Operators
and
Matrices
Volume 2, Number 2 (2008), 233–247

A CLASS OF TRIDIAGONAL REPRODUCING KERNELS

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(*communicated by S. McCullough*)

Abstract. The class of analytic reproducing kernels

$$K_p(z,w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)}$$

is considered where $f_n(z) = (1 - b_n z)z^n$ with $b_n = (\frac{n+1}{n+2})^p$ and $p > 0$. In this case $H(K_p)$ consists of functions with domain $\mathbb{D} \cup \{1\}$. For each p, a concrete realization of $H(K_p)$ is provided. For the case $p > 1/2$, $H(K_p)$ is shown to have the factorization property and the operator of multiplication by z is shown to be similar to a rank one perturbation of the unilateral shift. A characterization of the multiplier algebra of $H(K_p)$ is given for all values of $p > 0$.

1. Introduction

The function $K(z,w)$ is *positive definite* (denoted $K \gg 0$) on the set $E \times E$ if for any finite collection $z_1, z_2, \cdots, z_n$ in $E \subset \mathbb{C}$ and any complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$, the sum $\sum_{i,j=1}^{n} \bar{\alpha}_i \alpha_j K(z_i, z_j)$ is non-negative. It is well known that if $K \gg 0$ on $E \times E$, then the set of functions in z given by

$$\left\{ \sum_{j=1}^{n} \alpha_j K(z, w_j) : \alpha_1, \cdots, \alpha_n \in \mathbb{C}, w_1, \cdots, w_n \in E \right\}$$

has dense span in a Hilbert space $H(K)$ of functions on E with

$$\left\| \sum_{j=1}^{n} \alpha_j K(z, w_j) \right\|^2 = \sum_{i,j=1}^{n} \bar{\alpha}_i \alpha_j K(w_i, w_j).$$

A fundamental property of $H(K)$ is the *Reproducing Property* which states that $f(w) = \langle f(z), K(z,w) \rangle$ for every w in E and f in $H(K)$. Thus evaluation at w is a bounded linear functional for each w in E.

Mathematics subject classification (2000): 47B38, 46E20.
Key words and phrases: Tridiagonal reproducing kernel, multiplication operator.

Paper OAM-02-15

Conversely, it is well known that if F is a Hilbert space of functions defined on E such that evaluation at w is a bounded linear functional for each w in E, then there is a unique K defined on $E\times E$ such that $F = H(K)$. It follows from the reproducing property that $K(z,w) = \overline{K(w,z)}$. Hence if K is analytic in the first variable, then K is coanalytic in the second variable. In this case K is an *analytic* kernel. It is well known, see Adams, McGuire, and Paulsen [2], that if K is an analytic kernel with series expansion $K(z,w) = \sum_{i,j=0}^{\infty} a_{i,j} z^i \bar{w}^j$ about $(0,0)$ and $A = [a_{i,j}]$ is factored as $A = BB^*$, then $H(K)$ is identifiable with the range space of B in l^2. Recall the range space of B is given by $\mathscr{R}(B) = \{B\vec{x} : \vec{x} \in l^2\}$ with $||B\vec{x}||_{\mathscr{R}(B)} = ||\vec{x}||_{l^2}$. The column vectors $\{b_j\}_{j=0}^{\infty}$ of B given by $\vec{b}_j = \begin{pmatrix} b_{0,j} \\ b_{1,j} \\ b_{2,j} \\ \vdots \end{pmatrix}$ correspond to an orthonormal basis $\{f_j(z)\}_{j=0}^{\infty}$ of $H(K)$ where $f_j(z) = \sum_{i=0}^{\infty} b_{i,j} z^j$. An important observation is that if K_1 and K_2 are two such analytic kernels with associated factorizations $A_1 = B_1 B_1^*$ and $A_2 = B_2 B_2^*$, then $H(K_1) \subset H(K_2)$ if and only if the range of B_1 is contained in the range of B_2.

In Shields [13], multiplication operators on analytic reproducing kernel Hilbert spaces with kernels of the form $K(z,w) = \sum_{n=0}^{\infty} a_n z^n \bar{w}^n$ were extensively studied. In these spaces the monomials $\{\sqrt{a_n} z^n\}$ form an orthonormal basis, and the operator M_z of multiplication by z is a forward unilateral shift. Richter [12] extended the work of Shields [13] to study the invariant subspace structure of multiplication by z on certain Banach spaces, $\mathscr{B}$, of analytic functions in which evaluation is continuous and for which the following *Factorization Property* holds: if $f \in \mathscr{B}$ and $f(\lambda) = 0$, then there exists $g \in \mathscr{B}$ such that $(z-\lambda)g = f$.

In Adams and McGuire [1], a study was begun of the spaces with kernels of the form $K(z,w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)}$ where $f_n(z) = (a_{n,0} + a_{n,1}z + \cdots + a_{n,J}z^J)z^n$ and J is fixed. These spaces are known as bandwidth J spaces since the Taylor series expansion of $K(z,w) = \sum_{i,j=0}^{\infty} a_{i,j} z^i \bar{w}^j$ satisfies $a_{i,j} = 0$ outside the band $|i-j| \leqslant J$. In this case, the polynomials $\{f_n(z)\}$ form an orthonormal basis for $H(K)$. It was shown in Adams and McGuire [1] that the behavior of the multiplication operators on these spaces can be markedly different from the Shields case ($J = 0$).

