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STIELTJES LIKE FUNCTIONS AND INVERSE PROBLEMS FOR SYSTEMS WITH SCHRÖDINGER OPERATOR

SERGEY BELYI AND EDUARD TSEKANOVSKIĬ

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Abstract. A class of scalar Stieltjes like functions is realized as linear-fractional transformations of transfer functions of conservative systems based on a Schrödinger operator T_h in $L_2[a,+\infty)$ with a non-selfadjoint boundary condition. In particular it is shown that any Stieltjes function of this class can be realized in the unique way so that the main operator $\mathbb{A}$ of a system is an accretive $(*)$-extension of a Schrödinger operator T_h. We derive formulas that restore the system uniquely and allow to find the exact value of a non-real parameter h in the definition of T_h as well as a real parameter μ that appears in the construction of the elements of the realizing system. An elaborate investigation of these formulas shows the dynamics of the restored parameters h and μ in terms of the changing free term γ from the integral representation of the realizable function. It turns out that the parametric equations for the restored parameter h represent different circles whose centers and radii are determined by the realizable function. Similarly, the behavior of the restored parameter μ are described by hyperbolas.

1. Introduction

Realizations of different classes of holomorphic operator-valued functions in the open right half-plane, unit circle, and upper half-plane, as well as inverse spectral problems, play an important role in the spectral analysis of non-self-adjoint operators, interpolation problems, and system theory. The literature on realization theory and inverse spectral problems is too extensive to be discussed exhaustively in this note. We refer, however, to [2], [3], [7], [8], [9], [10], [11], [12], [18], [21], [24], [28] and the literature therein. A class of Herglotz-Nevanlinna functions is a rich source for many types of realization problems. An operator-valued function $V(z)$ acting on a finite-dimensional Hilbert space E belongs to the class of operator-valued Herglotz-Nevanlinna functions if it is holomorphic on $\mathbb{C}\setminus\mathbb{R}$, if it is symmetric with respect to the real axis, i.e., $V(z)^*=V(\bar z)$, $z\in\mathbb{C}\setminus\mathbb{R}$, and if it satisfies the positivity condition

$$\operatorname{Im} V(z)\geqslant 0,\quad z\in\mathbb{C}_+.$$

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It is well known (see e.g. [16], [17]) that operator-valued Herglotz-Nevanlinna functions admit the following integral representation:

$$V(z) = Q + Lz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) dG(t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{1.1}$$

where $Q = Q^*$, $L \geqslant 0$, and $G(t)$ is a nondecreasing operator-valued function on $\mathbb{R}$ with values in the class of nonnegative operators in E such that

$$\int_{\mathbb{R}} \frac{(dG(t)x,x)_E}{1+t^2} < \infty, \quad x \in E. \tag{1.2}$$

The realization of a selected class of Herglotz-Nevanlinna functions is provided by a linear conservative system Θ of the form

$$\begin{cases} (\mathbb{A} - zI)x = KJ\varphi_- \\ \varphi_+ = \varphi_- - 2iK^*x \end{cases} \tag{1.3}$$

or

$$\Theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & E \end{pmatrix}. \tag{1.4}$$

In this system $\mathbb{A}$, the *main operator* of the system, is a so-called $(*)$-extension, which is a bounded linear operator from $\mathcal{H}_+$ into $\mathcal{H}_-$ extending a symmetric operator A in $\mathcal{H}$, where $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ is a rigged Hilbert space. Moreover, K is a bounded linear operator from the finite-dimensional Hilbert space E into $\mathcal{H}_-$, while $J = J^* = J^{-1}$ is acting on E, are such that $\operatorname{Im} \mathbb{A} = KJK^*$. Also, $\varphi_- \in E$ is an input vector, $\varphi_+ \in E$ is an output vector, and $x \in \mathcal{H}_+$ is a vector of the state space of the system Θ. The system described by (1.3)-(1.4) is called a rigged canonical system of the Livšic type [22] or the Brodskiĭ-Livšic rigged operator colligation, cf., e.g. [11], [12], [13]. The operator-valued function

$$W_\Theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}KJ \tag{1.5}$$

is a transfer function (or characteristic function) of the system Θ. It was shown in [11] that an operator-valued function $V(z)$ acting on a Hilbert space E of the form (1.1) can be represented and realized in the form

$$V(z) = i[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I] = K^*(\mathbb{A}_R - zI)^{-1}K, \tag{1.6}$$

where $W_\Theta(z)$ is a transfer function of some canonical scattering ($J = I$) system Θ, and where the "*real part*" $\mathbb{A}_R = \frac{1}{2}(\mathbb{A} + \mathbb{A}^*)$ of $\mathbb{A}$ satisfies $\mathbb{A}_R \supset \hat{A} = \hat{A}^* \supset A$ if and only if the function $V(z)$ in (1.1) satisfies the following two conditions:

$$\begin{cases} L = 0, \\ Qx = \int_{\mathbb{R}} \frac{t}{1+t^2}\, dG(t)x \quad \text{when} \quad \int_{\mathbb{R}} (dG(t)x,x)_E < \infty. \end{cases} \tag{1.7}$$

In the current paper we are going to focus on an important subclass of Herglotz-Nevanlinna functions, the so called Stieltjes like functions that also includes Stieltjes functions. In Section 4 we specify a subclass of realizable Stieltjes operator-functions and show that any member of this subclass can be realized by a system of the form (1.4) whose main operator $\mathbb{A}$ is accretive.

