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THE ASYMPTOTIC BEHAVIOR OF FREE ADDITIVE CONVOLUTION

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Abstract. We provide a new proof of the limit theorem for sums of free random variables in a general infinitesimal triangular array. This result was proved by Chistyakov and Götze using subordination functions. Our proof does not depend on subordination, and is close to the approach used in the case of arrays with identically distributed rows [5].

1. Introduction

Given two probability measures μ, ν on the real line $\mathbb{R}$, we will denote by $\mu * \nu$ their classical convolution, and by $\mu \boxplus \nu$ their free additive convolution. Thus, $\mu * \nu$ is the distribution of the sum $X + Y$, where X and Y are classically independent random variables with distributions μ and ν, respectively. Analogously, $\mu \boxplus \nu$ is the distribution of $X + Y$, where X and Y are freely independent random variables with distributions μ and ν.

A triangular array $\{\mu_{nk} : n \geqslant 1,\ 1 \leqslant k \leqslant k_n\}$ of probability measures on $\mathbb{R}$ is said to be *infinitesimal* if

$$\lim_{n\to\infty} \max_{1\leqslant k\leqslant k_n} \mu_{nk}(\{t \in \mathbb{R} : |t| \geqslant \varepsilon\}) = 0,$$

for every $\varepsilon > 0$. The classical limit distribution theory for sums of independent random variables is concerned with the study of the asymptotic behavior of the measures

$$\mu_n = \mu_{n1} * \mu_{n2} * \ldots * \mu_{nk_n} * \delta_{c_n}, \qquad n \geqslant 1,$$

where δ_{c_n} is the point mass at $c_n \in \mathbb{R}$. Hinčin [12] proved that any weak limit of such a sequence $\{\mu_n\}_{n=1}^{\infty}$ is an infinitely divisible measure. Later Gnedenko (see [11] and [15]) found necessary and sufficient conditions for the convergence of the sequence $\{\mu_n\}_{n=1}^{\infty}$ to a given infinitely divisible measure.

The analogous free convolutions

$$\nu_n = \mu_{n1} \boxplus \mu_{n2} \boxplus \ldots \boxplus \mu_{nk_n} \boxplus \delta_{c_n}$$

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have been also the subject of several investigations. The first result in this direction was an analogue of the central limit theorem proved by Voiculescu [16]. Later, Pata [14] proved that the free central limit theorem holds precisely under the same conditions as the classical central limit theorem. The analogue of Hinčin's theorem, i.e. the fact that the possible weak limits of ν_n are $\boxplus$-infinitely divisible, was proved in [6]. Then it was shown in [5] that, in case $c_n = 0$ and $\mu_{n1} = \mu_{n2} = \ldots = \mu_{nk_n}$, the measures μ_n have a weak limit if and only if the measures ν_n do. The correspondence between the limit of μ_n and the limit of ν_n was thoroughly studied in [1, 2].

The result of [5] was extended in [10] to arbitrary arrays and centering constants c_n. The argument in [10] depends on two ingredients. The first is the fact that the classical centering of the measures in an infinitesimal array balances the real and the imaginary parts of the Cauchy transforms of the measures. The second is the existence of subordination functions as in [18, 9, 3].

We will provide a proof of the main result of [10] which makes no use of subordination functions, and is close to the argument of [5]. This approach also works for multiplicative free convolutions, as shown in [8].

The remainder of this paper is organized as follows. In Section 2 we describe the calculation of free convolution via Cauchy transform, and we provide some useful approximation results. The proof of the main result is in Section 3.

2. Preliminaries

Let $\mathscr{M}$ be the collection of all Borel probability measures on $\mathbb{R}$. The free analogue of the Fourier transform was discovered by Voiculescu [17] (see also [13] and [7]). The details are as follows. Denote by $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$ the upper half-plane, and set $\mathbb{C}^- = -\mathbb{C}^+$. For a measure $\mu \in \mathscr{M}$, one defines its *Cauchy transform* $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ by

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{1}{z-t}\, d\mu(t), \qquad z \in \mathbb{C}^+.$$

