Operators
and
Matrices
Volume 2, Number 1 (2008), 125–136

CHARACTERIZING JORDAN AUTOMORPHISMS OF MATRIX ALGEBRAS THROUGH PRESERVING PROPERTIES

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(*communicated by C.-K. Li*)

Abstract. Let M_n be the algebra of all $n \times n$ complex matrices, $n \geqslant 3$. We prove that a map $\phi : M_n \to M_n$ is a Jordan automorphism if and only if ϕ is a continuous spectrum and commutativity preserving map (no linearity is assumed). Examples are given showing that this characterization is optimal

1. Introduction and statement of the main result

Let M_n be the algebra of all $n \times n$ complex matrices. By D_n we will denote the subalgebra of all diagonal matrices. If X is an arbitrary complex matrix (not necessarily a square matrix), then X^t stands for the transpose of X.

It is well-known that every automorphism of the algebra M_n is inner. More precisely, if $\phi : M_n \to M_n$ is a bijective linear multiplicative map, then there exists an invertible matrix $A \in M_n$ such that $\phi(X) = AXA^{-1}$ for all $X \in M_n$. If $\phi : M_n \to M_n$ is an anti-automorphism (a bijective linear map satisfying $\phi(XY) = \phi(Y)\phi(X)$, $X, Y \in M_n$), then the map $X \mapsto \phi(X)^t$ is obviously an automorphism. Hence, every such ϕ is of the form $\phi(X) = AX^tA^{-1}$ for some invertible $A \in M_n$. Automorphisms and anti-automorphisms of algebras are special cases of Jordan automorphisms. Recall that a map ϕ defined on an algebra $\mathscr{A}$ is a Jordan automorphism if it is a bijective linear map satisfying $\phi(a^2) = \phi(a)^2$ for every $a \in \mathscr{A}$. It is well-known that every Jordan automorphism of the matrix algebra M_n is either an automorphism, or an anti-automorphism. In other words, every Jordan automorphism of M_n is an inner automorphism possibly composed with the transposition.

Jordan automorphisms of M_n have many preserving properties. For example, every Jordan automorphism of M_n preserves rank, spectrum, commutativity, and nilpotents, that is, if $\phi : M_n \to M_n$ is a Jordan automorphism, then for every pair $X, Y \in M_n$ we have $\operatorname{rank} \phi(X) = \operatorname{rank} X$, $\sigma(\phi(X)) = \sigma(X)$, $\phi(X)\phi(Y) = \phi(Y)\phi(X)$ whenever $XY = YX$, and $\phi(X)$ is nilpotent providing that X is nilpotent. Here, $\sigma(X)$ denotes the spectrum of X, that is, the set of all eigenvalues of X.

Mathematics subject classification (2000): 15A04, 15A18, 15A27, 16S50.
Key words and phrases: Jordan automorphism, matrix algebra, commutativity, spectrum, preserver.
Supported in part by a grant from the Ministry of Science of Slovenia.

Paper OAM-02-07

There is a vast literature on linear preserver problems (see the survey paper [4]). A linear preserver $\phi : M_n \to M_n$ is a linear map having a certain preserving property. It often turns out that such a map must be an automorphism or an anti-automorphism. Hence, linear preserver results can be considered as characterizations of Jordan automorphisms. Among the most studied linear preserver problems are the one dealing with linear maps preserving spectrum (or more generally, preserving invertibility) and the one treating commutativity preserving maps. Both of them were studied on much more general algebras and rings than matrix algebras. The study of linear maps preserving spectrum was initiated by Kaplansky. One of the reasons to study commutativity preserving maps is that the assumption of preserving commutativity can be equivalently reformulated as a property of preserving zero Lie products.

Only very recently the first results on non-linear preservers appeared in the literature [1, 5, 7, 9, 10]. It turns out that non-linear preservers of spectrum or commutativity may have a rather wild form. So, if we want to characterize Jordan automorphisms of matrix algebras using only preserving properties (without assuming linearity or multiplicativity), then we have to impose more than one preserving property on the maps under consideration. The first attempt in this direction was made in [6]. The result there was not optimal. It was only conjectured that one can characterize Jordan automorphisms of M_n as continuous maps preserving spectrum and commutativity. We will confirm this conjecture and show by examples that this is an optimal result. Moreover, the proof of the weaker result given in [6] (in that paper Jordan automorphisms were characterized as continuous maps preserving spectrum and commutativity in both directions, or as continuous maps preserving spectrum, commutativity, and rank one) was rather long and computational. The approach here is simpler and is based on a recent simple result from projective geometry.