This paper is focused on a special class of tridiagonal kernels (J $= 1$) that bring this difference into a sharper focus. The class is defined for each $p > 0$ and $n \in \mathbb{N}$ by setting $f_n(z) = (1 - b_n z)z^n$ where $b_n = (\frac{n+1}{n+2})^p$, resulting in the kernel $K_p(z,w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)}$. It is straightforward to verify that the domain of K_p is given by $\mathscr{D}(K_p) = \{(z,w) : z,w \in \mathbb{D} \cup \{1\}\}$, that $\{b_n\}$ is a sequence of positive numbers that increases to 1, and that $K_p(z,1) = \sum_{n=0}^{\infty}(1-b_n)(1-b_n z)z^n$.

The principle result of this paper is a functional decomposition of the space $H(K_p)$

for $p > 0$. This decomposition allows us to determine that the operator M_z is bounded if and only if $p > \frac{1}{2}$ and, in this case, to completely characterize the multiplier algebra of $H(K_p)$. For $0 < p < \frac{1}{2}$, we provide necessary and sufficient conditions for a function ϕ to be a multiplier of $H(K_p)$. Additionally, we show that for $p > \frac{1}{2}$, $H(K_p)$ satisfies the factorization property of Richter [12]. From this it easily follows from [12] that M_z^* is in the Cowen-Dougas class B_1, M_z is a cellular indecomposable operator, and that the invariant subspaces of M_z are either of the form $(1-z)M$ where $M = \psi H^2$ for some inner function ψ or the span of $(1-z)M$ and the function $K_p(z,1)$.

2. Main Results

Our first result shows that the functions in $H(K_p)$ can be decomposed into $(1-z)$ times an H^2 function plus a scalar multiple of the function $K_p(z,1)$. Our second and more difficult result determines precisely which H^2 functions can occur in the factorization and the dependency on p. Before proceeding, we include without proof a lemma that contains a few obvious facts that will be useful in the proofs of these results.

LEMMA 2.1. *Let $\mathscr{P}$ denote the collection of matrices with non-negative components, let $A * B$ denote the Schur or Hadamard product of the matrices A and B, and let $\mathscr{V}_+$ denote the collection of unit vectors in l^2_+ whose components are non-negative.*

(1) *If $A \in \mathscr{P}$, then $||A|| = \sup\limits_{\vec{v} \in \mathscr{V}_+} ||A\vec{v}||$.*

(2) *If A_1, $A_2 \in \mathscr{P}$, then $||A_1|| \leqslant ||A_1 + A_2|| \leqslant ||A_1|| + ||A_2||$.*

(3) *If A, $B \in \mathscr{P}$ with $B = [b_{j,k}]$, and $0 < \lambda \leqslant b_{j,k} \leqslant \gamma < \infty$ for each j, k, then $\lambda ||A|| \leqslant ||A * B|| \leqslant \gamma ||A||$.*

THEOREM 2.2. *If $f \in H(K_p)$, then $f(z) = (1-z)g(z) + \alpha K_p(z,1)$ for some g in the Hardy space $H^2(\mathbb{D})$ and $\alpha \in \mathbb{C}$.*

Proof. First note that if $f \in H(K_p)$ and Q is the projection of $H(K_p)$ onto the one dimensional span of $K_p(z,1)$, then $f = (I-Q)f + Qf$. Since

$$Qf = \left\langle f, \frac{K_p(z,1)}{\sqrt{K_p(1,1)}} \right\rangle \frac{K_p(z,1)}{\sqrt{K_p(1,1)}} = \frac{f(1)}{K_p(1,1)} K_p(z,1),$$

$(Qf)(1) = \langle Qf, K_p(z,1)\rangle = f(1) = \langle f, K_p(z,1)\rangle$ and $((I-Q)f)(1) = \langle (I-Q)f, K_p(z,1)\rangle = 0$. Thus it suffices to show that if $f \in H(K_p)$ and $f(1) = \langle f, K_p(z,1)\rangle = 0$, then $f(z) = (1-z)g(z)$ for some $g \in H^2(\mathbb{D})$. Writing $f(z) = \sum_{n=0}^{\infty} \alpha_n f_n(z) = \sum_{n=0}^{\infty} \alpha_n (1 - b_n z) z^n$ we note that the condition that $f(1) = 0$ implies $\sum_{n=0}^{\infty} \alpha_n (1 - b_n) = 0$. In order that

$$f(z) = (1-z)g(z) = (1-z)\sum_{n=0}^{\infty} g_n z^n = g_0 + \sum_{n=1}^{\infty} (g_n - g_{n-1}) z^n$$

for some $g \in H^2(\mathbb{D})$ we must produce a sequence $\{g_n\}$ in l^2 such that

$$g_0 + \sum_{n=1}^{\infty} (g_n - g_{n-1}) z^n = \sum_{n=0}^{\infty} \alpha_n (1 - b_n z) z^n = \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n - \alpha_{n-1} b_{n-1}) z^n.$$

This leads to the recursion $g_0 = \alpha_0$ and $g_n = g_{n-1} + \alpha_n - \alpha_{n-1}b_{n-1}$ for $n \geqslant 1$. Thus

$$g_1 = g_0 + \alpha_1 - \alpha_0 b_0 = \alpha_0(1-b_0) + \alpha_1,$$

$$g_2 = g_1 + \alpha_2 - \alpha_1 b_1 = \alpha_0(1-b_0) + \alpha_1(1-b_1) + \alpha_2,$$

and the n^{th} term is given by

$$g_n = \left(\sum_{k=0}^{n-1} \alpha_k(1-b_k)\right) + \alpha_n.$$

Since $\sum_{k=0}^{\infty} \alpha_k(1-b_k) = 0$, for $n \geqslant 1$ the sum

$$\sum_{k=0}^{n-1} \alpha_k(1-b_k) = -\sum_{k=n}^{\infty} \alpha_k(1-b_k) = 0$$

and hence $g_n = \alpha_n - \sum_{k=n}^{\infty} \alpha_k(1-b_k)$. Since $\{\alpha_n\}$ is an l^2 sequence, it suffices to show $\{\sum_{k=n}^{\infty} \alpha_k(1-b_k)\}_{n=1}^{\infty}$ is an l^2 sequence. Since

$$\left\{\sum_{k=n}^{\infty} \alpha_k(1-b_k)\right\}_{n=1}^{\infty} = B_p\{\alpha_n\}_{n=1}^{\infty}$$

where

$$B_p = \begin{pmatrix} 1-b_0 & 1-b_1 & 1-b_2 & \cdots \\ 0 & 1-b_1 & 1-b_2 & \cdots \\ 0 & 0 & 1-b_2 & \ddots \\ 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

it is enough to show that B_p is a bounded matrix.