In Section 5 we introduce a class of Stieltjes like scalar functions. Then we rely on the general realization results developed in Section 4 (see also [15]) to restore a system Θ of the form (1.4) containing the Schrödinger operator in $L_2[a,+\infty)$ with non-self-adjoint boundary conditions

$$\begin{cases} T_h y = -y'' + q(x)y \\ y'(a) = hy(a) \end{cases}, \quad \left(q(x) = \overline{q(x)},\, \operatorname{Im} h \neq 0\right).$$

We show that if a non-decreasing function $\sigma(t)$ is the spectral distribution function of positive self-adjoint boundary value problem

$$\begin{cases} A_\theta y = -y'' + q(x)y \\ y'(a) = \theta y(a) \end{cases}$$

and satisfies conditions

$$\int_0^\infty d\sigma(t) = \infty, \qquad \int_0^\infty \frac{d\sigma(t)}{1+t} < \infty,$$

then for every real γ a Stieltjes like function

$$V(z) = \gamma + \int_0^\infty \frac{d\sigma(t)}{t-z}$$

can be realized in the unique way as a $V_\Theta(z)$ function of a rigged canonical system Θ containing some Schrödinger operator T_h. In particular, it is shown that for every $\gamma \geqslant 0$ a Stieltjes function $V(z)$ with integral representation above can be realized by a system Θ whose main operator $\mathbb{A}$ is an accretive $(*)$-extension of a Schrödinger operator T_h.

On top of the general realization results, Section 5 provides the reader with formulas that allow to find the exact value of a non-real parameter h in the definition of T_h of the realizing system Θ. Similar investigation is presented in Section 6 to describe the real parameter μ that appears in the construction of the elements of the realizing system. A detailed study of these formulas shows the dynamics of the restored parameters h and μ in terms of a changing free term γ in the integral representation of $V(z)$ above. It will be shown and graphically presented that the parametric equations for the restored parameter h represent different circles whose centers and radii are completely determined by the function $V(z)$. Similarly, the behavior of the restored parameter μ are described by hyperbolas.

2. Some preliminaries

For a pair of Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$ we denote by $[\mathscr{H}_1, \mathscr{H}_2]$ the set of all bounded linear operators from $\mathscr{H}_1$ to $\mathscr{H}_2$. Let A be a closed, densely defined, symmetric operator in a Hilbert space $\mathscr{H}$ with inner product $(f,g), f,g \in \mathscr{H}$. Consider the rigged Hilbert space

$$\mathscr{H}_+ \subset \mathscr{H} \subset \mathscr{H}_-,$$

where $\mathscr{H}_+ = D(A^*)$ and

$$(f,g)_+ = (f,g) + (A^*f, A^*g), \quad f,g \in D(A^*).$$

Note that identifying the space conjugate to $\mathscr{H}_\pm$ with $\mathscr{H}_\mp$, we get that if $\mathbb{A} \in [\mathscr{H}_+, \mathscr{H}_-]$ then $\mathbb{A}^* \in [\mathscr{H}_+, \mathscr{H}_-]$.

DEFINITION 2.1. An operator $\mathbb{A} \in [\mathscr{H}_+, \mathscr{H}_-]$ is called a self-adjoint bi-extension of a symmetric operator A if $\mathbb{A} = \mathbb{A}^*$, $\mathbb{A} \supset A$, and the operator

$$\widehat{A}f = \mathbb{A}f,\ f \in D(\widehat{A}) = \{f \in \mathscr{H}_+ : \mathbb{A}f \in \mathscr{H}\}$$

is self-adjoint in $\mathscr{H}$.

The operator $\widehat{A}$ in the above definition is called a *quasi-kernel* of a self-adjoint bi-extension $\mathbb{A}$ (see [27]).

DEFINITION 2.2. An operator $\mathbb{A} \in [\mathscr{H}_+, \mathscr{H}_-]$ is called a $(*)$-extension (or correct bi-extension) of an operator T (with non-empty set $\rho(T)$ of regular points) if

$$\mathbb{A} \supset T \supset A, \mathbb{A}^* \supset T^* \supset A$$

and the operator $\mathbb{A}_R = \frac{1}{2}(\mathbb{A} + \mathbb{A}^*)$ is a self-adjoint bi-extension of an operator A.