A map $\phi : M_n \to M_n$ preserves commutativity if for every pair of matrices $X, Y \in M_n$ we have

$$XY = YX \Rightarrow \phi(Y)\phi(X) = \phi(X)\phi(Y).$$

It preserves spectrum if $\sigma(\phi(X)) = \sigma(X)$ for every $X \in M_n$. It should be mentioned that instead of spectrum preserving property we could also consider the assumption of preserving eigenvalues, that is, the property that $\text{e.v.}(\phi(X)) = \text{e.v.}(X)$ for every $X \in M_n$. Here, $\sigma(X)$ denotes the set of eigenvalues (the set with at least one element and at most n elements), while $\text{e.v.}(X)$ denotes the unordered n-tuple of eigenvalues where each eigenvalue λ appears $m(\lambda, A)$ times. Here, $m(\lambda, A)$ denotes the algebraic multiplicity of λ as an eigenvalue of A. As we will consider only continuous maps, these two preserving properties are equivalent. Clearly, if ϕ preserves eigenvalues, then it preserves spectrum. If we assume that ϕ preserves spectrum, then obviously, $\text{e.v.}(\phi(X)) = \text{e.v.}(X)$ for every $X \in M_n$ with n distinct eigenvalues. The set of such matrices is dense in M_n and ϕ is continuous. Thus, [2, Theorem 20.4] yields that $\text{e.v.}(\phi(X)) = \text{e.v.}(X)$ for every $X \in M_n$.

THEOREM 1.1. *Let $\phi : M_n \to M_n$, $n \geqslant 3$, be a map. Then the following conditions are equivalent:*

1. ϕ is a Jordan automorphism.

2. *ϕ is a continuous commutativity and spectrum preserving map.*
3. *There exists an invertible matrix $A \in M_n$ such that either $\phi(X) = AXA^{-1}$ for all $X \in M_n$, or $\phi(X) = AX^tA^{-1}$ for all $X \in M_n$.*

All we need to do to prove this theorem is to show that the second condition implies the last one. The proof combines linear algebra techniques with some tools from analysis and geometry.

Let us conclude with some notation. We will identify vectors in $\mathbb{C}^n$ with $n \times 1$ matrices. Hence, if $u, v \in \mathbb{C}^n$ are nonzero vectors, then $uv^t \in M_n$ is a rank one matrix. Note that every rank one matrix can be written in this form. The rank one matrix uv^t is an idempotent if and only if $v^tu = 1$, and uv^t is square-zero if and only if $v^tu = 0$. If $X = uv^t$ and $Y = wz^t$ are two rank one matrices, then we will write $X \sim Y$ if u and w are linearly dependent or v and z are linearly dependent. Note that this relation is well-defined, because $uv^t = u_1v_1^t \neq 0$ yields that u and u_1 as well as v and v_1 are linearly dependent. We further denote by $E_{ij} \in M_n$, $1 \leqslant i, j \leqslant n$, the elements of the standard basis of M_n, that is, E_{ij} is the $n \times n$ matrix whose entries are all zero except the (i, j)-entry which is equal to 1.

2. Preliminary results

Throughout this section we will assume that $\phi : M_n \to M_n$, $n \geqslant 3$, is a continuous commutativity and spectrum preserving map.

LEMMA 2.1. *Let $T \in M_n$ be an invertible matrix. Then there exists an invertible matrix S such that*

$$\phi(TDT^{-1}) = SDS^{-1}, \quad D \in D_n. \tag{1}$$

In particular, if X and Y are simultaneously diagonalizable matrices, then $\phi(XY) = \phi(X)\phi(Y)$.

Proof. Because the map ϕ preserves spectrum, the matrix

$$\phi(T\,\mathrm{diag}(1, 2, \ldots, n)T^{-1})$$

has n different eigenvalues, and is therefore diagonalizable. It follows that there exists an invertible $S \in M_n$ such that

$$\phi(T\,\mathrm{diag}(1, 2, \ldots, n)T^{-1}) = S\,\mathrm{diag}(1, 2, \ldots, n)S^{-1}.$$