The tangent line approximation to $f(x) = 1 - x^p$ at $x = 1$ is given by $-p(x-1)$. Since $\lim_{n\to\infty} \frac{n+1}{n+2} = 1$, for large n, $1 - b_n = 1 - (\frac{n+1}{n+2})^p$ can be approximated by $-p(\frac{n+1}{n+2} - 1) = \frac{p}{n+2}$. A straightforward application of part (3) of Lemma 2.1 shows that B_p is bounded if and only if the matrix

$$C_p = \begin{pmatrix} \frac{p}{2} & \frac{p}{3} & \frac{p}{4} & \cdots \\ 0 & \frac{p}{3} & \frac{p}{4} & \cdots \\ 0 & 0 & \frac{p}{4} & \ddots \\ 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

is bounded. It is well known that the Cesaro operator

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

is bounded, see Brown, Halmos, Shields [7]. Let Q_0 denote the projection onto the first canonical basis vector of l^2 and note $C_p = p(I-Q_0)C^*(I-Q_0)$ is bounded which establishes the result. □

Our next result provides a more explicit description of the nature of the decomposition of $H(K_p)$ that was obtained in Theorem 2.2. For convenience we will denote the diagonal operator with diagonal entries given by the sequence $\{a_n\}$ by either of $D[a_1, a_2, a_3, \ldots]$ or $D[\{a_n\}]$.

THEOREM 2.3. *If* $\mathscr{A}_p = \{g \in H^2(\mathbb{D}) : (1-z)g(z) \in H(K_p)\}$*, then*
(1) *for* $p > \frac{1}{2}$, $\mathscr{A}_p = H^2(\mathbb{D})$;
(2) *for* $p = \frac{1}{2}$, $\mathscr{A}_p$ *is dense in* $H^2(\mathbb{D})$*, but not equal to* $H^2(\mathbb{D})$;
(3) *for* $0 < p < \frac{1}{2}$, $\mathscr{A}_p$ *is the orthogonal complement in* $H^2(\mathbb{D})$ *of the span of*

$$\{g_p(z)\} \text{ where } g_p(z) = \sum_{n=0}^{\infty}(1-b_n)(n+2)^p z^n.$$

Proof. We begin with the case where $p > \frac{1}{2}$. In order to show that $\mathscr{A}_p = H^2$, it suffices to show that the range of A is contained in the range of B where

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ -1 & 1 & 0 & \cdots \\ 0 & -1 & 1 & \cdots \\ 0 & 0 & -1 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ -(\frac{1}{2})^p & 1 & 0 & \cdots \\ 0 & -(\frac{2}{3})^p & 1 & \cdots \\ 0 & 0 & -(\frac{3}{4})^p & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

By the Range Inclusion Theorem of Douglas [9], it suffices to show there exists a bounded operator $R = [r_{i,j}]$ such that $A = BR$. It is a straightforward computation to show that R must be lower triangular, $r_{i,i} = 1$ for each i, and $r_{i,j} = (\frac{i+2}{i+1})^p[(\frac{j+1}{j+2})^p - 1]$ for $0 \leqslant j \leqslant i-1$. To determine the values of p for which R is bounded, we will produce a sequence of matrices, beginning with R and ending with a block Toeplitz matrix, such that each matrix in the sequence is bounded if and only if its predecessor in the sequence is. To begin, write R as $I - M$ where $M = [m_{i,j}]$ satisfies

$$m_{i,j} = \begin{cases} 0, & \text{if } j \geqslant i; \\ (\frac{i+2}{i+1})^p[1-(\frac{j+1}{j+2})^p], & \text{if } j < i. \end{cases}$$

Let $\alpha_j = \frac{\frac{p}{j+2}}{1-(\frac{j+1}{j+2})^p} = \frac{\frac{p}{j+2}}{1-b_j}$ and note that $\lim_{j\to\infty}\alpha_j = 1$. Since the diagonal matrix $D[\{\alpha_j\}]$ is bounded and invertible the matrix M is bounded if and only if $MD[\{\alpha_j\}]$ is bounded. Note that

$$MD[\{\alpha_j\}] = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \frac{p}{2} & 0 & 0 & 0 & \cdots \\ \frac{p}{2}(\frac{2}{3})^p & \frac{p}{3} & 0 & 0 & \cdots \\ \frac{p}{2}(\frac{2}{4})^p & \frac{p}{3}(\frac{3}{4})^p & \frac{p}{4} & 0 & \cdots \\ \frac{p}{2}(\frac{2}{5})^p & \frac{p}{3}(\frac{3}{5})^p & \frac{p}{4}(\frac{4}{5})^p & \frac{p}{5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $L_1 = D[\{(\frac{n+1}{n})^{p-1}\}_{n=1}^{\infty}]$, $L_2 = D[\{\frac{1}{n}\}_{n=1}^{\infty}]$, and $D_p = D[\{n^{1-p}\}_{n=1}^{\infty}]$. It is straightforward to verify that $MD[\{\alpha_n\}] = pD_p(C-L_2)L_1D_p^{-1}$ where C is the Cesaro matrix. Since $pD_pL_2L_1D_p^{-1}$ is a bounded matrix, it easily follows that $MD[\{\alpha_n\}]$ is bounded if and only if

$$D_pCD_p^{-1} =$$

$$\begin{pmatrix} 1^{1-p} & 0 & 0 & \cdots \\ 0 & 2^{1-p} & 0 & \cdots \\ 0 & 0 & 3^{1-p} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1^{p-1} & 0 & 0 & \cdots \\ 0 & 2^{p-1} & 0 & \cdots \\ 0 & 0 & 3^{p-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is bounded. Our next goal is to show that $D_pCD_p^{-1}$ is bounded if and only if $p > \frac{1}{2}$.