The existence, description, and analog of von Neumann's formulas for self-adjoint bi-extensions and $(*)$-extensions were discussed in [27] (see also [4], [5], [11]). For instance, if Φ is an isometric operator from the defect subspace $\mathfrak{N}_i$ of the symmetric operator A onto the defect subspace $\mathfrak{N}_{-i}$, then the formulas below establish a one-to one correspondence between $(*)$-extensions of an operator T and Φ

$$\mathbb{A}f = A^*f + iR(\Phi - I)x,\ \mathbb{A}^*f = A^*f + iR(\Phi - I)y, \tag{2.1}$$

where $x, y \in \mathfrak{N}_i$ are uniquely determined from the conditions

$$f - (\Phi + I)x \in D(T),\ f - (\Phi + I)y \in D(T^*)$$

and R is the Riesz-Berezanskii operator of the triplet $\mathscr{H}_+ \subset \mathscr{H} \subset \mathscr{H}_-$ that maps $\mathscr{H}_+$ isometrically onto $\mathscr{H}_-$ (see [27]). If the symmetric operator A has deficiency indices (n,n), then formulas (2.1) can be rewritten in the following form

$$\mathbb{A}f = A^*f + \sum_{k=1}^{n} \Delta_k(f) V_k, \quad \mathbb{A}^*f = A^*f + \sum_{k=1}^{n} \delta_k(f) V_k, \tag{2.2}$$

where $\{V_j\}_1^n \in \mathscr{H}_-$ is a basis in the subspace $R(\Phi - I)\mathfrak{N}_i$, and $\{\Delta_k\}_1^n$, $\{\delta_k\}_1^n$, are bounded linear functionals on $\mathscr{H}_+$ with the properties

$$\Delta_k(f) = 0, \quad \forall f \in D(T), \qquad \delta_k(f) = 0, \quad \forall f \in D(T^*). \tag{2.3}$$

Let $\mathscr{H} = L_2[a,+\infty)$ and $l(y) = -y'' + q(x)y$ where q is a real locally summable function. Suppose that the symmetric operator

$$\begin{cases} Ay = -y'' + q(x)y \\ y(a) = y'(a) = 0 \end{cases} \tag{2.4}$$

has deficiency indices $(1,1)$. Let D^* be the set of functions locally absolutely continuous together with their first derivatives such that $l(y) \in L_2[a,+\infty)$. Consider $\mathscr{H}_+ = D(A^*) = D^*$ with the scalar product

$$(y,z)_+ = \int_a^\infty \left(y(x)\overline{z(x)} + l(y)\overline{l(z)} \right) dx, \quad y,\ z \in D^*.$$

Let

$$\mathscr{H}_+ \subset L_2[a,+\infty) \subset \mathscr{H}_-$$

be the corresponding triplet of Hilbert spaces. Consider operators

$$\begin{cases} T_h y = l(y) = -y'' + q(x)y \\ hy(a) - y'(a) = 0 \end{cases}, \qquad \begin{cases} T_h^* y = l(y) = -y'' + q(x)y \\ \overline{h}y(a) - y'(a) = 0 \end{cases}, \tag{2.5}$$

$$\begin{cases} \widehat{A}y = l(y) = -y'' + q(x)y \\ \mu y(a) - y'(a) = 0 \end{cases}, \qquad \operatorname{Im}\mu = 0.$$

It is well known [1] that $\widehat{A} = \widehat{A^*}$. The following theorem was proved in [6].

THEOREM 2.3. *The set of all* $(*)$*-extensions of a non-self-adjoint Schrödinger operator* T_h *of the form* (2.5) *in* $L_2[a,+\infty)$ *can be represented in the form*

$$\begin{aligned} \mathbb{A}y &= -y'' + q(x)y - \frac{1}{\mu - h}\,[y'(a) - hy(a)]\,[\mu\delta(x-a) + \delta'(x-a)], \\ \mathbb{A}^*y &= -y'' + q(x)y - \frac{1}{\mu - \overline{h}}\,[y'(a) - \overline{h}y(a)]\,[\mu\delta(x-a) + \delta'(x-a)]. \end{aligned} \tag{2.6}$$

In addition, the formulas (2.6) *establish a one-to-one correspondence between the set of all* $(*)$*-extensions of a Schrödinger operator* T_h *of the form* (2.5) *and all real numbers* $\mu \in [-\infty,+\infty]$.

DEFINITION 2.4. An operator T with the domain $D(T)$ and $\rho(T) \neq \emptyset$ acting on a Hilbert space $\mathscr{H}$ is called *accretive* if

$$\operatorname{Re}(Tf,f) \geqslant 0, \quad \forall f \in D(T).$$

DEFINITION 2.5. An accretive operator T is called [20] α*-sectorial* if there exists a value of $\alpha \in (0, \pi/2)$ such that

$$\cot \alpha \, |\mathrm{Im}\,(Tf, f)| \leqslant \mathrm{Re}\,(Tf, f), \qquad f \in \mathscr{D}(T).$$

An accretive operator is called *extremal accretive* if it is not α-sectorial for any $\alpha \in (0, \pi/2)$.

Consider the symmetric operator A of the form (2.4) with defect indices (1,1), generated by the differential operation $l(y) = -y'' + q(x)y$. Let $\varphi_k(x, \lambda)$ $(k = 1, 2)$ be the solutions of the following Cauchy problems:

$$\begin{cases} l(\varphi_1) = \lambda \varphi_1 \\ \varphi_1(a, \lambda) = 0 \\ \varphi_1'(a, \lambda) = 1 \end{cases}, \qquad \begin{cases} l(\varphi_2) = \lambda \varphi_2 \\ \varphi_2(a, \lambda) = -1 \\ \varphi_2'(a, \lambda) = 0 \end{cases}.$$

It is well known [1] that there exists a function $m_\infty(\lambda)$ (called the Weyl-Titchmarsh function) for which

$$\varphi(x, \lambda) = \varphi_2(x, \lambda) + m_\infty(\lambda)\varphi_1(x, \lambda)$$

belongs to $L_2[a, +\infty)$.