It is straightforward to verify that $X \in M_n$ commutes with $T\,\mathrm{diag}(1, 2, \ldots, n)T^{-1}$ if and only if $X = TDT^{-1}$ for some diagonal matrix D. It follows that ϕ maps every TDT^{-1}, $D \in D_n$, into a matrix of the form $SD'S^{-1}$, where D' is a diagonal matrix. Moreover, as ϕ preserves spectrum, D and D' have the same (possibly permuted) diagonal entries. Observe that in order to prove (1), it is enough to verify this equation only for diagonal matrices with distinct diagonal entries. Choose such a matrix $\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. We can find disjoint Jordan curves $\mu_k : [0, 1] \to \mathbb{C}$, $k = 1, \ldots, n$, such that $\mu_k(0) = k$ and $\mu_k(1) = \lambda_k$. We define $t_0 \in [0, 1]$ to be the supremum of the set of all real numbers $t \in [0, 1]$ satisfying

$$\phi(T\,\mathrm{diag}(\mu_1(t), \ldots, \mu_n(t))T^{-1}) = S\,\mathrm{diag}(\mu_1(t), \ldots, \mu_n(t))S^{-1}.$$

It is clear that the set of all real numbers $t \in [0,1]$ with the above property is open and closed in $[0,1]$, and hence, $t_0 = 1$. This completes the proof. $\square$

COROLLARY 2.2. *Let p be an arbitrary polynomial. Then*

$$\phi(p(X)) = p(\phi(X)), \quad X \in M_n.$$

Proof. Define a continuous map $\psi : M_n \to M_n$ by

$$\psi(X) = \phi(p(X)) - p(\phi(X)), \quad X \in M_n.$$

Then, by the previous lemma, $\psi(X) = 0$ for every diagonalizable matrix. The set of all diagonalizable matrices is dense in M_n, and thus, $\psi(X) = 0$ for every $X \in M_n$, as desired. $\square$

COROLLARY 2.3. *If $X, Y \in M_n$ are diagonalizable matrices and $XY = YX = 0$, then $\phi(X)\phi(Y) = \phi(Y)\phi(X) = 0$.*

Proof. All we have to do is to observe that if X, Y is a commuting pair of diagonalizable matrices, then X and Y are simultaneously diagonalizable, that is, there exists an invertible $T \in M_n$ such that both $T^{-1}XT$ and $T^{-1}YT$ are diagonal. $\square$

Let us now specialize to the 3×3-case.

LEMMA 2.4. *Let $\phi : M_3 \to M_3$ be a continuous commutativity and spectrum preserving map. Then for every pair of rank one idempotents $P, Q \in M_3$ the relation $P \sim Q$ implies that $\phi(P) \sim \phi(Q)$.*

Proof. Assume on the contrary that there exist rank one idempotents $P, Q \in M_3$ such that $P \sim Q$ and $\phi(P) \not\sim \phi(Q)$. After replacing ϕ by the map $X \mapsto \phi(SXS^{-1})$, where S is an appropriate invertible matrix, we may assume that either $P = E_{33}$ and $Q = E_{32} + E_{33}$, or $P = E_{33}$ and $Q = E_{23} + E_{33}$. The second case can be reduced to the first one if we compose ϕ with the transposition. So, we may, and will assume that $P = E_{33}$, $Q = E_{32} + E_{33}$, and $\phi(P) \not\sim \phi(Q)$.

Further, Lemma 2.1 tells us that after replacing ϕ by the map $X \mapsto T\phi(X)T^{-1}$, where T is an appropriate invertible matrix, we may and will assume that $\phi(E_{ii}) = E_{ii}$, $i = 1, 2, 3$. Since $E_{11}(E_{32} + E_{33}) = (E_{32} + E_{33})E_{11} = 0$, Corollary 2.3 yields that

$$\phi(E_{32} + E_{33}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix},$$

where the $*$'s stand for some complex numbers. We also know that the $(2,2)$-entry of this matrix is nonzero, since otherwise $\phi(E_{32} + E_{33})$ would be an idempotent of rank one satisfying $\phi(E_{32} + E_{33}) \sim \phi(E_{33})$, a contradiction.

Now, for every complex number λ the idempotent $E_{11} + \lambda E_{12}$ is orthogonal to both idempotents E_{33} and $E_{32} + E_{33}$, that is,

$$E_{33}(E_{11} + \lambda E_{12}) = (E_{11} + \lambda E_{12})E_{33} = 0$$

and

$$(E_{32}+E_{33})(E_{11}+\lambda E_{12}) = (E_{11}+\lambda E_{12})(E_{32}+E_{33}) = 0$$

(from now on we will denote the fact that X and Y are simultaneously diagonalizable matrices satisfying $XY = YX = 0$ by $X \perp Y$). Hence, by Corollary 2.3, we have $\phi(E_{32}+E_{33}), \phi(E_{33}) \perp \phi(E_{11}+\lambda E_{12})$ for every complex number λ. It follows that

$$\phi(E_{11}+\lambda E_{12}) = E_{11}$$

for every $\lambda \in \mathbb{C}$.