Define the analytic function $F_\mu : \mathbb{C}^+ \to \mathbb{C}^+$ by $F_\mu(z) = 1/G_\mu(z)$. Given $\alpha, \beta > 0$, set $\Gamma_\alpha = \{z = x + iy \in \mathbb{C}^+ : |x| < \alpha y\}$ and $\Gamma_{\alpha,\beta} = \{z = x + iy \in \Gamma_\alpha : y > \beta\}$. In [7] it is shown that $F_\mu(z)/z$ tends to 1 as $z \to \infty$ *nontangentially* to $\mathbb{R}$ (i.e., $|z| \to \infty$ but $\Re z/\Im z$ stays bounded; in other words, $z \in \Gamma_\alpha$ for some $\alpha > 0$), and this implies that for every $\alpha > 0$, there exists a $\beta = \beta(\alpha, \mu) > 0$ such that F_μ has a left inverse F_μ^{-1} defined on $\Gamma_{\alpha,\beta}$. The *Voiculescu transform* of the measure μ is then defined as

$$\phi_\mu(z) = F_\mu^{-1}(z) - z, \qquad z \in \Gamma_{\alpha,\beta}.$$

We have $\Im \phi_\mu(z) \leqslant 0$ for $z \in \Gamma_{\alpha,\beta}$, and $\phi_\mu(z) = o(|z|)$ as $z \to \infty$ nontangentially.

The most important property of the Voiculescu transform is that it linearizes the free convolution. More precisely, if $\mu, \nu \in \mathscr{M}$ then

$$\phi_{\mu \boxplus \nu}(z) = \phi_\mu(z) + \phi_\nu(z),$$

for all z in any truncated cone $\Gamma_{\alpha,\beta}$ where all three functions involved are defined.

It was first noted in [6] that for any given cone $\Gamma_{\alpha,\beta}$, ϕ_μ is defined on $\Gamma_{\alpha,\beta}$ as long as the measure μ puts most of its mass around the origin.

LEMMA 2.1. *For every $\alpha,\beta > 0$, there exists $\varepsilon > 0$ with the property that if $\mu \in \mathscr{M}$ is such that $\mu(\{t \in \mathbb{R} : |t| \geqslant \varepsilon\}) < \varepsilon$, then ϕ_μ is defined in $\Gamma_{\alpha,\beta}$.*

The following theorem from [4] translates weak convergence of probability measures in terms of convergence properties of their Voiculescu transforms.

THEOREM 2.2. *Given a sequence $\{\mu_n\}_{n=1}^{\infty} \subset \mathscr{M}$. The following statements are equivalent:*

(1) *μ_n converges weakly to a measure $\mu \in \mathscr{M}$ as $n \to \infty$;*

(2) *there exists a truncated cone $\Gamma_{\alpha,\beta}$ such that functions ϕ_{μ_n} and ϕ_μ are defined in $\Gamma_{\alpha,\beta}$, $\lim_{n\to\infty} \phi_{\mu_n}(iy) = \phi_\mu(iy)$ for all $y > \beta$, and $\phi_{\mu_n}(iy) = o(y)$ uniformly in n as $y \to \infty$.*

The following result is a reformulation of Propositions 2.6 and 2.7 in [5] which is more appropriate for our purposes.

PROPOSITION 2.3. *Given an infinitesimal array $\{\mu_{nk} : n \geqslant 1,\ 1 \leqslant k \leqslant k_n\} \subset \mathscr{M}$, and $\alpha,\beta > 0$, the functions $\phi_{\mu_{nk}}$ are defined in $\Gamma_{\alpha,\beta}$ for sufficiently large n, and*

$$\phi_{\mu_{nk}}(z) = z^2 \left[G_{\mu_{nk}}(z) - \frac{1}{z} \right] (1 + v_{nk}(z)),$$

where the sequence

$$v_n(z) = \max_{1 \leqslant k \leqslant k_n} |v_{nk}(z)|$$

satisfies $\lim_{n\to\infty} v_n(z) = 0$ for all $z \in \Gamma_{\alpha,\beta}$, and $v_n(z) = o(1)$ uniformly in n as $z \to \infty$, $z \in \Gamma_{\alpha,\beta}$.