Suppose that the symmetric operator A of the form (2.4) with deficiency indices (1,1) is nonnegative, i.e., $(Af, f) \geqslant 0$ for all $f \in D(A))$. It was shown in [25] that the Schrödinger operator T_h of the form (2.5) is accretive if and only if

$$\mathrm{Re}\, h \geqslant -m_\infty(-0). \tag{2.7}$$

For real h such that $h \geqslant -m_\infty(-0)$ we get a description of all nonnegative self-adjoint extensions of an operator A. For $h = -m_\infty(-0)$ the corresponding operator

$$\begin{cases} A_K\, y = -y'' + q(x)y \\ y'(a) + m_\infty(-0)y(a) = 0 \end{cases} \tag{2.8}$$

is the Kreĭn-von Neumann extension of A and for $h = +\infty$ the corresponding operator

$$\begin{cases} A_F\, y = -y'' + q(x)y \\ y(a) = 0 \end{cases} \tag{2.9}$$

is the Friedrichs extension of A (see [25], [6]).

3. Rigged canonical systems with Schrödinger operator

Let $\mathbb{A}$ be $(*)$ - extension of an operator T, i.e.,

$$\mathbb{A} \supset T \supset A, \quad \mathbb{A}^* \supset T^* \supset A$$

where A is a symmetric operator with deficiency indices (n, n) and $D(A) = D(T) \cap D(T^*)$. In what follows we will only consider the case when the symmetric operator A has dense domain, i.e., $\overline{\mathscr{D}(A)} = \mathscr{H}$.

DEFINITION 3.1. A system of equations

$$\begin{cases} (\mathbb{A}-zI)x = KJ\varphi_- \\ \varphi_+ = \varphi_- - 2iK^*x \end{cases},$$

or an array

$$\Theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & E \end{pmatrix} \tag{3.1}$$

is called a *rigged canonical system of the Livsic type* or the *Brodskiĭ-Livsic rigged operator colligation* if:

1) E is a finite-dimensional Hilbert space with scalar product $(\cdot,\cdot)_E$ and the operator J in this space satisfies the conditions $J = J^* = J^{-1}$,
2) $K \in [E, \mathcal{H}_-]$, $\ker K = \{0\}$,
3) $\operatorname{Im}\mathbb{A} = KJK^*$, where $K^* \in [\mathcal{H}_+, E]$ is the adjoint of K.

In the definition above $\varphi_- \in E$ stands for an input vector, $\varphi_+ \in E$ is an output vector, and x is a state space vector in $\mathcal{H}$. An operator $\mathbb{A}$ is called a *main operator* of the system Θ, J is a *direction operator*, and K is a *channel operator*. An operator-valued function

$$W_\Theta(\lambda) = I - 2iK^*(\mathbb{A}-\lambda I)^{-1}KJ \tag{3.2}$$

defined on the set $\rho(T)$ of regular points of an operator T is called the *transfer function* (*characteristic function*) of the system Θ, i.e., $\varphi_+ = W_\Theta(\lambda)\varphi_-$. It is known [25],[27] that any $(*)$-extension $\mathbb{A}$ of an operator T $(A^* \supset T \supset A)$, where A is a symmetric operator with deficiency indices (n,n) $(n < \infty)$, $D(A) = D(T) \cap D(T^*)$, can be included as a main operator of some rigged canonical system with $\dim E < \infty$ and invertible channel operator K.

It was also established [25], [27] that

$$V_\Theta(\lambda) = K^*(\operatorname{Re}\mathbb{A} - \lambda I)^{-1}K \tag{3.3}$$

is a Herglotz-Nevanlinna operator-valued function acting on a Hilbert space E, satisfying the following relation for $\lambda \in \rho(T)$, $\operatorname{Im}\lambda \neq 0$

$$V_\Theta(\lambda) = i[W_\Theta(\lambda) - I][W_\Theta(\lambda) + I]^{-1}J. \tag{3.4}$$

Alternatively,

$$\begin{aligned} W_\Theta(\lambda) &= (I + iV_\Theta(\lambda)J)^{-1}(I - iV_\Theta(\lambda)J) \\ &= (I - iV_\Theta(\lambda)J)(I + iV_\Theta(\lambda)J)^{-1}. \end{aligned} \tag{3.5}$$

Let us recall (see [27], [6]) that a symmetric operator with dense domain $\mathcal{D}(A)$ is called *prime* if there is no reducing, nontrivial invariant subspace on which A induces a self-adjoint operator. It was established in [26] that a symmetric operator A is prime if and only if

$$\underset{\lambda \neq \bar{\lambda}}{c.l.s.}\, \mathfrak{N}_\lambda = \mathcal{H}. \tag{3.6}$$

We call a rigged canonical system of the form (3.1) *prime* if

$$\underset{\lambda \neq \bar{\lambda},\, \lambda \in \rho(T)}{c.l.s.}\, \mathfrak{N}_\lambda = \mathcal{H}.$$

One easily verifies that if system Θ is prime, then a symmetric operator A of the system is prime as well.