By applying item (3) of Lemma 2.1, the boundedness of $D_pCD_p^{-1}$ can be shown to be equivalent to the boundedness of $E_pC_1E_p^{-1}$ where $E_p = D[1^{1-p}, 2^{1-p}, 2^{1-p}, 4^{1-p}, \ldots, 4^{1-p}, 8^{1-p}, \ldots, 8^{1-p}, \ldots]$ and C_1 is the lower triangular matrix

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

whose i,jth entry is 2^{-k} provided $2^k \leqslant i < 2^{k+1}$ and $1 \leqslant j \leqslant i$ for $k = 0,1,2,\ldots$.

By applying item (2) of Lemma 2.1 we can augment C_1 to obtain the equivalent problem of the boundedness of the matrix $E_pC_2E_p^{-1}$ where $E_p = D[1^{1-p}, 2^{1-p}, 2^{1-p},$

$4^{1-p},\dots,4^{1-p},8^{1-p},\dots,8^{1-p},\dots]$ and C_2 is the matrix

$$C_2=\left[\begin{array}{c|cc|cccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ \hline \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right].$$

The matrix $E_pC_2E_p^{-1}$ can be better expressed in block lower triangular form as the matrix $C_p=[(\frac{2^j}{2^k})^{1-p}\frac{1}{2^j}M_{j,k}]$ where $M_{j,k}$ is the $2^j\times 2^k$ matrix each of whose entries is 1. That is

$$C_p=\left[\begin{array}{c|c|c|c|c} (\frac{1}{1})^{1-p}M_{0,0} & 0 & 0 & 0 & \cdots \\ \hline (\frac{2}{1})^{1-p}\frac{1}{2}M_{1,0} & (\frac{2}{2})^{1-p}\frac{1}{2}M_{1,1} & 0 & 0 & \cdots \\ \hline (\frac{4}{1})^{1-p}\frac{1}{4}M_{2,0} & (\frac{4}{2})^{1-p}\frac{1}{4}M_{2,1} & (\frac{4}{4})^{1-p}\frac{1}{4}M_{2,2} & 0 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right].$$

Let $\vec{v}_k$ be the unit vector $\vec{v}_k=\frac{1}{\sqrt{2^k}}(1,1,\dots,1)^T$ and note that, for each j and k, $M_{j,k}:\mathbb{C}^{2^k}\to\mathbb{C}^{2^j}$ is rank 1 with $ker(M_{j,k})^{\perp}=\mathbb{C}\vec{v}_k$. Thus if $P_k:\mathbb{C}^{2^k}\to\mathbb{C}^{2^k}$ is the projection $P_k\vec{w}_k=\langle\vec{w}_k,\vec{v}_k\rangle\,\vec{v}_K$, then

$$C_p\begin{pmatrix}\vec{w}_0\\ \vec{w}_1\\ \vec{w}_2\\ \vdots\end{pmatrix}=C_p\begin{pmatrix}P_0\vec{w}_0\\ P_1\vec{w}_1\\ P_2\vec{w}_2\\ \vdots\end{pmatrix}.$$

Hence, for the purposes of determining the boundedness of C_p, it suffices to consider the action of C_p on vectors of the form $\vec{x}=\begin{pmatrix}\alpha_0\vec{v}_0\\ \alpha_1\vec{v}_1\\ \alpha_2\vec{v}_2\\ \vdots\end{pmatrix}$ where $\{\alpha_k\}\in l^2$. Since

$M_{j,k}\vec{v}_k = 2^{\frac{k+j}{2}}\vec{v}_j$ we see that

$$C_p\vec{x} = \begin{bmatrix} (\frac{1}{1})^{1-p}\ \alpha_0\ \vec{v}_0 \\ \left((\frac{2}{1})^{1-p}\ \frac{1}{2}\,2^{\frac{0+1}{2}}\ \alpha_0 + (\frac{2}{2})^{1-p}\ \frac{1}{2}\,2^{\frac{1+1}{2}}\ \alpha_1\right)\vec{v}_1 \\ \left((\frac{4}{1})^{1-p}\ \frac{1}{4}\,2^{\frac{0+2}{2}}\ \alpha_0 + (\frac{4}{2})^{1-p}\ \frac{1}{4}\,2^{\frac{1+2}{2}}\ \alpha_1 + (\frac{4}{4})^{1-p}\ \frac{1}{4}\,2^{\frac{2+2}{2}}\ \alpha_2\right)\vec{v}_2 \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} (\alpha_0\ \vec{v}_0 \\ \left((2^{-p+\frac{1}{2}}\ \alpha_0 + \alpha_1\right)\vec{v}_1 \\ \left(2^{-2p+1}\ \alpha_0 + 2^{-p+\frac{1}{2}}\ \alpha_1 + \alpha_2\right)\vec{v}_2 \\ \left(2^{-3p+\frac{3}{2}}\ \alpha_0 + 2^{-2p+1}\ \alpha_1 + 2^{-p+\frac{1}{2}}\alpha_2 + \alpha_3\right)\vec{v}_3 \\ \vdots \end{bmatrix}.$$

