Operators and Matrices
Volume 2, Number 2 (2008), 249–264

HIGHER–RANK NUMERICAL RANGE IN INFINITE–DIMENSIONAL HILBERT SPACE

RUBÉN A. MARTÍNEZ-AVENDAÑO

(*communicated by C.-K. Li*)

Abstract. In this paper we calculate the higher-rank numerical range, as defined by Choi, Kribs and Życzkowski, of selfadjoint operators and of nonunitary isometries on infinite-dimensional Hilbert space.

1. Introduction

Numerical ranges of operators have been studied since the early 20th century. One of the earliest results is that of Toeplitz and Hausdorff, who proved that the range of the quadratic form associated with an operator, restricted to the unit sphere of Hilbert space (i.e., the numerical range) is a convex subset of the complex plane (cf. [7]). The numerical range has been the subject of much research and a lot is known about it. There have also been several generalizations of the numerical range that have been studied; see, for example, [5, 7, 11].

In the context of "quantum error correction", Choi, Kribs and Życzkowski [3] defined the *rank-* k *numerical range* of an $n \times n$ matrix A to be the the set

$$\{\lambda \in \mathbb{C} : PAP = \lambda P, \text{ for some projection } P \text{ of rank } k\}.$$

This set is also called the set of compression values. Evidently, λ is in this higher-rank numerical range of a matrix A if and only if there exists an orthonormal basis such that, in the matrix representation corresponding of A in this basis, the $k \times k$ submatrix in the upper-left corner is just λ times the identity. Hence, the case $k = 1$ reduces to the classical numerical range.

Studies of this rank- k numerical range have developed rapidly since [3] was first circulated. Choi, Kribs and Życzkowski had calculated this higher-rank numerical range for Hermitian matrices and put forward a conjecture on what the higher-rank numerical range would be for the case of normal matrices [3]. Partial results on their conjecture and a number of related results can be found in [1, 2]. The CKZ conjecture was settled by Li and Sze in [10], where, among other things, they give an explicit expression for

Mathematics subject classification (2000): 47A12, 47B15, 15A60.

Key words and phrases: Higher-rank numerical range, selfadjoint operator, nonunitary isometry.

Paper OAM-02-16

the higher-rank numerical range as the intersection of closed half-planes. This provided an alternative proof to the question of convexity of the higher-rank numerical ranges, which had been settled affirmatively by Woerdeman [16]. Other interesting results can be found in [6, 8, 9, 10].

The purpose of this paper is to extend several of the above results to infinite-dimensional Hilbert space and, also, to allow the projection to be of infinite rank. A quick glance at the papers above will show the reader that many of the basic properties of the higher-rank numerical range for matrices hold for operators on infinite-dimensional Hilbert space as well. However, there are (at least) two cases in which a straight application of the finite-dimensional techniques do not work: selfadjoint operators and nonunitary isometries. We deal with those cases henceforth.

We should point out that there has been some research on extending some finite-dimensional results to the infite-dimensional setting. For example, in [10], Li and Sze prove convexity of the (finite) higher-rank numerical range for a bounded operator in infinite dimensional Hilbert space (this had also been shown by Woerdeman [16]) and they also give a concrete description of it as an intersection of planes determined by the eigenvalues of certain compressions of the operator. Compare this to our Theorem 3.4, in which we give a description of the higher-rank numerical range of a selfadjoint operator in terms of its spectral measure.

The present paper is divided as follows. In Section 2 we give the basic definitions and theorems that we will use throughout. In Section 3, we extend the calculations in [3] to obtain an expression for the higher-rank numerical range in the case of a selfadjoint operator on infinite-dimensional Hilbert space. Our main theorem (Theorem 3.4) gives an expression for the higher-rank numerical range of a selfadjoint operator A in terms of the projection-valued spectral measure associated to A by one form of the spectral theorem. Applications of this result include the calculation of the higher-rank numerical range for the operator of multiplication by the independent variable on some L^2 spaces.

In Section 4, we calculate the higher-rank numerical range for all nonunitary isometries on Hilbert space (Theorem 4.5). We apply this to the calculation of the higher-rank numerical range of some analytic Toeplitz operators (Corollary 4.6). We also observe how this calculation extends to some unitary operators.

Lastly, we would like to point out that, with the appropriate modifications, many of the results in Section 3 apply to unbounded operators. For simplicity, we only deal with bounded operators throughout this paper.

2. Preliminaries

Throughout this paper $\mathscr{H}$ will denote a separable and infinite-dimensional Hilbert space. Let us define $\mathbb{N}_\infty$ as the set $\mathbb{N} \cup \{\infty\}$. If $n \in \mathbb{N}_\infty$, whenever we say, for example, that an operator is of rank n, we mean that its range is of dimension n if n is finite, and we mean that its range is infinite-dimensional if $n = \infty$. If $n \in \mathbb{N}_\infty$, a space has codimension less that n if it has codimension at most $n-1$ when n is finite, and it has finite codimension when $n = \infty$. The reader should interpret other uses of $n = \infty$ accordingly.

DEFINITION 2.1. Let $A \in \mathbf{B}(\mathscr{H})$ and let $n \in \mathbb{N}_\infty$. We define the *numerical range of rank* n to be the set

$$\Lambda_n(A) = \{\lambda \in \mathbb{C} \,:\, PAP = \lambda P, \text{ for some projection } P \text{ of rank } n\}.$$

Observe that $\lambda \in \Lambda_n(A)$ if and only if there exists an orthonormal set $\{f_j\}_{j=1}^n$ such that

$$\langle Af_j, f_k\rangle = \lambda\delta_{j,k}.$$

Therefore, if a vector g of norm 1 is in the span of the vectors $\{f_j\}_{j=1}^n$, then $\langle Ag, g\rangle = \lambda$.

This means that $\lambda \in \Lambda_n(A)$ if and only if there is a basis of $\mathscr{H}$ in which the matrix of A has λI in its an upper-left corner, where I is the identity operator on (sub)space of dimension n.

Observe that $\Lambda_1(A)$ is the classical numerical range; i.e., $\Lambda_1(A) = W(A)$, where

$$W(A) = \{\langle Af, f\rangle \,:\, \|f\| = 1\}.$$

DEFINITION 2.2. Let $\mathscr{H}$ be a Hilbert space and $n \in \mathbb{N}_\infty$. We denote by $\mathscr{V}_n$ the set of all isometries $V : \mathscr{H} \longrightarrow \mathscr{H}$ such that the codimension of $\operatorname{ran} V$ is less than n.

The following proposition was proved for finite-dimensional Hilbert space in [3]. We include here the proof for the sake of completeness.

