\\ \varphi\end{pmatrix}\right\|^2\right\}=||D_{L^*}D_{F^*}g||^2,\ \inf_{\varphi\in\mathfrak{N}}\left\{\left\|D_{P_{\mathfrak{M}}T^*}\begin{pmatrix} g\\ \varphi\end{pmatrix}\right\|^2\right\}=||D_{F^*}g||^2.$$

Now (2.1) yields the following equalities for the shorted operators

$$\begin{cases}\left(D^2_T\right)_{\mathfrak{H}}=D_GD^2_LD_GP_{\mathfrak{H}},\\ \left(D^2_{P_{\mathfrak{N}}T}\right)_{\mathfrak{H}}=D^2_GP_{\mathfrak{H}},\\ \left(D^2_{T^*}\right)_{\mathfrak{K}}=D_{F^*}D^2_{L^*}D_{F^*}P_{\mathfrak{K}},\\ \left(D^2_{P_{\mathfrak{M}}T^*}\right)_{\mathfrak{K}}=D^2_{F^*}P_{\mathfrak{K}}.\end{cases} \tag{3.5}$$

From (3.5) it follows that

$$\left(D^2_T\right)_{\mathfrak{H}}=\left(D^2_{P_{\mathfrak{N}}T}\right)_{\mathfrak{H}}\iff LD_G=0\iff D_{F^*}LD_G=0\iff L^*D_{F^*}=0.$$

Thus

$$\begin{aligned}&\left(D^2_T\right)_{\mathfrak{H}}=\left(D^2_{P_{\mathfrak{N}}T}\right)_{\mathfrak{H}}\iff\left(D^2_{T^*}\right)_{\mathfrak{K}}=\left(D^2_{P_{\mathfrak{M}}T^*}\right)_{\mathfrak{K}}\\ &\iff T=\begin{pmatrix}-FD^*G & FD_D\\ D_{D^*}G & D\end{pmatrix}.\end{aligned} \tag{3.6}$$

The next statement easily follows from (3.1) and (3.2).

COROLLARY 3.2. *Let*

$$T=\begin{pmatrix}-FD^*G+D_{F^*}LD_G & FD_D\\ D_{D^*}G & D\end{pmatrix}:\begin{pmatrix}\mathfrak{H}\\ \mathfrak{M}\end{pmatrix}\to\begin{pmatrix}\mathfrak{K}\\ \mathfrak{N}\end{pmatrix}.$$

be a contraction. Then

(1) *T is isometric if and only if*

$$D_F D_D = 0,\ D_L D_G = 0,$$

(2) *T is co-isometric if and only if*

$$D_{G^*} D_{D^*} = 0,\ D_{L^*} D_{F^*} = 0.$$

Note that if D is isometric (co-isometric) then T takes the form

$$T = \begin{pmatrix} LD_G & 0 \\ D_{D^*}G & D \end{pmatrix},\ \left(T = \begin{pmatrix} D_{F^*}L & FD_D \\ 0 & D \end{pmatrix}\right).$$

Let $D \in \mathbf{L}(\mathfrak{M},\mathfrak{N})$ be a contraction with nonzero defect operators and let $Q = \begin{pmatrix} S & F \\ G & 0 \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_D \end{pmatrix} \to \begin{pmatrix} \mathfrak{K} \\ \mathfrak{D}_{D^*} \end{pmatrix}$ be a bounded operator. Define the transformation

$$\mathscr{M}_D(Q) = \begin{pmatrix} -FD^*G & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} I_{\mathfrak{K}} & 0 \\ 0 & D_{D^*} \end{pmatrix} \begin{pmatrix} S & F \\ G & 0 \end{pmatrix} \begin{pmatrix} I_{\mathfrak{H}} & 0 \\ 0 & D_D \end{pmatrix}. \tag{3.7}$$

Clearly, the operator $T = \mathscr{M}_D(Q)$ has the following matrix form

$$T = \begin{pmatrix} S - FD^*G & FD_D \\ D_{D^*}G & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{K} \\ \mathfrak{N} \end{pmatrix}.$$

PROPOSITION 3.3. *Let $\mathfrak{H},\mathfrak{M},\mathfrak{N}$ be separable Hilbert spaces, $D \in \mathbf{L}(\mathfrak{M},\mathfrak{N})$ be a contraction with nonzero defect operators, $Q = \begin{pmatrix} S & F \\ G & 0 \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_D \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_{D^*} \end{pmatrix}$ be a bounded operator, and let*

$$T = \mathscr{M}_D(Q) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}.$$

Then

(1) *hold the equalities*

$$\begin{aligned} \bigcap_{n=0}^{\infty} \ker\left(B^*A^{*n}\right) &= \bigcap_{n=0}^{\infty} \ker\left(F^*S^{*n}\right), \\ \bigcap_{n=0}^{\infty} \ker\left(CA^n\right) &= \bigcap_{n=0}^{\infty} \ker\left(GS^n\right), \end{aligned} \tag{3.8}$$

(2) *T is a contraction if and only if Q is a contraction. T is isometric (co-isometric) if and only if Q is isometric (co-isometric). Moreover, hold the equalities*

$$\begin{aligned} &\left(D_Q^2\right)_{\mathfrak{H}} = \left(D_T^2\right)_{\mathfrak{H}},\ \left(D^2_{P_{\mathfrak{D}_*}Q}\right)_{\mathfrak{H}} = \left(D^2_{P_{\mathfrak{N}}T}\right)_{\mathfrak{H}}, \\ &\left(D_{Q^*}^2\right)_{\mathfrak{K}} = \left(D_{T^*}^2\right)_{\mathfrak{K}},\ \left(D^2_{P_{\mathfrak{D}}Q^*}\right)_{\mathfrak{K}} = \left(D^2_{P_{\mathfrak{M}}T^*}\right)_{\mathfrak{K}}, \end{aligned}$$

where $\mathfrak{D} := \mathfrak{D}_D$, $\mathfrak{D}_* := \mathfrak{D}_{D^*}$.

Proof. Observe that

$$A=-FD^*G+S,\ B=FD_D,\ C=D_{D^*}G.$$

Let Ω be a neighborhood of the origin such that the resolvents $(I_{\mathfrak{H}}-\lambda A^*)^{-1}$, $(I_{\mathfrak{H}}-\lambda A)^{-1}$, $(I_{\mathfrak{H}}-\lambda S^*)^{-1}$, and $(I_{\mathfrak{H}}-\lambda S)^{-1}$ exist. Then

$$\begin{aligned} F^*\left(I_{\mathfrak{H}}-\lambda A^*\right)^{-1} &= F^*\left(I_{\mathfrak{H}}-\lambda S^*+\lambda G^*DF^*\right)^{-1}\\ &= F^*\left(I_{\mathfrak{H}}+\lambda\left(I_{\mathfrak{H}}-\lambda S^*\right)^{-1}G^*DF^*\right)^{-1}\left(I_{\mathfrak{H}}-\lambda S^*\right)^{-1}\\ &= \left(I_{\mathfrak{H}}+\lambda F^*\left(I_{\mathfrak{H}}-\lambda S^*\right)^{-1}G^*D\right)^{-1}F^*\left(I_{\mathfrak{H}}-\lambda S^*\right)^{-1}. \end{aligned}$$

It follows that

$$\bigcap_{\lambda\in\Omega}\ker\left(F^*\left(I_{\mathfrak{H}}-\lambda A^*\right)^{-1}\right)=\bigcap_{\lambda\in\Omega}\ker\left(F^*\left(I_{\mathfrak{H}}-\lambda S^*\right)^{-1}\right).$$

Similarly

$$\bigcap_{\lambda\in\Omega}\ker\left(G\left(I_{\mathfrak{H}}-\lambda A\right)^{-1}\right)=\bigcap_{\lambda\in\Omega}\ker\left(G\left(I_{\mathfrak{H}}-\lambda S\right)^{-1}\right).$$

For a bounded operator H in $\mathfrak{H}$ holds

$$(I_{\mathfrak{H}}-\lambda H)^{-1}=\sum_{n=0}^{\infty}\lambda^nH^n,\ |\lambda|<||H||^{-1}.$$

It follows that

$$\begin{aligned} \bigcap_{\lambda\in\Omega}\ker\left(F^*\left(I_{\mathfrak{H}}-\lambda S^*\right)^{-1}\right) &= \bigcap_{n=0}^{\infty}\ker\left(F^*S^{*n}\right),\\ \bigcap_{\lambda\in\Omega}\ker\left(F^*\left(I_{\mathfrak{H}}-\lambda A^*\right)^{-1}\right) &= \bigcap_{n=0}^{\infty}\ker\left(F^*A^{*n}\right). \end{aligned}$$

Since $\operatorname{ran}F^*$ is contained in $\mathfrak{D}_D$ on which D_D is injective and $B^*=D_DF^*$, we get

$$\bigcap_{n=0}^{\infty}\ker\left(F^*A^{*n}\right)=\bigcap_{n=0}^{\infty}\ker\left(D_DF^*A^{*n}\right)=\bigcap_{n=0}^{\infty}\ker\left(B^*A^{*n}\right).$$

Therefore, hold the relations in (3.8). Statement (2) is the consequence of Theorem 3.1 and formulas (3.1)–(3.5). □

PROPOSITION 3.4. *Let $D\in\mathbf{L}(\mathfrak{M},\mathfrak{N})$ be a contraction with nonzero defect operators, let $Q=\begin{pmatrix}S & F\\ G & 0\end{pmatrix}:\begin{pmatrix}\mathfrak{H}\\ \mathfrak{D}_D\end{pmatrix}\to\begin{pmatrix}\mathfrak{H}\\ \mathfrak{D}_{D^*}\end{pmatrix}$ be a contraction, and let $T=\mathscr{M}_D(Q)$. Then for every nonnegative selfadjoint contraction X in $\mathfrak{H}$ hold the following equalities*

$$\left(D_T^2+T^*(I_{\mathfrak{H}}-X)P'_{\mathfrak{H}}T\right)_{\mathfrak{H}}=\left(D_Q^2+Q^*(I_{\mathfrak{H}}-X)P'_{\mathfrak{H}}Q\right)_{\mathfrak{H}},\tag{3.9}$$

$$\begin{aligned} &\left(D_Q^2+Q^*(I_{\mathfrak{H}}-X)P'_{\mathfrak{H}}Q\right)_{\mathfrak{H}}\\ &\quad=\left(D_G^2-D_GL^*D_{F^*}X^{1/2}(I_{\mathfrak{H}}-X^{1/2}FF^*X^{1/2})^{-1}X^{1/2}D_{F^*}LD_G\right)P_{\mathfrak{H}}, \end{aligned}\tag{3.10}$$

where $P_{\mathfrak{H}}$ ($P'_{\mathfrak{H}}$) is the orthogonal projection in $\mathscr{H}=\mathfrak{H}\oplus\mathfrak{M}$ ($\mathscr{H}'=\mathfrak{H}\oplus\mathfrak{N}$) onto $\mathfrak{H}$.

Here

$$D_{F^*}X^{1/2}(I_{\mathfrak{H}} - X^{1/2}FF^*X^{1/2})^{-1}X^{1/2}D_{F^*}$$
$$:= \left((I_{\mathfrak{H}}-X^{1/2}FF^*X^{1/2})^{-1/2}X^{1/2}D_{F^*}\right)^* \left((I_{\mathfrak{H}}-X^{1/2}FF^*X^{1/2})^{-1/2}X^{1/2}D_{F^*}\right).$$

Proof. Define the contraction

$$\widetilde{Q} = \begin{pmatrix} X^{1/2}S & X^{1/2}F \\ G & 0 \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_D \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_{D^*} \end{pmatrix}. \tag{3.11}$$

Then

$$D^2_{\widetilde{Q}} = D^2_Q + Q^*(I_{\mathfrak{H}} - X)P'_{\mathfrak{H}}Q.$$

By Proposition 3.3 the operator $\widetilde{T} = \mathscr{M}_D(\widetilde{Q})$ is a contraction as well. Clearly,

$$D^2_{\widetilde{T}} = D^2_T + T^*(I_{\mathfrak{H}} - X)P'_{\mathfrak{H}}T$$

Applying once again Proposition 3.3 we arrive to (3.9).

