Operators
and
Matrices
Volume 2, Number 1 (2008), 143–151

COMMUTATORS, PINCHINGS, AND SPECTRAL VARIATION

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(*communicated by A. Böttcher*)

Abstract. The three topics of the title have been studied by several authors. The main aim of this note is to point out interesting connections between them.

1. Introduction

Let $\mathscr{B}(\mathscr{H})$ be the space of bounded linear operators on a Hilbert space $\mathscr{H}$. For convenience we assume $\mathscr{H}$ is finite-dimensional. Let $\|A\|$, $\|A\|_p$, and $|||A|||$ denote, respectively, the usual operator norm, the Schatten p-norm $(1 \leqslant p \leqslant \infty)$, and an arbitrary unitarily invariant norm of an operator A. Properties of these norms, and other basic facts that we use in this paper, can be found in [1]. We will repeatedly use the inequality $|||XYZ||| \leqslant \|X\|\,|||Y|||\,\|Z\|$ valid for any three operators X, Y, and Z. We call this property *submultiplicativity.*

An operator of the form $AX - XA$ is called a *commutator,* and one of the form $AX - XB$ is called a *generalized commutator.* The triangle inequality, and the submultiplicative property show that

$$|||AX - XB||| \leqslant (\|A\| + \|B\|)\,|||X|||\,. \tag{1}$$

If we choose $A = B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, then

$$AX - XB = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix},$$

and

$$|||AX - XB||| = 2\,|||X||| = (\|A\| + \|B\|)\,|||X|||\,.$$

So the inequality (1) is sharp. Improvements are possible in special cases. If A and B are positive (semidefinite), then we have

$$|||AX - XB||| \leqslant \max(\|A\|\,,\ \|B\|)\,|||X|||\,. \tag{2}$$

Mathematics subject classification (2000): 15A42, 15A60, 47A30, 47B47.
Key words and phrases: Commutator, pinching, spectral variation, inequality.

Paper OAM-02-09

Further, if A and X are positive, then

$$|||AX - XA||| \leqslant \frac{1}{2} \|A\| \; |||X \oplus X||| , \tag{3}$$

where $X \oplus X$ represents the operator $\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}$ on $\mathscr{H} \oplus \mathscr{H}$. Note that for the operator norm, and the special case $A = B$, the inequality (1) says that $\|AX - XA\| \leqslant 2\|A\|\,\|X\|$; the inequality (2) improves this to $\|AX - XA\| \leqslant \|A\|\,\|X\|$ in the special case when A is positive; and (3) gives the further improvement $\|AX - XA\| \leqslant \frac{1}{2}\|A\|\,\|X\|$ when A and X are positive.

These results for the operator norm are corollaries of general theorems on norms of derivations proved by J. G. Stampfli [10]; the results for all unitarily invariant norms follow from inequalities for singular values recently obtained by one of the present authors [8].

Let T be an operator on $\mathscr{H} \oplus \mathscr{K}$ partitioned as $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. The diagonal operator $\mathscr{C}(T) = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ is called a *pinching* of T. The operator $\mathscr{O}(T) = T - \mathscr{C}(T) = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ is called the *off-diagonal part* of T. It is well-known that $|||\mathscr{C}(T)||| \leqslant |||T|||$ and $|||\mathscr{O}(T)||| \leqslant |||T|||$ for all T. A general study of these operators was made by R. Bhatia, M.-D. Choi, and C. Davis in [4]. In particular, they observed that in the special case when T is positive

$$\|\mathscr{O}(T)\| \leqslant \frac{1}{2} \|T\| , \tag{4}$$

and

$$\|\mathscr{O}(T)\|_2 \leqslant \frac{1}{\sqrt{2}} \|T\|_2 . \tag{5}$$

Both (4) and (5) are subsumed in the inequality

$$|||\mathscr{O}(T)||| \leqslant \frac{1}{2} |||T \oplus T||| \tag{6}$$

that we prove as Theorem 1 below.