PROPOSITION 2.3. *Let $n \in \mathbb{N}_\infty$ and let $A \in \mathbf{B}(\mathscr{H})$. Then*

$$\Lambda_n(A) \subseteq \bigcap_{V \in \mathscr{V}_n} W(V^*AV).$$

Proof. Let $\lambda \in \Lambda_n(A)$. Choose orthonormal vectors $\{f_j\}_{j=1}^n$ such that $\langle Af_j, f_k\rangle = \lambda\delta_{j,k}$. Let $V \in \mathscr{V}_n$. Since $\operatorname{ran} V$ has codimension less than n, there exists a nonzero vector g in the span of $\{f_j\}_{j=1}^n$ such that $g = Vh$ for some $h \in \mathscr{H}$. In fact, take g to be of norm one. Then, $\|h\| = \|Vh\| = \|g\| = 1$ and also,

$$\langle V^*AVh, h\rangle = \langle AVh, Vh\rangle = \langle Ag, g\rangle = \lambda,$$

since g is in the span of $\{f_j\}_{j=1}^n$. Hence $\lambda \in W(V^*AV)$. Since V was an arbitrary element of $\mathscr{V}_n$, the result follows. $\square$

We need to make a few comments on the classical numerical range. It is easily seen and well-known (see, for example, Halmos [7]) that if A is an operator with $\langle Af, f\rangle = \lambda$ for some $|\lambda| = \|A\|$ and $\|f\| = 1$, then $Af = \lambda f$. We obtain the following useful proposition quickly as a corollary.

PROPOSITION 2.4. *Let $n \in \mathbb{N}_\infty$ and $A \in \mathbf{B}(\mathscr{H})$. If $\lambda \in \Lambda_n(A)$ is such that $|\lambda| = \|A\|$, then λ is an eigenvalue of multiplicity at least n.*

Proof. Choose an orthonormal set $\{f_j\}_{j=1}^n$ such that $\langle Af_j, f_k\rangle = \lambda\delta_{j,k}$. It then follows from the above remark that $Af_j = \lambda f_j$ for all j. $\square$

The following observation will also be useful. Recall that the numerical range of a selfadjoint operator is a bounded interval in the real line.

LEMMA 2.5. *Let $T \in \mathbf{B}(\mathscr{H})$ be selfadjoint, and let c be one of the endpoints of $W(T)$. If $c \in W(T)$, then c is an eigenvalue of T.*

Proof. Let d be the other endpoint of $W(T)$. The operator $T-d$ is selfadjoint, and $W(T-d)$ is an interval having as its endpoints the set $\{0, c-d\}$. These two facts imply that the norm of $T-d$ equals $|c-d|$. Since $c-d \in W(T-d)$ it follows from the remark before Proposition 2.4 that $c-d$ is an eigenvalue of $T-d$ and hence that c is an eigenvalue of T. □

Recall that a version of the spectral theorem says that, for a selfadjoint operator T, there exists a projection-valued measure E (called a *spectral measure*), defined on Borel subsets of $\mathbb{R}$ and supported on $\sigma(T)$, such that

$$T = \int_{\mathbb{R}} x\, dE(x).$$

Recall also that, for f and g in $\mathscr{H}$, we can define a complex measure $E_{f,g}$ by $E_{f,g}(\Delta) = \langle E(\Delta)f, g\rangle$ for every Borel subset $\Delta \subseteq \mathbb{R}$. In this case, we also have

$$\langle Tf, g\rangle = \int_{\mathbb{R}} x\, dE_{f,g}(x).$$

Familiarity with the definition of the spectral measure and this version of the spectral theorem is assummed in this paper. A good reference is Conway [4].

We will frequently use several properties of the spectral measure. Recall that $E(\mathbb{R}) = I$; that for each measurable set Δ we have $E(\Delta) + E(\mathbb{R} \setminus \Delta) = I$; that if $\Delta_1 \subseteq \Delta_2$ are measurable sets, then $\operatorname{ran} E(\Delta_1) \subseteq \operatorname{ran} E(\Delta_2)$; and that, for each increasing sequence a_n converging to a, we have that $E(-\infty, a_n]f \to E(-\infty, a)f$ for all $f \in \mathscr{H}$. All these properties can be found in [4].

We need the following technical lemma.

LEMMA 2.6. *Let $c < d$. Assume that $\phi \in \operatorname{ran} E(-\infty, c]$ and $\psi \in \operatorname{ran} E[d, \infty)$ are both of norm one. Then,*

$$\int_{\mathbb{R}} x\, dE_{\phi,\phi}(x) \leqslant c \quad \text{and} \quad \int_{\mathbb{R}} x\, dE_{\psi,\psi}(x) \geqslant d.$$

Also, we have

$$\int_{\mathbb{R}} x\, dE_{\phi,\psi}(x) = 0 \quad \text{and} \quad \int_{\mathbb{R}} x\, dE_{\psi,\phi}(x) = 0.$$

Proof. Since ϕ is in $\operatorname{ran} E(-\infty, c]$, it follows that the (in this case, positive) measure $E_{\phi,\phi}$ is supported inside $(-\infty, c]$. Since $x \leqslant c$ in the support of $E_{\phi,\phi}$ we have

$$\int_{\mathbb{R}} x\, dE_{\phi,\phi}(x) \leqslant c \int_{\mathbb{R}} dE_{\phi,\phi}(x) = c.$$

Analogously, since $\psi \in \operatorname{ran} E[d, \infty)$, it follows that the measure $E_{\psi,\psi}$ is supported inside $[d, \infty)$ and hence

$$\int_{\mathbb{R}} x\, dE_{\psi,\psi}(x) \geqslant d.$$

Now, to prove the last part of the lemma, it is enough to check that the measure $E_{\phi,\psi}$ is zero everywhere. Indeed, observe that

$$\begin{aligned}\langle E(\Delta)\phi,\psi\rangle &= \langle E(\Delta\cap(-\infty,d))\phi,\psi\rangle + \langle E(\Delta\cap[d,\infty))\phi,\psi\rangle \\ &= \langle \phi, E(\Delta\cap(-\infty,d))\psi\rangle + \langle E(\Delta\cap[d,\infty))\phi,\psi\rangle \\ &= 0+0,\end{aligned}$$

since

$$\begin{aligned}\psi &\in \operatorname{ran} E[d,\infty) = (\operatorname{ran} E(-\infty,d)))^{\perp} \\ &\subseteq (\operatorname{ran} E(\Delta\cap(-\infty,d)))^{\perp} \\ &= \ker E(\Delta\cap(-\infty,d))\end{aligned}$$

and

$$\begin{aligned}\phi &\in \operatorname{ran} E(-\infty,c] = (\operatorname{ran} E(c,\infty))^{\perp} \\ &\subseteq (\operatorname{ran} E[d,\infty))^{\perp} \subseteq (\operatorname{ran} E(\Delta\cap[d,\infty)))^{\perp} \\ &= \ker E(\Delta\cap[d,\infty)).\end{aligned}$$

Hence,

$$\int_{\mathbb{R}} x\, dE_{\phi,\psi}(x) = 0.$$

Similarly, one shows that

$$\int_{\mathbb{R}} x\, dE_{\psi,\phi}(x) = 0.$$

□

Observe that first part of above lemma still holds if we have $c = d$. The second part also holds for $c = d$ if, in addition, we ask that $\phi \in \operatorname{ran} E(-\infty,c)$ and $\psi \in \operatorname{ran} E[d,\infty)$ or we ask that $\phi \in \operatorname{ran} E(-\infty,c]$ and $\psi \in \operatorname{ran} E(d,\infty)$.