Since Q is a contraction, by Theorem 3.1 the operator S takes the form $S = D_{F^*}LD_G$, where $L \in \mathbf{L}(\mathfrak{D}_G, \mathfrak{D}_{F^*})$ is a contraction. Because $\widetilde{Q}$ given by (3.11) is a contraction, we get $X^{1/2}S = D_{\widetilde{F}^*}\widetilde{L}D_G$, where $\widetilde{F} = X^{1/2}F$ and $\widetilde{L} \in \mathbf{L}(\mathfrak{D}_G, \mathfrak{D}_{\widetilde{F}^*})$ is a contraction. Since

$$\begin{aligned} D^2_{\widetilde{F}^*} &= I_{\mathfrak{H}} - \widetilde{F}\widetilde{F}^* = I_{\mathfrak{H}} - X^{1/2}FF^*X^{1/2} \\ &= I_{\mathfrak{H}} - X + X - X^{1/2}FF^*X^{1/2} = I_{\mathfrak{H}} - X + X^{1/2}D^2_{F^*}X^{1/2}, \end{aligned}$$

we have $||D_{\widetilde{F}^*}f||^2 \geqslant ||D_{F^*}X^{1/2}f||^2$ for all $f \in \mathfrak{H}$. Using Theorem 1.2 we conclude that $\operatorname{ran} D_{\widetilde{F}^*} \supset \operatorname{ran}(X^{1/2}D_{F^*})$. Let $D^{-1}_{\widetilde{F}^*} = (I_{\mathfrak{H}} - X^{1/2}FF^*X^{1/2})^{-1/2}$ be the Moore-Penrose inverse for $D_{\widetilde{F}^*}$. Then we obtain the equality

$$\widetilde{L} = D^{-1}_{\widetilde{F}^*}(X^{1/2}D_{F^*})L.$$

The first equality in (3.5) yields $\left(D^2_{\widetilde{Q}}\right)_{\mathfrak{H}} = D_G D^2_{\widetilde{L}} D_G P_{\mathfrak{H}}$. Since

$$\left(D^2_{\widetilde{Q}}\right)_{\mathfrak{H}} = \left(D^2_Q + Q^*(I_{\mathfrak{H}} - X)P'_{\mathfrak{H}}Q\right)_{\mathfrak{H}},$$

we get (3.10). □

4. The Möbius representations

Let $T : H_1 \to H_2$ be a contraction and let $\mathscr{V}_{T^*}$ be the set of all contractions $Z \in \mathbf{L}(\mathfrak{D}_T, \mathfrak{D}_{T^*})$ such that $-1 \in \rho(T^*Z)$. In [28] the fractional-linear transformation

$$\mathscr{V}_{T^*} \ni Z \longmapsto Q = T + D_{T^*}Z(I_{\mathfrak{D}_T} + T^*Z)^{-1}D_T \tag{4.1}$$

was studied and the following result has been established.

THEOREM 4.1. [28]. *Let* $T \in \mathbf{L}(H_1, H_2)$ *be a contraction and let* $Z \in \mathcal{V}_{T^*}$. *Then* $Q = T + D_{T^*}Z(I_{\mathfrak{D}_T} + T^*Z)^{-1}D_T$ *is a contraction,*

$$||D_Q f||^2 = ||D_Z(I_{\mathfrak{D}_T} + T^*Z)^{-1}D_T f||,\ f \in H_1,$$

$\operatorname{ran} D_Q \subset \operatorname{ran} D_T$, *and* $\operatorname{ran} D_Q = \operatorname{ran} D_T$ *if and only if* $||Z|| < 1$. *Moreover, if* $Q \in \mathbf{L}(H_1, H_2)$ *is a contraction and* $Q = T + D_{T^*}XD_T$, *where* $X \in \mathbf{L}(\mathfrak{D}_T, \mathfrak{D}_{T^*})$ *then*

$$2\operatorname{Re}\left((I_{\mathfrak{D}_T} - T^*X)f, f\right) \geqslant ||f||^2$$

for all $f \in \mathfrak{D}_T$, *the operator* $Z = X(I_{\mathfrak{D}_T} - T^*X)^{-1}$ *belongs to* $\mathcal{V}_{T^*}$, *and*

$$Q = T + D_{T^*}Z(I_{\mathfrak{D}_T} + T^*Z)^{-1}D_T.$$

The transformation (4.1) is called in [28] the unitary linear-fractional transformation. If $||T|| < 1$ then the closed unit operator ball in $\mathbf{L}(H_1, H_2)$ belongs to the set $\mathcal{V}_{T^*}$ and, moreover

$$\begin{aligned} T + D_{T^*}Z(I_{H_1} + T^*Z)^{-1}D_T &= D_{T^*}^{-1}(Z + T)(I_{H_1} + T^*Z)^{-1}D_T \\ &= D_{T^*}(I_{H_2} + ZT^*)^{-1}(Z + T)D_T^{-1} \end{aligned}$$

for all $Z \in \mathbf{L}(H_1, H_2)$, $||Z|| \leqslant 1$. Thus, the transformation (4.1) is an operator analog of the well known Möbius transformation of the complex plane

$$z \to \frac{z + t}{1 + \bar{t}z},\ |t| \leqslant 1.$$

The next theorem is a version of the more general result established by Yu.L. Shmul'yan in [29].

THEOREM 4.2. *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be Hilbert spaces and let the function* $\Theta(\lambda)$ *be from the Schur class* $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$. *Then*

(1) *the linear manifolds* $\operatorname{ran} D_{\Theta(\lambda)}$ *and* $\operatorname{ran} D_{\Theta^*(\lambda)}$ *do not depend on* $\lambda \in \mathbb{D}$,
(2) *for arbitrary* λ_1, λ_2, λ_3 *in* $\mathbb{D}$ *the function* $\Theta(\lambda)$ *admits the representation*

$$\Theta(\lambda) = \Theta(\lambda_1) + D_{\Theta^*(\lambda_2)}\Psi(\lambda)D_{\Theta(\lambda_3)},$$

where $\Psi(\lambda)$ *is* $\mathbf{L}\left(\mathfrak{D}_{\Theta(\lambda_3)}, \mathfrak{D}_{\Theta^*(\lambda_2)}\right)$ *-valued function holomorphic in* $\mathbb{D}$.

Now using Theorems 4.1 and 4.2 we obtain the following result (cf. [14]).

THEOREM 4.3. *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be Hilbert spaces and let the function* $\Theta(\lambda)$ *be from the Schur class* $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$. *Then there exists a unique function* $Z(\lambda)$ *from the Schur class* $\mathbf{S}(\mathfrak{D}_{\Theta(0)}, \mathfrak{D}_{\Theta^*(0)})$ *such that*

$$\Theta(\lambda) = \Theta(0) + D_{\Theta^*(0)}Z(\lambda)(I_{\mathfrak{D}_{\Theta(0)}} + \Theta^*(0)Z(\lambda))^{-1}D_{\Theta(0)},\ \lambda \in \mathbb{D}. \tag{4.2}$$

We will say that the right hand side of (4.2) is the *Möbius representation* and the function $Z(\lambda)$ is the *Möbius parameter* of $\Theta(\lambda)$. Clearly, $Z(0) = 0$ and by Schwartz's lemma we obtain

$$||Z(\lambda)|| \leqslant |\lambda|,\ \lambda \in \mathbb{D}.$$

The next result provides the connections between the realizations of $\Theta(\lambda)$ and $Z(\lambda)$.

THEOREM 4.4.

(1) *Let* $\tau = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\right\}$ *be a passive system. Suppose* $D_D \neq 0$, $D_{D^*} \neq 0$, *and let*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -FD^*G + D_{F^*}LD_G & FD_D \\ D_{D^*}G & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}.$$

Let $\Theta(\lambda)$ *be the transfer function of* τ. *Then*

(a) *the Möbius parameter* $Z(\lambda)$ *of the function* $\Theta(\lambda)$ *is the transfer function of the passive system*

$$\nu = \left\{\begin{pmatrix} D_{F^*}LD_G & F \\ G & 0 \end{pmatrix}; \mathfrak{H}, \mathfrak{D}_D, \mathfrak{D}_{D^*}\right\};$$

(b) *the system* τ *is isometric (co-isometric)* $\Rightarrow$ *the system* ν *is isometric (co-isometric);*

(c) *the equalities* $\mathfrak{H}^c_\nu = \mathfrak{H}^c_\tau$, $\mathfrak{H}^o_\nu = \mathfrak{H}^o_\tau$ *hold and hence the system* τ *is controllable (observable)* $\Rightarrow$ *the system* ν *is controllable (observable), the system* τ *is simple (minimal)* $\Rightarrow$ *the system* ν *is simple (minimal).*

(2) *Let a nonconstant function* $\Theta(\lambda)$ *be from the Schur class* $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$ *and let* $Z(\lambda)$ *be the Möbius parameter of* $\Theta(\lambda)$. *Suppose that the transfer function of the linear system*

$$\nu' = \left\{\begin{pmatrix} S & F \\ G & 0 \end{pmatrix}; \mathfrak{H}, \mathfrak{D}_{\Theta(0)}, \mathfrak{D}_{\Theta^*(0)}\right\}$$

coincides with $Z(\lambda)$ *in a neighborhood of the origin. Then the transfer function of the linear system*

$$\tau' = \left\{\begin{pmatrix} -F\Theta^*(0)G + S & FD_{\Theta(0)} \\ D_{\Theta^*(0)}G & \Theta(0) \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\right\}$$

coincides with $\Theta(\lambda)$ *in a neighborhood of the origin. Moreover*

(a) *the equalities* $\mathfrak{H}^c_{\tau'} = \mathfrak{H}^c_{\nu'}$, $\mathfrak{H}^o_{\tau'} = \mathfrak{H}^o_{\nu'}$ *hold, and hence the system* ν' *is controllable (observable)* $\Rightarrow$ *the system* τ' *is controllable (observable), the system* ν' *is simple (minimal)* $\Rightarrow$ *the system* τ' *is simple (minimal),*

(b) *the system* ν' *is passive* $\Rightarrow$ *the system* τ' *is passive,*

(c) *the system* ν' *is isometric (co-isometric)* $\Rightarrow$ *the system* τ' *is isometric (co-isometric).*

Proof. Suppose $D \in \mathbf{L}(\mathfrak{M}, \mathfrak{N})$ is a contraction with nonzero defects. Given the operator matrix $Q = \begin{pmatrix} S & F \\ G & 0 \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_D \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_{D^*} \end{pmatrix}$. Let

$$T = \mathscr{M}_D(Q) = \begin{pmatrix} S - FD^*G & FD_D \\ D_{D^*}G & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}$$

and let Ω be a sufficiently small neighborhood of the origin. Consider the linear systems

$$\left\{\begin{pmatrix} S & F \\ G & 0 \end{pmatrix}; \mathfrak{H}, \mathfrak{D}_D, \mathfrak{D}_{D^*}\right\} \text{ and } \left\{\begin{pmatrix} -FD^*G + S & FD_D \\ D_{D^*}G & D \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\right\}$$

and define the transfer functions

$$\begin{aligned}&Z(\lambda)=\lambda G(I_{\mathfrak{H}}-\lambda S)^{-1}F \quad \text{and}\\ &\Theta(\lambda)=D+\lambda D_{D^*}G\left(I_{\mathfrak{H}}-\lambda(S-FD^*G)\right)^{-1}FD_D\end{aligned}$$

Since $\Theta(0)=D$, we have for $\lambda\in\Omega$

$$\begin{aligned}Z(\lambda)(I_{\mathfrak{D}_D}+\Theta^*(0)Z(\lambda))^{-1}&=\lambda G(I_{\mathfrak{H}}-\lambda S)^{-1}F\left(I_{\mathfrak{D}_D}+\lambda D^*G(I_{\mathfrak{H}}-\lambda S)^{-1}F\right)^{-1}\\&=\lambda G(I_{\mathfrak{H}}-\lambda S)^{-1}\left(I_{\mathfrak{H}}+\lambda FD^*G(I_{\mathfrak{H}}-\lambda S)^{-1}\right)^{-1}F\\&=\lambda G\left(I_{\mathfrak{H}}-\lambda S+\lambda FD^*G\right)^{-1}F.\end{aligned}$$

Hence

$$\Theta(\lambda)=\Theta(0)+D_{\Theta^*(0)}Z(\lambda)(I_{\mathfrak{D}_{\Theta(0)}}+\Theta^*(0)Z(\lambda))^{-1}D_{\Theta(0)},\ \lambda\in\Omega.$$

According to Theorem 3.1 the operator Q is a contraction if and only if F,G are contractions and $S=D_{F^*}LD_G$, where $L\in\mathbf{L}(\mathfrak{D}_G,\mathfrak{D}_{F^*})$ is a contraction. Now from Proposition 3.3 we get that all statements of Theorem 4.4 hold true. □

PROPOSITION 4.5. *Let $\Theta(\lambda)$ be a function from the Schur class $\mathbf{S}(\mathfrak{M},\mathfrak{N})$. Suppose that the Möbius parameter $Z(\lambda)$ of $\Theta(\lambda)$ is a linear function of the form $Z(\lambda)=\lambda K,\ ||K||\leqslant 1$. Then there exists a passive realization $\tau=\left\{\begin{pmatrix}A&B\\C&D\end{pmatrix};\mathfrak{H},\mathfrak{M},\mathfrak{N}\right\}$ such that*

$$\left(D^2_{P_{\mathfrak{N}}T}\right)_{\mathfrak{H}}=\left(D^2_T\right)_{\mathfrak{H}},\tag{4.3}$$

where $T=\begin{pmatrix}A&B\\C&D\end{pmatrix}$.