Recall that a measure $\nu \in \mathscr{M}$ is said to be $\boxplus$*-infinitely divisible* if for each $n \in \mathbb{N}$, there exists a measure $\mu_n \in \mathscr{M}$ such that

$$\nu = \underbrace{\mu_n \boxplus \mu_n \boxplus \ldots \boxplus \mu_n}_{n \text{ times}}.$$

The notion of $*$- *infinite divisibility* is defined similarly. The well-known Lévy-Hinčin formula characterizes the $*$-infinitely divisible measures in terms of their Fourier transform as follows: a measure ν is $*$-infinitely divisible if and only if there exist $\gamma \in \mathbb{R}$ and a finite positive Borel measure σ on $\mathbb{R}$ such that the Fourier transform $\hat{\nu}$ is given by

$$\hat{\nu}(t) = \exp\left[i\gamma t + \int_{-\infty}^{\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) \frac{1+x^2}{x^2}\, d\sigma(x) \right], \qquad t \in \mathbb{R}.$$

Here $\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}$ is interpreted as $-t^2/2$ for $x = 0$. We will denote by $\nu_*^{\gamma,\sigma}$ the $*$-infinitely divisible measure determined by γ and σ. The free analogue of the Lévy-Hinčin formula is proved in [7]. A measure $\nu \in \mathscr{M}$ is $\boxplus$-infinitely divisible if and only if there exist $\gamma \in \mathbb{R}$ and a finite positive Borel measure σ on $\mathbb{R}$ such that

$$\phi_\nu(z) = \gamma + \int_{-\infty}^{\infty} \frac{1+tz}{z-t}\, d\sigma(t), \qquad z \in \mathbb{C}^+. \tag{2.1}$$

We will denote the above measure ν by $\nu_{\boxplus}^{\gamma,\sigma}$. The following result is from [8]. We reproduce the short proof here because we actually require inequalities (2.2) and (2.3).

LEMMA 2.4. *Consider a sequence* $\{r_n\}_{n=1}^{\infty} \subset \mathbb{R}$ *and two triangular arrays* $\{z_{nk} : n \geqslant 1,\ 1 \leqslant k \leqslant k_n\}$, $\{w_{nk} : n \geqslant 1,\ 1 \leqslant k \leqslant k_n\}$ *of complex numbers. Assume that*

(1) $\Im w_{nk} \geqslant 0$, *for* $n \geqslant 1$ *and* $1 \leqslant k \leqslant k_n$.

(2)
$$z_{nk} = w_{nk}(1 + \varepsilon_{nk}),$$
where
$$\varepsilon_n = \max_{1 \leqslant k \leqslant k_n} |\varepsilon_{nk}|$$
converges to zero as $n \to \infty$.

(3) *There exists a positive constant* M *such that for sufficiently large* n,
$$|\Re w_{nk}| \leqslant M \Im w_{nk}.$$

Then the sequence $\{r_n + \sum_{k=1}^{k_n} z_{nk}\}_{n=1}^{\infty}$ *converges if and only if the sequence* $\{r_n + \sum_{k=1}^{k_n} w_{nk}\}_{n=1}^{\infty}$ *converges. Moreover, the two sequences have the same limit.*

Proof. The assumptions on $\{z_{nk}\}_{n,k}$ and $\{w_{nk}\}_{n,k}$ imply
$$\left| \left(r_n + \sum_{k=1}^{k_n} z_{nk} \right) - \left(r_n + \sum_{k=1}^{k_n} w_{nk} \right) \right| \leqslant 2(1+M)\varepsilon_n \left(\sum_{k=1}^{k_n} \Im w_{nk} \right), \tag{2.2}$$
and
$$(1 - \varepsilon_n - M\varepsilon_n) \left(\sum_{k=1}^{k_n} \Im w_{nk} \right) \leqslant \left| \sum_{k=1}^{k_n} \Im z_{nk} \right|, \tag{2.3}$$
for sufficiently large n. If the sequence $\{r_n + \sum_{k=1}^{k_n} z_{nk}\}_{n=1}^{\infty}$ converges to a complex number z, (2.3) implies that $\{\sum_{k=1}^{k_n} \Im w_{nk}\}_{n=1}^{\infty}$ is bounded, and then (2.2) shows that the sequence $\{r_n + \sum_{k=1}^{k_n} w_{nk}\}_{n=1}^{\infty}$ also converges to z. Conversely, if $\{r_n + \sum_{k=1}^{k_n} w_{nk}\}_{n=1}^{\infty}$ converges to z, then the sequence $\{\sum_{k=1}^{k_n} \Im w_{nk}\}_{n=1}^{\infty}$ is bounded and hence by (2.2) the sequence $\{r_n + \sum_{k=1}^{k_n} z_{nk}\}_{n=1}^{\infty}$ converges to z as well. □

