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NUMERICAL RADIUS INEQUALITIES FOR SQUARE–ZERO AND IDEMPOTENT OPERATORS

HWA-LONG GAU, CHIN-YING HUANG AND PEI YUAN WU

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Abstract. We show that if A is a square-zero or an idempotent operator on a Hilbert space and B commutes with A, then $w(AB) \leqslant \min\{w(A)\|B\|, \|A\|w(B)\}$ holds, where $w(\cdot)$ and $\|\cdot\|$ denote, respectively, the numerical radius and operator norm of an operator

Let A be a bounded linear operator on a complex Hilbert space H. The *numerical range* $W(A)$ and *numerical radius* $w(A)$ of A are, by definition,

$$W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$$

and

$$w(A) = \sup\{|z| : z \in W(A)\},$$

respectively, where $\langle \cdot, \cdot\rangle$ and $\|\cdot\|$ are the inner product and its corresponding norm in H. By the celebrated Hausdorff–Toeplitz theorem, $W(A)$ is always a convex subset of the plane. It is bounded and, when H is finite dimensional, it is even compact. Its closure contains the spectrum of A. The numerical radius satisfies $\|A\|/2 \leqslant w(A) \leqslant \|A\|$. For other properties, the reader may consult [2, Chapter 22] or [1].

In 1969, Holbrook [3] asked whether, for commuting operators A and B, the inequality $w(AB) \leqslant w(A)\|B\|$ holds. He showed that this is indeed the case when A and B doubly commute (i.e., $AB = BA$ and $AB^* = B^*A$) (cf. [3, Theorem 3.4]). There are many other cases in which we do have this inequality. It came as a surprise when in 1988 Müller [5] gave an example of two 12-by-12 commuting matrices A and B with $w(AB) > w(A)\|B\|$. Recently, we prove, as a consequence of a more general result, that if A is a quadratic operator and B commutes with A, then $w(AB) \leqslant w(A)\|B\|$ is true (cf. [8, Theorem 5]). Whether the inequality $w(AB) \leqslant \|A\|w(B)$ also holds is left open as the arguments in [8] do not seem to be extendable to cover this case. The purpose of the present paper is to show that $w(AB) \leqslant \min\{w(A)\|B\|, \|A\|w(B)\}$ is true

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for A a square-zero or an idempotent operator and B commuting with A. Recall that an operator A is *quadratic* if it satisfies $A^2+\alpha A+\beta I=0$ for some scalars α and β. A is *square-zero* (resp., *idempotent*) if $A^2=0$ (resp., $A^2=A$). The proofs here are more down-to-earth, not involving, e.g., the Sz.-Nagy–Foiaş functional model for contractions.

We start with the following result which describes the spectrum, canonical model, norm and numerical range of a quadratic operator. It is from [7].

PROPOSITION 1. *Let A be a quadratic operator satisfying $A^2+\alpha A+\beta I=0$. Then*

(a) *the spectrum of A consists of the zeros a and b of the quadratic polynomial $z^2+\alpha z+\beta$ as eigenvalues,*

(b) *A is unitarily equivalent to an operator of the form*

$$aI_1 \oplus bI_2 \oplus \begin{bmatrix} aI_3 & D \\ 0 & bI_3 \end{bmatrix},$$

where $D>0$ (i.e., $\langle Dx,x\rangle>0$ for any nonzero vector x),

(c) $\|A\| = \left\|\begin{bmatrix} a & \|D\| \\ 0 & b \end{bmatrix}\right\|$ *if A is not a scalar operator, and*

(d) *$W(A)$ is the (open or closed) elliptic disc with foci a and b and the length of the minor axis $\|D\|$.*

The numerical radius inequality for a square-zero operator is easier to prove.

PROPOSITION 2. *If A is square-zero and B commutes with A, then*

$$w(AB) \leqslant \min\{w(A)\|B\|, \|A\|w(B)\}.$$

Proof. Since $(AB)^2=A^2B^2=0$, Proposition 1 (c) and (d) imply that $W(AB)$ is the (open or closed) circular disc centered at the origin with radius $\|AB\|/2$. Thus

$$w(AB)=\frac{1}{2}\|AB\| \leqslant \left(\frac{1}{2}\|A\|\right)\|B\| \leqslant w(A)\|B\|.$$

Similarly, we have $w(AB)\leqslant \|A\|w(B)$. $\square$

The next result, our main theorem, gives the numerical radius inequality for idempotent operators.

THEOREM 3. *If A is idempotent and B commutes with A, then*

$$w(AB) \leqslant \min\{w(A)\|B\|, \|A\|w(B)\}.$$

Proof. Assume that $A\neq 0, I$ and $\dim\ker(A-I)\leqslant \dim\ker A$. The case for $\dim\ker(A-I) > \dim\ker A$ can be dealt with analogously. From Proposition 1 (b), we may further assume that $A = \begin{bmatrix} I & D' \\ 0 & 0 \end{bmatrix}$ on $H = H_1 \oplus H_2$ with $H_1 \subseteq H_2$, where $D' = [D\ 0] : H_2 = H_1 \oplus (H_2 \ominus H_1) \to H_1$ and $D > 0$ on H_1. Let

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \quad \text{on} \quad H = H_1 \oplus H_2.$$

Since A and B commute, we have

$$AB = \begin{bmatrix} B_{11} + D'B_{21} & B_{12} + D'B_{22} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{11}D' \\ B_{21} & B_{21}D' \end{bmatrix} = BA.$$

Hence $B_{21} = 0$ and $B_{12} = B_{11}D' - D'B_{22}$ and thus

$$B = \begin{bmatrix} B_{11} & B_{11}D' - D'B_{22} \\ 0 & B_{22} \end{bmatrix}.$$