Conversely, if a passive system $\tau=\left\{\begin{pmatrix}A&B\\C&D\end{pmatrix};\mathfrak{H},\mathfrak{M},\mathfrak{N}\right\}$ possesses the property (4.3) *then the Möbius parameter $Z(\lambda)$ of the transfer function $\Theta(\lambda)$ of τ is a linear function of the form λK.*

Proof. Let $Z(\lambda)=\lambda K,\ \lambda\in\mathbb{D}$, where $K\in\mathbf{L}(\mathfrak{D}_{\Theta(0)},\mathfrak{D}_{\Theta^*(0)})$ is a contraction. Then $Z(\lambda)$ can be realized as the transfer function of a passive system of the form

$$\nu=\left\{\begin{pmatrix}0&F\\G&0\end{pmatrix};\mathfrak{H},\mathfrak{D}_{\Theta(0)},\mathfrak{D}_{\Theta^*(0)}\right\}.$$

Actually, take

$$\mathfrak{H}=\overline{\operatorname{ran}}\,K,\ F=K,\ G=j,$$

where j is the embedding of $\overline{\operatorname{ran}}\,K$ into $\mathfrak{D}_{\Theta^*(0)}$. It follows that $GF=K$. By Theorem 4.4 the system $\tau=\left\{\begin{pmatrix}-F\Theta^*(0)G&FD_{\Theta(0)}\\D_{\Theta^*(0)}G&\Theta(0)\end{pmatrix};\mathfrak{H},\mathfrak{M},\mathfrak{N}\right\}$ is a passive realization of the function $\Theta(\lambda)$. From (3.6) it follows that $\left(D^2_{P_{\mathfrak{N}}T}\right)_{\mathfrak{H}}=\left(D^2_T\right)_{\mathfrak{H}}$ for $T=\begin{pmatrix}-F\Theta^*(0)G&FD_{\Theta(0)}\\D_{\Theta^*(0)}G&\Theta(0)\end{pmatrix}$.

Assume now $\left(D^2_{P_{\mathfrak{N}}T}\right)_{\mathfrak{H}} = \left(D^2_T\right)_{\mathfrak{H}}$, where $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}$ is a contraction and let $\Theta(\lambda)$ be the transfer function of the system $\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\}$. Then the entries $A,\ B,$ and C take the form (1.6), and $D = \Theta(0)$. According to (3.6) we have $D_{F^*}LD_G = 0$, i.e. $T = \begin{pmatrix} -F\Theta^*(0)G & FD_{\Theta(0)} \\ D_{\Theta^*(0)}G & \Theta(0) \end{pmatrix}$. By Theorem 4.4 the Möbius parameter $Z(\lambda)$ of $\Theta(\lambda)$ takes the form $Z(\lambda) = \lambda GF$. □

REMARK 4.6. Suppose $\Theta(\lambda) = D \in \mathbf{L}(\mathfrak{M}, \mathfrak{N})$ for all λ. Let

$$\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\}$$

be a realization of Θ. Since

$$CA^nB = 0,\ B^*A^{*n}C^* = 0 \quad \text{for} \quad n = 0, 1, \ldots,$$

the minimal realization with a nontrivial state space $\mathfrak{H}$ does not exist.

Let $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ and suppose $D = \Theta(0)$ is isometric (co-isometric) but non-unitary. Then $\Theta(\lambda) = D$ for all $\lambda \in \mathbb{D}$ and $Z(\lambda) = 0 \in \mathbf{S}(0, \mathfrak{D}_{D^*})$ ($\in \mathbf{S}(\mathfrak{D}_D, 0)$). A passive and observable (controllable) realization of Θ is of the form

$$\tau = \left\{ \begin{pmatrix} A & 0 \\ D_{D^*}G & D \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\} \ \left(\tau = \left\{ \begin{pmatrix} A & FD_D \\ 0 & D \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\} \right),$$

where $G \in \mathbf{L}(\mathfrak{H}, \mathfrak{D}_{D^*})$, $F \in \mathbf{L}(\mathfrak{D}_D, \mathfrak{H})$ are contractions, $A = LD_G$ ($A = D_{F^*}L$), $L \in \mathbf{L}(\mathfrak{D}_G, \mathfrak{H})$ ($L \in \mathbf{L}(\mathfrak{H}, \mathfrak{D}_{F^*})$) is a contraction.

EXAMPLE 4.7. Let A be completely non-unitary contraction in the Hilbert space $\mathfrak{H}$ and let $\Phi(\lambda)$ be the Sz.-Nagy–Foias characteristic function of A^* [33]:

$$\Phi(\lambda) = \left(-A^* + \lambda D_A(I_{\mathfrak{H}} - \lambda A)^{-1}D_{A^*}\right) \upharpoonright \mathfrak{D}_{A^*},\ |\lambda| < 1.$$

Then $\Phi(\lambda) \in \mathbf{S}(\mathfrak{D}_{A^*}, \mathfrak{D}_A)$ and is the transfer function of the conservative and simple system

$$\tau = \left\{ \begin{pmatrix} A & D_{A^*} \\ D_A & -A^* \end{pmatrix}; \mathfrak{H}, \mathfrak{D}_{A^*}, \mathfrak{D}_A \right\}.$$

Let

$$\Phi(\lambda) = \Phi(0) + D_{\Phi^*(0)}Z(\lambda)(I_{\mathfrak{D}_{\Phi(0)}} + \Phi^*(0)Z(\lambda))^{-1}D_{\Phi(0)},\ \lambda \in \mathbb{D}$$

be the Möbius representation of the function $\Phi(\lambda)$. Since F and G^* are imbedding of the subspaces $\mathfrak{D}_{A^*}$ and $\mathfrak{D}_A$ into $\mathfrak{H}$, we get that

$$D_{F^*} = P_{\ker D_{A^*}},\ D_G = P_{\ker D_A}$$

and $L = A \upharpoonright \ker \mathfrak{D}_A$ is isometric operator. Let

$$\nu = \left\{ \begin{pmatrix} AP_{\ker D_A} & I \\ P_{\mathfrak{D}_A} & 0 \end{pmatrix}; \mathfrak{H}, \mathfrak{D}_{A^*}, \mathfrak{D}_A \right\}.$$

By Theorem 4.4

$$Z(\lambda) = \lambda P_{\mathfrak{D}_A}\left(I_{\mathfrak{H}} - \lambda A P_{\ker D_A}\right)^{-1} \upharpoonright \mathfrak{D}_{A^*},\ |\lambda| < 1$$

and this function is the transfer function of ν. Note that this function is precisely the Sz.-Nagy–Foias characteristic function of the partial isometry $A^*P_{\ker \mathfrak{D}_{A^*}}$.

5. The KYP inequality and the Riccati equation

Let $\mathfrak{H}$, $\mathfrak{M}$ and $\mathfrak{N}$ be Hilbert spaces and let T be a bounded linear operator from the Hilbert space $\mathscr{H} = \mathfrak{H} \oplus \mathfrak{M}$ into the Hilbert space $\mathscr{H}' = \mathfrak{H} \oplus \mathfrak{N}$ given by the block matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}.$$

Suppose that X is a positive selfadjoint operator in the Hilbert space $\mathfrak{H}$ such that

$$A \operatorname{dom} X^{1/2} \subset \operatorname{dom} X^{1/2},\ \operatorname{ran} B \subset \operatorname{dom} X^{1/2}.$$

As it was mentioned in Introduction the inequality (1.3)

$$\left\|\begin{pmatrix} X^{1/2} & 0 \\ 0 & I_{\mathfrak{M}} \end{pmatrix}\begin{pmatrix} x \\ u \end{pmatrix}\right\|^2 - \left\|\begin{pmatrix} X^{1/2} & 0 \\ 0 & I_{\mathfrak{N}} \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ u \end{pmatrix}\right\|^2 \geqslant 0$$
$$\text{for all} \quad x \in \operatorname{dom} X^{1/2},\ u \in \mathfrak{M}$$

is called the generalized KYP inequality for T with respect to X [7], [8]. For a bounded solution X the KYP inequality (1.3) takes the form (1.4).

Put

$$\widehat{\mathbf{X}} := \begin{pmatrix} X & 0 \\ 0 & I_{\mathfrak{M}} \end{pmatrix},\ \widetilde{\mathbf{X}} := \begin{pmatrix} X & 0 \\ 0 & I_{\mathfrak{N}} \end{pmatrix}.$$

Operators $\widehat{\mathbf{X}}$ and $\widetilde{\mathbf{X}}$ are positive selfadjoint operators in Hilbert spaces $\mathscr{H}$ and $\mathscr{H}'$ respectively, $\operatorname{dom} \widehat{\mathbf{X}} = \operatorname{dom} X \oplus \mathfrak{M}$, $\operatorname{dom} \widetilde{\mathbf{X}} = \operatorname{dom} X \oplus \mathfrak{N}$. Let the operator X satisfy the KYP inequality. Define the operator T_X :

$$\begin{aligned} &\operatorname{dom} T_X := \operatorname{ran} \widehat{\mathbf{X}}^{1/2} = \operatorname{ran} X^{1/2} \oplus \mathfrak{M}, \\ &T_X := \widetilde{\mathbf{X}}^{1/2} T \widehat{\mathbf{X}}^{-1/2} = \begin{pmatrix} X^{1/2} & 0 \\ 0 & I_{\mathfrak{N}} \end{pmatrix} T \begin{pmatrix} X^{-1/2} & 0 \\ 0 & I_{\mathfrak{M}} \end{pmatrix} \\ &\quad = \begin{pmatrix} X^{1/2} A X^{-1/2} & X^{1/2} B \\ C X^{-1/2} & D \end{pmatrix}. \end{aligned} \tag{5.1}$$

Clearly, the following statements are equivalent:

(1) X is a solution of the KYP inequality (1.3);

(2) the operator T_X is a densely defined contraction, i.e.,

$$\left\|\begin{pmatrix} X^{1/2}x \\ u \end{pmatrix}\right\|^2 - \left\|T_X \begin{pmatrix} X^{1/2}x \\ u \end{pmatrix}\right\|^2 \geqslant 0,\ x \in \operatorname{dom} X^{1/2},\ u \in \mathfrak{M};$$

(3) the operator T is a contraction acting from a pre-Hilbert space $\operatorname{dom}\widehat{\mathbf{X}}$ into a pre-Hilbert space $\operatorname{dom}\widetilde{\mathbf{X}}$ equipped by the inner products

$$\left\langle \begin{pmatrix} x_1 \\ u_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ u_2 \end{pmatrix} \right\rangle = (X^{1/2}x_1, X^{1/2}x_2)_{\mathfrak{H}} + (u_1, u_2)_{\mathfrak{M}},$$
$$\left\langle \begin{pmatrix} x_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ v_2 \end{pmatrix} \right\rangle = (X^{1/2}x_1, X^{1/2}x_2)_{\mathfrak{H}} + (v_1, v_2)_{\mathfrak{N}},$$
$$x_1, x_2 \in \operatorname{dom} X^{1/2},\ u_1, u_2 \in \mathfrak{M},\ v_1, v_2 \in \mathfrak{N};$$

(4) $Z = X^{-1}$ is the solution of the generalized KYP inequality for the adjoint operator

$$T^* = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix},$$

i.e.,

$$\left\| \begin{pmatrix} Z^{1/2} & 0 \\ 0 & I_{\mathfrak{N}} \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} Z^{1/2} & 0 \\ 0 & I_{\mathfrak{M}} \end{pmatrix} \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \right\|^2 \geqslant 0 \quad \text{for all} \quad x \in \operatorname{dom} Z^{1/2},\ v \in \mathfrak{N}. \tag{5.2}$$

In addition, if

$$T_X = \begin{pmatrix} A_1 & B_1 \\ C_1 & D \end{pmatrix}$$

is defined by (5.1) then

$$X^{1/2}Af = A_1 X^{1/2} f,\ C_1 X^{1/2} f = Cf,\ f \in \operatorname{dom} X^{1/2},\ B_1 = X^{1/2}B.$$

It follows that

$$X^{1/2}A^n B = A_1^n B_1,\ X^{1/2}A_1^{*n} C_1 = A^{*n} C,\ n = 0, 1, \dots, \tag{5.3}$$

the transfer functions of the systems $\tau = \{T; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ and $\tau_X = \{T_X; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ coincide in a neighborhood of the origin, and

$$\begin{aligned} \left\| \sum_{k=0}^n A_1^k B_1 u_k \right\| &= \left\| X^{1/2} \left(\sum_{k=0}^n A^k B u_k \right) \right\|, \\ \left\| \sum_{k=0}^n A_1^{*k} C_1^* u_k \right\| &= \left\| X^{-1/2} \left(\sum_{k=0}^n A^{*k} C^* u_k \right) \right\| \end{aligned} \tag{5.4}$$

for all $n = 0, 1, \dots,$ and for each input sequence $u_0, u_1, u_2, \dots$ in $\mathfrak{M}$.