Consider now any nonzero matrix of the form

$$\begin{bmatrix} 0 & \lambda & \mu \\ 0 & \alpha\lambda & \alpha\mu \\ 0 & \beta\lambda & \beta\mu \end{bmatrix}$$

with $\alpha\lambda+\beta\mu \neq 0$ and $\alpha \neq 0$. Observe that the set of all such matrices is dense in the set of all rank one matrices with the first column equal to zero and that every such matrix is diagonalizable and orthogonal to the diagonalizable matrix $E_{11} - \alpha^{-1}E_{12}$ which is mapped by ϕ into E_{11}. We conclude that if X is any rank one matrix of the form

$$\begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix},$$

then $\phi(X)$ is of the form

$$\phi(X) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

We continue by showing that any idempotent P of rank one of the form

$$P = \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is mapped into E_{11}. We can write P as

$$P = \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix},$$

where x is a 1×2 matrix. We take any pair of rank one idempotents $T, S \in M_2$ with $T \perp S$. Then

$$\begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -xT \\ 0 & T \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & -xS \\ 0 & S \end{bmatrix}$$

are pairwise orthogonal rank one idempotents and we know that they are mapped into pairwise orthogonal rank one idempotents. But the last two are mapped into matrices having nonzero entries only in the bottom-right 2×2 corner, and consequently, P is mapped into E_{11}, as desired. Using Corollary 2.2 and the continuity of the map ϕ we conclude that any matrix X of the form

$$X = \begin{bmatrix} \lambda & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is mapped into λE_{11}. Here, λ is any complex number.

In the next step we will show that if X is any matrix of the form

$$\begin{bmatrix} \lambda & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix},$$

then $\phi(X)$ is of the form

$$\phi(X) = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

By the continuity, it is enough to show that this is true for every such X with three different eigenvalues. But every such X can be written as $X = \lambda P_1 + \mu P_2 + \eta P_3$, where pairwise orthogonal rank one idempotents P_1, P_2, and P_3 are of the form

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix},$$

respectively. Indeed, each of the P_i's is equal to $q(X)$ for some polynomial q, and consequently, each of the P_i's is of the same upper block triangular form as X. We conclude this step of the proof by observing that Lemma 2.1 yields that $\phi(X) = \lambda\phi(P_1) + \mu\phi(P_2) + \eta\phi(P_3)$.

We define a map $\varphi : M_2 \to M_2$ by

$$\phi\left(\begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & \varphi(Z) \end{bmatrix}.$$

Here, Z is any 2×2 matrix and the zeroes represent zero matrices of appropriate sizes. It is our aim now to show that for every $\lambda \in \mathbb{C}$, every 1×2 matrix x and every $Z \in M_2$ the matrix

$$X = \begin{bmatrix} \lambda & x \\ 0 & Z \end{bmatrix}$$

is mapped by ϕ into

$$\begin{bmatrix} \lambda & 0 \\ 0 & \varphi(Z) \end{bmatrix}.$$

Once again, it is enough to show this only for matrices X with three different eigenvalues, that is, for matrices X of the form

$$X = \lambda \begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & z \\ 0 & P \end{bmatrix} + \eta \begin{bmatrix} 0 & w \\ 0 & Q \end{bmatrix},$$

where the matrices in the above expression are pairwise orthogonal rank one idempotents. In particular, $P \perp Q$ and therefore, $\varphi(\mu P + \eta Q) = \mu\varphi(P) + \eta\varphi(Q)$. By what we already know we only need to show that every rank one idempotent of the form

$$R = \begin{bmatrix} 0 & z \\ 0 & P \end{bmatrix}$$

is mapped by ϕ into

$$\begin{bmatrix} 0 & 0 \\ 0 & \varphi(P) \end{bmatrix}.$$

As R is an idempotent, we have $zP = z$. Thus, $\phi(R)$ is orthogonal to

$$\phi\left(\begin{bmatrix} 0 & 0 \\ 0 & I-P \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & \varphi(I-P) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & I-\varphi(P) \end{bmatrix}.$$

Moreover, $\phi(R)$ is a rank one idempotent having nonzero entries only in the 2×2 bottom-right corner, and hence,

$$\phi(R) = \begin{bmatrix} 0 & 0 \\ 0 & \varphi(P) \end{bmatrix},$$

as desired.

