Operators and Matrices
Volume 2, Number 1 (2008), 53–65

A NOTE ON FREE PRODUCTS

Don Hadwin and Xiujuan Ma

Dedicated to the memory of Gruf T.

(*communicated by F. Gesztesy*)

Abstract. We answer two questions raised by I. M. Singer concerning free products. We prove that in a free product of separable unital C*-algebras, states on each algebra can be simultaneously extended to a pure state on the free product. We also show that the second dual of the free product of unital C*-algebras is the von Neumann algebra free product of their second duals. We give a proof that the extreme points of the set of tracial states of a C*-algebra is the set of factor tracial states.

1. Introduction

Free products appear in many different branches of mathematics, as do a number of other constructions of objects characterized by certain defining properties. In this paper we give a simple way to mentally verify that such objects exist. Later we focus on free products in the category of unital C*-algebras and free products in the category of von Neumann algebras. In particular, we affirmatively answer two questions asked informally by I. M. Singer concerning free products. The first asks whether the second dual of a free product of a family of unital C*-algebras is the von Neumann algebra free product of their second duals. We prove that this is true for amalgamated free products. The second asks if $\{\mathscr{A}_i : i \in I\}$ is a family of separable unital C*-algebras and if φ_i is a state on $\mathscr{A}_i$ for each $i \in I$, then does there exist a pure state φ on the free product $*_{i\in I}\mathscr{A}_i$ such that, for each $i \in I$, we have $\varphi|_{\mathscr{A}_i} = \varphi_i$. (Here we assume that $\dim \mathscr{A}_i > 1$ for each $i \in I$ and $\operatorname{Card}(I) > 1$.)

For basic facts about C*-algebras and von Neumann algebras we use [1], [5] and [6].

This work began in a seminar on free probability run by Liming Ge and the first author at M.I.T. in 1995-96.

Mathematics subject classification (2000): 46L09, 46L30, 46L05.
Key words and phrases: Free product, C*-algebra, state, pure state.

2. Constructing Objects with Defining properties

In many areas of mathematics there are objects that are characterized by certain defining properties, e.g., group algebras, free products, Stone-Cech compactifications. We present here a very simple way to visualize these constructions. However, there are set-theoretic difficulties with this visualization, and we give a simple recipe for showing how these difficulties can be overcome. The desired result is that the reader can obtain an intuitive feeling for such constructions and mentally verify their existence with little difficulty.

In category theory these ideas are essentially contained in the adjoint functor theorem [7]. However, these ideas do not seem to appear in the operator theory literature; moreover, verifying the hypotheses of the adjoint functor theorem is no easier than applying the ideas below.

We use the example of group C^*-algebras. Suppose $(G,\cdot)$ is a (discrete) group. The *group C*-algebra* $\mathrm{C}^*[G]$ is defined by the universal property that $\mathrm{C}^*[G]$ is a unital C^*-algebra and there is a homomorphism τ from G to the unitary group $\mathscr{U}(\mathrm{C}^*[G])$ of $\mathrm{C}^*[G]$ such that, for any unital C^*-algebra $\mathscr{A}$ and any homomorphism $\sigma : G \to \mathscr{U}(\mathscr{A})$ there is a unique unital algebra homomorphism $\pi : \mathrm{C}^*[G] \to \mathscr{A}$ such that $\sigma = \pi \circ \tau$. A direct construction of $C^*[G]$ is not difficult, but we want to look at a more general idea.

Let $\{(\sigma_\lambda, \mathscr{A}_1) : \lambda \in \Lambda\}$ denote the set of all pairs $(\sigma, \mathscr{A})$ such that $\mathscr{A}$ is a unital C^*-algebra and $\sigma : G \to \mathscr{U}(\mathscr{A})$ is a group homomorphism. Let $\mathscr{B} = \prod_{\lambda \in \Lambda} \mathscr{A}_\lambda$ (the ℓ^∞-product), and define $\tau : G \to \mathscr{U}\left(\prod_{\lambda \in \Lambda} \mathscr{A}_\lambda\right)$ by

$$\tau(g)(\lambda) = \sigma_\lambda(g).$$

We let $\mathrm{C}^*[G]$ be the unital algebra generated by $\tau(G)$. If $\mathscr{A}$ is a unital C^*-algebra and $\sigma : G \to \mathscr{U}(\mathscr{A})$ is a group homomorphism, then $(\sigma, \mathscr{A}) = (\sigma_\lambda, \mathscr{A}_\lambda)$ for some $\lambda \in \Lambda$, so $\sigma = \pi_\lambda \circ \tau$, where $\pi_\lambda : \mathscr{B} \to \mathscr{A}_\lambda$ is the λ^{th} coordinate homomorphism. The uniqueness of the morphism π is ensured by the fact that $\mathrm{C}^*[G]$ is generated by the union of the ranges of the τ_i's.

The trouble with this construction is that

$$\{(\sigma, \mathscr{A}) : \mathscr{A} \text{ is a unital } \mathrm{C}^*\text{-algebra}, \sigma : G \to \mathscr{U}(\mathscr{A}) \text{ a homomorphism}\}$$

is not really a set, because the union of the second coordinates (i.e., the $\mathscr{A}$'s) would have cardinality larger than any cardinal. However, the above construction still works if we restrict ourselves to one algebra in each isomorphism class. More important is the fact that if we are given $\sigma : G \to \mathscr{U}(\mathscr{A})$, we can replace $\mathscr{A}$ with the unital C^*-subalgebra $\mathscr{A}_0$ of $\mathscr{A}$ generated by $\sigma(G)$. Thus we can restrict ourselves to one algebra in each isomorphism class of an $\mathscr{A}_0$. However, there is dense subset $\mathscr{D}$ of $\mathscr{A}_0$ consisting of the elements of the the form $a_1\sigma(g_1) + \cdots + a_n\sigma(g_n)$ where $n \in \mathbb{N}$,

$a_k \in \mathbb{C}$, and $g_k \in G$ for $1 \leqslant k \leqslant n$. Thus the cardinality of $\mathscr{D}$ is no more than

$$\sum_{n=1}^{\infty} (\operatorname{Card} \mathbb{C})^n \operatorname{Card}(G)^n \leqslant 2^{\aleph_0} \operatorname{Card}(G).$$

The elements of $\mathscr{A}_0$ are limits of sequences in $\mathscr{D}$, the the cardinality of $\mathscr{A}_0$ is no larger that the number of sequences in $\mathscr{D}$, which is no larger than $(2\operatorname{Card}(G))^{\aleph_0}$.