3. Selfadjoint Operators

For each fixed $n \in \mathbb{N}_\infty$, we define the sets A_n and B_n as

$$A_n := \{a \in \mathbb{R} : \dim \operatorname{ran} E(-\infty,a] < n\}$$

and

$$B_n := \{b \in \mathbb{R} : \dim \operatorname{ran} E[b,\infty) < n\}.$$

Observe that A_n and B_n are nonempty since all real numbers to the left of $\sigma(T)$ are in A_n and all real numbers to the right of $\sigma(T)$ are in B_n. In fact, more is true.

PROPOSITION 3.1. *Let $T \in \mathbf{B}(\mathscr{H})$ be a selfadjoint operator. For $n \in \mathbb{N}_\infty$, let A_n and B_n be as defined above. Then $a \leqslant b$ for all $a \in A_n$ and all $b \in B_n$.*

Proof. Suppose not. Then there exist $a \in A_n$ and $b \in B_n$ such that $a > b$. Since $a \in A_n$ we have $\dim \operatorname{ran} E(-\infty, a] < n$. Since $\operatorname{ran} E(-\infty, b) \subseteq \operatorname{ran} E(-\infty, a]$, we have that $\dim \operatorname{ran} E(-\infty, b) < n$. Since $b \in B_n$ we have $\dim \operatorname{ran} E[b, \infty) < n$.

Since $I = E(-\infty, b) + E[b, \infty)$, this would imply that the dimension of $\mathscr{H}$ is less than $2n$. This is a contradiction. □

We now define $\alpha_n = \sup A_n$ and $\beta_n = \inf B_n$. The previous proposition guarantees that $\alpha_n \leqslant \beta_n$. The following observation turns out to be helpful.

LEMMA 3.2. *Let $n \in \mathbb{N}$ and let α_n and β_n defined as above. If $\alpha_n = \beta_n$ then $\alpha_n \notin A_n$ and $\beta_n \notin B_n$.*

Proof. Assume that $\alpha_n \in A_n$ and $\beta_n \in B_n$. Then $\operatorname{ran} E(-\infty, \alpha_n]$ and $\operatorname{ran} E[\beta_n, \infty)$ have dimension less than n. But this implies that $\operatorname{ran} E(-\infty, \alpha_n)$ and $\operatorname{ran} E[\beta_n, \infty)$ have dimension less than n. Since $\alpha_n = \beta_n$ it follows that $I = E(-\infty, \alpha_n) + E[\beta_n, \infty)$ which in turn implies that the dimension of $\mathscr{H}$ is less than $2n$. This is a contradiction.

Assume $\alpha_n \notin A_n$ and $\beta_n \in B_n$. Since $\alpha_n = \sup A_n$, there exists an increasing sequence $\{a_k\}$ in A_n such that $a_k \to \alpha_n$. But since $\dim \operatorname{ran} E(-\infty, a_k] < n$ and $E(-\infty, a_k]f \to E(-\infty, \alpha_n)f$ for all $f \in \mathscr{H}$, it follows that $\dim \operatorname{ran} E(-\infty, \alpha_n) < n$ as well. We also have that $\beta_n \in B_n$ which implies that $\dim \operatorname{ran} E[\beta_n, \infty) < n$. But again, we have that $I = E(-\infty, \alpha_n) + E[\beta_n, \infty)$ which implies that the dimension of $\mathscr{H}$ is less than $2n$, a contradiction.

The case $\beta_n \notin B_n$ and $\alpha_n \in A_n$ is handled similarly. □

Observe that only the first paragraph of proof above works if $n = \infty$. So we can only conclude that if $\alpha_\infty = \beta_\infty$ then $\alpha_\infty \notin A_\infty$ or $\beta_\infty \notin B_\infty$. We will see an example where the conclussion of the lemma above fails for $n = \infty$.

DEFINITION 3.3. For $n \in \mathbb{N}_\infty$, let A_n, B_n, α_n and β_n be as before. We define the interval Ω_n as $A_n^c \cap B_n^c$. Equivalently,

$$\Omega_n = \begin{cases} [\alpha_n, \beta_n] & \text{if } \alpha_n \notin A_n \text{ and } \beta_n \notin B_n, \\ [\alpha_n, \beta_n) & \text{if } \alpha_n \notin A_n \text{ and } \beta_n \in B_n, \\ (\alpha_n, \beta_n] & \text{if } \alpha_n \in A_n \text{ and } \beta_n \notin B_n, \\ (\alpha_n, \beta_n) & \text{if } \alpha_n \in A_n \text{ and } \beta_n \in B_n. \end{cases}$$

Observe that the previous lemma guarantees that Ω_n is never empty for $n \in \mathbb{N}$. We will see an example where Ω_∞ is empty.

We are now ready to prove our main result. A finite-dimensional version of this was proved in [3].

THEOREM 3.4. *Let $T \in \mathbf{B}(\mathscr{H})$ be a selfadjoint operator and let $n \in \mathbb{N}_\infty$. Then*

$$\Lambda_n(T) = \bigcap_{V \in \mathscr{V}_n} W(V^*TV) = \Omega_n.$$

In particular, $\Lambda_n(T)$ is always a convex set (nonempty if $n \neq \infty$).

Proof. The containment

$$\Lambda_n(T) \subseteq \bigcap_{V \in \mathscr{V}_n} W(V^*TV)$$

is the statement of Proposition 2.3 (observe that we do not need the hypothesis of selfadjointness there). We need to prove two more containments.

$$\bigcap_{V \in \mathscr{V}_n} W(V^*TV) \subseteq \Omega_n \tag{$\clubsuit$}$$

We prove this in several cases.

- Assume that $\alpha_n \in A_n$ and $\beta_n \in B_n$.

In this case we have that $\operatorname{ran} E(-\infty, \alpha_n]$ and $\operatorname{ran} E[\beta_n, \infty)$ have dimension less than n and hence $\operatorname{ran} E(\alpha_n, \infty)$ and $\operatorname{ran} E(-\infty, \beta_n)$ have codimension less than n. Thus there exists isometries V_{α_n} and V_{β_n} with ranges the (closed) infinite dimensional subspaces $\operatorname{ran} E(\alpha_n, \infty)$ and $\operatorname{ran} E(-\infty, \beta_n)$ respectively. Hence V_{α_n} and V_{β_n} are in $\mathscr{V}_n$. Observe also that $V_{\alpha_n} V_{\alpha_n}^* = E(\alpha_n, \infty)$ and $V_{\beta_n} V_{\beta_n}^* = E(-\infty, \beta_n)$. Notice also that

$$V_{\alpha_n}^* T V_{\alpha_n} = V_{\alpha_n}^* E(\alpha_n, \infty)\, T\, E(\alpha_n, \infty)\, V_{\alpha_n}.$$

Since V_{α_n} is an isometry, we have

$$W(V_{\alpha_n}^* E(\alpha_n, \infty)\, T\, E(\alpha_n, \infty)\, V_{\alpha_n}) \subseteq W(E(\alpha_n, \infty)\, T\, E(\alpha_n, \infty)).$$

These two facts imply that $W(V_{\alpha_n}^* T V_{\alpha_n}) \subseteq W(E(\alpha_n, \infty)\, T\, E(\alpha_n, \infty))$. But, since we know that

$$E(\alpha_n, \infty)\, T\, E(\alpha_n, \infty) = \int_{(\alpha_n, \infty)} x\, dE(x),$$

it follows tha α_n cannot be an eigenvalue of the operator $E(\alpha_n, \infty)\, T\, E(\alpha_n, \infty)$.