Because $\phi(E_{22} - E_{32}) = \phi(E_{22} + E_{33}) - \phi(E_{32} + E_{33}) = E_{22} + E_{33} - \phi(E_{32} + E_{33})$ and because $\phi(E_{32} + E_{33}) \not\sim \phi(E_{33}) = E_{33}$, the $(3,3)$-entry of rank one idempotent $\phi(E_{22} - E_{32})$ is non-zero. It follows that the $(3,3)$-entry of every rank one idempotent $\phi(\lambda E_{12} + E_{22} - E_{32})$, where λ is any complex number, is nonzero.

Consider any rank one idempotent R of the form

$$\begin{bmatrix} * & 0 & * \\ 0 & 0 & 0 \\ * & 0 & * \end{bmatrix},$$

where the $*$'s are all nonzero scalars. Each such idempotent is orthogonal to E_{22} as well as to some rank one idempotent of the form $\lambda E_{12} + E_{22} - E_{32}$. As $\phi(E_{22}) = E_{22}$ and $\phi(\lambda E_{12} + E_{22} - E_{32})$ has a nonzero $(3,3)$-entry, we conclude that $\phi(R) = E_{11}$ for every R as above. But we can find two such idempotents R_1 and R_2 with the additional property that $R_1 \perp R_2$. It follows that $E_{11} = \phi(R_1) \perp \phi(R_2) = E_{11}$, a contradiction. This completes the proof. $\square$

3. Proof of the main result

Assume that $\phi : M_n \to M_n$, $n \geqslant 3$, is a continuous commutativity and spectrum preserving map. In the first step we will prove that for every pair of rank one idempotents $P, Q \in M_n$ the relation $P \sim Q$ implies that $\phi(P) \sim \phi(Q)$. Indeed, if $n = 3$, then we are done by Lemma 2.4. So, assume that $n > 3$ and let $P, Q \in M_n$ be rank one idempotents such that $P \sim Q$. We can find pairwise orthogonal rank one idempotents $P_4, \ldots, P_n$ such that $P \perp P_j$ and $Q \perp P_j$, $j = 4, \ldots, n$. Set $R = I - P_4 - \ldots - P_n$.

After composing ϕ with a similarity transformation we may, and will assume that $\phi(R) = R$ and $\phi(P_j) = P_j$, $j = 4,\dots,n$. Obviously, R is an idempotent of rank three, $P, Q \in RM_nR$, and RM_nR can be identified with M_3. So, if we show that $\phi(RM_nR) \subset RM_nR$, then we can apply Lemma 2.4 to conclude that $\phi(P) \sim \phi(Q)$. If $T \in RM_nR$ is any idempotent of rank one, then T is orthogonal to P_j, $j = 4,\dots,n$, and consequently, $\phi(T) \in RM_nR$. It follows that $\phi(X) \in RM_nR$ for every diagonalizable $X \in RM_nR$, and by the continuity of ϕ, we have $\phi(X) \in RM_nR$ whenever $X \in RM_nR$.

For nonzero vectors $x, y \in \mathbb{C}^n$ we define

$$L_x = \{xu^t \,:\, u \in \mathbb{C}^n,\ u^t x = 1\}$$

and

$$R_y = \{wy^t \,:\, w \in \mathbb{C}^n,\ y^t w = 1\}.$$

These are both subsets of M_n consisting of rank one idempotents. Clearly, $P \sim Q$ whenever $P, Q \in L_x$ ($P, Q \in R_y$).

In the next step we will prove that either for every nonzero $x \in \mathbb{C}^n$ there exists a nonzero $u \in \mathbb{C}^n$ such that $\phi(L_x) \subset L_u$, or for every nonzero $x \in \mathbb{C}^n$ there exists a nonzero $y \in \mathbb{C}^n$ such that $\phi(L_x) \subset R_y$. Assume for a moment that we have already proved this. Then we can assume with no loss of generality that we have the first possibility, since in the second case we can replace the map ϕ by the map $X \mapsto (\phi(X))^t$, $X \in M_n$.