If we choose a set X with $\operatorname{Card}(X) = (2\operatorname{Card}(G))^{\aleph_0}$, then, given any $(\sigma, \mathscr{A})$, we can always find an algebra $\mathscr{B} \subset X$ with $\mathscr{B}$ isomorphic to $\mathscr{A}_0$. Therefore we can fix our construction by letting $\{(\sigma_\lambda, \mathscr{A}_\lambda) : \lambda \in \Lambda\}$ denote the set of all $(\sigma, \mathscr{A})$ such that $\mathscr{A} \subset X$, $\mathscr{A}$ is a unital C^*-algebra and $\sigma : G \to \mathscr{U}(\mathscr{A})$ is a homomorphism.

Thus once we check the required cardinality restrictions, the "incorrect" construction is always correct, and we can freely use the "incorrect" construction as our mental image of the construction, because it is so easy to visualize.

In cases where we are dealing with purely algebraic objects the ℓ^∞-product is replaced with the Cartesian product, and when we are dealing with closures in nonmetrizable spaces,we use the fact that if Y is a Hausdorff topological space, and $E \subset Y$, then

$$\operatorname{Card}\left(\overline{E}\right) \leqslant 2^{2^{\operatorname{Card}(E)}}.$$

This is because each element of $\overline{E}$ is the limit of some ultrafilter in E, and, since Y is Hausdorff, different points in Y cannot be limits of the same ultrafilter.

3. Free Products

In general a *free product* $\underset{i\in I}{*} G_i$ of objects $\{G_i : i \in I\}$ in a category $\mathscr{C}$ is an object G in the category along with a family $\{\pi_i : i \in I\}$ of morphisms with $\pi_i : G_i \to G$ for each i in I, such that, given any object G in $\mathscr{C}$ and a family $\{\rho_i : i \in I\}$ of morphisms, $\rho_i : G_i \to H$, there is a unique morphism $\rho : G \to H$ such that, for every $i \in I$, $\rho_i = \rho \circ \pi_i$. In category theory free products are called *coproducts* [7].

Free products can often be constructed using the techniques of the preceding section; we call this the *direct product* construction. Sometimes there is an "internal" construction that comes closer to describing the elements of the free product.

In the example of groups, if each G_i is generated by E_i with a set R_i of relations, then the free product is isomorphic to the group generated by the disjoint union of the E_i's with the set $R = \underset{i\in I}{\cup} R_i$ of relations. In the category of unital C*-algebras it was shown [4] that each unital C*-algebra can be defined with generators and relations (defined as equations with noncommutative continuous functions). The free product is again the unital C*-algebra generated by the disjoint union of the generators with the union of the relations.

In the category of von Neumann algebras with morphisms defined as unital normal $*$-homomorphisms free products also exist. This is because the ℓ^∞-product of dual Banach spaces is the dual of the ℓ^1-product of their preduals. The following is an answer to a question of I. Singer, and the simple proof is obtained by showing that either object satisfies the defining property of the other.

THEOREM 1. *Suppose* $\{\mathscr{A}_i : i \in I\}$ *is a family of unital* C^* *-algebras. Then the natural inclusion map from the free product* $\underset{i\in I}{*}\mathscr{A}_i$ *of* C^* *-algebras into the free product* $\underset{i\in I}{*}\mathscr{A}_i^{\#\#}$ *of von Neumann algebras extends uniquely to a normal isomorphism between* $\left(\underset{i\in I}{*}\mathscr{A}_i\right)^{\#\#}$ *and* $\underset{i\in I}{*}\mathscr{A}_i^{\#\#}$.

It is well-known that the second dual of a C*-algebra is a von Neumann algebra. If $\mathscr{D}$ and $\mathscr{A}$ are unital C*-algebras and $\rho : \mathscr{D} \to \mathscr{A}$ is a unital $*$-homomorphism, then $\rho^{\#\#} : \mathscr{A}^{\#\#} \to \mathscr{B}^{\#\#}$ is a normal $*$-homomorphism. More generally, if $\mathscr{D}$ is a unital C*-algebra $\mathscr{M}$ is a von Neumann algebra, and $\rho : \mathscr{D} \to \mathscr{M}$ is a unital $*$-homomorphism, then ρ extends uniquely to a normal $*$-homomorphism $\widehat{\rho} : \mathscr{D}^{\#\#} \to \mathscr{M}$. The same ideas used to prove Theorem 1 apply to prove the following result, which was suggested to us by Liguang Wang.

COROLLARY 1. *Suppose* $\{\mathscr{A}_i : i \in I\}$ *is a family of unital* C^* *-algebras. Then the natural inclusion map from the amalgamated free product* $*_{i\in I,\mathscr{D}}\mathscr{A}_i$ *of* C^* *-algebras into the amalgamated free product* $*_{i\in I,\mathscr{D}^{\#\#}}\mathscr{A}_i^{\#\#}$ *of von Neumann algebras extends uniquely to a normal isomorphism between* $(*_{i\in I,\mathscr{D}}\mathscr{A}_i)^{\#\#}$ *and* $*_{i\in I,\mathscr{D}^{\#\#}}\mathscr{A}_i^{\#\#}$.