The much-studied problem of spectral variation is concerned with finding upper and lower bounds for various distances between the eigenvalues of two operators. Let A be Hermitian and let $\lambda_1(A) \geqslant \cdots \geqslant \lambda_n(A)$ be its eigenvalues arranged in decreasing order. We denote by $\mathrm{Eig}^{\downarrow}(A)$ the diagonal matrix with entries $\lambda_1(A), \ldots, \lambda_n(A)$ down its diagonal, and by $\mathrm{Eig}^{\uparrow}(A)$ the diagonal matrix with entries $\lambda_n(A), \ldots, \lambda_1(A)$ down its diagonal. A well-known inequality in perturbation theory [1, p. 101], [2, p. 45] says that whenever A and B are Hermitian, we have

$$|||A - B||| \leqslant \left|\left|\left|\mathrm{Eig}^{\downarrow}(A) - \mathrm{Eig}^{\uparrow}(B)\right|\right|\right| . \tag{7}$$

In Section 2, we give new proofs of the inequalities (2) and (3), and we give a generalization of the inequality (2) to Hermitian operators. Our analysis demonstrates the connections between commutators, pinchings, and spectral variation. Though we confine our discussion to operators on finite-dimensional Hilbert spaces, by slight modifications the inequalities we obtain here can be extended to the infinite-dimensional setting.

2. Proofs and connections

A very simple proof of the inequality (2) can be obtained using an idea going back to Stampfli. For any two complex numbers z and w, and for all A and X,

$$\begin{aligned} |||AX - XA||| &= |||(A-z)(X-w) - (X-w)(A-z)||| \\ &\leqslant 2\,\|A-z\|\,|||X-w|||\,. \end{aligned} \tag{8}$$

Let

$$\triangle\,(X,\ |||\cdot|||) = \inf_{w\in\mathbb{C}} |||X-w||| \tag{9}$$

be the distance of X from scalar operators. Then the inequality (8) implies that

$$|||AX - XA||| \leqslant 2\,\triangle(A,\ \|\cdot\|)\,\triangle(X,\ |||\cdot|||). \tag{10}$$

Let $c(A)$ be the diameter of the smallest disk in the complex plane that contains all the eigenvalues of A. If A is normal, then

$$c(A) = 2\,\triangle(A,\ \|\cdot\|) \tag{11}$$

and the inequality (10) reduces to

$$|||AX - XA||| \leqslant c(A)\,\triangle(X,\ |||\cdot|||). \tag{12}$$

We do not know any good estimate for $\triangle(X,\ |||\cdot|||)$, but in any case it is not bigger than $|||X|||$. So if A is normal, then

$$|||AX - XA||| \leqslant c(A)\ |||X|||\,. \tag{13}$$

For any operator A, the *spread* of A, denoted $\operatorname{spd}(A)$, is the diameter of the spectrum of A, i.e., the maximum distance between any two eigenvalues of A. Clearly, $\operatorname{spd}(A) \leqslant c(A)$. If A is Hermitian, then

$$\operatorname{spd}(A) = c(A) = \lambda_1(A) - \lambda_n(A). \tag{14}$$

If A is positive, then $\lambda_n(A) \geqslant 0$ and

$$c(A) \leqslant \lambda_1(A) = \|A\|\,. \tag{15}$$

So for positive A, the inequality (13) implies

$$|||AX - XA||| \leqslant \|A\|\,|||X|||\,. \tag{16}$$

If A and B are positive, and X arbitrary, we apply this inequality to the operators $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$ to obtain (2).

We remark that if A is positive definite, then

$$\lambda_1(A) - \lambda_n(A) = \|A\| - \left\|A^{-1}\right\|^{-1}. \tag{17}$$

This leads to some improvement of (16), and consequently of (2).

For the operator norm alone, the inequality (10) and the relation (11) show that when A and X both are normal, we have

$$\|AX - XA\| \leqslant \frac{1}{2} c(A) c(X). \tag{18}$$

In particular, if A and X are positive, then

$$\|AX - XA\| \leqslant \frac{1}{2} \|A\| \, \|X\|, \tag{19}$$

and this is the inequality (3) for this special norm.

Now let us turn to pinchings, and for the convenience of the reader, give a proof of the inequality (6), from which (4) and (5) follow as special cases.