By Theorem 1.1 the subset $\mathscr{X}_\tau$ of all solutions of the generalized KYP inequality (1.3) for a minimal system $\tau = \{T; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ contains the minimal and maximal elements $X_{\min}$ and $X_{\max}$, respectively. If, in addition, τ is a passive then $I_{\mathfrak{H}}$ is a solution of (1.3). Therefore, $X_{\min}$ and $X_{\max}^{-1}$ are positive contractions.

PROPOSITION 5.1. *Let*

$$\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\}$$

be a minimal and passive system. Suppose that X *is solution of* (1.3) *and* $X \leqslant X_{\min}$. *Then* $X = X_{\min}$.

Proof. Let $\Theta(\lambda)$ be the transfer function of τ and let $\dot{\tau} = \left\{\dot{T}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\right\}$, where

$$\dot{T} = \begin{pmatrix} \dot{A} & \dot{B} \\ \dot{C} & D \end{pmatrix}$$

is defined by (1.5). Then the system $\dot{\tau}$ is a passive minimal and optimal realization of Θ. Since $X_{\min} \leqslant I_{\mathfrak{H}}$, we have for all $n = 0, 1, \ldots,$ and for each input sequence $u_0, u_1, u_2, \ldots$ in $\mathfrak{M}$ the inequalities

$$\left\|\sum_{k=0}^{n} \dot{A}^k \dot{B} u_k\right\| \leqslant \left\|\sum_{k=0}^{n} A^k B u_k\right\|.$$

Construct the passive system $\tau_X = \{T_X; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$, where $T_X = \begin{pmatrix} A_1 & B_1 \\ C_1 & D \end{pmatrix}$ is defined by (5.1). From (5.3) it follows that τ_X is a passive and controllable realization of $\Theta(\lambda)$. Since $X \leqslant X_{\min}$, from (5.4) we get

$$\begin{aligned} \left\|\sum_{k=0}^{n} A_1^k B_1 u_k\right\| &= \left\|X^{1/2}\left(\sum_{k=0}^{n} A^k B u_k\right)\right\| \leqslant \left\|X_{\min}^{1/2}\left(\sum_{k=0}^{n} A^k B u_k\right)\right\| \\ &= \left\|\sum_{k=0}^{n} \dot{A}^k \dot{B} u_k\right\|. \end{aligned}$$

for all $n = 0, 1, \ldots,$ and for each input sequence $u_0, u_1, u_2, \ldots$ in $\mathfrak{M}$. Because $\dot{\tau}$ is optimal, we get $X = X_{\min}$. $\square$

Assume that the positive selfadjoint operator X in $\mathfrak{H}$ satisfies the KYP inequality. If

$$\inf_{u \in \mathfrak{M}} \left\{ \left\|\widehat{\mathbf{X}}^{1/2}\begin{pmatrix} x \\ u \end{pmatrix}\right\|^2 - \left\|\widetilde{\mathbf{X}}^{1/2} T \begin{pmatrix} x \\ u \end{pmatrix}\right\|^2 \right\} = 0 \quad \text{for all} \quad x \in \operatorname{dom} X^{1/2} \tag{5.5}$$

we will say that the operator X satisfies *the Riccati equation.*

PROPOSITION 5.2. *A positive selfadjoint operator X satisfies the Riccati equation* (5.5) *if and only if the continuation of the operator T_X defined by* (5.1) *meets the condition*

$$\left(D_{T_X}^2\right)_{\mathfrak{H}} = 0.$$

Proof. By (5.1) we get for all $\vec{f} \in \operatorname{dom} \widehat{\mathbf{X}}^{1/2}$

$$\left\|D_{T_X} \widehat{\mathbf{X}}^{1/2} \vec{f}\right\|^2 = ||\widehat{\mathbf{X}}^{1/2} \vec{f}||^2 - ||T_X \widehat{\mathbf{X}}^{1/2} \vec{f}||^2 = ||\widehat{\mathbf{X}}^{1/2} \vec{f}||^2 - ||\widetilde{\mathbf{X}}^{1/2} T \vec{f}||^2.$$

It follows that (5.5) holds if and only if

$$\inf_{u \in \mathfrak{M}} \left\{ \left\| D_{T_X} \begin{pmatrix} X^{1/2} x \\ u \end{pmatrix} \right\|^2 \right\} = 0 \quad \text{for all} \quad x \in \operatorname{dom} X^{1/2}.$$

Because $\operatorname{ran} X^{1/2}$ is dense in $\mathfrak{H}$, the latter is equivalent to $\left(D_{T_X}^2\right)_{\mathfrak{H}} = 0.$ $\square$

PROPOSITION 5.3. *Let $D \in \mathbf{L}(\mathfrak{M},\mathfrak{N})$ be a contraction with nonzero defect operators. Let*

$$Q = \begin{pmatrix} S & F \\ G & 0 \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_D \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_{D^*} \end{pmatrix}$$

and let

$$T = \mathscr{M}_D(Q) = \begin{pmatrix} S - FD^*G & FD_D \\ D_{D^*} & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}.$$

Then if X is a solution of the generalized KYP inequality (the Riccati equation) for Q then X is a solution of the generalized KYP inequality (the Riccatti equation) for T. Moreover, the KYP inequalities for T and Q are equivalent on the set of bounded solutions.

Proof. If X is a solution of the generalized KYP inequality for Q then $\operatorname{ran} F \subset \operatorname{dom} X^{1/2}$ and $S\operatorname{dom} X^{1/2} \subset \operatorname{dom} X^{1/2}$. It follows that

$$\operatorname{ran} FD_D \subset \operatorname{dom} X^{1/2},\ (S - FD^*G)\operatorname{dom} X^{1/2} \subset \operatorname{dom} X^{1/2}.$$

From (3.7) we have the relation

$$\mathscr{M}_D\left(\begin{pmatrix} X^{1/2} & 0 \\ 0 & I_{\mathfrak{D}_{D^*}} \end{pmatrix} Q \begin{pmatrix} X^{-1/2} & 0 \\ 0 & I_{\mathfrak{D}_D} \end{pmatrix}\right) = \begin{pmatrix} X^{1/2} & 0 \\ 0 & I_{\mathfrak{N}} \end{pmatrix} \mathscr{M}_D(Q) \begin{pmatrix} X^{-1/2} & 0 \\ 0 & I_{\mathfrak{M}} \end{pmatrix}.$$

Now the result follows from Propositions 3.3 and 5.2. □

6. Equivalent forms of the KYP inequality and the Riccati equation for a passive system

THEOREM 6.1. *Let*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -FD^*G + D_{F^*}LD_G & FD_D \\ D_{D^*}G & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}$$

be a contraction and let

$$Q = \begin{pmatrix} D_{F^*}LD_G & F \\ G & 0 \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_D \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_{D^*} \end{pmatrix}.$$

Then the following inequalities are equivalent

$$\begin{cases} \begin{pmatrix} X & 0 \\ 0 & I_{\mathfrak{M}} \end{pmatrix} - T^* \begin{pmatrix} X & 0 \\ 0 & I_{\mathfrak{N}} \end{pmatrix} T \geqslant 0 \\ 0 < X \leqslant I_{\mathfrak{H}} \end{cases}, \tag{6.1}$$

$$\begin{cases} \begin{pmatrix} X - A^*XA - C^*C & -A^*XB - C^*D \\ -B^*XA - D^*C & I_{\mathfrak{M}} - B^*XB - D^*D \end{pmatrix} \geqslant 0 \\ 0 < X \leqslant I_{\mathfrak{H}} \end{cases},$$

$$\begin{cases} (I_{\mathfrak{H}} - X)P_{\mathfrak{H}} \leqslant \left(D_T^2 + T^*(I_{\mathfrak{H}} - X)P'_{\mathfrak{H}}T\right)_{\mathfrak{H}} \\ 0 < X \leqslant I_{\mathfrak{H}} \end{cases}, \tag{6.2}$$

$$\begin{cases} \begin{pmatrix} X-G^*G & D_GL^*D_{F^*}X^{1/2} \\ X^{1/2}D_{F^*}LD_G & I_{\mathfrak H}-X^{1/2}FF^*X^{1/2} \end{pmatrix} \geqslant 0 \\ 0<X\leqslant I_{\mathfrak H} \end{cases}, \tag{6.3}$$

$$\begin{cases} X\geqslant G^*G+D_GL^*D_{F^*}X^{1/2}(I_{\mathfrak H}-X^{1/2}FF^*X^{1/2})^{-1}X^{1/2}D_{F^*}LD_G \\ 0<X\leqslant I_{\mathfrak H} \end{cases}, \tag{6.4}$$

$$\begin{cases} \begin{pmatrix} X & 0 \\ 0 & I_{\mathfrak D_D} \end{pmatrix} - Q^* \begin{pmatrix} X & 0 \\ 0 & I_{\mathfrak D_{D^*}} \end{pmatrix} Q \geqslant 0 \\ 0<X\leqslant I_{\mathfrak H} \end{cases}, \tag{6.5}$$

$$\begin{cases} \begin{pmatrix} X-G^*G-D_GL^*D_{F^*}XD_{F^*}LD_G & -D_GL^*D_{F^*}XF \\ -F^*XD_{F^*}LD_G & I_{\mathfrak D_D}-F^*XF \end{pmatrix} \geqslant 0 \\ 0<X\leqslant I_{\mathfrak H} \end{cases}, \tag{6.6}$$

$$\begin{cases} (I_{\mathfrak H}-X)P_{\mathfrak H}\leqslant \left(D_Q^2+Q^*(I_{\mathfrak H}-X)P'_{\mathfrak H}Q\right)_{\mathfrak H} \\ 0<X\leqslant I_{\mathfrak H} \end{cases}. \tag{6.7}$$

Recall (see Introduction) that for a bounded selfadjoint and nonnegative M and for a bounded N in the case $\operatorname{ran} N\subset \operatorname{ran} M^{1/2}$ the operator $N^*M^{-1}N$ is defined as $(M^{-1/2}N)^*M^{-1/2}N$. Here $M^{-1/2}$ is the Moore–Penrose pseudo-inverse to $M^{1/2}$.

Proof. Note that (6.6) is (6.5) rewritten in terms of the entries. By Proposition 5.3 the inequalities (6.1) and (6.5) are equivalent. Let us prove the equivalence of (6.1) and (6.2). Suppose that X satisfies (6.1) and put $Y=I_{\mathfrak H}-X$. The operator Y is nonnegative contraction and $\ker(I_{\mathfrak H}-Y)=\{0\}$. In terms of the operator Y we have

$$\begin{aligned} 0 &\leqslant \begin{pmatrix} X & 0 \\ 0 & I_{\mathfrak M} \end{pmatrix} - T^*\begin{pmatrix} X & 0 \\ 0 & I_{\mathfrak N} \end{pmatrix}T = \begin{pmatrix} I_{\mathfrak H}-Y & 0 \\ 0 & I_{\mathfrak M} \end{pmatrix} - T^*\begin{pmatrix} I_{\mathfrak H}-Y & 0 \\ 0 & I_{\mathfrak N} \end{pmatrix}T \\ &= I-T^*T+T^*YP'_{\mathfrak H}T-YP_{\mathfrak H}, \end{aligned}$$

i.e.,

$$(YP_{\mathfrak H}f,f)\leqslant \left(\left(D_T^2+T^*YP'_{\mathfrak H}T\right)(f+u),f+u\right),\; f\in\mathfrak H\oplus\mathfrak M,\; u\in\mathfrak M. \tag{6.8}$$

The equality (2.1) for the shorted operator yields that the operator Y is a solution of the system

$$\begin{cases} YP_{\mathfrak H}\leqslant \left(D_T^2+T^*YP'_{\mathfrak H}T\right)_{\mathfrak H}, \\ 0\leqslant Y<I_{\mathfrak H} \end{cases} \tag{6.9}$$

If X is a solution of the system (6.2) then $Y=I_{\mathfrak H}-X$ satisfies (6.8) and therefore X satisfies (6.1). Similarly (6.5) is equivalent to (6.7). Note that by Proposition 3.4 the right hand sides of (6.2) and (6.7) are equal. Using (3.10) we get that (6.7) is equivalent to (6.4). (6.3) is equivalent to (6.4) in accordance with (2.2). □