4. Pure States on Free Products

We next look at states on free products of C^* -algebras. If we are given a family $\{\mathscr{A}_i : i \in I\}$ of unital C^* -algebras and a family $\{\varphi_i : i \in I\}$ of states (each φ_i is a state on $\mathscr{A}_i$), there are many states on $\underset{i\in I}{*}\mathscr{A}_i$ such that, for every $i \in I$, $\varphi|\mathscr{A}_i = \varphi_i$. It was asked by I. M. Singer whether φ can always be chosen to be a pure state. We will provide an affirmative answer if each $\mathscr{A}_i$ is separable with dimension greater than 1 and if I contains more than one element.

We first note a simple reduction of the problem.

LEMMA 1. *Suppose* φ *is a pure state on* $\underset{i\in I}{*}\mathscr{A}_i$ *and* $\varphi_i = \varphi|\mathscr{A}_i$ *for each* $i \in I$. *Suppose also, for each* $i \in I$, *that* $\alpha_i : \mathscr{A}_i \to \mathscr{A}_i$ *is a* $*$*-automorphism, and the state* τ_i *is defined on* $\mathscr{A}_i$ *by* $\tau_i(x) = \varphi_i(\alpha(x))$. *Then there is a pure state* τ *on* $\underset{i\in I}{*}\mathscr{A}_i$ *such that* $\tau_i = \tau|\mathscr{A}_i$ *for every* $i \in I$.

Another reduction involves unital separable C*-algebras whose second duals do not contain 3 nonzero mutually orthogonal projections.

LEMMA 2. *Suppose* $\mathscr{A}$ *is a separable unital C*-algebra,* $0 \neq P \neq 1$ *is a projection in* $\mathscr{A}^{\#\#}$ *such that* P *and* $1-P$ *are minimal projections in* $\mathscr{A}^{\#\#}$. *Then* $\mathscr{A}$ *is isomorphic to exactly one of* $\mathbb{C}\oplus\mathbb{C}$, *or* $\mathscr{M}_2(\mathbb{C})$.

Proof. We know $\mathscr{A}^{\#\#}$ is a von Neumann algebra. If $\mathscr{A}^{\#\#}$ is not a factor, then $\mathscr{A}^{\#\#}$ contains a minimal projection Q. Hence QP, $Q(1-P)$, $(1-Q)P$, $(1-Q)(1-P)$

are pairwise orthogonal projections, which, by the minimality assumption, implies that $P = Q$ or $P = 1 - Q$. Thus $\mathscr{A}^{\#\#} = \mathbb{C} \oplus \mathbb{C}$. If $\mathscr{A}^{\#\#}$ is a factor, then the hypothesis implies that $\mathscr{A}^{\#\#}$ is isomorphic to $\mathscr{M}_2(\mathbb{C})$. □

LEMMA 3. *Suppose $\mathscr{M}$ is a von Neumann algebra contained in $B(H)$, e is a unit vector in H and $\varphi : \mathscr{M} \to \mathbb{C}$ is defined by $\varphi(T) = (Te, e)$. The following are equivalent.*

(1) *φ is multiplicative*

(2) *$\|Qe\| = 0$ or $\|Qe\| = 1$ for every projection Q in $\mathscr{M}$.*

Proof. $(1) \Rightarrow (2)$. If φ is multiplicative and $Q \in \mathscr{M}$ is a projection, then $\varphi(Q) = 0$ or $\varphi(1 - Q) = 0$. If $\varphi(Q) = 0$, then

$$0 = (Qe, e) = (Qe, Qe) = \|Qe\|^2.$$

On the other hand $\varphi(1 - Q) = 0$ implies $(1 - Q)e = 0$, which implies $\|Qe\| = \|e\| = 1$.

$(2) \Rightarrow (1)$. Let $\mathscr{R}$ denote the set of all operators T in $B(H)$ such that T and T^* both leave $\mathbb{C}e$ invariant. It is clear that $\mathscr{R}$ is a von Neumann algebra, and, (2) implies that $\mathscr{R}$ contains every projection in $\mathscr{M}$. Since a von Neumann algebra is generated by its projections, $\mathscr{R}$ contains $\mathscr{M}$. It is now easy to see that φ is multiplicative. □

We begin with the case of $\mathscr{A}_1 * \mathscr{A}_2$ where each $\mathscr{A}_i$ is either $\mathscr{M}_2(\mathbb{C})$ or the algebra $\mathscr{D} = \mathbb{C} \oplus \mathbb{C}$ of all diagonal 2×2 matrices. Suppose $0 \leqslant t \leqslant \frac{1}{2}$. We define states $\varphi_t : \mathscr{M}_2(\mathbb{C}) \to \mathbb{C}$ and $\psi_t : \mathscr{D} \to \mathbb{C}$ by

$$\varphi_t\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ta + (1 - t)b,$$

and

$$\psi_t = \varphi|_{\mathscr{D}}.$$

The following lemma is well-known [6].

LEMMA 4. *Every state on $\mathscr{D}$ has the form $\psi_t \circ \alpha$ for some automorphism α of $\mathscr{D}$ and some $t \in \left[0, \frac{1}{2}\right]$, and every state on $\mathscr{M}_2(\mathbb{C})$ has the form $\varphi_t \circ \beta$ for some automorphism β of $\mathscr{M}_2(\mathbb{C})$ and some $t \in [0, 1]$.*

We first consider the case of $\mathscr{D} * \mathscr{D}$.