3. Proof of the Main Result

Given an infinitesimal triangular array $\{\mu_{nk} : n \geqslant 1,\ 1 \leqslant k \leqslant k_n\} \subset \mathscr{M}$, define constants
$$a_{nk} = \int_{|t|<1} t\, d\mu_{nk}(t),$$
and measures $\overline{\mu}_{nk}$ by
$$d\overline{\mu}_{nk}(t) = d\mu_{nk}(t + a_{nk}).$$
Note that $\max_{1 \leqslant k \leqslant k_n} |a_{nk}| \to 0$ as $n \to \infty$, and this implies that $\{\overline{\mu}_{nk}\}_{n,k}$ is also an infinitesimal array. Define the analytic function
$$f_{nk}(z) = z^2 \left[G_{\overline{\mu}_{nk}}(z) - \frac{1}{z} \right], \qquad z \in \mathbb{C}^+,$$

and the real-valued function

$$b_{nk}(y) = \int_{|t|\geqslant 1} a_{nk}\, d\mu_{nk}(t) + \int_{|t|\geqslant 1} \frac{(t-a_{nk})y^2}{y^2+(t-a_{nk})^2}\, d\mu_{nk}(t), \qquad y \geqslant 1.$$

Observe that $\Im f_{nk}(z) \leqslant 0$ if $\Im z > 0$, and $f_{nk}(z) = o(|z|)$ as $z \to \infty$ nontangentially.

The following lemma is related with a calculation in Section 4 of [10].

LEMMA 3.1. *For $y \geqslant 1$, we have for all $n \in \mathbb{N}$,*

$$|\Re\,[f_{nk}(iy) - b_{nk}(y)]| \leqslant 2\,|\Im f_{nk}(iy)|\,,$$

and for sufficiently large n,

$$|\Re f_{nk}(iy)| \leqslant (3+6y)\,|\Im f_{nk}(iy)|\,,$$

where $1 \leqslant k \leqslant k_n$.

Proof. Note that for $y \geqslant 1$, and $n \in \mathbb{N}$,

$$f_{nk}(iy) = \int_{-\infty}^{\infty} \frac{(t-a_{nk})y^2}{y^2+(t-a_{nk})^2}\, d\mu_{nk}(t) - i\int_{-\infty}^{\infty} \frac{(t-a_{nk})^2 y}{(t-a_{nk})^2+y^2}\, d\mu_{nk}(t).$$

Moreover, since $\int_{|t|<1}(t-a_{nk})\, d\mu_{nk}(t) = \int_{|t|\geqslant 1} a_{nk}\, d\mu_{nk}(t)$, we have

$$\begin{aligned}
|\Re\,[f_{nk}(iy) - b_{nk}(y)]| &= \left|\int_{|t|<1} \left[\frac{(t-a_{nk})y^2}{y^2+(t-a_{nk})^2}\, d\mu_{nk}(t) - (t-a_{nk})\right] d\mu_{nk}(t)\right| \\
&= \left|\int_{|t|<1} \frac{-(t-a_{nk})^3}{y^2+(t-a_{nk})^2}\, d\mu_{nk}(t)\right| \leqslant 2\,|\Im f_{nk}(iy)|\,.
\end{aligned}$$

Note that the infinitesimality of the family $\{\mu_{nk}\}_{n,k}$, implies that there exists $N \in \mathbb{N}$ such that $|a_{nk}| \leqslant 1/2$, for all $n \geqslant N$, $1 \leqslant k \leqslant k_n$. Therefore, for $n \geqslant N$,

$$\begin{aligned}
\left|\int_{|t|\geqslant 1} a_{nk}\, d\mu_{nk}(t)\right| &\leqslant |a_{nk}|\,(1+4y^2)\int_{|t|\geqslant 1} \frac{(t-a_{nk})^2}{y^2+(t-a_{nk})^2}\, d\mu_{nk}(t) \\
&\leqslant \frac{1+4y^2}{y}\,|\Im f_{nk}(iy)| \leqslant (1+4y)\,|\Im f_{nk}(iy)|\,,
\end{aligned}$$

and since $2x^2 \geqslant |x|$ when $|x| \geqslant \frac{1}{2}$, we have

$$\left|\int_{|t|\geqslant 1} \frac{(t-a_{nk})y^2}{y^2+(t-a_{nk})^2}\, d\mu_{nk}(t)\right| \leqslant 2y\,|\Im f_{nk}(iy)|\,.$$