The integral above also gives

$$\sigma(E(\alpha_n, \infty)\, T\, E(\alpha_n, \infty)) \subseteq [\alpha_n, \infty).$$

Also, since the closure of the numerical range of a selfadjoint operator equals the convex hull of the spectrum of the operator, it follows that

$$W(E(\alpha_n, \infty)\, T\, E(\alpha_n, \infty)) \subseteq [\alpha_n, \infty).$$

Now, Lemma 2.5 implies that if α_n was in $W(E(\alpha_n, \infty)\, T\, E(\alpha_n, \infty))$, then α_n would be an eigenvalue of $E(\alpha_n, \infty)\, T\, E(\alpha_n, \infty)$, which, as noted above, does not happen. Hence

$$W(E(\alpha_n, \infty)\, T\, E(\alpha_n, \infty)) \subseteq (\alpha_n, \infty),$$

and thus $W(V_{\alpha_n}^* T V_{\alpha_n}) \subseteq (\alpha_n, \infty)$.

Analogously, we have that $W(V_{\beta_n}^* T V_{\beta_n}) \subseteq (-\infty, \beta_n)$. Then,

$$\bigcap_{V \in \mathscr{V}_n} W(V^*TV) \subseteq W(V_{\alpha_n}^* T V_{\alpha_n}) \cap W(V_{\beta_n}^* T V_{\beta_n}) \subseteq (\alpha_n, \beta_n) = \Omega_n,$$

which shows ($\clubsuit$) in this case.

- Assume that $\alpha_n \in A_n$ and $\beta_n \notin B_n$.

  In this case we have that $\operatorname{ran} E(-\infty, \alpha_n]$ has dimension less than n. Also, there exists a decreasing sequence $\{b_k\}$ in B_n such that $b_k \to \beta_n$. Hence, $\dim \operatorname{ran} E[b_k, \infty)$ has dimension less than n. Thus we have that $\operatorname{ran} E(\alpha_n, \infty)$ and $\operatorname{ran} E(-\infty, b_k)$ have codimension less than n.

  As we did before, we find isometries $V_{\alpha_n}, V_{b_k} \in \mathbf{B}(\mathscr{H})$ such that $V_{\alpha_n} V_{\alpha_n}^* = E(\alpha_n, \infty)$ and $V_{b_k} V_{b_k}^* = E(-\infty, b_k)$. As in the previous case, we can conclude that

$$W(V_{\alpha_n}^* T V_{\alpha_n}) \subseteq (\alpha_n, \infty)$$

and

$$W(V_{b_k}^* T V_{b_k}) \subseteq (-\infty, b_k).$$

Hence,

$$\bigcap_{V \in \mathscr{V}_n} W(V^* T V) \subseteq W(V_{\alpha_n}^* T V_{\alpha_n}) \cap \bigcap_{k=1}^{\infty} W(V_{b_k}^* T V_{b_k}) \subseteq (\alpha_n, \beta_n] = \Omega_n,$$

which shows (♣) also in this case.

- Assume that $\alpha_n \notin A_n$ and $\beta_n \in B_n$.

  This case is done exactly as the previous case.

- Assume that $\alpha_n \notin A_n$ and $\beta_n \notin B_n$.

  This case is done combining the techniques used in the previous two cases.

$$\Omega_n \subseteq \Lambda_n(T). \tag{♠}$$

We also prove this in several cases.

- Assume that $\alpha_n \in A_n$ and $\beta_n \in B_n$.

  In this case $\Omega_n = (\alpha_n, \beta_n)$. Take $\lambda \in \Omega_n = (\alpha_n, \beta_n)$. Choose a and b such that $\alpha_n < a < \lambda < b < \beta_n$. We have that $\operatorname{ran} E(-\infty, a]$ and $\operatorname{ran} E[b, \infty)$ have dimension at least n. Let Φ be the n-dimensional space $\operatorname{ran} E(-\infty, a]$ and Ψ be the n-dimensional space $\operatorname{ran} E[b, \infty)$.

  We choose an orthonormal set $\{\phi_j\}_{j=1}^n$ in Φ in such a way that $\langle T\phi_j, \phi_k \rangle = 0$ for $j \neq k$. Indeed, if n is finite this can be achieved by observing that the operator T compressed to the space Φ is a selfadjoint operator on a finite-dimensional space and hence diagonalizable. If $n = \infty$ the condition can be achieved by choosing ϕ_1 to be any unit vector in Φ and then, inductively, choosing $\phi_{s+1} \in \Phi$ in such a way that $\{\phi_1, \phi_2, \ldots, \phi_s, \phi_{s+1}\}$ is orthonormal and such that ϕ_{s+1} is orthogonal to $\{T\phi_1, T\phi_2, \ldots, T\phi_s\}$; clearly, T compressed to the space spanned by $\{\phi_j\}_{j=1}^{\infty}$ is upper-triangular and hence, since T is selfadjoint, this matrix representation of the compression of T is diagonal.

  Analogously, choose an orthonormal set $\{\psi_j\}_{j=1}^n$ in Ψ in such a way that $\langle T\psi_j, \psi_k \rangle = 0$ for $j \neq k$. Observe that since $\operatorname{ran} E(-\infty, a]$ and $\operatorname{ran} E[b, \infty)$ are orthogonal subspaces, we have that ϕ_k is orthogonal to ψ_j for every j and k.

As seen in Lemma 2.6, we have that

$$\int_{\mathbb{R}} x\,dE_{\phi_j,\phi_j}(x) \leqslant a \quad \text{and} \quad \int_{\mathbb{R}} x\,dE_{\psi_j,\psi_j}(x) \geqslant b.$$

Since $\lambda \in (a,b)$ we know that, for each natural $j < n+1$ there exist real numbers s_j and t_j, with $s_j^2 + t_j^2 = 1$ such that

$$\lambda = s_j^2 \int_{\mathbb{R}} x\,dE_{\phi_j,\phi_j}(x) + t_j^2 \int_{\mathbb{R}} x\,dE_{\psi_j,\psi_j}(x).$$

Define the set $\{f_j\}_{j=1}^n$ to consist of the vectors $f_j := s_j\phi_j + t_j\psi_j$. It is easily seen that $\{f_j\}_{j=1}^n$ is an orthonormal set.

We now have

$$\begin{aligned} \langle Tf_j, f_j \rangle &= s_j^2 \langle T\phi_j, \phi_j \rangle + s_jt_j \langle T\phi_j, \psi_j \rangle + t_js_j \langle T\psi_j, \phi_j \rangle + t_j^2 \langle T\psi_j, \psi_j \rangle \\ &= s_j^2 \int_{\mathbb{R}} x\,dE_{\phi_j,\phi_j}(x) + s_jt_j \int_{\mathbb{R}} x\,dE_{\phi_j,\psi_j}(x) \\ &\qquad + t_js_j \int_{\mathbb{R}} x\,dE_{\psi_j,\phi_j}(x) + t_j^2 \int_{\mathbb{R}} x\,dE_{\psi_j,\psi_j}(x). \end{aligned}$$

Since $\int_{\mathbb{R}} x\,dE_{\phi_j,\psi_j}(x) = 0$ and $\int_{\mathbb{R}} x\,dE_{\psi_j,\phi_j}(x) = 0$ by Lemma 2.6, we have that

$$\langle Tf_j f_j \rangle = s_j^2 \int_{\mathbb{R}} x\,dE_{\phi_j,\phi_j}(x) + t_j^2 \int_{\mathbb{R}} x\,dE_{\psi_j,\psi_j}(x) = \lambda,$$

as desired.