LEMMA 5. *If $0 \leqslant s, t \leqslant \frac{1}{2}$, then there is a pure state φ on $\mathscr{D}_1 * \mathscr{D}_2$ (where $\mathscr{D}_1 = \mathscr{D}_2 = \mathscr{D}$) such that, $\varphi|_{\mathscr{D}_1} = \psi_s$, and $\varphi|_{\mathscr{D}_2} = \psi_t$.*

Proof. Case 1.Assume both ψ_s and ψ_t are not multiplicative, i.e., $0 < s, t \leqslant \frac{1}{2}$. Let $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathscr{D}_1$, and $P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathscr{D}_2$. We define unital representations $\pi_1, \pi_2 : \mathscr{D} \to \mathscr{M}_2(\mathbb{C})$ by

$$\pi_1(P_1) = \begin{pmatrix} s & \sqrt{s - s^2} \\ \sqrt{s - s^2} & 1 - s \end{pmatrix}$$

and

$$\pi_2(P_2) = \begin{pmatrix} t & i\sqrt{t-t^2} \\ -i\sqrt{t-t^2} & 1-t \end{pmatrix}.$$

From the definition of free product there is a representation $\pi : \mathscr{D}_1 * \mathscr{D}_2 \to \mathscr{M}_2(\mathbb{C})$ such that $\pi|_{\mathscr{D}_i} = \pi_i$ for $i = 1,2$. We let $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and define a state ψ on $\mathscr{D}_1 * \mathscr{D}_2$ by

$$\psi(A) = (\pi(A)e, e).$$

It is clear that $\psi|_{\mathscr{D}_1} = \psi_s$ and $\psi|_{\mathscr{D}_2} = \psi_t$. Since the the only proper C*-subalgebras of $\mathscr{M}_2(\mathbb{C})$ are commutative and $\pi(P_1)$ and $\pi(P_2)$ do not commute, then $\pi(\mathscr{D}_1 * \mathscr{D}_2) = \mathscr{M}_2(\mathbb{C})$, which means that π is irreducible. This, in turn, implies that ψ is a pure state.

Case 2. Consider the case where exactly one of ψ_s or ψ_t is multiplicative. Without loss of generality, we can assume that $0 < s \leqslant \frac{1}{2}$ and $t = 0$. If, as in Case 1, we take $\pi_1(P_1) = \begin{pmatrix} s & \sqrt{s-s^2} \\ \sqrt{s-s^2} & 1-s \end{pmatrix}$ and $\pi_2(P_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and let

$$\psi(A) = (\pi(A)e, e).$$

Arguing as in the preceding case, we see that ψ is the required state.

Case 3. Now consider the case where both ψ_s and ψ_t are multiplicative. This means that $\psi_s, \psi_t : \mathscr{D} \to \mathbb{C}$ are unital $*$- homomorphisms. It follows from the definition of free product that there is a unital $*$- homomorphism $\psi : \mathscr{D}_1 * \mathscr{D}_2 \to \mathbb{C}$ such that $\psi|_{\mathscr{D}_1} = \psi_s$ and $\psi|_{\mathscr{D}_2} = \psi_t$. Clearly, ψ is a pure state. $\square$

We now consider the case of $\mathscr{D} * \mathscr{M}_2(\mathbb{C})$.

LEMMA 6. *Suppose* $0 \leqslant s \leqslant \frac{1}{2}$, $0 \leqslant t \leqslant \frac{1}{2}$. *Then there is a pure state* φ *on* $\mathscr{D} * \mathscr{M}_2(\mathbb{C})$ *such that* $\varphi|\mathscr{D} = \psi_s$ *and* $\varphi|\mathscr{M}_2(\mathbb{C}) = \varphi_t$.

Proof. Case 1. Suppose $s > 0$. Define $\pi_2 : \mathscr{M}_2(\mathbb{C}) \to \mathscr{M}_4(\mathbb{C})$ by

$$\pi_2\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} aI_2 & bI_2 \\ cI_2 & dI_2 \end{pmatrix},$$

and let $e = \begin{pmatrix} \sqrt{t} \\ 0 \\ 0 \\ \sqrt{1-t} \end{pmatrix}$. Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathscr{D}$, and define $\pi_1 : \mathscr{D} \to \mathscr{M}_4(\mathbb{C})$ by

$$\pi_1(P) = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix},$$

where $P_1 = \begin{pmatrix} s & \sqrt{s-s^2} \\ \sqrt{s-s^2} & 1-s \end{pmatrix}$ and $P_2 = \begin{pmatrix} 1-s & i\sqrt{s-s^2} \\ -i\sqrt{s-s^2} & s \end{pmatrix}$. It is easy to show that the range of π, which must be the C*-algebra generated by the union of the ranges of π_1 and π_2, is $\mathscr{M}_2(\mathscr{A})$ where $\mathscr{A}$ is the unital C*-algebra generated by

P_1 and P_2. Since P_1 and P_2 do not commute, and the only proper C*-subalgebras of $\mathcal{M}_2(\mathbb{C})$ are commutative, we know that $\mathcal{A} = \mathcal{M}_2(\mathbb{C})$ and, therefore, that the range of π is all of $\mathcal{M}_4(\mathbb{C})$. Hence π is irreducible and φ is a pure state.

Case 2. Suppose $s = 0$ and $0 < t \leqslant \frac{1}{2}$. Define π_2 and e as in Case 1, and define

$$\pi_1(P) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A = \begin{pmatrix} 1-t & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -\sqrt{t-t^2} \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ -\sqrt{t-t^2} & 0 \end{pmatrix}$, and $D = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$.

If $\pi : \mathcal{D} * \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C})$ is the unique extension of π_1 and π_2, then the state $\varphi : \mathcal{D} * \mathcal{M}_2(\mathbb{C}) \to \mathbb{C}$ defined by

$$\varphi(A) = (\pi(A)e, e)$$

is and extension of both ψ_s and φ_t. As in Case 1, the range of π is $\mathcal{M}_2(\mathcal{A})$, where $\mathcal{A}$ is the unital C*-algebra generated by A, B, C, D. However, $C^*(B) = \mathcal{M}_2(\mathbb{C})$, so φ is a pure state.