PROPOSITION 6.2. *Suppose $\Theta(\lambda)\in\mathbf S(\mathfrak M,\mathfrak N)$ and let $Z(\lambda)$ be the Möbius parameter of Θ. Then the passive minimal realization*

$$\nu=\left\{\begin{pmatrix} S & F \\ G & 0 \end{pmatrix}; \mathfrak H, \mathfrak D_{\Theta(0)}, \mathfrak D_{\Theta^*(0)}\right\}$$

of $Z(\lambda)$ *is optimal ((*)-optimal) if and only if the passive minimal realization*

$$\tau = \left\{ \begin{pmatrix} -F\Theta^*(0)G + S & FD_{\Theta(0)} \\ D_{\Theta^*(0)}G & \Theta(0) \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\}$$

of $\Theta(\lambda)$ *is optimal ((*)-optimal).*

Proof. According to Theorem 6.1 the set of all solutions of the KYP inequality (6.1) for $T = \begin{pmatrix} -F\Theta^*(0)G + S & FD_{\Theta(0)} \\ D_{\Theta^*(0)}G & \Theta(0) \end{pmatrix}$ coincides with the set of all solutions of the KYP inequality (6.5) for $Q = \begin{pmatrix} S & F \\ G & 0 \end{pmatrix}$. If the system ν (τ) is the optimal realization of $Z(\lambda)$ ($\Theta(\lambda)$) then by (5.4) the minimal solution of (6.5) in the set $\mathscr{X}_Q$ (of (6.1) in set $\mathscr{X}_T$, respectively) is $X_{\min} = I_{\mathfrak{H}}$. Therefore, the minimal solution of (6.1) (6.5) is $I_{\mathfrak{H}}$ as well. Thus, the system τ (ν) is the optimal realization of $\Theta(\lambda)$ ($Z(\lambda)$). Passing to the adjoint systems

$$\nu^* = \left\{ \begin{pmatrix} S^* & G^* \\ F^* & 0 \end{pmatrix}; \mathfrak{H}, \mathfrak{D}_{\Theta^*(0)}, \mathfrak{D}_{\Theta(0)} \right\}$$

and

$$\tau^* = \left\{ \begin{pmatrix} -G^*\Theta(0)F^* + S^* & G^*D_{\Theta^*(0)} \\ D_{\Theta(0)}F^* & \Theta^*(0) \end{pmatrix}; \mathfrak{H}, \mathfrak{N}, \mathfrak{M} \right\}$$

and their transfer functions $Z^*(\overline{\lambda})$ and $\Theta^*(\overline{\lambda})$, respectively, we get that ν is (*)-optimal if and only if τ is (*)-optimal. □

The next theorem is an immediate consequence of Theorem 6.1.

THEOREM 6.3. *Let*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -FD^*G + D_{F^*}LD_G & FD_D \\ D_{D^*}G & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}$$

be a contraction and let $Q := \begin{pmatrix} D_{F^*}LD_G & F \\ G & 0 \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_D \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_{D^*} \end{pmatrix}$. *Then the following equations are equivalent on the operator interval* $(0, I_{\mathfrak{H}}]$*:*

$$X - A^*XA - C^*C - (A^*XB + C^*D)(I_{\mathfrak{M}} - B^*XB - D^*D)^{-1}(B^*XA + D^*C) = 0, \quad (6.10)$$

$$(I_{\mathfrak{H}} - X)P_{\mathfrak{H}} = \left(D_T^2 + T^*(I_{\mathfrak{H}} - X)P'_{\mathfrak{H}}T\right)_{\mathfrak{H}}, \quad (6.11)$$

$$(I_{\mathfrak{H}} - X)P_{\mathfrak{H}} = \left(D_Q^2 + Q^*(I_{\mathfrak{H}} - X)P'_{\mathfrak{H}}Q\right)_{\mathfrak{H}}, \quad (6.12)$$

$$X - G^*G - S^*XS - S^*XF(I_{\mathfrak{H}} - F^*XF)^{-1}F^*XS = 0, \quad (6.13)$$

$$X = G^*G + S^*X^{1/2}(I_{\mathfrak{H}} - X^{1/2}FF^*X^{1/2})^{-1}X^{1/2}S, \quad (6.14)$$

where $S = D_{F^*}LD_G$. *Moreover, the equations* (6.11) – (6.14) *are equivalent to the equation* (5.5).

The equivalent equations (6.11) – (6.14) will be called the Riccati equations.

REMARK 6.4. Let $Q = \begin{pmatrix} S & F \\ G & 0 \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{L}_1 \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{L}_2 \end{pmatrix}$ be an isometric operator. Then F is isometry, D_{F^*} is the orthogonal projection, $S = D_{F^*}LD_G$, and $D_L = 0$. Denote $\mathscr{K} = \ker F^* = \operatorname{ran} D_{F^*}$. Since $D_{F^*} = P_{\mathscr{K}}$, we get

$$\begin{aligned} &D_G^2 - D_G L^* D_{F^*} X^{1/2}(I_{\mathfrak{H}} - X^{1/2}FF^*X^{1/2})^{-1}X^{1/2}D_{F^*}LD_G \\ &\qquad = S^* \left(P_{\mathscr{K}} - P_{\mathscr{K}}X^{1/2}(I_{\mathfrak{H}} - X^{1/2}P_{\mathscr{K}^\perp}X^{1/2})^{-1}X^{1/2}P_{\mathscr{K}}\right) S. \end{aligned}$$

Taking into account Theorem 2.2 and 6.4 we get the corresponding KYP inequality

$$I_{\mathfrak{H}} - X \leqslant S^* \left(I_{\mathfrak{H}} - X\right)_{\mathscr{K}} S.$$

REMARK 6.5. Suppose $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ and $D = \Theta(0)$ is isometric (co-isometric) but non-unitary. Then this function admits a passive realization of the form

$$\tau = \left\{ \begin{pmatrix} A & 0 \\ D_{D^*}G & D \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\}, \ \left(\tau = \left\{ \begin{pmatrix} A & FD_D \\ 0 & D \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\} \right),$$

where $A = LD_G$ $(A = D_{F^*}L)$, $L \in \mathbf{L}(\mathfrak{D}_G, \mathfrak{H})$ $(L \in \mathbf{L}(\mathfrak{H}, \mathfrak{D}_{F^*}))$ is a contraction (see Remark 4.6). The corresponding KYP inequality (6.4) is of the form

$$X \geqslant G^*G + D_G L^* X L D_G \qquad (X \geqslant L^* D_{F^*} X^{1/2}(I_{\mathfrak{H}} - X^{1/2}FF^*X^{1/2})^{-1}X^{1/2}D_{F^*}L).$$

EXAMPLE 6.6. Let $\mathfrak{H}$, $\mathfrak{M}$, and $\mathfrak{N}$ be separable Hilbert spaces. Suppose $G \in \mathbf{L}(\mathfrak{H}, \mathfrak{N})$ and $F \in \mathbf{L}(\mathfrak{M}, \mathfrak{H})$ are such that $G^*G = FF^* = \alpha I_{\mathfrak{H}}$, where $\alpha \in (0,1)$. Then $D_G = D_{F^*} = (1-\alpha)^{1/2} I_{\mathfrak{H}}$. Let L be a unitary operator in $\mathfrak{H}$. By Theorem 3.1 the operator

$$Q = \begin{pmatrix} (1-\alpha)L & F \\ G & 0 \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}$$

is a contraction. Consider the passive system

$$\nu = \left\{ \begin{pmatrix} (1-\alpha)L & F \\ G & 0 \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\}.$$

Because $\operatorname{ran} F = \operatorname{ran} G^* = \mathfrak{H}$, the system ν is minimal. The corresponding Riccati equation (6.14) takes the form

$$X = \alpha I_{\mathfrak{H}} + (1-\alpha)^2 L^* X (I_{\mathfrak{H}} - \alpha X)^{-1} L, \ 0 < X \leqslant I_{\mathfrak{H}}. \tag{6.15}$$

We will prove that $X = I_{\mathfrak{H}}$ is the unique solution of (6.15).

Put $W = (1-\alpha)(I_{\mathfrak{H}} - \alpha X)^{-1}$. Then $(1-\alpha)I_{\mathfrak{H}} < W \leqslant I_{\mathfrak{H}}$. From (6.15) we obtain the equation

$$L^*WL + W^{-1} = 2I_{\mathfrak{H}}, \tag{6.16}$$

Clearly, (6.16) has a solution $W = I_{\mathfrak{H}}$. Let W be any solution of (6.16) such that $(1-\alpha)I_{\mathfrak{H}} < W \leqslant I_{\mathfrak{H}}$, i.e., $\sigma(W) \subset [1-\alpha, 1]$. Since L is a unitary operator, from

(6.16) it follows that $\sigma(2I_{\mathfrak{H}} - W^{-1}) = \sigma(W)$. Let $\lambda_0 \in \sigma(W)$ then $2 - \lambda_0^{-1} \in \sigma(2I_{\mathfrak{H}} - W^{-1}) = \sigma(W)$. Since $2 - \lambda_0^{-1} > 0$, we get

$$\frac{1}{2} < \lambda_0 \leqslant 1.$$

Because $\mu_0 = 2 - \lambda_0^{-1} \in \sigma(W)$ and $1/2 < \mu_0 \leqslant 1$, we get

$$\frac{2}{3} < \lambda_0 \leqslant 1.$$

Thus

$$\frac{2}{3} < 2 - \lambda_0^{-1} \leqslant 1.$$

It follows that

$$\frac{3}{4} < \lambda_0 \leqslant 1.$$

Continuing these reasonings, we get

$$\frac{n}{n+1} < \lambda_0 \leqslant 1,\ n = 1, 2, \ldots .$$

It follows that $\lambda_0 = 1$, i.e. $\sigma(W) = \{1\}$. Because W is a selfadjoint operator we have $W = I_{\mathfrak{H}}$. Thus, $X = I_{\mathfrak{H}}$ is the unique solution of (6.15).

Let

$$\nu_* = \left\{ \begin{pmatrix} (1-\alpha)L^* & G^* \\ F^* & 0 \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\}.$$

be the adjoint system. Then the Riccati equation is of the form

$$Z = \alpha I_{\mathfrak{H}} + (1-\alpha)^2 LZ(I_{\mathfrak{H}} - \alpha Z)^{-1} L^*,\ 0 < Z \leqslant I_{\mathfrak{H}}.$$

Hence, $Z = I_{\mathfrak{H}}$ is its unique solution. It follows that the system ν is the optimal and $(*)$-optimal realization of the Schur class function $\Theta(\lambda) = \lambda G(I_{\mathfrak{H}} - \lambda(1-\alpha)L)^{-1}F$.

7. Properties of solutions of the KYP inequality and the Riccati equation

PROPOSITION 7.1. *Let* $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}$ *be a contraction. Suppose that the passive system* $\tau = \{T; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ *is observable. If a nonnegative contraction* X *in* $\mathfrak{H}$ *is a solution of the inequality*

$$(I_{\mathfrak{H}} - X)P_{\mathfrak{H}} \leqslant \left(D_T^2 + T^*(I_{\mathfrak{H}} - X)P'_{\mathfrak{H}} T\right)_{\mathfrak{H}} \tag{7.1}$$

then $\ker X = \{0\}$.

Proof. Suppose that X satisfies (7.1) and $\ker X \neq \{0\}$. Then there is a nonzero vector x in $\mathfrak{H}$ such that $(I_{\mathfrak{H}} - X)x = x$. Since $D_T^2 + T^*(I_{\mathfrak{H}} - X)P'_{\mathfrak{H}} T$ is a contraction, we obtain $\left(D_T^2 + T^*(I_{\mathfrak{H}} - X)P'_{\mathfrak{H}} T\right)_{\mathfrak{H}} x = x$ and hence $D_T^2 x + T^*(I_{\mathfrak{H}} - X)P'_{\mathfrak{H}} Tx = x$. It follows that

$$P_{\mathfrak{N}} Tx = 0,\ XP'_{\mathfrak{H}} Tx = 0.$$

This means that $Cx = 0$ and $XAx = 0$. Replacing x by Ax we get $CAx = 0$ and $XA^2x = 0$. By induction $CA^nx = 0$ for all $n = 0, 1, \ldots$. Since the system τ is observable, we get $x = 0$. □

THEOREM 7.2. *Let* $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}$ *be a contraction. The following statements are valid:*

(1) *each X from the operator interval*

$$[I_{\mathfrak{H}} - \left(D_T^2\right)_{\mathfrak{H}} \upharpoonright \mathfrak{H}, I_{\mathfrak{H}}] \tag{7.2}$$

is a solution of (7.1) *and even a solution of* (6.2) *when the system* $\tau = \{T; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ *is observable;*

(2) *each solution X of the inequality* (7.1) *satisfies the estimate*

$$X \geqslant I_{\mathfrak{H}} - \left(D^2_{P_{\mathfrak{N}}T}\right)_{\mathfrak{H}} \upharpoonright \mathfrak{H},$$

hence, in the case

$$\left(D_T^2\right)_{\mathfrak{H}} = \left(D^2_{P_{\mathfrak{N}}T}\right)_{\mathfrak{H}},$$

the operator $X = I_{\mathfrak{H}} - \left(D^2_{P_{\mathfrak{N}}T}\right)_{\mathfrak{H}} \upharpoonright \mathfrak{H}$ *is the minimal solution of* (7.1)*;*

(3) *if the system* $\tau = \{T; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ *is minimal and* $\left(D^2_{T^*}\right)_{\mathfrak{H}} \neq 0$*, then each positive selfadjoint X possessing properties* $I_{\mathfrak{H}} \leqslant X \leqslant (I_{\mathfrak{H}} - \left(D^2_{T^*}\right)_{\mathfrak{H}} \upharpoonright \mathfrak{H})^{-1}$ *(in the sense of quadratic forms) is a solution of the generalized KYP inequality* (1.3).