It is now apparent that C_p is bounded if and only if the Toeplitz operator

$$T_\phi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2^{-(p-\frac{1}{2})} & 1 & 0 & 0 \\ 2^{-2(p-\frac{1}{2})} & 2^{-(p-\frac{1}{2})} & 1 & 0 \\ 2^{-3(p-\frac{1}{2})} & 2^{-2(p-\frac{1}{2})} & 2^{-(p-\frac{1}{2})} & 1 \\ \vdots & & & \end{bmatrix}$$

with symbol $\phi(z) = \sum_{n=0}^{\infty} 2^{-n(p-\frac{1}{2})}z^n = \dfrac{1}{1-2^{-(p-\frac{1}{2})}z}$ is bounded. Since T_ϕ is bounded if and only if $p > \frac{1}{2}$, the proof that $\mathscr{A}_p = H^2(\mathbb{D})$ for $p > \frac{1}{2}$ is now complete. Additionally, we have shown that $\mathscr{A}_p \neq H^2(\mathbb{D})$ if $p \leqslant \frac{1}{2}$. We next show that $\mathscr{A}_p^{\perp}$ is $\{0\}$ if $p = \frac{1}{2}$ and $\mathbb{C}g_p$ if $p < \frac{1}{2}$.

To this end, first note that

$$\begin{aligned} f_{n+1}(1)f_n(z) - f_n(1)f_{n+1}(z) &= (1-b_{n+1})(1-b_nz)z^n - (1-b_n)(1-b_{n+1}z)z^{n+1} \\ &= (1-b_{n+1})z^n + (b_nb_{n+1}-1)z^{n+1} + (1-b_n)b_{n+1}z^{n+2} \\ &= (1-z)z^n(\,(1-b_{n+1}) - (1-b_n)b_{n+1}z\,) \\ &= (1-z)g_n(z), \end{aligned}$$

where $g_n(z) = z^n((1-b_{n+1}) - (1-b_n)b_{n+1}z)$ is in $\mathscr{A}_p \subset H^2(\mathbb{D})$. Suppose now that $\phi_p = \sum_{n=0}^{\infty} \gamma_n z^n \in \mathscr{A}_p^{\perp}$. Taking the $H^2(\mathbb{D})$ inner product of ϕ_p with g_n yields

$$0 = \langle\, \phi_p, g_n \rangle = \gamma_n(1-b_{n+1}) - \gamma_{n+1}(1-b_n)b_{n+1}.$$

Thus, for $n = 0, 1, 2, \dots$,

$$\gamma_{n+1} = \frac{(1-b_{n+1})}{(1-b_n)} \frac{1}{b_{n+1}} \gamma_n$$

which leads to

$$\gamma_n = \frac{(1-b_n)}{(1-b_0)} \frac{1}{b_1 b_2 \cdots b_n} \gamma_0.$$

Since $b_1 b_2 \cdots b_n = (\frac{2}{n+2})^p$ and $\frac{1-b_n}{1-b_0} \approx \frac{p}{(1-b_0)} \frac{1}{n+2}$, we obtain

$$\gamma_n \approx \frac{p}{(1-b_0)} 2^{-p} (n+2)^{p-1} \gamma_0.$$

It is now apparent that $\{\gamma_n\} \in l^2$ if and only if $p < \frac{1}{2}$. Since γ_n is comparable to $(1-b_n)(n+2)^p$, if we let $g_p = \sum_{n=0}^{\infty} (1-b_n)(n+2)^p z^n$ for $p < \frac{1}{2}$, then we have that

$$\mathscr{A}_p^{\perp} \subset \begin{cases} 0, & \text{if } p \geqslant \frac{1}{2} \\ \mathbb{C} g_p, & \text{if } p < \frac{1}{2} \end{cases} .$$

To complete the proof, it remains to show that $\mathscr{A}_p$ is the orthogonal complement of $\{\mathbb{C} g_p\}$ if $0 < p < \frac{1}{2}$. If $g \in \{g_p\}^{\perp}$, then $g(z) = \sum_{n=0}^{\infty} a_n z^n$ with

$$\sum_{n=0}^{\infty} a_n (1-b_n)(n+2)^p = 0.$$

We must show that $f(z) = (1-z)g(z) = a_0 + \sum_{n=1}^{\infty} (a_n - a_{n-1}) z^n$ is in $H(K_p)$. Note that if $f \in H(K_p)$, then $f(z) = \sum_{n=0}^{\infty} \beta_n f_n(z) = \sum_{n=0}^{\infty} \beta_n (1 - b_n z) z^n = \beta_0 + \sum_{n=0}^{\infty} (\beta_n - \beta_{n-1} b_{n-1}) z^n$ for some sequence $\{\beta_n\}_{n=0}^{\infty} \in l^2$.