The following theorem [6] establishes the connection between two rigged canonical systems with equal transfer functions.

THEOREM 3.2. *Let* $\Theta_1 = \begin{pmatrix} \mathbb{A}_1 & K_1 & J \\ \mathcal{H}_{+1} \subset \mathcal{H}_1 \subset \mathcal{H}_{-1} & & E \end{pmatrix}$ *and* $\Theta_2 = \begin{pmatrix} \mathbb{A}_2 & K_2 & J \\ \mathcal{H}_{+2} \subset \mathcal{H}_2 \subset \mathcal{H}_{-2} & & E \end{pmatrix}$ *be two prime rigged canonical systems of the Livsic type with*

$$\begin{aligned} &\mathbb{A}_1 \supset T_1 \supset A_1, \quad \mathbb{A}_1^* \supset T_1^* \supset A_1, \\ &\mathbb{A}_2 \supset T_2 \supset A_2, \quad \mathbb{A}_2^* \supset T_2^* \supset A_2, \end{aligned} \tag{3.7}$$

and such that A_1 *and* A_2 *have finite and equal defect indices.*

If

$$W_{\Theta_1}(\lambda) = W_{\Theta_2}(\lambda), \tag{3.8}$$

then there exists an isometric operator U *from* $\mathcal{H}_1$ *onto* $\mathcal{H}_2$ *such that* $U_+ = U|_{\mathcal{H}_{+1}}$ *is an isometry*[1] *from* $\mathcal{H}_{+1}$ *onto* $\mathcal{H}_{+2}$, $U_-^* = U_+^*$ *is an isometry from* $\mathcal{H}_{-1}$ *onto* $\mathcal{H}_{-2}$, *and*

$$UT_1 = T_2U, \quad \mathbb{A}_2 = U_-\mathbb{A}_1U_+^{-1}, \quad U_-K_1 = K_2. \tag{3.9}$$

COROLLARY 3.3. *Let* Θ_1 *and* Θ_2 *be the two prime systems from the statement of theorem 3.2. Then the mapping* U *described in the conclusion of the theorem is unique.*

Proof. First let us make an observation that if $\Theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & E \end{pmatrix}$ is a prime rigged canonical system such that $U_-\mathbb{A} = \mathbb{A}U_+$ and $U_-K = K$, where U is an isometry mapping described in theorem 3.2, then $U = I$. Indeed, it is well known [27] that

$$(\mathrm{Re}\,\mathbb{A} - \lambda I)^{-1}KE = \mathfrak{N}_\lambda. \tag{3.10}$$

We have

$$\begin{aligned} U(\mathrm{Re}\,\mathbb{A} - \lambda I)^{-1}Ke &= U_+(\mathrm{Re}\,\mathbb{A} - \lambda I)^{-1}Ke \\ &= (\mathrm{Re}\,\mathbb{A} - \lambda I)^{-1}U_-Ke \\ &= (\mathrm{Re}\,\mathbb{A} - \lambda I)^{-1}Ke, \qquad \forall e \in E,\ \lambda \neq \bar{\lambda}. \end{aligned}$$

Combining the above equation with (3.6) and (3.10) we obtain $U = I$.

Now let Θ_1 and Θ_2 be the two prime systems from the statement of theorem 3.2. Suppose there are two isometric mappings U_1 and U_2 guaranteed by theorem 3.2. Then the relations

$$\mathbb{A}_2 = U_{-,1}\mathbb{A}_1U_{+,1}^{-1}, \quad U_{-,1}K_1 = K_2, \quad \mathbb{A}_2 = U_{-,2}\mathbb{A}_1U_{+,2}^{-1}, \quad U_{-,2}K_1 = K_2,$$

lead to

$$\mathbb{A}_1U_{+,1}^{-1}U_{+,2} = U_{-,1}^{-1}U_{-,2}\mathbb{A}_1, \qquad U_{-,1}^{-1}U_{-,2}K = K.$$

[1] It was shown in [6] that the operator U_+ defined this way is an isometry from $\mathcal{H}_{+1}$ onto $\mathcal{H}_{+2}$. It is also shown there that the isometric operator $U^* : \mathcal{H}_{+2} \to \mathcal{H}_{+1}$ uniquely defines operator $U_- = (U^*)^* : \mathcal{H}_{-1} \to \mathcal{H}_{-2}$.