Case 3. Suppose $s = t = 0$. Let $\pi_1 : \mathcal{D} \to \mathcal{M}_2(\mathbb{C})$ be the inclusion map, let $\pi_2 : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C})$ be the identity map, and let $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and define π and φ as in the preceding cases. Since π_2 is surjective, π is irreducible. Hence, φ is a pure state. $\square$

LEMMA 7. *Suppose $\mathcal{A}_1 = \mathcal{A}_2 = M_2(\mathbb{C})$, and $0 \leqslant s, t \leqslant \frac{1}{2}$. Then there is a pure state φ on $\mathcal{A}_1 * \mathcal{A}_2$ such that, $\varphi|_{\mathcal{A}_1} = \varphi_s$, and $\varphi|_{\mathcal{A}_2} = \varphi_t$.*

Proof. Case 1. Suppose $s = t = 0$. Let $\pi_1 = \pi_2 = id_{\mathcal{M}_2(\mathbb{C})}$, and let $e = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then the induced π and φ (as in the preceding lemma) have the required properties.

Case 2. Suppose at least one of s, t is positive. We can assume that $s > 0$. Define $\pi_2 : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C})$ and e as in the the proof of the first case of the preceding lemma. Choose a unitary $U = \begin{pmatrix} \sqrt{st} & 0 & -\sqrt{1-t} & \sqrt{t(1-s)} \\ -\sqrt{1-s} & 0 & 0 & \sqrt{s} \\ 0 & 1 & 0 & 0 \\ \sqrt{s(1-t)} & 0 & \sqrt{t} & \sqrt{(1-t)(1-s)} \end{pmatrix}$. Then for every $T \in \mathcal{M}_2(\mathbb{C})$ we have

$$\varphi_s(T) = \left(U \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} U^* e, e \right).$$

Define a representation π_1 on $\mathcal{A}_1$ by $\pi_1(T) = U \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} U^*$. If $\pi : \mathcal{A}_1 * \mathcal{A}_2 \to \mathcal{M}_4(\mathbb{C})$ induced by π_1 and π_2 and if $\varphi(T) = (\pi(T)e, e)$, then $\varphi|_{\mathcal{A}_1} = \varphi_s$ and $\varphi|_{\mathcal{A}_2} = \varphi_t$.

Since

$$\pi_1\left(\left(\begin{array}{cc}0 & 1\\ 0 & 0\end{array}\right)\right)=\left(\begin{array}{cc}A & B\\ C & D\end{array}\right),$$

with $A=\left(\begin{array}{cc}-\sqrt{t\,(1-t)\,(1-s)} & -\sqrt{s\,(1-t)}\\ 0 & 0\end{array}\right)$, and the range of π contains $\mathscr{M}_2\,(C^*\,(A))=\mathscr{M}_2\,(\mathscr{M}_2\,(\mathbb{C}))$ (since $-\sqrt{s\,(1-t)}\neq 0$), we see that π must be irreducible and, hence, φ is a pure state. □

Recall [6] that the second dual $\mathscr{A}^{\#\#}$ of a separable unital C*-algebra $\mathscr{A}$ is a von Neumann algebra. Moreover, if φ is a state on $\mathscr{A}$, then there is a separable Hilbert space H and a faithful representation $\pi:\mathscr{A}^{\#\#}\to B\,(H)$ and a unit vector $e\in H$ such that

(1) $\pi\,(\mathscr{A})''=\pi\left(\mathscr{A}^{\#\#}\right)$,

(2) $\varphi\,(a)=(\pi\,(a)\,e,e)$ for every $a\in\mathscr{A}$.

Note that e need not be a cyclic vector for π.

LEMMA 8. *Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ are separable unital C*-algebras and $\mathscr{A}_2^{\#\#}$ has 3 orthogonal non-zero projections whose sum is 1. If φ_i is a state on $\mathscr{A}_i$, for $i=1,2$, respectively, then there is an irreducible representation $\pi:\mathscr{A}_1*\mathscr{A}_2\to B\left(\ell^2\right)$ and a unit vector f such that for any $a\in\mathscr{A}_i$, $\varphi_i\,(a)=(\pi\,(a)f,f)$ for $i=1,2$.*

Proof. Since $H=\ell^2\oplus\ell^2$ is a separable infinite-dimensional Hilbert space, there is a unit vector e and, for $i=1,2$, a faithful representation $\pi_i:\mathscr{A}_i^{\#\#}\to B(H)$ with $\pi_i\,(\mathscr{A}_i)''=\pi_i\left(\mathscr{A}_i^{\#\#}\right)$ such that, for each $a\in\mathscr{A}_i$,

$$\varphi_i(a)=(\pi_i(a)e,e).$$

We can assume also that, for $i=1,2$, π_i is unitarily equivalent to $\pi_i\oplus\pi_i\oplus\cdots$; this implies that every nontrivial projection in $\pi_i(\mathscr{A}_i)''$ has infinite rank and nullity. Since $\mathscr{A}_1\neq\mathbb{C}$, there is a projection $Q\in\pi_1(\mathscr{A}_1)''$ such that $0\neq Q\neq 1$. Since $\mathscr{A}_2^{\#\#}$ has 3 orthogonal non-zero projections and π_2 is faithful, we can choose an orthogonal family $\{P_1,P_2,P_3\}$ of nonzero projections in $\pi_2(\mathscr{A}_2)''$ whose sum is 1. Let $t=\|Qe\|$ and let $t_j=\|P_je\|$ for $j=1,2,3$. By replacing Q with $1-Q$ if necessary, we can assume that $0\leqslant t\leqslant 1/2$.