Proof. Let X be a solution of (7.1) then $Y = I_{\mathfrak{H}} - X$ is a solution of the inequality

$$YP_{\mathfrak{H}} \leqslant \left(D_T^2 + T^*YP'_{\mathfrak{H}}T\right)_{\mathfrak{H}}.$$

In view of

$$D_T^2 + T^*YP'_{\mathfrak{H}}T \leqslant D_T^2 + T^*P'_{\mathfrak{H}}T = I - T^*P_{\mathfrak{N}}T = D^2_{P_{\mathfrak{N}}T}$$

we get

$$Y \leqslant \left(D^2_{P_{\mathfrak{N}}T}\right)_{\mathfrak{H}} \upharpoonright \mathfrak{H}. \tag{7.3}$$

Hence $X \geqslant I_{\mathfrak{H}} - \left(D^2_{P_{\mathfrak{N}}T}\right)_{\mathfrak{H}} \upharpoonright \mathfrak{H}$.

By Proposition 2.1 we have

$$\left(D_T^2 + T^*\left(D_T^2\right)_{\mathfrak{H}} P'_{\mathfrak{H}}T\right)_{\mathfrak{H}} \geqslant \left(D_T^2\right)_{\mathfrak{H}} P_{\mathfrak{H}}.$$

If $Y \in [0, \left(D_T^2\right)_{\mathfrak{H}} \upharpoonright \mathfrak{H}]$ then

$$YP_{\mathfrak{H}} \leqslant \left(D_T^2\right)_{\mathfrak{H}} P_{\mathfrak{H}} \leqslant \left(D_T^2 + T^*YP'_{\mathfrak{H}}T\right)_{\mathfrak{H}}.$$

It follows that each X from the operator interval (7.2) is a solution of (7.1). If, in addition, the system τ is observable, then by Proposition 7.1 any X from (7.2) is positive.

Statement (3) follows from Statement (1) and from the fact that if Z is a solution of the KYP inequality for T^* then $X = Z^{-1}$ is the solution of the KYP inequality for T. $\square$

COROLLARY 7.3. *Let* $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ *and let*

$$\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\}$$

be a passive and minimal realization of Θ. *The following statements are valid:*

(1) *if the system* τ *is optimal then*

$$(D_T^2)_{\mathfrak{H}} = 0; \tag{7.4}$$

(2) *if* $(D^2_{P_{\mathfrak{N}}T})_{\mathfrak{H}} = 0$ *then the system* τ *is optimal.*

Proof. If τ is optimal realization of $\Theta(\lambda)$ then $X_{\min} = I_{\mathfrak{H}}$. By Theorem 7.2 the operator $X = I_{\mathfrak{H}} - (D_T^2)_{\mathfrak{H}} \upharpoonright \mathfrak{H}$ is a solution of (6.2). Since $X \leqslant X_{\min}$, by Proposition 5.1 $X = X_{\min}$, i.e., $X = I_{\mathfrak{H}}$. Hence, $(D_T^2)_{\mathfrak{H}} = 0$.

Suppose $(D^2_{P_{\mathfrak{N}}T})_{\mathfrak{H}} = 0$. Since $D_T^2 \leqslant D^2_{P_{\mathfrak{N}}T}$ and $(D^2_{P_{\mathfrak{N}}T})_{\mathfrak{H}} = 0$, we obtain $(D_T^2)_{\mathfrak{H}} = 0$. By Theorem 7.2 in this case the minimal solution of (1.4) is $X_{\min} = I_{\mathfrak{H}}$. This means that τ is the optimal realization of Θ. $\square$

PROPOSITION 7.4. *Let* $\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\}$ *be a passive minimal system. Then the minimal solution of the KYP inequality* (6.1) *satisfies the Riccati equations* (6.10) – (6.14).

Proof. Since the system τ is passive and minimal, the minimal solution $X_{\min}$ of (1.3) satisfies $0 < X_{\min} \leqslant I_{\mathfrak{H}}$. Let the operator $\dot{T}$ be defined by (1.5), i.e.,

$$\dot{T} := \begin{pmatrix} X_{\min}^{1/2} & 0 \\ 0 & I_{\mathfrak{N}} \end{pmatrix} T \begin{pmatrix} X_{\min}^{-1/2} & 0 \\ 0 & I_{\mathfrak{M}} \end{pmatrix}, \ \operatorname{dom} \dot{T} = \operatorname{ran} X_{\min}^{1/2} \oplus \mathfrak{M}.$$

The operator $\dot{T}$ is a densely defined contraction. We preserve the notation $\dot{T}$ for its continuation. The system $\dot{\tau} = \{\dot{T}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ is the passive minimal and optimal realization of the transfer function $\Theta(\lambda) = D + \lambda C(I - \lambda A)^{-1}B$, $|\lambda| < 1$ for the system τ. According to Corollary 7.3 the operator $\dot{T}$ satisfies the condition $(D^2_{\dot{T}})_{\mathfrak{H}} = 0$. It follows that

$$\inf_{u \in \mathfrak{M}} \left\{ \left\| D_{\dot{T}} \begin{pmatrix} g \\ u \end{pmatrix} \right\|^2 \right\} = 0$$

for all $g \in \mathfrak{H}$. In particular

$$\inf_{u \in \mathfrak{M}} \left\{ \left\| D_{\dot{T}} \begin{pmatrix} X_{\min}^{1/2} & 0 \\ 0 & I_{\mathfrak{M}} \end{pmatrix} \begin{pmatrix} g \\ u \end{pmatrix} \right\|^2 \right\} = 0, \ g \in \mathfrak{H}.$$

Since

$$\begin{pmatrix} X_{\min} & 0 \\ 0 & I_{\mathfrak{M}} \end{pmatrix} - T^* \begin{pmatrix} X_{\min} & 0 \\ 0 & I_{\mathfrak{N}} \end{pmatrix} T = \begin{pmatrix} X_{\min}^{1/2} & 0 \\ 0 & I_{\mathfrak{M}} \end{pmatrix} D_{\bf T}^2 \begin{pmatrix} X_{\min}^{1/2} & 0 \\ 0 & I_{\mathfrak{M}} \end{pmatrix},$$

we get

$$\left(\begin{pmatrix} X_{\min} & 0 \\ 0 & I_{\mathfrak{M}} \end{pmatrix} - T^* \begin{pmatrix} X_{\min} & 0 \\ 0 & I_{\mathfrak{N}} \end{pmatrix} T\right)_{\mathfrak{H}} = 0.$$

Hence, if $Y_0 = I_{\mathfrak{H}} - X_{\min}$ then

$$\begin{pmatrix} I_{\mathfrak{H}} - Y_0 & 0 \\ 0 & I_{\mathfrak{M}} \end{pmatrix} - T^* \begin{pmatrix} I_{\mathfrak{H}} - Y_0 & 0 \\ 0 & I_{\mathfrak{N}} \end{pmatrix} T = \\ = D_T^2 + T^* Y_0 P'_{\mathfrak{H}} T - Y_0 P_{\mathfrak{H}}.$$

Since

$$\left(\begin{pmatrix} I_{\mathfrak{H}} - Y_0 & 0 \\ 0 & I_{\mathfrak{M}} \end{pmatrix} - T^* \begin{pmatrix} I_{\mathfrak{H}} - Y_0 & 0 \\ 0 & I_{\mathfrak{N}} \end{pmatrix} T\right)_{\mathfrak{H}} = 0,$$

we get

$$0 = (D_T^2 + T^* Y_0 P'_{\mathfrak{H}} T - Y_0 P_{\mathfrak{H}})_{\mathfrak{H}} = (D_T^2 + T^* Y_0 P'_{\mathfrak{H}} T)_{\mathfrak{H}} - Y_0 P_{\mathfrak{H}}.$$

Thus, Y_0 satisfies the equation $Y P_{\mathfrak{H}} = (D_T^2 + T^* Y P'_{\mathfrak{H}} T)_{\mathfrak{H}}$ and $X_{\min} = I_{\mathfrak{H}} - Y_0$ satisfies the equation (6.11). □

REMARK 7.5. Let $\tau = \{T; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ be a minimal passive system and let

$$T = \begin{pmatrix} -FD^*G + D_{F^*} L D_G & F D_D \\ D_{D^*} G & D \end{pmatrix}.$$

Then the statements of Theorem 7.2 and Corollary 7.3 can be reformulated as follows:

(1) each X from the operator interval $[G^*G + D_G L^* L D_G, I_{\mathfrak{H}}]$ is a solution of (6.3) and every solution of (6.3) satisfies the estimate $X \geqslant G^*G$;
(2) if $L = 0$ then $X_0 = G^*G$ is the minimal solution of (6.3) (cf. [9]);
(3) if $D_{L^*} D_{F^*} \neq 0$ then each selfadjoint X in $\mathfrak{H}$ possessing the properties $I_{\mathfrak{H}} \leqslant X \leqslant (FF^* + D_{F^*} L L^* D_{F^*})^{-1}$ is a solution of the generalized KYP inequality (1.3);
(4) if the system τ is minimal and optimal realization of the function $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ then $D_L D_G = 0$;
(5) if the system τ is minimal realization of the function $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ and if G is isometry then the system τ is optimal.

REMARK 7.6. Let $\tau = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix}; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\right\}$ be a minimal system with the transfer function $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$. Suppose that the bounded positive selfadjoint operator X is such that the operator

$$\delta(X) = I_{\mathfrak{M}} - D^*D - B^*XB$$

is positive definite. Then the KYP inequality

$$L(X)=\begin{pmatrix} X-A^*XA-C^*C & -A^*XB-C^*D \\ -B^*XA-D^*C & I-B^*XB-D^*D \end{pmatrix}\geqslant 0$$

is equivalent to the inequality $R(X)\geqslant 0$, where

$$R(X)=X-A^*XA-C^*C-(B^*XA+D^*C)(I_{\mathfrak{M}}-D^*D-B^*XB)^{-1}(A^*XB+C^*D)$$

is the corresponding Schur complement. Then for the minimal solution $X_{\min}$ of the KYP inequality we have $\delta(X_{\min})\geqslant\delta(X)$ and $R(X_{\min})\geqslant 0$. For a finite dimensional $\mathfrak{H}$ it was shown in [26] that the minimal solution $X_{\min}$ satisfies the algebraic Riccati equation $R(X_{\min})=0$. Thus, the statement of Proposition 7.4 is the generalization of the result in [26].

PROPOSITION 7.7. *Let $\Theta(\lambda)\in\mathbf{S}(\mathfrak{M},\mathfrak{N})$ and let the Möbius parameter $Z(\lambda)$ of Θ be of the form $Z(\lambda)=\lambda K$, $K\in\mathbf{L}(\mathfrak{D}_{\Theta(0)},\mathfrak{D}_{\Theta^*(0)})$, $K\neq 0$. Then*

(1) *the minimal passive and optimal realization τ of Θ is unitarily equivalent to the system*

$$\tau=\left\{\begin{pmatrix} -K\Theta^*(0) & KD_{\Theta(0)} \\ D_{\Theta^*(0)} & \Theta(0)\end{pmatrix};\overline{\operatorname{ran}}\,K,\mathfrak{M},\mathfrak{N}\right\};$$

(2) *the minimal passive and $(*)$- optimal realization τ of Θ is unitarily equivalent to the system*

$$\eta=\left\{\begin{pmatrix} -P_{\overline{\operatorname{ran}}\,K^*}\Theta^*(0)K\upharpoonright\overline{\operatorname{ran}}\,K^* & P_{\overline{\operatorname{ran}}\,K^*}D_{\Theta(0)} \\ D_{\Theta^*(0)}K\upharpoonright\overline{\operatorname{ran}}\,K^* & \Theta(0)\end{pmatrix};\overline{\operatorname{ran}}\,K^*,\mathfrak{M},\mathfrak{N}\right\};$$

Proof. Let j be the embedding of $\overline{\operatorname{ran}}\,K$ into $\mathfrak{D}_{\Theta^*(0)}$. Then the system

$$\nu=\left\{\begin{pmatrix} 0 & K \\ j & 0\end{pmatrix};\overline{\operatorname{ran}}\,K,\mathfrak{D}_{\Theta(0)},\mathfrak{D}_{\Theta^*(0)}\right\}.$$

is a passive and minimal realization of the function $Z(\lambda)=\lambda K$ (see Proposition 4.5). The corresponding Riccati equation (6.13) takes the form $X=I_{\overline{\operatorname{ran}}\,K}$. By Remark 7.5, the system ν is the optimal realization of $Z(\lambda)=\lambda K$. From Proposition 6.2 it follows that the system

$$\tau=\left\{\begin{pmatrix} -K\Theta^*(0) & KD_{\Theta(0)} \\ D_{\Theta^*(0)} & \Theta(0)\end{pmatrix};\overline{\operatorname{ran}}\,K,\mathfrak{M},\mathfrak{N}\right\}$$

is the minimal passive and optimal realization of Θ.