Hence the second inequality follows. □

LEMMA 3.2. *Given $\beta \geqslant 1$, let $\Gamma_{\alpha,\beta}$ be a truncated cone where all the functions $\phi_{\mu_{nk}}$ are defined, and let $\{c_n\}_{n=1}^{\infty}$ be a sequence of real numbers.*

(1) *For any* $y > \beta$, *the sequence* $\{c_n + \sum_{k=1}^{k_n} \phi_{\mu_{nk}}(iy)\}_{n=1}^{\infty}$ *converges if and only the sequence* $\{c_n + \sum_{k=1}^{k_n} [a_{nk} + f_{nk}(iy)]\}_{n=1}^{\infty}$ *converges. Moreover, the two sequences have the same limit.*

(2) *If*

$$L = \sup_{n \geqslant 1} \sum_{k=1}^{k_n} \int_{-\infty}^{\infty} \frac{t^2}{1+t^2}\, d\overline{\mu}_{nk}(t) < \infty,$$

then $c_n + \sum_{k=1}^{k_n} \phi_{\mu_{nk}}(iy) = o(y)$ *uniformly in* n *as* $y \to \infty$ *if and only if* $c_n + \sum_{k=1}^{k_n} [a_{nk} + f_{nk}(iy)] = o(y)$ *uniformly in* n *as* $y \to \infty$.

Proof. Fix $y > \beta$. Applying Proposition 2.3 to $\{\overline{\mu}_{nk}\}_{n,k}$, we have

$$-\phi_{\mu_{nk}}(iy) + a_{nk} = -\phi_{\overline{\mu}_{nk}}(iy) = -f_{nk}(iy) \cdot (1 + v_{nk}(iy)), \tag{3.1}$$

where functions

$$v_n(iy) = \max_{1 \leqslant k \leqslant k_n} |v_{nk}(iy)|$$

converge to zero as $n \to \infty$. Then (1) follows from Lemma 3.1 and Lemma 2.4 by choosing $z_{nk} = -\phi_{\mu_{nk}}(iy) + a_{nk}$, $w_{nk} = -f_{nk}(iy)$ and $r_n = -c_n - \sum_{k=1}^{k_n} a_{nk}$.

We next prove (2). From (3.1), we have

$$z'_{nk}(iy) = w'_{nk}(iy) \cdot (1 + v_{nk}(iy)),$$

where $z'_{nk}(iy) = -\phi_{\mu_{nk}}(iy) + a_{nk} + b_{nk}(y) + b_{nk}(y)v_{nk}(iy)$, and $w'_{nk}(iy) = -f_{nk}(iy) + b_{nk}(y)$. Lemma 3.1 implies that for all $n \in \mathbb{N}$, and $1 \leqslant k \leqslant k_n$,

$$|\Re w'_{nk}(iy)| \leqslant 2\,|\Im f_{nk}(iy)|, \qquad y > \beta.$$

Since $\max_{1 \leqslant k \leqslant k_n} |a_{nk}| \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that $|a_{nk}| \leqslant 1/2$, for all $n \geqslant N$, $1 \leqslant k \leqslant k_n$. Therefore, for $n \geqslant N$, and $y > \beta \geqslant 1$,

$$\begin{aligned}
\sum_{k=1}^{k_n} |b_{nk}(y)| &\leqslant \sum_{k=1}^{k_n} \int_{|t| \geqslant 1} \frac{1}{2}\, d\mu_{nk}(t) + y \sum_{k=1}^{k_n} \int_{|t| \geqslant 1} \left| \frac{(t-a_{nk})y}{y^2 + (t-a_{nk})^2} \right| d\mu_{nk}(t) \\
&\leqslant (1+y) \sum_{k=1}^{k_n} \int_{|t| \geqslant 1} \frac{1}{2}\, d\mu_{nk}(t) \leqslant 5y \sum_{k=1}^{k_n} \int_{|t| \geqslant 1} \frac{1}{5}\, d\mu_{nk}(t) \\
&\leqslant 5y \sum_{k=1}^{k_n} \int_{|t| \geqslant 1} \frac{(t-a_{nk})^2}{1+(t-a_{nk})^2}\, d\mu_{nk}(t) \leqslant 5yL.
\end{aligned}$$