Assume first that for every nonzero $x \in \mathbb{C}^n$ there exists an idempotent P_x of rank one such that $\phi(Q) = P_x$ for every $Q \in L_x$. If $x, y \in \mathbb{C}^n$ are linearly independent, then we can find $u, v \in \mathbb{C}^n$ such that $u^t x = 1$, $u^t y = 0$, $v^t x = 0$, and $v^t y = 1$. Thus, $xu^t \perp yv^t$, and consequently, $P_x \perp P_y$. It follows that there exist infinitely many pairwise orthogonal rank one idempotents, a contradiction.

We have shown that there exists a nonzero $x \in \mathbb{C}^n$ and $u, v \in \mathbb{C}^n$ satisfying $u^t x = v^t x = 1$ such that $\phi(xu^t) \neq \phi(xv^t)$. On the other hand, we know that $\phi(xu^t) \sim \phi(xv^t)$. So, we have either

$$\phi(xu^t) = zw_1^t \quad \text{and} \quad \phi(xv^t) = zw_2^t$$

for some $z \in \mathbb{C}^n$ and linearly independent vectors $w_1, w_2 \in \mathbb{C}^n$ such that $w_1^t z = w_2^t z = 1$, or

$$\phi(xu^t) = z_1 w^t \quad \text{and} \quad \phi(xv^t) = z_2 w^t$$

for some $w \in \mathbb{C}^n$ and linearly independent vectors $z_1, z_2 \in \mathbb{C}^n$ such that $w^t z_1 = w^t z_2 = 1$. We will consider only the second case. If $y \in \mathbb{C}^n$ is any vector such that $y^t x = 1$ then $\phi(xy^t) \sim \phi(xu^t)$ and $\phi(xy^t) \sim \phi(xv^t)$, and consequently, $\phi(xy^t) \in R_w$. Thus, $\phi(L_x) \subset R_w$ and we will prove that for every nonzero $x_1 \in \mathbb{C}^n$ there exists a nonzero w_1 such that $\phi(L_{x_1}) \subset R_{w_1}$. There is nothing to prove in the case when all members of L_{x_1} are mapped into the same idempotent of rank one. So, we may assume that $\phi(L_{x_1})$ contains two distinct elements. Then, as above we see that either $\phi(L_{x_1}) \subset L_{u_1}$ for some nonzero u_1, or $\phi(L_{x_1}) \subset R_{w_1}$ for some nonzero w_1.

All we have to do is to show that the first possibility cannot occur. Assume on the contrary that $\phi(L_x) \subset R_w$ and $\phi(L_{x_1}) \subset L_{u_1}$ where both $\phi(L_x)$ and $\phi(L_{x_1})$ contain more than just one element. Then x and x_1 are linearly independent, and therefore, we can find $z, z_1 \in \mathbb{C}^n$ such that $z^t x = 1$, $z^t x_1 = 0$, $z_1^t x = 0$, and $z_1^t x_1 = 1$. It follows that $xz^t \perp x_1 z_1^t$. Thus, $\phi(xz^t) \perp \phi(x_1 z_1^t)$, and consequently, $w^t u_1 = 0$. Now, take any $u \in \mathbb{C}^n$ such that $u^t x = u^t x_1 = 1$. Then $\phi(xu^t) \in R_w$, $\phi(x_1 u^t) \in L_{u_1}$, and $\phi(xu^t) \sim \phi(x_1 u^t)$. It follows that $\phi(xu^t)$ belongs to the linear span of the rank one matrix $u_1 w^t$ or $\phi(x_1 u^t)$ belongs to the linear span of the rank one matrix $u_1 w^t$. But this is impossible as $u_1 w^t$ is a square-zero matrix.