The system

$$\sigma=\left\{\begin{pmatrix} 0 & P_{\overline{\operatorname{ran}}\,K^*} \\ K\upharpoonright\overline{\operatorname{ran}}\,K^* & 0\end{pmatrix};\overline{\operatorname{ran}}\,K^*,\mathfrak{D}_{\Theta(0)},\mathfrak{D}_{\Theta^*(0)}\right\}$$

is the passive and minimal realization of the function λK. The KYP inequality (6.4) for the adjoint system σ^* takes the form

$$\begin{cases} X\geqslant I_{\overline{\operatorname{ran}}\,K^*}, \\ 0\leqslant X\leqslant I_{\overline{\operatorname{ran}}\,K^*}\end{cases}.$$

So, $X = I_{\overline{\mathrm{ran}}\,K^*}$ is the minimal solution. It is the minimal solution of the generalized KYP inequality for σ^*. Hence, $X = I_{\overline{\mathrm{ran}}\,K^*}$ is the maximal solution of the generalized KYP inequality for σ. It follows that σ is a $(*)$-optimal realization of $Z(\lambda) = \lambda K$ and by Proposition 6.2 the system η is a $(*)$-optimal realization of $\Theta(\lambda)$. $\square$

The next theorems provide the sufficient uniqueness conditions for the solutions of the Riccati equation.

THEOREM 7.8. *Assume that a contraction* $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}$ *possesses the properties*

$$\begin{cases} \left(D_T^2\right)_{\mathfrak{H}} = 0, \\ \operatorname{ran}\left(\left(D^2_{P_{\mathfrak{N}}T}\right)_{\mathfrak{H}}\right)^{1/2} \bigcap \operatorname{ran}\left(\left(D^2_{P_{\mathfrak{M}}T^*}\right)_{\mathfrak{H}}\right)^{1/2} \subset \operatorname{ran}\left(\left(D^2_{T^*}\right)_{\mathfrak{H}}\right)^{1/2}. \end{cases} \tag{7.5}$$

Then the passive system $\tau = \{T; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ *is observable and* $X = I_{\mathfrak{H}}$ *is the unique solution of the Riccati equation* (6.11).

Proof. The operator T takes the form

$$T = \begin{pmatrix} -FD^*G + D_{F^*}LD_G & FD_D \\ D_{D^*}G & D \end{pmatrix}$$

with contractions $D,\ F,\ G$, and L. By (3.5) the conditions (7.5) are equivalent to the following

$$\begin{cases} D_L D_G = 0 \\ \operatorname{ran} D_G \cap \operatorname{ran} D_{F^*} \subset \operatorname{ran}\left(D_{F^*} D_{L^*}\right) \end{cases}. \tag{7.6}$$

If $\left(D^2_{P_{\mathfrak{N}}T}\right)_{\mathfrak{H}} = 0$ then $D_G = 0$ and $X = I_{\mathfrak{H}}$ is a unique solution of (6.11). Moreover, because G is an isometry, the system τ is observable.

Assume $\left(D^2_{P_{\mathfrak{N}}T}\right)_{\mathfrak{H}} \neq 0$. Since $(D_T^2)_{\mathfrak{H}} = 0$, from the equivalences (3.6) it follows that

$$\left(D^2_{P_{\mathfrak{M}}T^*}\right)_{\mathfrak{H}} \neq 0.$$

From (7.5) and (3.5) we have $D_G \neq 0,\ D_{F^*} \neq 0,\ D_L = 0$.

Let

$$Q = \begin{pmatrix} D_{F^*}LD_G & F \\ G & 0 \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_D \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{D}_{D^*} \end{pmatrix}$$

and let $\nu = \{Q, \mathfrak{H}, \mathfrak{D}_D, \mathfrak{D}_{D^*}\}$ be the corresponding passive system. Let us show that ν is observable. Suppose $f \in \bigcap\limits_{n=0}^{\infty} \ker\left(G(D_{F^*}LD_G)^n\right)$. Then

$$Gf = 0 \Rightarrow D_G f = f,\ G(D_{F^*}LD_G)f = 0 \Rightarrow D_G D_{F^*}Lf = D_{F^*}Lf.$$

From (7.6)

$$D_{F^*}Lf \in \operatorname{ran} D_{F^*}D_{L^*} \Rightarrow Lf \in \operatorname{ran} D_{L^*}.$$

Since L is an isometry, $\operatorname{ran} L \perp \mathfrak{D}_{L^*}$. Now we get $Lf = 0 \Rightarrow f = 0$. Thus, the system ν is observable and by Theorem 4.4 the system τ is observable too.

According to Proposition 6.3 the equation (6.11) is equivalent to the equation (6.14). We will prove that (6.14) has the unique solution $X = I_{\mathfrak{H}}$. Suppose X is a solution. Define $\Psi := I_{\mathfrak{H}} - X^{1/2}FF^*X^{1/2}$. Since $\Psi = I_{\mathfrak{H}} - X + X^{1/2}D^2_{F^*}X^{1/2}$, we have $\Psi \geqslant I_{\mathfrak{H}} - X$ and $\Psi \geqslant X^{1/2}D^2_{F^*}X^{1/2}$. Therefore

$$(I_{\mathfrak{H}} - X)^{1/2} = U\Psi^{1/2},\ D_{F^*}X^{1/2} = V\Psi^{1/2},$$

where $U : \overline{\operatorname{ran}}\,\Psi^{1/2} \to \overline{\operatorname{ran}}\,(I_{\mathfrak{H}} - X)^{1/2}$, $V : \overline{\operatorname{ran}}\,\Psi^{1/2} \to \overline{\operatorname{ran}}\,D_{F^*} = \overline{\operatorname{ran}}\,(D_{F^*}X^{1/2})$, and $U^*U + V^*V = I_{\overline{\operatorname{ran}}\,\Psi^{1/2}}$. Hence $U^*U = D^2_V$. Since $X^{1/2}D_{F^*} = \Psi^{1/2}V^*$, we get

$$X^{1/2}D_{F^*}D_{V^*} = \Psi^{1/2}V^*D_{V^*} = \Psi^{1/2}D_V V^*.$$

From

$$I_{\mathfrak{H}} - X = \Psi^{1/2}U^*U\Psi^{1/2} = \Psi^{1/2}D^2_V\Psi^{1/2}$$

we get that $\operatorname{ran}(I_{\mathfrak{H}} - X)^{1/2} = \Psi^{1/2}\operatorname{ran}D_V$. Therefore,

$$\operatorname{ran}(X^{1/2}D_{F^*}D_{V^*}) \subset \operatorname{ran}(I_{\mathfrak{H}} - X)^{1/2}. \tag{7.7}$$

Using the well known relation

$$\operatorname{ran}X^{1/2} \cap \operatorname{ran}(I_{\mathfrak{H}} - X)^{1/2} = \operatorname{ran}(X^{1/2}(I_{\mathfrak{H}} - X)^{1/2})$$

for every $X \in [0, I_{\mathfrak{H}}]$, from (7.7) we get

$$D_{F^*}\operatorname{ran}D_{V^*} \subset \operatorname{ran}(I_{\mathfrak{H}} - X)^{1/2}.$$

The equation (6.14) can be rewritten as follows

$$X = G^*G + D_G L^* VV^* L D_G.$$

Since $L^*L = I_{\mathfrak{D}_G}$, we get $I_{\mathfrak{H}} - X = D_G L^* D^2_{V^*} L D_G$. It follows that

$$\operatorname{ran}(I_{\mathfrak{H}} - X)^{1/2} = D_G L^* \operatorname{ran}D_{V^*} \subset \operatorname{ran}D_G.$$

Now we obtain

$$D_{F^*}\operatorname{ran}D_{V^*} \subset \operatorname{ran}D_G \cap \operatorname{ran}D_{F^*} \subset D_{F^*}\operatorname{ran}D_{L^*}.$$

Hence $\operatorname{ran}D_{V^*} \subset \operatorname{ran}D_{L^*}$. Since $L : \mathfrak{D}_G \to \mathfrak{D}_{F^*}$ is isometry, we get $\ker L^* = \operatorname{ran}D_{L^*}$. Therefore $L^* \upharpoonright \operatorname{ran}D_{V^*} = 0$. It follows $\operatorname{ran}(I_{\mathfrak{H}} - X)^{1/2} = \{0\}$, i.e., $X = I_{\mathfrak{H}}$. $\square$

Observe, Example 6.6 shows that conditions (7.5) are not necessary for the uniqueness of the solutions of the KYP inequality (6.2).

THEOREM 7.9. *Assume that a contraction* $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}$ *possesses the properties*

$$\begin{cases} \left(D^2_T\right)_{\mathfrak{H}} = 0,\ \left(D^2_{T^*}\right)_{\mathfrak{H}} = 0, \\ \operatorname{ran}\left(\left(D^2_{P_{\mathfrak{N}}T}\right)_{\mathfrak{H}}\right)^{1/2} \bigcap \operatorname{ran}\left(\left(D^2_{P_{\mathfrak{M}}T^*}\right)_{\mathfrak{H}}\right)^{1/2} = \{0\} \end{cases}. \tag{7.8}$$

Let $\Theta(\lambda)$ *be the transfer function of the passive system* $\tau = \{T; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$. *Then every minimal and passive realization of* Θ *is the unitarily equivalent to the system* τ.

Proof. Using Theorem 7.8 we see that under the conditions (7.8) the systems τ and $\tau^* = \{T^*, \mathfrak{H}, \mathfrak{N}, \mathfrak{M}\}$ are observable. So, τ is minimal.

By Theorem 7.8 the identity operator $I_{\mathfrak{H}}$ is the minimal solutions of the KYP inequality (6.2) for T and the KYP inequality

$$\begin{cases} (I_{\mathfrak{H}} - Z)P_{\mathfrak{H}} \leqslant \left(D^2_{T^*} + T(I_{\mathfrak{H}} - Z)P_{\mathfrak{H}}T^*\right)_{\mathfrak{H}} \\ 0 < Z \leqslant I_{\mathfrak{H}} \end{cases}$$

for T^*. It follows that $X_{\min} = X^{-1}_{\max} = I_{\mathfrak{H}}$, where $X_{\min}$ and $X_{\max}$ are the minimal and maximal solutions, respectively, of the generalized KYP inequality (1.3) for T. Therefore, the system τ is the optimal and $(*)$-optimal. Hence, any passive and minimal realization of the transfer function $\Theta(\lambda)$ of τ is unitarily equivalent to τ. □

REMARK 7.10. The conditions (7.5) are equivalent to the following:

$$\begin{cases} \mathfrak{H} \cap \operatorname{ran} D_T = \{0\}, \\ \left(\mathfrak{H} \cap \operatorname{ran} D_{P_{\mathfrak{N}}T}\right) \bigcap \left(\mathfrak{H} \cap \operatorname{ran} D_{P_{\mathfrak{M}}T^*}\right) \subset \mathfrak{H} \cap \operatorname{ran} D_{T^*}, \end{cases}$$

and

$$(7.8) \iff \begin{cases} \mathfrak{H} \cap \operatorname{ran} D_T = \mathfrak{H} \cap \operatorname{ran} D_{T^*} = \{0\}, \\ \left(\mathfrak{H} \cap \operatorname{ran} D_{P_{\mathfrak{N}}T}\right) \bigcap \left(\mathfrak{H} \cap \operatorname{ran} D_{P_{\mathfrak{M}}T^*}\right) = \{0\}. \end{cases}$$

8. Approximation of the minimal solution

The solutions of the Riccati equations (6.10)– (6.14) are fixed points of the corresponding maps. We will prove that extremal solutions can be obtained by iteration procedures with a special initial points.

THEOREM 8.1. *Let* $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a contraction. Suppose that the passive system* $\tau = \{T; \mathfrak{H}, \mathfrak{M}, \mathfrak{N}\}$ *is observable. Define the sequence of nonnegative contractions in* $\mathfrak{H}$*:*

$$Y^{(0)} := I_{\mathfrak{H}},\ Y^{(n+1)} := \left(D^2_T + T^* Y^{(n)} P'_{\mathfrak{H}} T\right)_{\mathfrak{H}} \upharpoonright \mathfrak{H},\ n = 0, 1, \dots. \tag{8.1}$$

Then

(1) *the sequence* $\{Y^{(n)}\}_{n=0}^{\infty}$ *is nonincreasing,*

(2) *the operator*

$$Y_0 := s - \lim_{n\to\infty} Y^{(n)}$$

satisfies the equality

$$Y_0 P_{\mathfrak{H}} = \left(D^2_T + T^* Y_0 P'_{\mathfrak{H}} T\right)_{\mathfrak{H}} \tag{8.2}$$

and $\ker(I_{\mathfrak{H}} - Y_0) = \{0\}$,

(3) *the operator* Y_0 *is a maximal solution of inequality* (6.9).