Since Θ_1 is prime then $U_1^{-1}U_2 = I$ and hence $U_1 = U_2$. This proves the uniqueness of U. □

Now we shall construct a rigged canonical system based on a non-self-adjoint Schrödinger operator. One can easily check that the $(*)$-extension

$$\mathbb{A}y = -y'' + q(x)y - \frac{1}{\mu - h}[y'(a) - hy(a)][\mu\delta(x-a) + \delta'(x-a)], \quad \operatorname{Im} h > 0$$

of the non-self-adjoint Schrödinger operator T_h of the form (2.5) satisfies the condition

$$\operatorname{Im}\mathbb{A} = \frac{\mathbb{A} - \mathbb{A}^*}{2i} = (.,g)g, \tag{3.11}$$

where

$$g = \frac{(\operatorname{Im} h)^{\frac{1}{2}}}{|\mu - h|}[\mu\delta(x-a) + \delta'(x-a)] \tag{3.12}$$

and $\delta(x-a), \delta'(x-a)$ are the delta-function and its derivative at the point a. Moreover,

$$(y,g) = \frac{(\operatorname{Im} h)^{\frac{1}{2}}}{|\mu - h|}[\mu y(a) - y'(a)], \tag{3.13}$$

where

$$y \in \mathscr{H}_+, g \in \mathscr{H}_-, \mathscr{H}_+ \subset L_2(a,+\infty) \subset \mathscr{H}_-$$

and the triplet of Hilbert spaces is as discussed in theorem 2.3. Let $E = \mathbb{C}$, $Kc = cg$ $(c \in \mathbb{C})$. It is clear that

$$K^*y = (y,g), \quad y \in \mathscr{H}_+ \tag{3.14}$$

and $\operatorname{Im}\mathbb{A} = KK^*$. Therefore, the array

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathscr{H}_+ \subset L_2[a,+\infty) \subset \mathscr{H}_- & & \mathbb{C} \end{pmatrix} \tag{3.15}$$

is a rigged canonical system with the main operator $\mathbb{A}$ of the form (2.6), the direction operator $J = 1$ and the channel operator K of the form (3.14). Our next logical step is finding the transfer function of (3.15). It was shown in [6] that

$$W_\Theta(\lambda) = \frac{\mu - h}{\mu - \overline{h}}\,\frac{m_\infty(\lambda) + \overline{h}}{m_\infty(\lambda) + h}, \tag{3.16}$$

and

$$V_\Theta(\lambda) = \frac{(m_\infty(\lambda) + \mu)\operatorname{Im} h}{(\mu - \operatorname{Re} h)\, m_\infty(\lambda) + \mu\operatorname{Re} h - |h|^2}. \tag{3.17}$$

4. Realization of Stieltjes functions

Let E be a finite-dimensional Hilbert space. The scalar versions of the following definition can be found in [19].

DEFINITION 4.1. We will call an operator-valued Herglotz-Nevanlinna function $V(z) \in [E,E]$ by a *Stieltjes function* if $V(z)$ admits the following integral representation

$$V(z) = \gamma + \int_0^\infty \frac{dG(t)}{t-z}, \tag{4.1}$$

where $\gamma \geqslant 0$ and $G(t)$ is a non-decreasing on $[0,+\infty)$ operator-valued function such that

$$\int_0^\infty \frac{(dG(t)e,e)_E}{1+t} < \infty, \quad \forall e \in E.$$

Alternatively (see [19]) an operator-valued function $V(z)$ is Stieltjes if it is holomorphic in $\operatorname{Ext}[0,+\infty)$ and

$$\frac{\operatorname{Im}[zV(z)]}{\operatorname{Im} z} \geqslant 0. \tag{4.2}$$

The theorem 4.2 below was stated in [14], [15] and we present its proof for the convenience of a reader.

THEOREM 4.2. *Let Θ be a prime system of the form* (3.1). *Then an operator-valued function $V_\Theta(z)$ defined by* (3.3), (3.4) *is a Stieltjes function if and only if the main operator $\mathbb{A}$ of the system Θ is accretive.*

Proof. Let us assume first that $\mathbb{A}$ is an accretive operator, i.e. $(\operatorname{Re}\mathbb{A}x,x) \geqslant 0$, for all $x \in \mathcal{H}_+$. Let $\{z_k\}$ $(k=1,...,n)$ be a sequence of non-real complex numbers and h_k be a sequence of vectors in E. Let us denote

$$Kh_k = \delta_k, \qquad x_k = (\operatorname{Re}\mathbb{A} - z_k I)^{-1}\delta_k, \qquad x = \sum_{k=1}^n x_k. \tag{4.3}$$

Since $(\operatorname{Re}\mathbb{A}x,x) \geqslant 0$, we have

$$\sum_{k,l=1}^n (\operatorname{Re}\mathbb{A}x_k, x_l) \geqslant 0. \tag{4.4}$$

By formal calculations one can have

$$\operatorname{Re}\mathbb{A}x_k = \delta_k + z_k(\operatorname{Re}\mathbb{A} - z_k I)^{-1}\delta_k,$$

and

$$\sum_{k,l=1}^n (\operatorname{Re}\mathbb{A}x_k, x_l) = \sum_{k,l=1}^n \left[(\delta_k, (\operatorname{Re}\mathbb{A}-z_l I)^{-1}\delta_l) + (z_k(\operatorname{Re}\mathbb{A}-z_k I)^{-1}\delta_k, (\operatorname{Re}\mathbb{A}-z_k I)^{-1}\delta_l)\right].$$

Using obvious equalities

$$\big((\mathrm{Re}\,\mathbb{A} - z_k I)^{-1} K h_k, K h_l\big) = \big(V_\theta(z_k) h_k, h_l\big)_E,$$

and

$$\big((\mathrm{Re}\,\mathbb{A} - \bar{z}_l I)^{-1} (\mathrm{Re}\,\mathbb{A} - z_k I)^{-1} K h_k, K h_l\big) = \left(\frac{V_\theta(z_k) - V_\theta(\bar{z}_l)}{z_k - \bar{z}_l} h_k, h_l\right)_E,$$

we obtain

$$\sum_{k,l=1}^{n} (\mathrm{Re}\,\mathbb{A} x_k, x_l) = \sum_{k,l=1}^{n} \left(\frac{z_k V_\theta(z_k) - \bar{z}_l V_\theta(\bar{z}_l)}{z_k - \bar{z}_l} h_k, h_l\right)_E \geqslant 0. \tag{4.5}$$

The choice of z_k was arbitrary, which means that $V_\Theta(z)$ is a Stieltjes function (see [3]).