Let $s_1(T) \geqslant \cdots \geqslant s_n(T)$ be the singular values of T. If T is Hermitian, we have $s_j(T) = \lambda_j(T \oplus -T)$ for $1 \leqslant j \leqslant n$. If A and B are positive, then $A - B \leqslant A$ and $B - A \leqslant B$. Hence, by Weyl's monotonicity principle [1, p. 63],

$$s_j(A-B) = \lambda_j((A-B) \oplus (B-A)) \leqslant \lambda_j(A \oplus B) = s_j(A \oplus B). \tag{20}$$

The inequality (20) was proved by X. Zhan [13]. A corollary of this is the inequality

$$|||A - B||| \leqslant |||A \oplus B||| \tag{21}$$

valid for all positive operators A and B. This inequality has been obtained earlier in [5].

As an application of the inequality (21), we present a simple proof of the inequality (6).

THEOREM 1. *Let* $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be a positive partitioned operator. Then*

$$|||\mathscr{O}(T)||| \leqslant \frac{1}{2} |||T \oplus T|||.$$

Proof. If $U = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$, then U is a unitary operator and

$$\mathscr{O}(T) = \frac{1}{2}(T - UTU^*). \tag{22}$$

Since T is positive, it follows that UTU^* is positive. So, in view of the unitary invariance of $|||\cdot|||$, the relation (22) and the inequality (21) yield the inequality (6). □

Now suppose X is any operator on $\mathscr{H}$ and P is an orthogonal projection. Then, in an appropriate coordinate system, we have $P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ and $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$. So, $PX - XP = \begin{bmatrix} 0 & X_{12} \\ -X_{21} & 0 \end{bmatrix}$, which in turn is the product of the unitary operator $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ and $\mathscr{O}(X)$. This shows that

$$|||PX - XP||| = |||\mathscr{O}(X)||| \leqslant |||X||| \tag{23}$$

for every X, and, by Theorem 1,

$$|||PX - XP||| \leqslant \frac{1}{2}\, |||X \oplus X||| \tag{24}$$

if X is positive. These two inequalities are special instances of (2) and (3), when $A = B = P$.

To go to the general case, we use a convexity argument. Suppose A is a positive contraction, i.e., $0 \leqslant A \leqslant I$. The eigenvalue n-tuple of A is a point $\alpha = (\alpha_1, \ldots, \alpha_n)$ in the hypercube $[0,1]^n$. The vertices of this hypercube are points in the set $\{0, 1\}^n$. So α is a convex combination of these vertices, and consequently A is a convex combination of orthogonal projections. Let $A = \sum_{j=1}^{m} t_j P_j$, where $t_1, \ldots, t_m$ are positive real numbers with $\sum_{j=1}^{m} t_j = 1$, and $P_1, \ldots, P_m$ are orthogonal projection operators. Then from (23) we obtain

$$\begin{aligned} |||AX - XA||| &\leqslant \sum_{j=1}^{m} t_j\, |||P_j X - XP_j||| \leqslant \sum_{j=1}^{m} t_j\, |||X||| \\ &= |||X|||\,. \end{aligned}$$

If A is any positive operator, then $\frac{A}{\|A\|}$ is positive and contractive. So using the inequality obtained above, we get

$$|||AX - XA||| \leqslant \|A\|\, |||X|||\,,$$

from which (2) follows as before. Using the same argument, we obtain the inequality (3) from (24).

The spectral variation inequality (7) can be employed to give commutator inequalities as follows. Let A be Hermitian and U unitary. Then, by unitary invariance and (7), we have

$$\begin{aligned} \|AU - UA\| = \|A - UAU^*\| &\leqslant \left\|\mathrm{Eig}^{\downarrow}(A) - \mathrm{Eig}^{\uparrow}(A)\right\| \\ &= |\lambda_1(A) - \lambda_n(A)| = \mathrm{spd}(A). \end{aligned} \tag{25}$$