Since $v_n(iy) = o(1)$ uniformly in n as $y \to \infty$, by decreasing the cone we may assume that $v_n(iy) < 1/6$, for all $y > \beta$, and $n \in \mathbb{N}$. Define $r'_n(y) = -c_n - \sum_{k=1}^{k_n} a_{nk} - \sum_{k=1}^{k_n} b_{nk}(y)$. Replacing r_n, z_{nk}, w_{nk} by r'_n, z'_{nk}, and w'_{nk} respectively in inequalities (2.2) and (2.3), we deduce that

$$\left| \left(\sum_{k=1}^{k_n} \phi_{\mu_{nk}}(iy) \right) - \left(\sum_{k=1}^{k_n} [a_{nk} + f_{nk}(iy)] \right) \right| \leqslant \left| \sum_{k=1}^{k_n} \Im f_{nk}(iy) \right| + 5yLv_n(iy),$$

and

$$\frac{1}{2}\left|\sum_{k=1}^{k_n}\Im f_{nk}(iy)\right| \leqslant \left|\sum_{k=1}^{k_n}\Im\phi_{\mu_{nk}}(iy)\right| + 5yL v_n(iy),$$

for all $n \geqslant N$, and $y > \beta$. Hence the result follows from the facts that $v_n(iy) = o(1)$ uniformly in n as $y \to \infty$, and that (2) holds uniformly for n in a finite subset of $\mathbb{N}$. □

Fix a real number γ and a finite positive Borel measure σ on $\mathbb{R}$. Recall that the measure $\nu_{\boxplus}^{\gamma,\sigma}$ (resp., $\nu_{*}^{\gamma,\sigma}$) is the $\boxplus$-infinitely divisible (resp., $*$-infinitely divisible) probability distribution discussed in Section 2.

THEOREM 3.3. *For an infinitesimal array $\{\mu_{nk}\}_{n,k} \subset \mathscr{M}$ and a sequence $\{c_n\}_{n=1}^{\infty} \subset \mathbb{R}$, the following statements are equivalent:*

(1) *The sequence $\mu_{n1} \boxplus \mu_{n2} \boxplus \ldots \boxplus \mu_{nk_n} \boxplus \delta_{c_n}$ converges weakly to $\nu_{\boxplus}^{\gamma,\sigma}$;*
(2) *The sequence $\mu_{n1} * \mu_{n2} * \ldots * \mu_{nk_n} * \delta_{c_n}$ converges weakly to $\nu_{*}^{\gamma,\sigma}$;*
(3) *The sequence of measures*

$$d\sigma_n(t) = \sum_{k=1}^{k_n}\frac{t^2}{1+t^2}\,d\overline{\mu}_{nk}(t)$$

converges weakly on $\mathbb{R}$ to σ, and the sequence of numbers

$$\gamma_n = c_n + \sum_{k=1}^{k_n}\left[a_{nk} + \int_{-\infty}^{\infty}\frac{t}{1+t^2}\,d\overline{\mu}_{nk}(t)\right]$$

converges to γ as $n \to \infty$.

Proof. The equivalence of (2) and (3) is classical (see [11, 15]). We will prove the equivalence of (1) and (3). Assume that (1) holds. By Theorem 2.2, there exist $\alpha > 0$ and $\beta \geqslant 1$ such that $\phi_{\mu_{nk}}$ are defined on $\Gamma_{\alpha,\beta}$, and we have

$$\lim_{n\to\infty}\phi_{\mu_{n1}\boxplus\mu_{n2}\boxplus\ldots\boxplus\mu_{nk_n}\boxplus\delta_{c_n}}(iy) = \phi_{\nu_{\boxplus}^{\gamma,\sigma}}(iy), \qquad y > \beta.$$