Proof. Let us show that the sequence defined by (8.1) is nonincreasing. Since $(D^2_{P_{\mathfrak{N}}T})_{\mathfrak{H}} \leqslant P_{\mathfrak{H}}$, we get

$$Y^{(1)}P_{\mathfrak{H}} = \left(D_T^2 + T^*P'_{\mathfrak{H}}T\right)_{\mathfrak{H}} = \left(D^2_{P_{\mathfrak{N}}T}\right)_{\mathfrak{H}}.$$

Hence $Y^{(1)} \leqslant Y^{(0)}$. Suppose that $Y^{(n)} \leqslant Y^{(n-1)}$ for given $n \geqslant 1$. Then

$$Y^{(n+1)}P_{\mathfrak{H}} = \left(D_T^2 + T^*Y^{(n)}P'_{\mathfrak{H}}T\right)_{\mathfrak{H}} \leqslant \left(D_T^2 + T^*Y^{(n-1)}P'_{\mathfrak{H}}T\right)_{\mathfrak{H}} = Y^{(n)}P_{\mathfrak{H}}.$$

Thus, the sequence $\{Y^{(n)}\}_{n=0}^{\infty}$ is nonincreasing. Because the operators $Y^{(n)}$ are non-negative, there exists a strong limit

$$Y_0 = s - \lim_{n\to\infty} Y^{(n)}.$$

Since

$$Y^{(n+1)} = \left(D_T^2 + T^*Y^{(n)}P'_{\mathfrak{H}}T\right)_{\mathfrak{H}} \upharpoonright \mathfrak{H},\ n = 0, 1, \ldots,$$

applying Proposition 2.1 we get (8.2).

Let us show that any solution Y of (6.9) satisfies the inequality $Y \leqslant Y_0$. Suppose that Y is a solution of (6.9). Taking into account (7.3) we get $Y \leqslant Y^{(1)}$. If it is proved that $Y \leqslant Y^{(n)}$ for some $n \geqslant 1$ then

$$Y \leqslant \left(D_T^2 + T^*YP'_{\mathfrak{H}}T\right)_{\mathfrak{H}} \upharpoonright \mathfrak{H} \leqslant \left(D_T^2 + T^*Y^{(n)}P'_{\mathfrak{H}}T\right)_{\mathfrak{H}} \upharpoonright \mathfrak{H} = Y^{(n+1)}.$$

By the induction it follows that $Y \leqslant Y_0$. Using Proposition 7.1 we get $\ker(I - Y_0) = \{0\}$. $\square$

REMARK 8.2. The nondecreasing sequence

$$X^{(0)} = I_{\mathfrak{H}} - Y^{(0)} = 0,\ X^{(n+1)} = I_{\mathfrak{H}} - Y^{(n+1)} = I_{\mathfrak{H}} - \left(D_T^2 + T^*(I_{\mathfrak{H}} - X^{(n)})P'_{\mathfrak{H}}T\right)_{\mathfrak{H}} \upharpoonright \mathfrak{H}$$

strongly converges to the minimal solution X_0 of the KYP inequalities (6.2) and the Riccati equations (6.10)– (6.14). From (3.9) and (3.10) we get

$$X^{(n+1)} = G^*G + D_GL^*D_{F^*}(X^{(n)})^{1/2}\left(I_{\mathfrak{H}} - (X^{(n)})^{1/2}FF^*(X^{(n)})^{1/2}\right)^{-1}(X^{(n)})^{1/2}D_{F^*}LD_G.$$

If the system τ is minimal then by Proposition 5.1, $X_0 = X_{\min}$, where $X_{\min}$ is the minimal solution of the generalized KYP inequality (1.3) for T. The nondecreasing sequence

$$\begin{aligned} Z^{(0)} &= 0,\\ Z^{(n+1)} &= I_{\mathfrak{H}} - \left(D_{T^*}^2 + T(I_{\mathfrak{H}} - Z^{(n)})P_{\mathfrak{H}}T^*\right)_{\mathfrak{H}} \upharpoonright \mathfrak{H}\\ &= FF^* + D_{F^*}LD_G(Z^{(n)})^{1/2}\left(I_{\mathfrak{H}} - (Z^{(n)})^{1/2}G^*G(Z^{(n)})^{1/2}\right)^{-1}(Z^{(n)})^{1/2}D_GL^*D_{F^*},\\ &\qquad n = 0, 1, \ldots \end{aligned}$$

strongly converges to the inverse $Z_0 = X_{\max}^{-1}$, where $X_{\max}$ is the maximal solution of the generalized KYP inequality (1.3) for T.

EXAMPLE 8.3. Let $F \in \mathbf{L}(\mathfrak{M},\mathfrak{H})$ be a strict contraction ($||Fh||_{\mathfrak{H}} < ||h||_{\mathfrak{M}}$ for all $h \in \mathfrak{M}\setminus\{0\}$) and $\ker F^* = \{0\}$. Then $\overline{\operatorname{ran}} D_{F^*} = \mathfrak{H}$. Let $\alpha \in (0,1)$ and suppose that the operator $G \in \mathbf{L}(\mathfrak{H},\mathfrak{N})$ is chosen such that $G^*G = \alpha FF^*$. Then

$$\ker G = \{0\},\ D_G = (I_{\mathfrak{H}} - \alpha FF^*)^{1/2}$$

and $\operatorname{ran} D_G = \mathfrak{H}$. Therefore $\operatorname{ran} D_{F^*} \subset \operatorname{ran} D_G$. Let $L = I_{\mathfrak{H}}$. By Theorem 3.1 the operator

$$Q = \begin{pmatrix} D_{F^*}D_G & F \\ G & 0 \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{M} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{N} \end{pmatrix}$$

is a contraction and from (3.5) we get that $\left(D_Q^2\right)_{\mathfrak{H}} = 0,\ \left(D^2_{P_{\mathfrak{N}}Q}\right)_{\mathfrak{H}} \neq 0,$ and

$$\operatorname{ran}\left(\left(D^2_{P_{\mathfrak{M}}Q^*}\right)_{\mathfrak{H}}\right)^{1/2} \subset \operatorname{ran}\left(\left(D^2_{P_{\mathfrak{N}}Q}\right)_{\mathfrak{H}}\right)^{1/2}.$$

The system

$$\nu = \left\{\begin{pmatrix} D_{F^*}D_G & F \\ G & 0 \end{pmatrix}; \mathfrak{H},\mathfrak{M},\mathfrak{N}\right\}$$

is passive. The condition $\ker F^* = \{0\}$ yields that

$$\bigcap_{n\geqslant 0} \ker\ (F^*(D_GD_{F^*})^n) = \bigcap_{n\geqslant 0} \ker\ (G(D_{F^*}D_G)^n) = \{0\}.$$

So, the system ν is minimal. Its transfer function $Z(\lambda)$ takes the form

$$Z(\lambda) = \lambda G\left(I_{\mathfrak{H}} - \lambda(I_{\mathfrak{H}} - FF^*)^{1/2}(I_{\mathfrak{H}} - \alpha FF^*)^{1/2}\right)^{-1} F.$$

The corresponding Riccati equation (6.14) takes the form

$$\begin{cases} X = \alpha FF^* + (I_{\mathfrak{H}} - \alpha FF^*)^{1/2} D_{F^*} X^{1/2} (I_{\mathfrak{H}} - X^{1/2}FF^*X^{1/2})^{-1} X^{1/2} D_{F^*} (I_{\mathfrak{H}} - \alpha FF^*)^{1/2} \\ 0 < X \leqslant I_{\mathfrak{H}} \end{cases} \tag{8.3}$$

and has the solution $X_0 = \alpha I_{\mathfrak{H}}$. Because $\alpha I_{\mathfrak{H}} < I_{\mathfrak{H}}$, the system ν is a non-optimal realization of $Z(\lambda)$. Let us show that $X_0 = \alpha I_{\mathfrak{H}}$ is the minimal solution of (8.3). Note that $X_{\min} \leqslant X_0 = \alpha I_{\mathfrak{H}}$. According to Remark 8.2 the sequence of operators

$$\begin{aligned} &X^{(0)} = 0,\\ &X^{(n+1)} = \alpha FF^* \\ &+ (I_{\mathfrak{H}} - \alpha FF^*)^{1/2} D_{F^*} (X^{(n)})^{1/2} (I_{\mathfrak{H}} - (X^{(n)})^{1/2} FF^* (X^{(n)})^{1/2})^{-1} (X^{(n)})^{1/2} D_{F^*} (I_{\mathfrak{H}} - \alpha FF^*)^{1/2},\\ &\qquad n = 0,1,\ldots \end{aligned}$$

is nondecreasing and strongly converges to the minimal solution $X_{\min}$ of (8.3). Hence $X^{(n)} \leqslant \alpha I_{\mathfrak{H}}$ and because $X^{(1)} = \alpha FF^*$, one has $X^{(n)}FF^* = FF^*X^{(n)}$ for all n. It follows that $X_{\min}FF^* = FF^*X_{\min}$ and

$$\begin{aligned} I_{\mathfrak{H}} - X_{\min} &= (I_{\mathfrak{H}} - \alpha FF^*)\left(I_{\mathfrak{H}} - D^2_{F^*}X_{\min}(I_{\mathfrak{H}} - X_{\min}FF^*)^{-1}\right) \\ &= (I - X_{\min})(I_{\mathfrak{H}} - \alpha FF^*)(I_{\mathfrak{H}} - X_{\min}FF^*)^{-1}. \end{aligned}$$

Hence

$$(I_{\mathfrak{H}} - X_{\min})(\alpha I_{\mathfrak{H}} - X_{\min})FF^* = 0.$$

Therefore $(\alpha I_{\mathfrak{H}} - X_{\min})FF^* = 0$. Taking into account that $\ker F^* = \{0\}$, we get $X_{\min} = \alpha I_{\mathfrak{H}}$.

Note that if the orthogonal projection P in $\mathfrak{H}$ commutes with FF^* then the operator $X = P + \alpha P^{\perp}$ is a solution of the Riccati equation (8.3).

Consider the adjoint system

$$\nu^* = \left\{ \begin{pmatrix} D_G D_{F^*} & G^* \\ F^* & 0 \end{pmatrix}; \mathfrak{H}, \mathfrak{N}, \mathfrak{M} \right\}.$$

We will show that $X = I_{\mathfrak{H}}$ is the minimal solution of the corresponding Riccati equation

$$\begin{cases} X = FF^* + D_{F^*}(I_{\mathfrak{H}} - \alpha FF^*)^{1/2} X^{1/2} (I_{\mathfrak{H}} - \alpha FF^*)^{-1} X^{1/2} (I_{\mathfrak{H}} - \alpha X^{1/2} FF^* X^{1/2})^{1/2} D_{F^*} \\ 0 < X \leqslant I_{\mathfrak{H}} \end{cases}.$$

According to Remark 8.2 the sequence of operators

$$\begin{aligned} &X^{(0)} = 0, \\ &X^{(n+1)} = FF^* \\ &+ D_{F^*}(I_{\mathfrak{H}} - \alpha FF^*)^{1/2} (X^{(n)})^{1/2} (I_{\mathfrak{H}} - \alpha (X^{(n)})^{1/2} FF^* (X^{(n)})^{1/2})^{-1} (X^{(n)})^{1/2} (I_{\mathfrak{H}} - \alpha FF^*)^{1/2} D_{F^*}, \\ &\qquad\qquad n = 0, 1, \ldots \end{aligned}$$

is nondecreasing and strongly converges to the minimal solution $X_{\min}$. It follows that $X^{(n)}FF^* = FF^*X^{(n)}$ for all n, $X_{\min}FF^* = FF^*X_{\min}$ and

$$\begin{aligned} I_{\mathfrak{H}} - X_{\min} &= (I_{\mathfrak{H}} - FF^*) \left(I_{\mathfrak{H}} - (I_{\mathfrak{H}} - \alpha FF^*) X_{\min} (I_{\mathfrak{H}} - \alpha X_{\min} FF^*)^{-1} \right) \\ &= (I_{\mathfrak{H}} - X_{\min})(I_{\mathfrak{H}} - FF^*)(I_{\mathfrak{H}} - \alpha FF^* X_{\min})^{-1}. \end{aligned}$$

Hence

$$(I_{\mathfrak{H}} - X_{\min})(I_{\mathfrak{H}} - \alpha X_{\min}) = 0.$$

Because $\operatorname{ran}(I_{\mathfrak{H}} - \alpha X_{\min}) = \mathfrak{H}$, we get $X_{\min} = I_{\mathfrak{H}}$. Thus, the minimal passive system ν is $(*)$-optimal realization of the function $Z(\lambda)$.

Observe that this example shows that the condition (7.4) for a passive minimal system is not sufficient for the optimality.

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