In order that this occur, we must have $\beta_0 = a_0$ and, for $n \geqslant 1$,

$$\beta_n = a_n + \beta_{n-1} b_{n-1} - a_{n-1}.$$

This recursion leads to $\beta_1 = a_1 - (1-b_0) a_0$ and, for $n > 1$,

$$\begin{aligned} \beta_n &= a_n - [(1-b_{n-1})a_{n-1} + b_{n-1}(1-b_{n-2})a_{n-2} \\ &\quad + \cdots + (b_{n-1} b_{n-2} \cdots b_1)(1-b_0)a_0] \\ &= a_n - \left[(1-b_{n-1})a_{n-1} + \left(\frac{n}{n+1} \right)^p (1-b_{n-2}) a_{n-2} \right. \\ &\quad \left. + \left(\frac{n-1}{n+1} \right)^p (1-b_{n-3}) a_{n-3} + \cdots + \left(\frac{2}{n+1} \right)^p (1-b_0) a_0 \right] \\ &= a_n - \left(\frac{1}{n+1} \right)^p \left[\sum_{k=0}^{n-1} a_k (1-b_k)(k+2)^p \right] \\ &= a_n + \left(\frac{1}{n+1} \right)^p \left[\sum_{k=n}^{\infty} a_k (1-b_k)(k+2)^p \right] . \end{aligned}$$

This last equality follows from the fact that $\sum_{n=0}^{\infty} a_n(1-b_n)(n+2)^p = 0$. We can express this in matrix form as

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \end{pmatrix} = (I+B_1) \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix}$$

where

$$B_1 = \begin{bmatrix} (1-b_0)\left(\frac{2}{1}\right)^p & (1-b_1)\left(\frac{3}{1}\right)^p & (1-b_2)\left(\frac{4}{1}\right)^p & (1-b_3)\left(\frac{5}{1}\right)^p & \cdots \\ 0 & (1-b_1)\left(\frac{3}{2}\right)^p & (1-b_2)\left(\frac{4}{2}\right)^p & (1-b_3)\left(\frac{5}{2}\right)^p & \cdots \\ 0 & 0 & (1-b_2)\left(\frac{4}{3}\right)^p & (1-b_3)\left(\frac{5}{3}\right)^p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

As we observed earlier in the proof, $1-b_n \approx \frac{p}{n+2}$ for large n. Hence B_1 is bounded if and only if the matrix B_2 is bounded where

$$B_2 = \begin{bmatrix} 2^{p-1} & 3^{p-1} & 4^{p-1} & 5^{p-1} & \cdots \\ 0 & 3^{p-1}2^{-p} & 4^{p-1}2^{-p} & 5^{p-1}2^{-p} & \cdots \\ 0 & 0 & 4^{p-1}3^{-p} & 5^{p-1}3^{-p} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Breaking B_2 into blocks in the same manner as was done earlier, we see that the boundedness of B_2 is equivalent to the boundedness of B_3 where

$$B_3 = \begin{bmatrix} 2^{p-1}M_{0,0} & 4^{p-1}M_{0,1} & 8^{p-1}M_{0,2} & 16^{p-1}M_{0,3} & \cdots \\ 0 & 4^{p-1}2^{-p}M_{1,1} & 8^{p-1}2^{-p}M_{1,2} & 16^{p-1}2^{-p}M_{1,3} & \cdots \\ 0 & 0 & 8^{p-1}4^{-p}M_{2,2} & 16^{p-1}4^{-p}M_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Recalling the estimate that $||M_{n,m}|| = 2^{\frac{n+m}{2}}$ reduces the boundedness of B_3 to the boundedness of the Toeplitz matrix

$$T_\psi = \begin{bmatrix} 2^{p-\frac{2}{2}} & 2^{2p-\frac{3}{2}} & 2^{3p-\frac{4}{2}} & 2^{4p-\frac{5}{2}} & \cdots \\ 0 & 2^{p-1} & 2^{2p-\frac{3}{2}} & 2^{3p-\frac{4}{2}} & \cdots \\ 0 & 0 & 2^{p-1} & 2^{2p-\frac{3}{2}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since the symbol $\psi(z) = \sum_{n=0}^{\infty} 2^{n(p-\frac{1}{2})-\frac{1}{2}} z^n$ is bounded for $p < \frac{1}{2}$, the Toeplitz matrix is bounded and the proof is complete. □

The complete decomposition can now be summarized in the following corollary.

COROLLARY 2.4. *The space $H(K_p)$ decomposes as follows.*

(1) *If $p > \frac{1}{2}$, then $H(K_p) = (1-z)H^2(\mathbb{D}) + \mathbb{C}K_p(z,1)$.*
(2) *If $p = \frac{1}{2}$, then $H(K_p) = (1-z)\mathscr{A}_p + \mathbb{C}K_p(z,1)$ where $\mathscr{A}_p$ is dense in $H^2(\mathbb{D})$, but not equal to $H^2(\mathbb{D})$.*
(3) *If $0 < p < \frac{1}{2}$, then $H(K_p) = (1-z)\mathscr{A}_p + \mathbb{C}K_p(z,1)$ where $\mathscr{A}_p$ is the orthogonal complement in $H^2(\mathbb{D})$ of the function*

$$g_p(z) = \sum_{n=0}^{\infty} (1-b_n)(n+2)^p z^n.$$

Recall that an analytic function ϕ is a multiplier of $H(K_p)$ if $\phi f \in H(K_p)$ whenever $f \in H(K_p)$. Our next goal is to give a characterization of the multipliers of $H(K_p)$. Before doing so we establish a few simple facts about $H(K_p)$.