Since

$$\phi_{\mu_{n1}\boxplus\mu_{n2}\boxplus\ldots\boxplus\mu_{nk_n}\boxplus\delta_{c_n}}(z) = c_n + \sum_{k=1}^{k_n}\phi_{\mu_{nk}}(z), \qquad z \in \Gamma_{\alpha,\beta},$$

we have

$$\lim_{n\to\infty}\left(c_n + \sum_{k=1}^{k_n}\phi_{\mu_{nk}}(iy)\right) = \phi_{\nu_{\boxplus}^{\gamma,\sigma}}(iy),$$

for all $y > \beta$, and $c_n + \sum_{k=1}^{k_n}\phi_{\mu_{nk}}(iy) = o(y)$ uniformly in n as $y \to \infty$. By Lemma 3.2,

$$\lim_{n\to\infty}\left(c_n + \sum_{k=1}^{k_n}[a_{nk} + f_{nk}(iy)]\right) = \phi_{\nu_{\boxplus}^{\gamma,\sigma}}(iy), \qquad y > \beta. \tag{3.2}$$

Note that for $z \in \mathbb{C}^+$, $n \in \mathbb{N}$,

$$
\begin{aligned}
z^2\left[G_{\overline{\mu}_{nk}}(z) - \frac{1}{z}\right] &= \int_{-\infty}^{\infty} \frac{tz}{z-t}\, d\overline{\mu}_{nk}(t) \\
&= \int_{-\infty}^{\infty} \frac{t}{1+t^2}\, d\overline{\mu}_{nk}(t) + \int_{-\infty}^{\infty} \left[\frac{tz}{z-t} - \frac{t}{1+t^2}\right] d\overline{\mu}_{nk}(t) \\
&= \int_{-\infty}^{\infty} \frac{t}{1+t^2}\, d\overline{\mu}_{nk}(t) + \int_{-\infty}^{\infty} \left[\frac{1+tz}{z-t}\right] \frac{t^2}{1+t^2}\, d\overline{\mu}_{nk}(t).
\end{aligned}
$$

We conclude that

$$
c_n + \sum_{k=1}^{k_n} [a_{nk} + f_{nk}(z)] = \gamma_n + \int_{-\infty}^{\infty} \frac{1+tz}{z-t}\, d\sigma_n(t). \tag{3.3}
$$

Since $\Im f_{nk}(z) \leqslant 0$ for $z \in \mathbb{C}^+$, $\{c_n + \sum_{k=1}^{k_n} [a_{nk} + f_{nk}(z)]\}_{n=1}^{\infty}$ is a normal family of analytic functions in $\mathbb{C}^+$, and from (3.2) the sequence has pointwise limit $\phi_{\nu_{\boxplus}^{\gamma,\sigma}}(z)$ for all $z = iy$, $y > \beta$. It is an easy application of the Montel Theorem that (3.2) holds uniformly on compact subsets of $\mathbb{C}^+$. Hence (3.3) and (2.1) imply, at $z = i$, that

$$
\begin{aligned}
\lim_{n\to\infty} \sigma_n(\mathbb{R}) &= \lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{1+t^2}{1+t^2}\, d\sigma_n(t) \\
&= \lim_{n\to\infty} -\Im\left(c_n + \sum_{k=1}^{k_n} [a_{nk} + f_{nk}(i)]\right) \\
&= -\Im \phi_{\nu_{\boxplus}^{\gamma,\sigma}}(i) \\
&= \sigma(\mathbb{R}).
\end{aligned}
$$

Thus,

$$
L = \sup_{n\geqslant 1} \sigma_n(\mathbb{R}) = \sup_{n\geqslant 1} \sum_{k=1}^{k_n} \int_{-\infty}^{\infty} \frac{t^2}{1+t^2}\, d\overline{\mu}_{nk}(t) < \infty.
$$

By Lemma 3.2, this implies that $c_n + \sum_{k=1}^{k_n} [a_{nk} + f_{nk}(iy)] = o(y)$ uniformly in n as $y \to \infty$. For $y > \beta$, $n \in \mathbb{N}$, note that

$$
\frac{1}{2}\sigma_n(\{|t| \geqslant y\}) \leqslant \int_{-\infty}^{\infty} \frac{1+t^2}{y^2+t^2}\, d\sigma_n(t) = -\frac{1}{y}\Im\left(c_n + \sum_{k=1}^{k_n} [a_{nk} + f_{nk}(iy)]\right).
$$