We are now in a position to use a nonbijective version of the fundamental theorem of projective geometry (see [3]). In fact, for our purpose it is more convinient to use one of the recently proved consequences of this theorem [8, Theorem 1.2]. We will show that for every pair of rank one idempotents $P, Q \in M_n$ we have $PQ = 0 \Rightarrow \phi(P)\phi(Q) = 0$. Indeed, let $P = xy^t$ and $Q = uv^t$ be rank one idempotents with $PQ = 0$, that is, $y^t u = 0$. As $y^t x = 1$, the vectors u and x are linearly independent. Hence, we can find $z \in \mathbb{C}^n$ such that $z^t x = 0$ and $z^t u = 1$. It follows that $xy^t \perp uz^t = R$. Thus, $\phi(P) \perp \phi(R)$, and since $\phi(Q)$ and $\phi(R)$ both belong to L_w for some nonzero $w \in \mathbb{C}^n$, we have $\phi(P)\phi(Q) = 0$. It follows [8, Theorem 1.2] that there exist a nonsingular matrix $A \in M_n$ and a nonzero endomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ such that $\phi(X) = AX_\varphi A^{-1}$ for every idempotent X of rank one. Here, X_φ denotes the matrix obtained from X by applying φ entrywise, $X_\varphi = [x_{ij}]_\varphi = [\varphi(x_{ij})]$. After replacing ϕ with the map $X \mapsto A^{-1}\phi(X)A$, $X \in M_n$, we may, and will assume that $\phi(X) = X_\varphi$ for every idempotent X of rank one. In particular, $\phi(E_{11} + \lambda E_{12}) = E_{11} + \varphi(\lambda)E_{12}$, $\lambda \in \mathbb{C}$. The continuity of ϕ yields the continuity od φ. It is well-known that the identity and the complex-conjugation are the only nonzero continuous endomorphisms of the complex field. Hence, we have either $\phi(X) = X$ for every idempotent X of rank one, or $\phi(X) = \overline{X}$ for every idempotent X of rank one. Here, $\overline{X}$ denotes the matrix obtained from X by applying the complex-conjugation entrywise, $\overline{X} = \overline{[x_{ij}]} = [\overline{x_{ij}}]$. In the first case we get using the same arguments as before first that $\phi(X) = X$ for every diagonalizable matrix and then by continuity we conclude that $\phi(X) = X$ for every $X \in M_n$. So, it remains to show that the second case cannot occur. Indeed, in this case we have

$$\begin{aligned}\phi(E_{12}) &= \lim_{\lambda\to 0} \phi(\lambda E_{11} + E_{12}) = \lim_{\lambda\to 0} \lambda\phi\left(E_{11} + \lambda^{-1}E_{12}\right) \\ &= \lim_{\lambda\to 0} \lambda\left(E_{11} + \overline{\lambda^{-1}}E_{12}\right) = \lim_{\lambda\to 0}\left(\frac{\lambda}{|\lambda|}\right)^2 E_{12},\end{aligned}$$

a contradiction because $\lim_{\lambda\to 0}\left(\frac{\lambda}{|\lambda|}\right)^2$ does not exist. $\square$

4. Final remarks

We have characterized Jordan automorphisms of M_n, $n \geqslant 3$, as continuous commutativity and spectrum preserving maps. It is clear that in such a characterization we need two preserving properties. Namely, if $X \mapsto p_X$, $X \in M_n$, is a continuous map from the algebra of all $n \times n$ matrices into the space of all polynomials of degree at most n, then the continuous map $\phi : M_n \to M_n$ defined by $\phi(X) = p_X(X)$ preserves commutativity. Of course, such a map is in general far from being a Jordan automorphism and it does not preserve the spectrum. Similarly, if $X \mapsto A_X$, $X \in M_n$, is a continuous map from M_n into the set of all invertible $n \times n$ matrices, then the continuous map $\phi : M_n \to M_n$ defined by $\phi(X) = A_X X A_X^{-1}$ preserves spectrum. But in general, it is not a commutativity preserver.

It is thus clear that we need two preserving properties to characterize Jordan automorphisms. To show that our characterization is optimal we must prove that the assumption of continuity and the condition $n \geqslant 3$ are indispensable.

Let us first give an example showing that the condition $n \geqslant 3$ is essential in our main result. Denote by sl_2 the space of all 2×2 complex matrices with trace zero and let $\varphi : sl_2 \to sl_2$ be any continuous map satisfying $\det \varphi(X) = \det X$ for every $X \in sl_2$, $\varphi(0) = 0$, and $\varphi(\mu X) = \mu\varphi(X)$ for every $X \in sl_2$ and every $\mu \in \mathbb{C}$. For any $\lambda \in \mathbb{C}$ and $X \in sl_2$ set $\phi(\lambda I + X) = \lambda I + \varphi(X)$. Obviously, ϕ is a well-defined continous map of M_2 into itself. It preserves commutativity. Indeed, assume that $X, Y \in M_2$ commute. If one of these two matrices, say X, is a scalar multiple of the identity then $\phi(X) = X$ is a scalar matrix as well, and consequently, $\phi(X)$ and $\phi(Y)$ commute. If none of them is a scalar matrix, then using the Jordan canonical form it is easy to see that $X = \lambda I + \mu Y$ for some scalars λ, μ. From

$$Y = \frac{\operatorname{tr} Y}{2} I + \left(Y - \frac{\operatorname{tr} Y}{2} I \right)$$

and

$$Y - \frac{\operatorname{tr} Y}{2} I \in sl_2$$

one can easily conclude that $\phi(X)$ and $\phi(Y)$ commute. If X is a trace zero matrix then the eigenvalues of X are λ and $-\lambda$, where $\det X = -\lambda^2$. Thus, $\phi(X)$ and X have the same spectra if X is a trace zero matrix. It is then easy to conclude that $\sigma(\phi(X)) = \sigma(X)$ for every $X \in M_2$. In general, ϕ is far from being a Jordan automorphism.