Also, for $j \neq k$, we have

$$\begin{aligned} \langle Tf_j, f_k \rangle &= s_js_k \langle T\phi_j, \phi_k \rangle + s_jt_k \langle T\phi_j, \psi_k \rangle + t_js_k \langle T\psi_j, \phi_k \rangle + t_jt_k \langle T\psi_j, \psi_k \rangle \\ &= s_js_k\, 0 + s_jt_k \int_{\mathbb{R}} x\,dE_{\phi_j,\psi_k}(x) + t_js_k \int_{\mathbb{R}} x\,dE_{\psi_j,\phi_k}(x) + t_jt_k\, 0. \end{aligned}$$

But as seen in Lemma 2.6, each of the integrals above is zero.

Therefore, $\langle Tf_j, f_k \rangle = \lambda\delta_{j,k}$ as desired, and hence $\lambda \in \Lambda_n(T)$ which proves (♠) in this case.

- Assume that $\alpha_n \in A$ and $\beta_n \notin B$.

In this case, $\Omega_n = (\alpha_n, \beta_n]$. If $\alpha_n = \beta_n$ there is nothing to prove, so assume $\alpha_n < \beta_n$. Let $\lambda \in \Omega_n$. Choose a such that $\alpha_n < a < \lambda \leqslant \beta_n$. We have that $\operatorname{ran} E(-\infty, a]$ and $\operatorname{ran} E[\beta_n, \infty)$ have dimension at least n. As in the previous case, choose orthonormal sets $\{\phi_j\}_{j=1}^n$ in $\Phi := \operatorname{ran} E(-\infty, a]$ and $\{\psi_j\}_{j=1}^n$ in $\Psi := \operatorname{ran} E[\beta_n, \infty)$ in such a way that $\langle T\phi_j, \phi_k \rangle = 0$ for all $j \neq k$ and $\langle T\psi_j, \psi_k \rangle = 0$ for all $j \neq k$.

As seen in Lemma 2.6, we have that

$$\int_{\mathbb{R}} x\,dE_{\phi_j,\phi_j}(x) \leqslant a \quad \text{and} \quad \int_{\mathbb{R}} x\,dE_{\psi_j,\psi_j}(x) \geqslant \beta_n.$$

Also, for each natural $j < n+1$ choose real numbers s_j and t_j with $s_j^2 + t_j^2 = 1$ such that

$$\lambda = s_j^2 \int_{\mathbb{R}} x\, dE_{\phi_j,\phi_j}(x) + t_j^2 \int_{\mathbb{R}} x\, dE_{\psi_j,\psi_j}(x).$$

As before, define the set $\{f_j\}_{j=1}^n$ as $f_j := s_j\phi_j + t_j\psi_j$. Then $\{f_j\}_{j=1}^n$ is an orthonormal set and one checks, exactly as before, that $\langle Tf_j, f_k\rangle = \lambda\delta_{j,k}$. Hence $\lambda \in \Lambda_n(T)$ which proves (♠) in this case.

- Assume that $\alpha_n \notin A$ and $\beta_n \in B$.

In this case, $\Omega_n = [\alpha_n, \beta_n)$. The proof is done as in the previous case.

- Assume that $\alpha_n \notin A$ and $\beta_n \notin B$.

In this case $\Omega_n = [\alpha_n, \beta_n]$. Choose $\lambda \in \Omega_n = [\alpha_n, \beta_n]$. If $\alpha_n < \beta_n$, this case can be treated exactly as the previous cases, taking $\Phi := E(-\infty, \alpha_n]$ and taking $\Psi := E[\beta_n, \infty)$.

Thus assume that $\alpha_n = \beta_n$. We know that $\operatorname{ran} E(-\infty, \alpha_n]$ and $\operatorname{ran} E[\beta_n, \infty)$ have both dimension at least n. If $\dim \operatorname{ran} E(-\infty, \alpha_n)$ is at least n, we can act as in previous cases, with $\Phi := \operatorname{ran} E(-\infty, \alpha_n)$ and $\Psi := \operatorname{ran} E[\beta_n, \infty)$.

Assume now that $\dim \operatorname{ran} E(-\infty, \alpha_n)$ is less than n. Then, if $\dim \operatorname{ran} E(\beta_n, \infty)$ is at least n, we can act as in previous cases, with $\Phi := \operatorname{ran} E(-\infty, \alpha_n]$ and $\Psi := \operatorname{ran} E(\beta_n, \infty)$.

Thus we may assume that both $\operatorname{ran} E(-\infty, \alpha_n)$ and $\operatorname{ran} E(\beta_n, \infty)$ have dimension less than n. But this implies that $\operatorname{ran} E(\{\alpha_n\})$ is infinite-dimensional and hence that $\lambda = \alpha_n = \beta_n$ is an eigenvalue of infinite multiplicity. Hence, taking $\{f_j\}_{j=1}^n$ to be an orthonormal set of eigenvectors corresponding to λ we see that $\langle Tf_s, f_t\rangle = \lambda\delta_{s,t}$, an hence we have shown (♠), as desired. □

We also obtain as a corollary the following surprising result. Notice that we do not need selfadjointness.

COROLLARY 3.5. *Let $A \in \mathbf{B}(\mathscr{H})$ and $n \in \mathbb{N}$. Then $\Lambda_n(A)$ is nonempty.*

Proof. Write A as $A = T_1 + iT_2$ with T_1 and T_2 selfadjoint. Choose $N \in \mathbb{N}$ such that $2n-1 \leqslant N$. By the theorem above, there exists a real number λ such that $\lambda \in \Lambda_N(T_1)$; i.e., there exists an orthonormal set $\{f_1, f_2, \ldots, f_N\}$ such that $\langle T_1 f_j, f_k\rangle = \lambda\delta_{j,k}$. Thus, if $\mathscr{M}$ is the subspace generated by $\{f_1, f_2, \ldots, f_N\}$, it follows that the compression of T_1 to $\mathscr{M}$ is just λ times the identity on $\mathscr{M}$.

But then, the compression of the operator T_2 to the N-dimensional subspace $\mathscr{M}$ is also selfadjoint and hence, since $2n-1 \leqslant N$, it follows by [3, Theorem 2.4] that there exists an orthonormal set $\{g_1, g_2, \ldots, g_n\}$ in $\mathscr{M}$ such that $\langle T_2 g_j, g_k\rangle = \mu\delta_{j,k}$ for some real number μ. But this implies that $\langle Ag_j, g_k\rangle = \langle T_1 g_j, g_k\rangle + i\langle T_2 g_j, g_k\rangle = \lambda\delta_{j,k} + i\mu\delta_{j,k} = (\lambda + i\mu)\delta_{j,k}$, which shows that $\lambda + i\mu \in \Lambda_n(A)$. □

The previous result can be obtained in a different way. Let N be large enough such that $4n-3 \leqslant N$. Taking the compression of T into an N-dimensional subspace and applying [3, Corollary 2.5] to this compression we obtain our corollary.