We begin with a construction on ℓ^2. Suppose

$$A=\left(\begin{array}{ccccc}\alpha_0 & & & & \\ & \alpha_1 & & & \\ & & \alpha_2 & & \\ & & & \alpha_4 & \\ & & & & \ddots\end{array}\right)$$

is a diagonal operator on ℓ^2, $0\leqslant A\leqslant 1$, with distinct eigenvalues, none of which is 0

or 1. Let $h = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{9} \\ -\frac{1}{9} \\ \frac{1}{81} \\ -\frac{1}{81} \\ \cdots \\ \frac{1}{9^n} \\ -\frac{1}{9^n} \\ \cdots \end{pmatrix}$, $f_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$, $f_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$, and $u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$. Since $h,f_1,f_2,$ and $u - \left(u, \frac{h}{\|h\|}\right)\frac{h}{\|h\|}$ are orthogonal, there is a projection $P \in B\left(\ell^2\right)$ with infinite rank and nullity such that $h,f_1 \in \operatorname{ran} P$ and $f_2, u - \left(u, \frac{h}{\|h\|}\right)\frac{h}{\|h\|} \in \ker P$. It follows that $Pu = h,$ so the first column of the matrix for P has no zero entries. Then any operator that commutes with A must be diagonal, and any diagonal operator that commutes with P must be a scalar. In other words $\{A,P\}' = \mathbb{C}1$. Now on $\ell^2 \oplus \ell^2$ let

$$\bar{P}_1 = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{P}_2 = \begin{pmatrix} 1-P & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{P}_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\bar{Q} = \begin{pmatrix} A & \sqrt{A-A^2} \\ \sqrt{A-A^2} & 1-A \end{pmatrix}.$$

Suppose $T \in B\left(\ell^2 \oplus \ell^2\right)$ commutes with $\bar{P}_1 + \bar{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $T = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$. If, in addition, $T\bar{Q} = \bar{Q}T$ and we have $BA = AB$, $CA = AC$, and $C\sqrt{A-A^2} = \sqrt{A-A^2}B$. However, $CA = AC$ implies $C\sqrt{A-A^2} = \sqrt{A-A^2}C$, so $\sqrt{A-A^2}C = \sqrt{A-A^2}B$. Since $0,1$ are not eigenvalues of A, $\sqrt{A-A^2}$ is injective. Thus $B = C$. If we now assume that T commutes with $\bar{P}_1$, we see that $B \in \{P,A\}'$, which implies $B = C \in \mathbb{C}1$. What we have shown is that $\left\{\bar{P}_1, \bar{P}_2, \bar{P}_3, \bar{Q}\right\}'' = B\left(\ell^2 \oplus \ell^2\right)$.

Next suppose we construct vectors $e_1 \in \operatorname{ran} P$, $e_2 \in \ker P$, $e_3 \in \ell^2$ such that $\|e_j\| = t_j$ for $1 \leqslant j \leqslant 3$, and we let $f = \begin{pmatrix} e_1 + e_2 \\ e_3 \end{pmatrix}$. Since $\|\bar{P}_j f\| = \|P_j e\|$ for $j = 1,2,3,$ and since all of the projections involved have infinite rank and nullity, there is unitary operator $U_1 \in B\left(\ell^2 \oplus \ell^2\right)$ such that $U_1 e = f$ and such that $U_1 P_j U_1^* = \bar{P}_j$ for $j = 1,2,3$. If in addition we can choose e_1, e_2, e_3 so that $\left\|\bar{Q}f\right\| = \|Qe\|$, then there is a unitary operator U_2 with $U_2 e = f$ such that $U_2 Q U_2^* = \bar{Q}$. If $\pi : \mathscr{A}_1 * \mathscr{A}_2 \to B\left(\ell^2 \oplus \ell^2\right)$ is the unique representation that extends $U_1 \pi_1 ()\, U_1^*$ and $U_2 \pi_2 ()\, U_2^*$, then the state defined by $\varphi(A) = (\pi(A)f, f)$ is an extension of φ_1 and φ_2. Moreover, since the double commutant of the range of π contains $\left\{\bar{P}_1, \bar{P}_2, \bar{P}_3, \bar{Q}\right\}'' = B\left(\ell^2 \oplus \ell^2\right)$, π is irreducible and φ is a pure state. Thus in each of the following cases, except the first one, we need only define $\alpha_0, \alpha_1, \ldots$ and e_1, e_2, e_3.

Case 1. Both φ_1 and φ_2 are multiplicative. In this case we can choose $\varphi = \pi : \mathscr{A}_1 * \mathscr{A}_2 \to \mathbb{C}$ to be the extension of φ_1 and φ_2 guaranteed by the definition of the free product.

Case 2. φ_2 is not multiplicative, and at most one of $\|P_1e\|$, $\|P_2e\|$, $\|P_3e\|$ is 0. Without loss of generality, we can assume $0 < \|P_1e\| = t_1, \|P_2e\| = t_2$. Then, by Lemma 3, we can choose $Q \in \pi_2(\mathscr{A})''$ such that

$0 < \|Qe\| = t \leqslant 1/2$. Choose $0 < \varepsilon_1 \neq \varepsilon_2 < t$, and let $\alpha_0 = 1-t,$. $\alpha_1 = t-\varepsilon_1$, $\alpha_2 = t+\varepsilon_1$, $\alpha_3 = t-\varepsilon_2$, and $\alpha_4 = t+\varepsilon_2$. Let $e_1 = \frac{t_1}{\sqrt{2}}f_2$, $e_2 = \frac{t_2}{\sqrt{2}}f_2$, and $e_3 = t_3u$.

Case 3. φ_2 is not multiplicative, and two of $\|P_1e\|$, $\|P_2e\|$, $\|P_3e\|$ are 0. Note that it is impossible for all three of $\|P_1e\|$, $\|P_2e\|$, $\|P_3e\|$ to be 0. We can assume that $\|P_1e\| = \|P_2e\| = 0$, which implies that $\|P_3e\| = 1$. Now we let $e_1 = e_2 = 0$ and let $e_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$, and we choose A so that $\alpha_0 = 1-t$.