Now we prove necessity. Since Θ is a prime system then A is a prime symmetric operator. Then the equivalence of (4.5) and (4.4) implies that $(\mathrm{Re}\,\mathbb{A}x, x) \geqslant 0$ for any x from c.l.s. $\{\mathfrak{N}_z\}$, $z \neq \bar{z}$. As we have already mentioned above, a symmetric operator A with the equal deficiency indices is prime if and only if for all $\lambda \neq \bar{\lambda}$

$$\text{c.l.s.}\,\{\mathfrak{N}_\lambda\} = \mathscr{H}.$$

Therefore we can conclude that $(\mathrm{Re}\,\mathbb{A}x, x) \geqslant 0$ for any $x \in \mathscr{H}_+$ and hence $\mathbb{A}$ is an accretive operator. □

A system Θ of the form (3.1) is called an *accretive system* if its main operator $\mathbb{A}$ is accretive.

Now we define a certain class $S_0(R)$ of realizable Stieltjes functions. At this point we need to note that since Stieltjes functions form a subset of Herglotz-Nevanlinna functions then we can utilize the conditions (1.7) to form a *class* $S(R)$ of all *realizable Stieltjes functions* (see also [15]). Clearly, $S(R)$ is a subclass of $N(R)$ of all realizable Herglotz-Nevanlinna functions described in details in [11] and [12]. To see the specifications of the class $S(R)$ we recall that aside of integral representation (4.1), any Stieltjes function admits a representation (1.1). Applying condition (1.7) we obtain

$$Q = \frac{1}{2}\left[V_\theta(-i) + V_\theta^*(-i)\right] = \gamma + \int_0^{+\infty} \frac{t}{1+t^2} dG(t). \tag{4.6}$$

Combining the second part of condition (1.7) and (4.6) we conclude that

$$\gamma e = 0, \tag{4.7}$$

for all $e \in E$ such that

$$\int_0^{\infty} (dG(t)e, e)_E < \infty. \tag{4.8}$$

holds. Consequently, (4.7)-(4.8) is precisely the condition for $V(z) \in S(R)$.

We are going to focus though on the subclass $S_0(R)$ of $S(R)$ whose definition is the following.

DEFINITION 4.3. An operator-valued Stieltjes function $V(z) \in [E,E]$ is said to be a member of the **class** $S_0(R)$ if in the representation (4.1) we have

$$\int_0^{\infty} (dG(t)e,e)_E = \infty. \tag{4.9}$$

for all non-zero $e \in E$.

We note that a function $V(z)$ can belong to class $S_0(R)$ and have an arbitrary constant $\gamma \geqslant 0$ in the representation (4.1).

The following statement [15] is the direct realization theorem for the functions of the class $S_0(R)$.

THEOREM 4.4. *Let* Θ *be an accretive system of the form* (3.1). *Then the operator-function* $V_\Theta(z)$ *of the form* (3.3), (3.4) *belongs to the class* $S_0(R)$.

Proof. To see that $V_\Theta(z)$ is a Stieltjes operator-function we merely apply theorem 4.2 to system Θ.

Now we will show that $V_\Theta(z)$ belongs to $S_0(R)$. It was shown in [11] and [12] that $E_\infty = K^{-1}\mathfrak{L}$, where $\mathfrak{L} = \mathcal{H} \ominus \overline{\mathcal{D}(A)}$ and

$$E_\infty = \left\{ e \in E : \int_0^{+\infty} (dG(t)e,e)_E < \infty \right\}.$$

But $\overline{\mathcal{D}(A)} = \mathcal{H}$ and consequently $\mathfrak{L} = \{0\}$. Next, $E_\infty = \{0\}$,

$$\int_0^{\infty} (dG(t)e,e)_E = \infty,$$

for all non-zero $e \in E$, and therefore $V_\theta(z) \in S_0(R)$. □

The inverse realization theorem can be stated and proved (see [15]) for the classes $S_0(R)$ as follows.

THEOREM 4.5. *Let an operator-valued function* $V(z)$ *belong to the class* $S_0(R)$. *Then* $V(z)$ *admits a realization by an accretive prime system* Θ *of the form* (3.1) *with* $\mathcal{D}(T) \neq \mathcal{D}(T^*)$ *and* $J = I$.