We should also point out that the above corollary was obtained independently in [8].

The following observation should be made.

COROLLARY 3.6. *Let $T \in \mathbf{B}(\mathscr{H})$ be selfadjoint. Then,* $\Lambda_\infty(T) = \bigcap_{n=1}^{\infty} \Lambda_n(T)$.

Proof. The inclusion

$$\bigcap_{n=1}^{\infty} \Lambda_n(T) \supseteq \Lambda_\infty(T)$$

is trivial. So let $\lambda \in \bigcap_{n=1}^{\infty} \Lambda_n(T)$. This means that $\lambda \in \Lambda_n(T)$ for all $n \in \mathbb{N}$ and by Theorem 3.4, it follows that $\lambda \notin A_n$ and $\lambda \notin B_n$ for each $n \in \mathbb{N}$, where A_n and B_n are as used in Theorem 3.4. This means that $\dim \operatorname{ran} E(-\infty, \lambda] \geqslant n$ and $\dim \operatorname{ran} E[\lambda, \infty) \geqslant n$. Since this occurs for all $n \in \mathbb{N}$ we have that $\operatorname{ran} E(-\infty, \lambda]$ and $\operatorname{ran} E[\lambda, \infty)$ are both infinite-dimensional.

If $\operatorname{ran} E(-\infty, \lambda)$ is infinite-dimensional, we can act as in Theorem 3.4 to construct orthonormal sets $\{\phi_j\}_{j=1}^{\infty}$ in $\Phi := \operatorname{ran} E(-\infty, \lambda)$ and $\{\psi_j\}_{j=1}^{\infty}$ in $\Psi := \operatorname{ran} E[\lambda, \infty)$, and construct the desired orthonormal set $\{f_j\}_{j=1}^{\infty}$ from them. Thus assume that $\operatorname{ran} E(-\infty, \lambda)$ is finite-dimensional.

If $\operatorname{ran} E(\lambda, \infty)$ is infinite-dimensional, we can act as before with $\Phi := \operatorname{ran} E(-\infty, \lambda]$ and $\Psi := \operatorname{ran} E(\lambda, \infty)$. Thus assume also that $\operatorname{ran} E(\lambda, \infty)$ is finite-dimensional.

But this implies that $\operatorname{ran} E(\{\lambda\})$ is infinite-dimensional. Hence λ is an eigenvalue of infinite multiplicity and if $\{f_j\}_{j=1}^{\infty}$ is an orthonormal set of eigenvectors corresponding to λ, we have that $\langle Tf_s, f_t \rangle = \lambda \delta_{s,t}$, as desired. □

Is the equality

$$\bigcap_{n=1}^{\infty} \Lambda_n(T) = \Lambda_\infty(T)$$

true also for nonselfadjoint operators T? We conjecture it is, but we have not been able to prove it.

Note added: After this paper was submitted, Li, Poon and Sze [9] answered the above question affirmatively.

One can calculate higher-rank numerical ranges for specific operators by using the above theorem.

For example, if P is an orthogonal projection onto some subspace $\mathscr{M}$ of the infinite-dimensional Hilbert space $\mathscr{H}$, then one easily checks (directly, or by applying Theorem 3.4) that $\Lambda_n(P) = [0,1]$ if $n \leqslant \min\{\dim \mathscr{M}, \dim \mathscr{M}^\perp\}$, that $\Lambda_n(P) = \{0\}$ if $\dim \mathscr{M} < n \leqslant \dim \mathscr{M}^\perp$ and $\Lambda_n(P) = \{1\}$ if $\dim \mathscr{M}^\perp < n \leqslant \dim \mathscr{M}$.

Another application is to a diagonal selfadjoint operator T. Suppose that T has negative eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ and positive eigenvalues $\mu_1 \geqslant \mu_2 \geqslant \mu_3 \geqslant \cdots$, where all eigenvalues are repeated according to multiplicity. One easily checks that $\lambda_n(T) = [\lambda_n, \mu_n]$ for $n \in \mathbb{N}$. On the other hand, if all eigenvalues of T are positive, say $\mu_1 \geqslant \mu_2 \geqslant \mu_3 \geqslant \cdots$ and $\mu_k \to 0$, then $\Lambda_n(T) = (0, \mu_n]$ for $n \in \mathbb{N}$ and $\Lambda_\infty(T) = \emptyset$. By the way, this provides the example promised after Lemma 3.2 and also the example promised before Theorem 3.4. Other distributions of eigenvalues for T are handled similarly.

Another interesting case is the operator $M_x : L^2[0,1] \longrightarrow L^2[0,1]$ of multiplication by the independent variable. Its is easy to show using Theorem 3.4 that $\Lambda_n(M_x)$ equals the open interval $(0,1)$ for any $n \in \mathbb{N}_\infty$.

4. Nonunitary Isometries

Throughout this section, let W denote a nonunitary isometry on $\mathscr{H}$. For each $k \in \mathbb{N}$, let $\mathscr{M}_k$ be the subspace defined by

$$\mathscr{M}_k := \left(\operatorname{ran} W^k\right)^{\perp},$$

and let $\mathscr{M} := \mathscr{M}_1 = (\operatorname{ran} W)^{\perp}$. We also define P_k to be the orthogonal projection onto $\mathscr{M}_k$

The statement of following lemma is a part of the proof of the Wold decomposition. It can be found in [14, p. 3].

LEMMA 4.1. *Let W be a nonunitary isometry, let $k \in \mathbb{N}$ and let $\mathscr{M}_k$ be as above. Then,*

$$\mathscr{M}_k = \mathscr{M} \oplus W\mathscr{M} \oplus W^2\mathscr{M} \oplus \cdots \oplus W^{k-1}\mathscr{M}.$$

Observe that $\mathscr{M}$, and hence $\mathscr{M}_k$ could be finite-dimensional. The following calculation will also be useful.

LEMMA 4.2. *Let $k \in \mathbb{N}$, $k \geqslant 2$. Let $f : \mathbb{R}^k \longrightarrow \mathbb{R}$ be a function defined by*

$$f(r_0, r_1, r_2, \ldots, r_{k-1}) = r_0 r_1 + r_1 r_2 + r_2 r_3 + \cdots + r_{k-2} r_{k-1},$$

and let

$$K := \left\{ (r_0, r_1, \ldots, r_{k-1}) \in \mathbb{R}^k \; : \; r_i \geqslant 0, \sum_{j=0}^{k-1} r_j^2 = 1 \right\}.$$

Then $f(K) = [0, q_k]$, for some number q_k such that $1 - \frac{1}{k} \leqslant q_k \leqslant 1$.