Case 4. φ_2 is multiplicative and φ_1 is not multiplicative. We know from Lemma 3 that $\pi_1(\mathscr{A}_1)''$ contains a projection E so that $0 < \|Ee\| < 1$, and therefore, $\|(1-E)e\| < 1$. Since $\mathscr{A}_1^{\#\#}$ has 3 nonzero orthogonal projections, $\mathscr{A}_1^{\#\#}$ is not isomorphic to $\mathscr{M}_2(\mathbb{C})$. Thus, by Lemma 2, at least one of E or $1-E$ is not minimal. Hence we can choose P_1, P_2, P_3 so that $0 < \|P_1e\|, \|P_3e\|$. Since φ_2 is multiplicative, we can also choose Q so that $t = \|Qe\| = 0$. The key here is that if $x, y > 0$ and $0 < \alpha < 1$, the equation

$$\alpha x = \sqrt{\alpha - \alpha^2}y$$

is equivalent to

$$\alpha = \frac{y^2}{y^2 + x^2}$$

is equivalent to

$$\sqrt{\alpha - \alpha^2}x = (1-\alpha)y.$$

First suppose $t_2 = \|P_2e\| = 0$. Then let $e_2 = 0$, $e_1 = \frac{t_1}{\sqrt{2}}f_1$, $e_3 = \begin{pmatrix} 0 \\ -t_3\sqrt{\frac{1}{3}} \\ -t_3\sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$,

and let $\alpha_1 = \frac{\left(\frac{1}{3}\right)t_3^2}{\left(\frac{1}{3}\right)t_3^2 + \left(\frac{1}{2}\right)t_1^2}$, $\alpha_2 = \frac{\left(\frac{2}{3}\right)t_3^2}{\left(\frac{2}{3}\right)t_3^2 + \left(\frac{1}{2}\right)t_1^2}$. If we define f as above, then $\bar{Q}f = 0$.

Things are slightly more complicated if $t_2 \neq 0$. In this case we let $e_j = \frac{t_j}{\sqrt{2}}f_j$ for $j = 1, 2$, and we define strictly negative numbers b_1, b_2, c_1, c_2 so that

$$b_1^2 + b_2^2 + c_1^2 + c_2^2 = t_3^2,$$

and so that the four numbers $\alpha_1 = \frac{b_1^2}{b_1^2+\left(\frac{1}{2}\right)t_1^2}$, $\alpha_2 = \frac{b_2^2}{b_2^2+\left(\frac{1}{2}\right)t_1^2}$, $\alpha_3 = \frac{2c_1^2}{2c_1^2+t_2^2}$, $\alpha_4 = \frac{2c_2^2}{2c_2^2+t_2^2}$ are distinct. We let $e_3 = \begin{pmatrix} 0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ 0 \\ \vdots \end{pmatrix}$. In this case we get $\bar{Q}f = 0$. □

THEOREM 2. *Suppose $\{\mathscr{A}_i : i \in I\}$ is a family of separable unital C^*-algebras with $\dim \mathscr{A}_i > 1$, and, for each i in I, φ_i is a state on $\mathscr{A}_i$. If* Card $(I) > 1$, *then there is a pure state φ on $\underset{i \in I}{*} \mathscr{A}_i$ such that, for each i in I, $\varphi | \mathscr{A}_i = \varphi_i$.*

Proof. If one of the φ_i's is a pure state whose GNS representation is infinite-dimensional, then we can represent the other $\mathscr{A}_j$'s arbitrarily on the same space with the same vector, and the corresponding π will be irreducible.

Next note that if $\mathscr{A}$ and $\mathscr{B}$ are separable unital C*-algebras with dimension greater than 1, then $\mathscr{A}^{\#\#}$ and $\mathscr{B}^{\#\#}$ contain nontrivial projections, and it follows from Theorem 1 that $(\mathscr{A} * \mathscr{B})^{\#\#}$ contains 3 nonzero orthogonal projections. This fact combined with the preceding remark implies that if the cardinality of I is at least 3, the theorem is true. The cases in which the cardinality of I is 2 are covered in the lemmas of this section. □

It is well-known that the set of pure states of a unital C*-algebra is the set of extreme points of the set of all states. Analogously, the set of extreme points of the set of tracial states is the set of factor tracial states, i.e., tracial states whose GNS representation generates a factor von Neumann algebra [3]. Since no proof was given in [3], we provide a proof here.

PROPOSITION 1. *The set of extreme points of the set of tracial states of a unital C*-algebra is the set of factor tracial states.*

Proof. Suppose $\mathscr{A}$ is a unital C*-algebra. Let $\mathscr{T}$ denote the set of tracial states of $\mathscr{A}$ and suppose φ is an extreme point of φ. Using the GNS construction there is a Hilbert space H, a unit vector e in H, a unital $*$-homomorphism $\pi : \mathscr{A} \to B(H)$ with $[\pi(\mathscr{A})e]^- = H$, such that

$$\varphi(a) = (\pi(a)e, e)$$

for every $a \in \mathscr{A}$. Assume, via contradiction, that $0 \neq P \neq 1$ is a projection in the center of $\pi(\mathscr{A})''$. Since $\pi(\mathscr{A})Pe = P\pi(\mathscr{A})e$, it is clear that $e_1 = Pe \neq 0$. Similarly, $e_2 = (1-P)e \neq 0$. If we define

$$\varphi_i(a) = \left(\pi(a)\frac{e_i}{\|e_i\|}, \frac{e_i}{\|e_i\|}\right)$$

for $a \in \mathscr{A}$ and $i = 1,2$, then φ_1 and φ_2 are tracial states on $\mathscr{A}$ and $\varphi = \|e_1\|^2 \varphi_1 + \|e_2\|^2 \varphi_2$. Moreover, there is a net $\{a_n\}$ in $\mathscr{A}$ such that $\pi(a_n) \to P$ in the weak operator topology. Thus $\varphi_1(a_n) \to 1$ and $\varphi_2(a_n) \to 0$, which means $\varphi_1 \neq \varphi_2$, which contradicts the fact that φ is an extreme point of $\mathscr{T}$.