Now suppose X is any operator with $\|X\| \leqslant 1$. Then there exist unitaries U and V such that $X = \frac{1}{2}(U + V)$. (One proof of this statement goes as follows: By the singular value decomposition $X = U_1 S U_2$, where U_1 and U_2 are unitary and S is diagonal with diagonal entries s_j all of which are in $[0,1]$. Each of these can be expressed as $s_j = \frac{1}{2}(e^{i\theta_j} + e^{-i\theta_j})$ for some θ_j). So, from the inequality (25), we get

$$\|AX - XA\| \leqslant \operatorname{spd}(A)\,\|X\| \tag{26}$$

for all X. Let $\delta_A(X) := AX - XA$. Then δ_A is a linear operator on $\mathscr{B}(\mathscr{H})$, and the inequality (26) says

$$\sup_{\|X\|\leqslant 1} \|\delta_A(X)\| \leqslant \operatorname{spd}(A).$$

In fact, equality holds here. This can be seen by choosing an orthonormal basis for $\mathscr{H}$ in which $A = \operatorname{diag}(\lambda_1(A), \ldots, \lambda_n(A))$ and X is a matrix with entry $x_{1n} = 1$ and all its other entries are equal to zero. In other words

$$\|\delta_A\| := \sup_{\|X\|\leqslant 1} \|\delta_A(X)\| = \operatorname{spd}(A). \tag{27}$$

The norm $\|\cdot\|_1$ is the dual of the norm $\|\cdot\|$. So we also have by a standard duality argument

$$\|\delta_A\|_1 := \sup_{\|X\|_1\leqslant 1} \|\delta_A(X)\|_1 = \operatorname{spd}(A). \tag{28}$$

A familiar characterisation of unitarily invariant norms (see Exercise 2.7.12 in [3]) enables us to obtain an alternative proof of the inequality (13) for the case when A is Hermitian. To do this, we need to recall some basic facts of unitarily invariant norms. The *Ky Fan norms* of an operator T are given by $\|T\|_{(k)} = \sum_{j=1}^{k} s_j(T)$, $1 \leqslant k \leqslant n$. Thus, $\|T\|_{(n)} = \|T\|_1$, the trace norm of T. It is well-known that the Ky Fan norms have the representation

$$\|T\|_{(k)} = \min\left\{\|S\|_1 + k\,\|R\| : T = S + R\right\}. \tag{29}$$

If $\|T\|_{(k)} \leqslant \|S\|_{(k)}$ for $1 \leqslant k \leqslant n$, then $|||T||| \leqslant |||S|||$ for all unitarily invariant norms. This is called the *Fan dominance theorem* [3, p. 58].

THEOREM 2. *Let A be a Hermitian operator and X be any operator. Then*

$$|||AX - XA||| \leqslant \operatorname{spd}(A)\,|||X|||. \tag{30}$$

Proof. We prove that

$$||\delta_A(X)||_{(k)} \leqslant \operatorname{spd}(A)\,||X||_{(k)} \quad \text{for} \quad 1 \leqslant k \leqslant n.$$

Use the relation (29) to write $X = Y + Z$ for some operators $Y,\ Z$ with $\|X\|_{(k)} = \|Y\|_1 + k\,\|Z\|$. Thus, by (27) and (28),

$$\|\delta_A(Y)\|_1 \leqslant \operatorname{spd}(A)\,\|Y\|_1,$$
$$\|\delta_A(Z)\| \leqslant \operatorname{spd}(A)\,\|Z\|.$$

Another application of (29) leads to the inequality

$$\begin{aligned}\|\delta_A(X)\|_{(k)} &\leqslant \|\delta_A(Y)\|_1 + k\,\|\delta_A(Z)\| \\ &= \mathrm{spd}(A)(\|Y\|_1 + k\,\|Z\|) \\ &= \mathrm{spd}(A)\,\|X\|_{(k)}\,.\end{aligned}$$

Now the inequality (30) follows by the Fan dominance theorem. □

The inequality (7) and the reasoning in the proof of Theorem 2 can be employed to obtain natural generalizations of the inequalities (30) and (21) to generalized commutators.