PROPOSITION 2.5. *The following statements hold.*

(1) *For $p > 0$, $K_p(z,1)$ extends continuously to $\partial\mathbb{D}$.*
(2) *If $\frac{1}{2} < p < \infty$, then $f(z) = 1$ belongs to $H(K_p)$.*
(3) *If $0 < p < \infty$, then $H(K_p) \subset H^2(\mathbb{D})$.*

Proof. Note that

$$\begin{aligned} K_p(z,1) &= \sum_{n=0}^{\infty} f_n(1) f(z) \\ &= \sum_{n=0}^{\infty} \left(1 - \left(\frac{n+1}{n+2}\right)^p\right)\left(1 - \left(\frac{n+1}{n+2}\right)^p z\right) z^n \\ &= 1 - \left(\frac{1}{2}\right)^p + \sum_{n=0}^{\infty}\left[1 - \left(\frac{n+1}{n+2}\right)^p - \left(\frac{n}{n+1}\right)^p + \left(\frac{n}{n+1}\right)^{2p}\right] z^n \\ &= 1 - \left(\frac{1}{2}\right)^p + \sum_{n=0}^{\infty}\left[\left(1 - \left(\frac{n}{n+1}\right)^p\right)^2 + \left(\frac{n}{n+1}\right)^p - \left(\frac{n+1}{n+2}\right)^p\right] z^n. \end{aligned}$$

Earlier it was observed that for large enough n, $1 - \left(\frac{n}{n+1}\right)^p < \frac{2p}{n+1}$. In similar fashion it is easy to verify that, for large n,

$$\left|\left(\frac{n}{n+1}\right)^p - \left(\frac{n+1}{n+2}\right)^p\right| < \frac{2p}{(n+1)(n+2)}.$$

Thus the series converges absolutely on $\partial\mathbb{D}$ and part (1) of the proposition follows.

To establish (2), note that $1 = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\right)^p f_n(z)$ where

$$\{f_n(z) = \left(1 - \left(\frac{n+1}{n+2}\right)^p z\right) z^n\}$$

is our set of orthonormal basis vectors. Since $\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\right)^{2p} < \infty$ for $\frac{1}{2} < p < \infty$, $1 \in H(K_P)$.

Likewise it is easy to see that $H(K_p) \subset H^2(\mathbb{D})$ since

$$\sum_{n=0}^{\infty} \alpha_n f_n(z) = \alpha_0 + \sum_{n=1}^{\infty}\left[\alpha_n - \left(\frac{n}{n+1}\right)^p \alpha_{n-1}\right] z^n$$

and the latter is in $H^2(\mathbb{D})$ whenever $\{\alpha_n\}$ is an l^2 sequence. □

THEOREM 2.6. *For $p > \frac{1}{2}$, the function ϕ is a multiplier of $H(K_p)$ if and only if $\phi \in H^\infty$ and*

$$\frac{\phi(z)-\phi(1)}{z-1} \in H^2(\mathbb{D}).$$

Proof. Assume ϕ is a multiplier. Since $1 \in \mathscr{H}(K_p)$, Corollary 2.4 allows us to write $\phi(z)-\phi(1) = (1-z)g(z) + \alpha K_p(z,1)$ for some $g \in H^2$. Evaluating at $z = 1$ implies $\alpha = 0$. Therefore $\frac{\phi(z)-\phi(1)}{z-1} \in H^2(\mathbb{D})$.

In general Hilbert function spaces it is well known, see section 2.3, page 21 of [4], that if ϕ is a multiplier, then the multiplication operator M_ϕ is bounded, $K_p(z,\lambda)$ is an eigenvector of the adjoint M_ϕ^* with eigenvalue $\overline{\phi(\lambda)}$, and consequently ϕ is bounded on the domain. Thus $\phi \in H^\infty$.

Conversely, assume that $\phi \in H^\infty$ and $\frac{\phi(z)-\phi(1)}{z-1} \in H^2(\mathbb{D})$. Clearly ϕ maps $(z-1)H^2(\mathbb{D})$ into $(z-1)H^2(\mathbb{D})$ and

$$\phi(z)K_p(z,1) = (z-1)\frac{\phi(z)-\phi(1)}{z-1}K_p(z,1) + \phi(1)K_p(z,1).$$

Since $K_p(z,1)$ is continuous on the closed disk $\overline{\mathbb{D}}$, $\frac{\phi(z)-\phi(1)}{z-1}K_p(z,1)$ is in $H^2(\mathbb{D})$ and it follows from Corollary 2.4 that $\phi(z)$ multiplies $K_p(z,1)$ into $\mathscr{H}(K_p)$. □

COROLLARY 2.7. *For $p > \frac{1}{2}$, the multiplication operator M_z on $H(K_p)$ is bounded and similar to a rank one perturbation of the unilateral shift.*

Proof. That M_z is bounded is immediate from the theorem above. Part (1) of Corollary 2.4 establishes that M_z is a rank one perturbation of the unilateral shift since M_z acts as the unilateral shift on $(1-z)H^2$. □

THEOREM 2.8. *For $p > \frac{1}{2}$, $H(K_p)$ has the factorization property for $\lambda \in \mathbb{D}$: $f(\lambda) = 0$ implies $f(z) = (z-\lambda)g(z)$ for $g \in H(K_p)$.*

Proof. Suppose $f(\lambda) = 0$ for some $\lambda \in \mathbb{D}$ and $f \in H(K_p)$. By Corollary 2.4

$$f(z) = (1-z)h(z) + f(1)\frac{K(z,1)}{K(1,1)}$$

for some $h \in H^2$. Hence $h(\lambda) = -\frac{f(1)}{(1-\lambda)}\frac{K(\lambda,1)}{K(1,1)}$. Note

$$g(z) = h(z) + \frac{f(1)}{(1-\lambda)}\frac{K(z,1)}{K(1,1)} \in H^2$$

and $g(\lambda) = 0$. Since H^2 has the factorization property, there exists $r \in H^2$ such that $g(z) = (z-\lambda)r(z)$. So

$$h(z) = (z-\lambda)r(z) - \frac{f(1)}{(1-\lambda)}\frac{K(z,1)}{K(1,1)}$$

and

$$\begin{aligned}(1-z)h(z) &= (z-\lambda)(1-z)r(z) - (1-z)\frac{f(1)}{(1-\lambda)}\frac{K(z,1)}{K(1,1)} \\ &= f(z) - f(1)\frac{K(z,1)}{K(1,1)}.\end{aligned}$$