Proof. Since K is compact and connected and f is continuous, it follows that $f(K)$ is a compact interval in $\mathbb{R}$. For $(r_0, r_1, \ldots, r_{k-1}) \in K$, clearly $f(r_0, r_1, \ldots, r_{k-1}) \geqslant 0$. Also, since $f(1,0,0,\ldots,0) = 0$, the interval $f(K)$ is of the form $[0, q_k]$ for some nonnegative number q_k. By the Cauchy–Schwarz inequality, we obtain

$$\sum_{j=0}^{k-2} r_j r_{j+1} \leqslant \left(\sum_{j=0}^{k-2} r_j^2\right)^{1/2} \left(\sum_{j=0}^{k-2} r_{j+1}^2\right)^{1/2} \leqslant 1,$$

for $(r_0, r_1, \ldots, r_{k-1}) \in K$. Thus $q_k \leqslant 1$. Lastly, observe that $(\frac{1}{\sqrt{k}}, \frac{1}{\sqrt{k}}, \ldots, \frac{1}{\sqrt{k}}) \in K$ and

$$f\left(\frac{1}{\sqrt{k}}, \frac{1}{\sqrt{k}}, \ldots, \frac{1}{\sqrt{k}}\right) = (k-1)\frac{1}{k} = 1 - \frac{1}{k},$$

from which it follows that $q_k \geqslant 1 - \frac{1}{k}$. $\square$

The number q_k in the lemma above can be explicitly calculated. It turns out that $q_k = \frac{1}{2}\sqrt{2+2\cos(\frac{2\pi}{k+1})}$. We will not use this expression here.

To find the higher-rank numerical range of a nonunitary isometry W, it will be useful to calculate the numerical range of some compressions of W.

PROPOSITION 4.3. *For each $k \in \mathbb{N}$, $k \geqslant 2$. Define the operator $R_k : \mathscr{M}_k \longrightarrow \mathscr{M}_k$ to be the restriction to $\mathscr{M}_k$ of the operator $P_k W$. Then the numerical range $\Lambda_1(R_k)$ contains the closed disk $\{z \in \mathbb{D} : |z| \leqslant 1 - \frac{1}{k}\}$.*

Proof. Let $\lambda \in \{z \in \mathbb{D} : |z| \leqslant 1 - \frac{1}{k}\}$. Then $\lambda = re^{i\theta}$ for some $0 \leqslant r \leqslant 1 - \frac{1}{k}$ and $\theta \in [0, 2\pi)$. By Lemma 4.2, there exists $(r_0, r_1, \ldots, r_{k-1})$, with $r_j \geqslant 0$ and $r_0^2 + r_1^2 + \cdots + r_{k-1}^2 = 1$, such that

$$r_0 r_1 + r_1 r_2 + r_2 r_3 + \cdots + r_{k-2} r_{k-1} = r.$$

Choose $\phi \in \mathscr{M}$ with $\|\phi\| = 1$ and, for each $s = 0, 1, 2, \ldots, k-1$, define the functions $f_s = r_s e^{-is\theta}\phi \in \mathscr{M}$. Define also $f = f_0 + W f_1 + W^2 f_2 + \cdots + W^{k-1} f_{k-1}$. By Lemma 4.1, the vector f is in $\mathscr{M}_k$ and

$$\|f\|^2 = \|f_0\|^2 + \|f_1\|^2 + \cdots + \|f_{k-1}\|^2 = r_0^2 + r_1^2 + \cdots + r_{k-1}^2 = 1.$$

But now

$$\langle R_k f, f \rangle = \langle P_k W f, f \rangle = \langle W f, f \rangle = \sum_{s=0}^{k-1} \sum_{t=0}^{k-1} \left\langle W^{s+1} f_s, W^t f_t \right\rangle.$$

By Lemma 4.1, since f_s and f_t are in $\mathscr{M}$, we have that $\left\langle W^{s+1} f_s, W^t f_t \right\rangle = 0$, unless $s + 1 = t$ in which case $\left\langle W^{s+1} f_s, W^t f_t \right\rangle = \langle f_s, f_t \rangle$. Thus

$$\begin{aligned}
\langle R_k f, f \rangle &= \sum_{s=0}^{k-2} \langle f_s, f_{s+1} \rangle \\
&= \sum_{s=0}^{k-2} \left\langle r_s e^{-is\theta}\phi, r_{s+1} e^{-i(s+1)\theta}\phi \right\rangle \\
&= \sum_{s=0}^{k-2} r_s r_{s+1} e^{i(-s+(s+1))\theta} \|\phi\|^2 \\
&= re^{i\theta},
\end{aligned}$$

and therefore, $\langle R_k f, f \rangle = \lambda$ for some $f \in \mathscr{M}_k$ of norm 1. $\square$

The ideas behind the above proposition and the following theorem came from the calculation of the numerical range for the unilateral shift on ℓ^2.

THEOREM 4.4. *Let $n \in \mathbb{N}_\infty$ be fixed and let W be a nonunitary isometry. Then $\mathbb{D} \subseteq \Lambda_n(W)$.*

Proof. Let $\lambda \in \mathbb{D}$. Choose k large enough such that $1-\frac{1}{k} \geqslant |\lambda|$. By Proposition 4.3, there exists $f \in \mathscr{M}_k$, $\|f\| = 1$ such that $\langle R_k f, f \rangle = \lambda$, where R_k is the operator $P_k W$ restricted to $\mathscr{M}_k$ as in the statement of that proposition. Observe that, since $f \in \mathscr{M}_k$, we have $\langle R_k f, f \rangle = \langle P_k W f, f \rangle = \langle Wf, f \rangle$ and thus $\langle Wf, f \rangle = \lambda$.

For natural $j < n+1$, define $g_j \in \mathscr{H}$ as $g_j = W^{(j-1)(k+1)} f$. We first observe that $\{g_j\}_{j=1}^n$ is an orthonormal set. Indeed

$$\langle g_s, g_s \rangle = \left\langle W^{(s-1)(k+1)} f, W^{(s-1)(k+1)} f \right\rangle = \langle f, f \rangle = 1,$$

and, for $s > t$,

$$\langle g_s, g_t \rangle = \left\langle W^{(s-1)(k+1)} f, W^{(t-1)(k+1)} f \right\rangle = \left\langle W^{(s-t)(k+1)} f, f \right\rangle = 0$$

since $f \in \mathscr{M}_k = (\operatorname{ran} W^k)^{\perp}$.

The theorem will be proved if we can show that $\langle W g_s, g_t \rangle = \lambda \delta_{s,t}$. First, observe that

$$\langle W g_s, g_s \rangle = \left\langle W W^{(s-1)(k+1)} f, W^{(s-1)(k+1)} f \right\rangle = \langle Wf, f \rangle = \lambda.$$

If $s > t$, we have

$$\begin{aligned} \langle W g_s, g_t \rangle &= \left\langle W W^{(s-1)(k+1)} f, W^{(t-1)(k+1)} f \right\rangle \\ &= \left\langle W^{(s-t)(k+1)+1} f, f \right\rangle \\ &= \left\langle W^k W^{(s-t)(k+1)+1-k} f, f \right\rangle \\ &= 0, \end{aligned}$$

since $f \in (\operatorname{ran} W^k)^{\perp}$. If $s < t$, we obtain

$$\begin{aligned} \langle W g_s, g_t \rangle &= \left\langle W W^{(s-1)(k+1)} f, W^{(t-1)(k+1)} f \right\rangle \\ &= \left\langle f, W^{(t-s)(k+1)-1} f \right\rangle \\ &= \left\langle f, W^k W^{(t-s-1)(k+1)} f \right\rangle \\ &= 0, \end{aligned}$$

since $f \in (\operatorname{ran} W^k)^{\perp}$. This finishes the proof. □

We are now ready to state our main result of this section.