By the spectral theorem for normal operators, there are diagonal operators D_n (meaning D_n is unitarily equivalent to a diagonal matrix) such that D_n converges to D in norm. For each n, let $D'_n = [D_n\ 0]$,

$$A_n = \begin{bmatrix} I & D'_n \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_n = \begin{bmatrix} B_{11} & B_{11}D'_n - D'_nB_{22} \\ 0 & B_{22} \end{bmatrix}.$$

Then A_n and B_n commute and $A_n \to A$ and $B_n \to B$ in norm. If we can show that $w(A_nB_n) \leqslant \min\{w(A_n)\|B_n\|, \|A_n\|w(B_n)\}$ for all n, then, since the norm and numerical radius of operators are continuous (cf. [2, Problem 220] for the latter), we will have the asserted inequality for A and B. Hence without loss of generality we may assume that $D = \operatorname{diag}(d_1, d_2, \ldots)$ is diagonal with $d_1 \geqslant d_n$ for all n. (Here we are also assuming that H_1 is separable; a slight modification of the arguments below to accommodate the uncountable sum applies to the nonseparable case.) Letting

$$\widetilde{A} = \begin{bmatrix} I & d_1 I & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix} \quad \text{and} \quad \widetilde{B} = \begin{bmatrix} B_{11} & & \\ & B_{11} & \\ & & 0 \end{bmatrix}$$

on $H = H_1 \oplus H_1 \oplus (H_2 \ominus H_1)$, we now show that $w(AB) \leqslant w(\widetilde{A}\widetilde{B})$. Indeed, note that

$$AB = \begin{bmatrix} B_{11} & B_{11}D' \\ 0 & 0 \end{bmatrix}$$

and hence

$$w(AB) = \sup\left\{ |\langle B_{11}x, x\rangle + \langle B_{11}D'y, x\rangle| : \|x\|^2 + \|y\|^2 = 1 \right\}.$$

Letting $u = \langle B_{11}x, x\rangle$, $y = [y_1\ y_2\ \ldots]^T$ and $B_{11}^*x = [z_1\ z_2\ \ldots]^T$, we have

$$|\langle B_{11}x, x\rangle + \langle B_{11}D'y, x\rangle| = \left| u + \sum_n d_n y_n \overline{z_n} \right|.$$

For each n, let θ_n be a real number such that $\arg\left(y_n\overline{z_n}e^{i\theta_n}\right) = \arg u$. Then

$$\begin{aligned}\left|u+\sum_n d_n y_n\overline{z_n}\right| &\leqslant |u|+\sum_n d_n|y_n\overline{z_n}| \\ &\leqslant |u|+d_1\sum_n \left|y_n\overline{z_n}e^{i\theta_n}\right| = \left|u+d_1\sum_n y_n\overline{z_n}e^{i\theta_n}\right|.\end{aligned}$$

Hence

$$\begin{aligned} w(AB) &\leqslant \sup\left\{\left|\langle B_{11}x,x\rangle + d_1\sum_n\left(y_ne^{i\theta_n}\right)\overline{z_n}\right| : \|x\|^2+\|y\|^2=1\right\} \\ &= \sup\left\{|\langle B_{11}x,x\rangle + d_1\langle B_{11}\widetilde{y},x\rangle| : \|x\|^2+\|\widetilde{y}\|^2=1\right\} \\ &= w(\widetilde{A}\widetilde{B}),\end{aligned}$$

where $\widetilde{y} = \left[y_1e^{i\theta_1}\ y_2e^{i\theta_2}\ \ldots\right]^T$. Since $\widetilde{A}$ and $\widetilde{B}$ doubly commute, [3, Theorem 3.4] implies that $w(\widetilde{A}\widetilde{B}) \leqslant \min\left\{w(\widetilde{A})\|\widetilde{B}\|, \|\widetilde{A}\|w(\widetilde{B})\right\}$. But

$$w(\widetilde{A}) = w\left(\begin{bmatrix}1 & d_1\\ 0 & 0\end{bmatrix}\right) = w(A)$$

and

$$\|\widetilde{A}\| = \left\|\begin{bmatrix}1 & d_1\\ 0 & 0\end{bmatrix}\right\| = \|A\|$$

by Proposition 1 (d) and (c), and $w(\widetilde{B}) = w(B_{11}) \leqslant w(B)$ and $\|\widetilde{B}\| = \|B_{11}\| \leqslant \|B\|$. We conclude from above that

$$w(AB) \leqslant w(\widetilde{A}\widetilde{B}) \leqslant \min\{w(A)\|B\|, \|A\|w(B)\}. \quad \square$$

In conclusion, three remarks are in order: (1) Proposition 2 and Theorem 3 for the finite matrix case were contained in the Master thesis [4] of the second author supervised by the third. (2) Although in the proof of Theorem 3 the reduction to the diagonal D is still valid for general quadratic operators, other parts there seem difficult to be extended to cover the general case. (3) It was shown in [6] that $w(AB) \leqslant \|A\|w(B)$ holds for A satisfying $A^2 = aI$ for some scalar a and B commuting with A. The proof there is quite different from above.

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Hwa-Long Gau
Dept. of Mathematics
National Central University
Chung-Li 320
Taiwan
e-mail: hlgau@math.ncu.edu.tw

Chin-Ying Huang
Dept. of Applied Mathematics
National Chiao Tung University
Hsinchu 300
Taiwan

Pei Yuan Wu
Dept. of Applied Mathematics
National Chiao Tung University
Hsinchu 300
Taiwan
e-mail: pywu@math.nctu.edu.tw