Conversely, suppose $\varphi \in \mathscr{T}$ is not an extreme point. Then there exist $\rho_1 \neq \rho_2$ in $\mathscr{T}$ such that $\varphi = \frac{1}{2}(\rho_1 + \rho_2)$. The GNS construction insures the existence of Hilbert spaces M_i, unit vectors e_i, unital representations $\tau_i : \mathscr{A} \to B(M_i)$ with $[\tau_i(\mathscr{A}) e_i]^- = M_i$ such that

$$\rho_i(a) = (\tau_i(a) e_i, e_i)$$

for $a \in \mathscr{A}$ and $i = 1, 2$. Let $M = M_1 \oplus M_2$, $e = (e_1 \oplus e_2)/\sqrt{2}$, $\tau = \tau_1 \oplus \tau_2$, and let $H = [\tau(\mathscr{A}) e]^-$, and define $\pi : \mathscr{A} \to B(H)$ by

$$\pi(a) = \tau(a)|_H.$$

Clearly (π, e) generates the GNS representation for φ. Assume via contradiction that $\pi(\mathscr{A})''$ is a factor. Write $\rho_i = \rho_{i1} \oplus \rho_{i2}$ relative to $M_i = M_{i1} \oplus M_{i2}$ with $\rho_{i1} << \pi$ and ρ_{i2} disjoint from π. Then there is a net $\{a_\lambda\}$ with $0 \leqslant a_\lambda \leqslant 1$ such that $\pi(a_\lambda) \to 1$ and $(\rho_{12} \oplus \rho_{22})(a_\lambda) \to 0$ in the strong operator topology. It follows that the projection P_i of M_i onto M_{i1} is a central projection in $\tau_i(\mathscr{A})''$ and that

$$1 = \lim_\lambda \varphi(a_\lambda) = \frac{1}{2}\left[(P_1 e_1, e_1) + (P_2 e_2, e_2)\right].$$

It follows that $M_i = M_{i1}$ for $i = 1, 2$ and that $\tau_i << \pi$ for $i = 1, 2$. It follows that the map

$$\pi(A) \mapsto (\tau_i(A) e_i, e_i)$$

extends to a tracial state on the factor $\pi(\mathscr{A})''$. However, every such factor has a unique tracial state, which implies that $\rho_1 = \rho_2$, which is the desired contradiction. Hence φ is not a factor tracial state. ☐

Note that every state on $\mathscr{D} = \mathbb{C} \oplus \mathbb{C}$ is tracial, but the only tracial factor state on $\mathscr{D} * \mathscr{D}$ (the universal C*-algebra generated by two projections) is a representation onto $\mathscr{M}_2(\mathbb{C})$ followed by the normalized trace on $\mathscr{M}_2(\mathbb{C})$ [6], which must send every projection to $0, \frac{1}{2}$, or 1. Hence, not every pair of tracial states on $\mathscr{D}$ can be extended to a factor tracial state on $\mathscr{D} * \mathscr{D}$. This observation and the preceding proposition leads us naturally to the following variant of Singer's question on pure states.

CONJECTURE 1. *If $\{\mathscr{A}_i : i \in I\}$ is a collection of separable unital C*-algebras of dimension greater than 2 and, for each $i \in I$, φ_i is a tracial state on $\mathscr{A}_i$, then there must exist a factor tracial state φ on $*_{i\in I}\mathscr{A}_i$ such that $\varphi|_{\mathscr{A}_i} = \varphi_i$ for each $i \in I$.*

It was proved by W.-M. Ching [2] that if each of a family of von Neumann algebras of dimension at least 2 and at least one with dimension greater than 2 along with given tracial states have an orthonormal basis consisting of unitaries, then the free-product tracial state is a factor state. This shows that our conjecture is true in many cases.

REFERENCES

[1] ARVESON, WILLIAM, *An invitation to C*-algebras, Graduate Texts in Mathematics, No. 39*, Springer-Verlag, New York-Heidelberg, 1976.
[2] CHING, WAI-MEE, *Free products of von Neumann algebras*, Trans. Amer. Math. Soc. 178 (1973) 147-163.
[3] HADWIN, DON, *A noncommutative moment problem*, Proc. Amer. Math. Soc. 129 (2001), no. 6, 1785–1791.
[4] HADWIN, DON; KAONGA, LLOLSTEN; MATHES, BEN, *Noncommutative continuous functions*, J. Korean Math. Soc. 40 (2003), no. 5, 789–830.
[5] KADISON, RICHARD V.; RINGROSE, JOHN R., *Fundamentals of the theory of operator algebras, Vol. I, Elementary theory, Graduate Studies in Mathematics, 15*, American Mathematical Society, Providence, RI, 1997.
[6] KADISON, RICHARD V.; RINGROSE, JOHN R., *Fundamentals of the theory of operator algebras. Vol. II. Advanced theory, Graduate Studies in Mathematics, 16*. American Mathematical Society, Providence, RI, 1997.
[7] MAC LANE, SAUNDERS, *Categories for the working mathematician, Graduate Texts in Mathematics, 5*, Springer-Verlag, New York, 1998.

(Received January 8, 2007)

Don Hadwin
Mathematics Department
University of New Hampshire
e-mail: don@math.unh.edu
URL: http://www.math.unh.edu/~don

Xiujuan Ma
Mathematics Department
Hebei University of Science and Technology
e-mail: mxjsusan@hotmail.com