THEOREM 4.5. *Let W be a nonunitary isometry and let $n \in \mathbb{N}_\infty$. Then*

$$\Lambda_n(W) = \mathbb{D} \cup \{\lambda \in S^1 \ : \ \dim \ker(W - \lambda) \geqslant n\}.$$

Proof. As proved above, $\mathbb{D} \subseteq \Lambda_n(W)$. Recall also that $\Lambda_n(W) \subseteq \Lambda_1(W)$, and that $\Lambda_1(W) \subseteq \overline{\mathbb{D}}$ since $\|W\| = 1$. Thus we have

$$\mathbb{D} \subseteq \Lambda_n(W) \subseteq \overline{\mathbb{D}}.$$

By Proposition 2.4, if $\lambda \in \Lambda_n(W)$ and $|\lambda| = 1$ then $\dim\ker(W-\lambda) \geqslant n$ thus we have

$$\Lambda_n(W) \subseteq \mathbb{D} \cup \{\lambda \in S^1 \ : \ \dim\ker(W-\lambda) \geqslant n\}.$$

But it is also clear that if $\lambda \in S^1$ is an eigenvalue of multiplicity n, then $\lambda \in \Lambda_n(W)$ and hence we obtain the reverse containment. □

An immediate corollary of the theorem above gives the higher-rank numerical range of Toeplitz operators on the Hardy-Hilbert space. The relevant definitions can be found, for example, in [12].

COROLLARY 4.6. *Let $\mathbf{H}^2$ be the classical Hardy–Hilbert space, let θ be a nonconstant inner function and let T_θ be the Toeplitz operator on $\mathbf{H}^2$ with symbol θ. Then, $\Lambda_n(T_\theta) = \mathbb{D}$ for all $n \in \mathbb{N}_\infty$.*

Proof. It is easily verified that T_ϕ is a nonunitary isometry and it is well known that analytic Toeplitz operators do not have eigenvalues. □

Let us close with one more consequence of the above theorem. It is easily seen that if $\mathscr{M}$ is an invariant subspace for an operator $T \in \mathbf{B}(\mathscr{H})$, then

$$\Lambda_n\big(T|_{\mathscr{M}}\big) \subseteq \Lambda_n(T).$$

In particular, if T is a unitary operator and $\mathscr{M}$ is a nonreducing invariant subspace, then $T|_{\mathscr{M}}$ is a nonunitary isometry and hence Theorem 4.5 implies that $\mathbb{D} \subseteq \Lambda_n(T)$ for all $n \in \mathbb{N}_\infty$.

A normal operator such that all its invariant subspaces are reducing is called *completely normal* (see [13, p. 22]). It can be shown that a unitary operator W is not completely normal if and only if there exists a reducing subspace $\mathscr{M}$ such that $W|_{\mathscr{M}}$ is a bilateral shift. See [15] (or consult [13, p. 24]).

The above two paragraphs imply the following result.

COROLLARY 4.7. *Let W be a unitary operator. Suppose there exists a reducing subspace $\mathscr{M}$ such that $W|_{\mathscr{M}}$ is a bilateral shift. Then*

$$\Lambda_n(W) = \mathbb{D} \cup \{\lambda \in S^1 \ : \ \dim\ker(W-\lambda) \geqslant n\},$$

for all $n \in \mathbb{N}_\infty$.

In particular, if W is the bilateral shift on $\ell^2(\mathbb{Z})$, we have $\lambda_n(W) = \mathbb{D}$ for all $n \in \mathbb{N}_\infty$.

Acknowledgements. The author would like to thank D. Kribs for a very stimulating conversation on the higher-rank numerical ranges while he visited Pachuca in January 2006. Also, thanks to an unknown referee for pointing out several references.

REFERENCES

[1] MAN-DUEN CHOI, MICHAEL GIESINGER, JOHN A. HOLBROOK AND DAVID W. KRIBS, *Geometry of higher-rank numerical ranges*, Linear and Multilinear Algebra **56** (2008), 53-64.

[2] MAN-DUEN CHOI, JOHN A. HOLBROOK, DAVID W. KRIBS, AND KAROL ŻYCZKOWSKI, *Higher-rank numerical ranges of unitary and normal matrices*, Operators and Matrices **1** (2007) 409–426.

[3] MAN-DUEN CHOI, DAVID W. KRIBS AND KAROL ŻYCZKOWSKI, *Higher-rank numerical ranges and compression problems*, Linear Algebra Appl. **418** (2006) 828–839.

[4] JOHN B. CONWAY, *A Course in Functional Analysis*, second edition, Springer, New York, 1990.

[5] DOUGLAS R. FARENICK, *Matricial extensions of the numerical range: a brief survey*, Linear Multilinear Algebra **34** (1993) 197–211.

[6] HWA-LONG GAU, CHI-KWONG LI AND PEI YUAN WU, *Higher-rank numerical ranges and dilations*, J. Operator Theory, to appear.

[7] PAUL R. HALMOS, *A Hilbert Space Problem Book*, second edition, Springer, New York, 1982.

[8] CHI-KWONG LI, YIU-TUNG POON AND NUNG-SING SZE, *Condition for the higher rank numerical range to be non-empty*, Linear and Multilinear Algebra, to appear.

[9] CHI-KWONG LI, YIU-TUNG POON AND NUNG-SING SZE, *Higher rank numerical ranges and low rank perturbations of quantum channels*, preprint.

[10] CHI-KWONG LI AND NUNG-SING SZE, *Canonical forms, higher rank numerical ranges, totally isomorphic subspaces and matrix equations*, Proc. Amer. Math. Soc., to appear.

[11] CHI-KWONG LI AND NAM-KIU TSING, *On the k-th matrix numerical range*, Linear Multilinear Algebra **28** (1991) 229–239.

[12] RUBÉN A. MARTÍNEZ-AVENDAÑO AND PETER ROSENTHAL, *An Introduction to Operators on the Hardy–Hilbert Space*, Springer, New York, 2007.

[13] PETER ROSENTHAL AND HEYDAR RADJAVI, *Invariant Subspaces*, second edition, Dover, Mineola, NY, 2002.

[14] BÉLA SZ.-NAGY AND CIPRIAN FOIAŞ, *Harmonic Analysis of Operators in Hilbert Space*, North-Holland, Amsterdam, 1970.

[15] J. WERMER, *On invariant subspaces of normal operators*, Proc. Amer. Math. Soc. **3** (1952) 270–277.

[16] HUGO J. WOERDEMAN, *The higher rank numerical range is convex*, Linear Multilinear Algebra, **56** (2008) 65–67.

(Received September 15, 2007)

Rubén A. Martínez-Avendaño
Centro de Investigación en Matemáticas
Universidad Autónoma del Estado de Hidalgo
Ciudad Universitaria
Carr. Pachuca–Tulancingo, km 4.5
Pachuca, Hidalgo
42184, Mexico
e-mail: rubenma@uaeh.edu.mx