Since $c_n + \sum_{k=1}^{k_n} [a_{nk} + f_{nk}(iy)] = o(y)$ uniformly in n as $y \to \infty$, we conclude that $\{\sigma_n\}_{n=1}^{\infty}$ is a tight family. Let σ' be a weak cluster point of $\{\sigma_n\}_{n=1}^{\infty}$ and consider a subsequence $\{\sigma_{n_j}\}_{j=1}^{\infty}$ that converges weakly to σ'. Hence, for any $z = x + iy \in \Gamma_{\alpha,\beta}$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}(1+t^2)\,d\sigma'(t) &= -\lim_{j\to\infty} \Im\left(c_{n_j} + \sum_{k=1}^{k_{n_j}} \left[a_{n_jk} + f_{n_jk}(x+iy)\right]\right) \\ &= -\Im\phi_{\nu_{\boxplus}^{\gamma,\sigma}}(x+iy) \\ &= \int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}(1+t^2)\,d\sigma(t).\end{aligned}$$

Therefore, the Poisson integrals of the measures $(1+t^2)\,d\sigma'(t)$ and $(1+t^2)\,d\sigma(t)$ are identical since they coincide on an open subset of $\mathbb{C}^+$. Thus, $\sigma' = \sigma$. Since the tight family $\{\sigma_n\}_{n=1}^{\infty}$ has a unique weak cluster point, they must converge weakly to σ. Moreover, we deduce that $\lim_{n\to\infty}\gamma_n = \gamma$ from (3.2) and (3.3).

To prove the converse, assume the sequence of measures $\{\sigma_n\}_{n=1}^{\infty}$ converges weakly to σ and the sequence $\{\gamma_n\}_{n=1}^{\infty}$ converges to γ as $n\to\infty$. Then, $\{\sigma_n\}_{n=1}^{\infty}$ is a tight family and in particular

$$L = \sup_n \sigma_n(\mathbb{R}) < \infty.$$

From Lemma 2.1 and the infinitesimality of the array $\{\mu_{nk}\}_{n,k}$, there exists a truncated cone $\Gamma_{\alpha',\beta'}$ with $\beta' \geqslant 1$ such that all $\phi_{\mu_{nk}}$ are defined in $\Gamma_{\alpha',\beta'}$. Combine the inequality

$$\left|\frac{1+ity}{iy-t}\right| \leqslant y, \qquad t\in\mathbb{R},\ \ y\geqslant 1.$$

with (3.3) to obtain

$$\lim_{n\to\infty}\left(c_n + \sum_{k=1}^{k_n}\left[a_{nk} + f_{nk}(iy)\right]\right) = \phi_{\nu_{\boxplus}^{\gamma,\sigma}}(iy), \qquad y > \beta'.$$

Hence by Lemma 3.2,

$$\lim_{n\to\infty}\left(c_n + \sum_{k=1}^{k_n}\phi_{\mu_{nk}}(iy)\right) = \phi_{\nu_{\boxplus}^{\gamma,\sigma}}(iy), \qquad y > \beta'.$$

Also, note that for any $M>0$ and $y>\beta'\geqslant 1$, we have

$$\begin{aligned}\frac{1}{y}\left|c_n + \sum_{k=1}^{k_n}\left[a_{nk} + f_{nk}(iy)\right]\right| &\leqslant \frac{|\gamma_n|}{y} + \frac{1}{y}\int_{-\infty}^{\infty}\left|\frac{1+ity}{iy-t}\right| d\sigma_n(t) \\ &\leqslant \frac{|\gamma_n|}{y} + \frac{1}{y}\int_{|t|<M}\frac{1+|t|\,y}{\sqrt{y^2+t^2}}\,d\sigma_n(t) + \sigma_n(\{|t|\geqslant M\}) \\ &\leqslant \frac{|\gamma_n|}{y} + \frac{L(1+My)}{y^2} + \sigma_n(\{|t|\geqslant M\}).\end{aligned}$$

Therefore, it follows from the convergence of $\{\gamma_n\}_{n=1}^{\infty}$ and the tightness of the family $\{\sigma_n\}_{n=1}^{\infty}$ that $c_n + \sum_{k=1}^{k_n}\left[a_{nk} + f_{nk}(iy)\right] = o(y)$ uniformly in n as $y\to\infty$. By Lemma

3.2 again, $c_n + \sum_{k=1}^{k_n} \phi_{\mu_{nk}}(iy) = o(y)$ uniformly in n as $y \to \infty$. Statement (1) now follows from Theorem 2.2. $\square$

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