Hence

$$\begin{aligned}f(z) &= (z-\lambda)(1-z)r(z) + \left[-(1-z)\frac{f(1)}{(1-\lambda)} + f(1)\right]\frac{K(z,1)}{K(1,1)} \\ &= (z-\lambda)(1-z)r(z) + f(1)\left[\frac{(-1+z+1-\lambda)}{(1-\lambda)}\right]\frac{K(z,1)}{K(1,1)} \\ &= (z-\lambda)\left[(1-z)r(z) + \frac{f(1)}{(1-\lambda)}\frac{K(z,1)}{K(1,1)}\right].\end{aligned}$$

Since Corollary 2.4 implies $(1-z)r(z) + \frac{f(1)}{(1-\lambda)}\frac{K(z,1)}{K(1,1)}$ is in $H(K_p)$, factorization holds for all $\lambda \in \mathbb{D}$. □

The following corollary follows at once from Theorem 2.10 and Section 3 of Richter[12].

COROLLARY 2.9. *If $p > \frac{1}{2}$, then*
(1) *M_z^* is in the Cowen-Dougas class B_1;*
(2) *M_z is a cellular indecomposable operator;*
(3) *The invariant subspaces of M_z are either of the form $(1-z)M$ where $M = \psi H^2$ for some inner function ψ or the span of the function $K_p(z,1)$ and a subspace of the form $(1-z)M$.*

THEOREM 2.10. *If $0 < p < \frac{1}{2}$, then ϕ is a non-trivial multiplier of $H(K_p)$ if and only if*
(1) *$\phi \in H^\infty$;*
(2) *$\frac{\phi(z)-\phi(1)}{z-1}K_p(z,1)$ is in A_p;*
(3) *there exists a constant $\lambda \in \mathbb{C}$ such that $\langle \phi-\lambda, M_z^{*n}g_p\rangle_{H^2} = 0$ for all $n \geqslant 0$, where $g_p(z) = \sum_{n=0}^{\infty}(1-b_n)(n+2)^p z^n$.*

Proof. First, assume that ϕ is a multiplier. As in the $p > \frac{1}{2}$ case, it is well known (see [4]) that ϕ is bounded. Since $H(K_p) = (1-z)A_p + \mathbb{C}K_p(z,1)$ where $A_p = H^2 \ominus \mathbb{C}g_p$, we can write

$$\phi(z)\frac{K_p(z,1)}{K_p(1,1)} = (z-1)h(z) + \phi(1)\frac{K_p(z,1)}{K_p(1,1)}.$$

Hence

$$\frac{\phi(z)-\phi(1)}{z-1}K_p(z,1) = K_p(1,1)h(z) \in A_p.$$

To establish the third condition, first note that if ϕ is a multiplier, then it is easy to see that $\phi A_p \subset A_p$. For simplicity we write the function $g_p = \sum_{n=0}^{\infty} c_n z^n$ where $c_n = (1-b_n)(n+2)^p$. Next, observe that for each $n \geqslant 0$, $h_n(z) = -\frac{c_n}{c_0} + z^n$ is in A_p since $\langle h_n, g_p\rangle_{H^2} = 0$. Hence

$$\phi(z)h_n(z) = -\frac{c_n}{c_0}\phi(z) + z^n\phi(z)$$

is in A_p which implies

$$\begin{aligned} 0 &= \langle \phi(z)h_n(z), g_p(z)\rangle_{H^2} \\ &= -\frac{c_n}{c_0}\langle \phi(z), g_p(z)\rangle_{H^2} + \langle z^n\phi(z), g_p(z)\rangle_{H^2}. \end{aligned}$$

Thus, for all $n \geqslant 0$,

$$\langle z^n\phi(z), g_p(z)\rangle_{H^2} = \frac{c_n}{c_0}\langle \phi(z), g_p(z)\rangle_{H^2}.$$

Note that ϕ is a multiplier if and only if $\phi - \lambda$ is also a multiplier for all $\lambda \in \mathbb{C}$. Condition (3) results on letting λ be such that $\langle \phi - \lambda, g_p\rangle_{H^2} = 0$.

For the converse, first note that since ϕ is a multiplier if and only if $\phi-\lambda$ is also a multiplier, we may reduce to the case where $\lambda = 0$. Next, note that conditions (1) and (3) imply that $\phi(z)p(z)$ is orthogonal in H^2 to g_p for every polynomial $p(z)$. Since the polynomials are dense in H^2, this means $\phi(z)h(z)$ is in A_p for every $h \in H^2$. In particular, $\phi A_p \subset A_p$. Since

$$H(K_p) = (1-z)A_p + \mathbb{C}K_p(z,1),$$

it suffices to show $\phi(z)K_p(z,1) \in H(K_p)$. By condition (2), $\frac{\phi(z)-\phi(1)}{z-1}K_p(z,1)$ is in A_p. Hence

$$[\phi(z)-\phi(1)]K_p(z,1) = (z-1)h(z)$$

for some $h \in A_p$. Thus $\phi(z)K_p(z,1) = (z-1)h(z) + \phi(1)K_p(z,1)$ is in $H(K_p)$ and ϕ is a multiplier. □

COROLLARY 2.11. *If $0 < p < \frac{1}{2}$, and g_p is a cyclic vector for M_z^*, then $H(K_p)$ has no non-trivial multipliers.*

Although characterizations of the cyclic vectors for the backward shift exist in the literature (see Garcia [11] and Douglas, Shapiro, and Shields [10]), applying the criteria to particular functions is often quite difficult. The authors were unable to determine whether or not g_p is a cyclic vector for M_z^* and must leave this as an open question.

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(Received July 31, 2007)

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