THEOREM 3. *Let A and B be Hermitian operators and X be any operator. Then*

$$|||AX - XB||| \leqslant \left\|Eig^{\downarrow}(A) - Eig^{\uparrow}(B)\right\| \, |||X|||\,. \tag{31}$$

It is easy to see that if A and B are Hermitian, then

$$\left\|\mathrm{Eig}^{\downarrow}(A) - \mathrm{Eig}^{\uparrow}(B)\right\| = \max(|\lambda_1(A) - \lambda_n(B)|\,,\ |\lambda_1(B) - \lambda_n(A)|). \tag{32}$$

THEOREM 4. *Let A and B be Hermitian operators and X be any operator. Then*

$$|||AX - XB||| \leqslant \|X\| \, \left|\left|\left|Eig^{\downarrow}(A) - Eig^{\uparrow}(B)\right|\right|\right|\,. \tag{33}$$

For the operator norm, the inequalities (31) and (33) reduce to the same inequality, which yields a recent related inequality in [12].

When A and B are positive, the inequality (33) implies that

$$|||AX - XB||| \leqslant \|X\| \, |||A \oplus B|||\,, \tag{34}$$

an inequality obtained, by different means, in [8].

For the operator norm, the inequalities (34) and (2) are the same, and this case has been considered in [7].

3. Remarks

Let A and B be normal operators with eigenvalues $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$, respectively. It is known that [1, p. 164]

$$\|A - B\| \leqslant \sqrt{2} \max_{i,j} |\lambda_i - \mu_j|\,. \tag{35}$$

Using this relation, we can see (as in Section 2) that if A is normal and X arbitrary, then

$$|||AX - XA||| \leqslant \sqrt{2}\,\mathrm{spd}(A)\, |||X|||\,. \tag{36}$$

This inequality can also be obtained from (13) by showing that

$$c(A) \leqslant \sqrt{2}\,\mathrm{spd}(A). \tag{37}$$

(This is the content of Problem VI.8.4 in [1]). For the special norm $\|\cdot\|_2$, one can easily improve upon (36). Choosing a basis in which $A = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$, we see that

$$\|AX - XA\|_2^2 = \sum_{i,j} |\lambda_i - \lambda_j|^2 |x_{ij}|^2 \leqslant (\operatorname{spd}(A))^2 \|X\|_2^2 . \tag{38}$$

Thus, for this norm the factor $\sqrt{2}$ in (36) can be replaced by 1. This is not generally possible for other norms. For example, when $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $X = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then

$$\|AX - XA\|_1 = 2, \ \|X\|_1 = 1, \ \operatorname{spd}(A) = \sqrt{3}.$$

However, an interpolation argument does give a better inequality than (36) for the Schatten p-norms. The linear operator δ_A on $\mathscr{B}(\mathscr{H})$ has norm at most $\sqrt{2}\operatorname{spd}(A)$ when $\mathscr{B}(\mathscr{H})$ is equipped with the norm $\|\cdot\|$, or with the norm $\|\cdot\|_1$. Its norm is at most $\operatorname{spd}(A)$ when the norm on $\mathscr{B}(\mathscr{H})$ is taken to be $\|\cdot\|_2$. A standard interpolation argument as used in [4] shows that the norm of δ_A on the space $(\mathscr{B}(\mathscr{H}), \|\cdot\|_p)$ is not bigger than $2^{\frac{1}{2}\left|\frac{1}{p}-\frac{1}{q}\right|}\operatorname{spd}(A)$, where $\frac{1}{p}+\frac{1}{q}=1$. In other words, when A is normal, we have for all X

$$\|AX - XA\|_p \leqslant 2^{\frac{1}{2}\left|\frac{1}{p}-\frac{1}{q}\right|}\operatorname{spd}(A) \ \|X\|_p , \tag{39}$$

$1 \leqslant p \leqslant \infty$.

In this context, we should mention that A. Böttcher and D. Wenzel have shown that if A and B are any two operators, then

$$\|AX - XA\|_2 \leqslant \sqrt{3}\, \|A\|_2 \, \|X\|_2 , \tag{40}$$

and conjectured that the factor $\sqrt{3}$ here can be replaced by $\sqrt{2}$. This conjecture has been proved for $n = 2$ in [6], for $n = 3$ in [9], and finally for general n in [11].

Acknowledgement. The authors thank Albrecht Böttcher for his suggestions that have led to an improved presentation.

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(Received November 19, 2007)

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