To show that the continuity assumption is essential we will consider only 4×4 case (the same idea can be used to produce counterexamples in other dimensions). We define $W_1, W_2, W_3, W_4 \subset M_4$ by

$$W_1 = \left\{ \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C},\ \lambda_i \neq \lambda_j \text{ whenever } i \neq j \right\},$$

$$W_2 = \left\{ \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & 0 & \lambda_1 & \lambda_2 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C},\ \lambda_2 \neq 0 \right\},$$

$$W_3 = \left\{ \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_1 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C},\ \lambda_1 \neq \lambda_4 \text{ and } \lambda_2 \neq 0 \right\},$$

and

$$W_4 = \left\{ \begin{bmatrix} \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & \lambda_4 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C},\ \lambda_1 \neq \lambda_3 \text{ and } \lambda_2, \lambda_4 \neq 0 \right\}.$$

Let $\phi : M_4 \to M_4$ be any map such that $\phi(X) = X$ for every $X \in M_4 \setminus (W_1 \cup W_2 \cup W_3 \cup W_4)$, ϕ maps W_j bijectively onto W_j, $j = 1, \dots, 4$, and the unordered 4-tuple of diagonal entries of $\phi(X)$ is the same as the unordered 4-tuple of diagonal entries of X for every $X \in W_1 \cup \dots \cup W_4$. Then, clearly, $\sigma(\phi(X)) = \sigma(X)$ for every $X \in M_4$. For every $X \in M_4$ we denote by X' the commutant of X, $X' = \{Y \in M_4 \,:\, XY = YX\}$. It is straightforward to verify that $\phi(X)' = X'$ for every $X \in M_4$. So, if $XY = YX$ and $X \notin W_1 \cup \dots \cup W_4$, then $\phi(X) = X \in Y' = \phi(Y)'$, and hence, $\phi(X)\phi(Y) = \phi(Y)\phi(X)$. To show that ϕ preserves commutativity it remains to consider the case when both $X, Y \in W_1 \cup \dots \cup W_4$. But then $XY = YX$ yields that both X, Y belong to the same W_j, $j \in \{1, \dots, 4\}$. It is clear that $\phi(X)$ commutes with $\phi(Y)$ in this case as well.

In the above example we have $\phi(X)' = X'$ for every $X \in M_4$. A slightly more complicated example of a non-continuous bijective spectrum and commutativity preserving map is the following one. We define subsets $W_1, W_2 \subset M_4$ by

$$W_1 = \left\{ A(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = \begin{bmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \lambda_0 & \lambda_1 & \lambda_2 \\ 0 & 0 & \lambda_0 & \lambda_1 \\ 0 & 0 & 0 & \lambda_0 \end{bmatrix} : \lambda_0, \dots, \lambda_3 \in \mathbb{C},\ \lambda_1 \neq 0 \right\}$$

and

$$W_2 = \left\{ B(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = \begin{bmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \lambda_0 & 2\lambda_1 & \lambda_2 \\ 0 & 0 & \lambda_0 & \lambda_1 \\ 0 & 0 & 0 & \lambda_0 \end{bmatrix} : \lambda_0, \dots, \lambda_3 \in \mathbb{C},\ \lambda_1 \neq 0 \right\}.$$

It is not too difficult to check that the bijective map $\phi : M_4 \to M_4$ defined by $\phi(X) = X$, $X \in M_4 \setminus (W_1 \cup W_2)$, $\phi(A(\lambda_0, \lambda_1, \lambda_2, \lambda_3)) = B(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$, $A(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in W_1$, and $\phi(B(\lambda_0, \lambda_1, \lambda_2, \lambda_3)) = A(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$, $B(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in W_2$, preserves spectrum and commutativity. We can produce further examples of bijective maps preserving spectrum and commutativity by composing maps of this type. All these examples show that the continuity assumption is indispensible.

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(Received August 25, 2007)

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