Proof. We have already noted that the class of Stieltjes function lies inside the wider class of all Herglotz-Nevanlinna functions. Thus all we actually have to show is that $S_0(R) \subset N_0(R)$, where $N_0(R)$ is subclass of realizable Herglotz-Nevanlinna functions described in [12], and that the realizing system constructed in [12] appears to be an accretive system. The former is rather obvious and follows directly from the definition of the class $S_0(R)$. To see that the realizing system is accretive we need to apply theorem 4.2 to $V_\theta(z) = V(z)$, where $V_\Theta(z)$ is related to the model system Θ that was constructed in [12]. As it was also shown in [11] and [12], the symmetric operator A of the model system Θ is prime and hence (3.6) takes place. We are going to show that in this case the system Θ is also prime, i.e.,

$$\underset{\lambda \neq \bar{\lambda},\, \lambda \in \rho(T)}{c.l.s.} \mathfrak{N}_\lambda = \mathcal{H}. \tag{4.10}$$

Consider the operator $U_{\lambda_0\lambda} = (\tilde{A}-\lambda_0 I)(\tilde{A}-\lambda I)^{-1}$, where $\tilde{A}$ is an arbitrary self-adjoint extension of A. By a simple check one confirms that $U_{\lambda_0\lambda}\mathfrak{N}_{\lambda_0} = \mathfrak{N}_\lambda$. To prove (4.10) we assume that there is a function $f \in \mathscr{H}$ such that

$$f \perp \underset{\lambda\neq\bar{\lambda},\, \lambda\in\rho(T)}{c.l.s.} \mathfrak{N}_\lambda.$$

Then $(f, U_{\lambda_0\lambda} g) = 0$ for all $g \in \mathfrak{N}_{\lambda_0}$ and all $\lambda \in \rho(T)$. But accretiveness of the system Θ implies that there are regular points of T in the upper and lower half-planes. This leads to a conclusion that the function $\phi(\lambda) = (f, U_{\lambda_0\lambda} g) \equiv 0$ for all $\lambda \neq \bar{\lambda}$. Combining this with (3.6) we conclude that $f = 0$ and thus (4.10) holds. □

5. Restoring a non-self-adjoint Schrödinger operator T_h

In this section we are going to use the realization results for Stieltjes functions developed in section 4 to obtain the solution of inverse spectral problem for Schrödinger operator of the form (2.5) in $L_2[a,+\infty)$ with non-self-adjoint boundary conditions

$$\begin{cases} T_h y = -y'' + q(x)y \\ y'(a) = hy(a) \end{cases}, \qquad \left(q(x) = \overline{q(x)},\ \mathrm{Im}\, h \neq 0\right). \tag{5.1}$$

In particular, we will show that if a non-decreasing function $\sigma(t)$ is the spectral function of positive self-adjoint boundary value problem

$$\begin{cases} A_\theta y = -y'' + q(x)y \\ y'(a) = \theta y(a) \end{cases} \tag{5.2}$$

and satisfies conditions

$$\int_0^\infty d\sigma(t) = \infty, \quad \int_0^\infty \frac{d\sigma(t)}{1+t} < \infty, \tag{5.3}$$

then for every $\gamma \geqslant 0$ a Stieltjes function

$$V(z) = \gamma + \int_0^\infty \frac{d\sigma(t)}{t-z}$$

can be realized in the unique way as a $V_\Theta(z)$ function of an accretive rigged canonical system Θ with some Schrödinger operator T_h.

Let $\mathscr{H} = L_2[a,+\infty)$ and $l(y) = -y'' + q(x)y$ where q is a real locally summable function. We consider a symmetric operator with defect indices $(1,1)$

$$\begin{cases} \tilde{B}y = -y'' + q(x)y \\ y'(a) = y(a) = 0 \end{cases} \tag{5.4}$$

together with its positive self-adjoint extension of the form

$$\begin{cases} \tilde{B}_\theta y = -y'' + q(x)y \\ y'(a) = \theta y(a) \end{cases} \tag{5.5}$$

defined in $\mathscr{H} = L_2[a,+\infty)$. A non-decreasing function $\sigma(\lambda)$ defined on $[0,+\infty)$ is called the *distribution function* (see [23]) *of an operator pair* $\tilde{B}_\theta$, $\tilde{B}$, where $\tilde{B}_\theta$ of the form (5.5) is a self-adjoint extension of symmetric operator $\tilde{B}$ of the form (5.4), and if the formulas

$$\begin{aligned} \varphi(\lambda) &= Uf(x), \\ f(x) &= U^{-1}\varphi(\lambda), \end{aligned} \tag{5.6}$$

establish one-to-one isometric correspondence U between $L_2^\sigma[0,+\infty)$ and $L_2[a,+\infty)$. Moreover, this correspondence is such that the operator $\tilde{B}_\theta$ is unitarily equivalent to the operator

$$\Lambda_\sigma\varphi(\lambda) = \lambda\varphi(\lambda), \qquad (\varphi(\lambda) \in L_2^\sigma[0,+\infty)) \tag{5.7}$$

in $L_2^\sigma[0,+\infty)$ while symmetric operator $\tilde{B}$ in (5.4) is unitarily equivalent to the symmetric operator

$$\Lambda_\sigma^0\varphi(\lambda) = \lambda\varphi(\lambda), \quad D(\Lambda_\sigma^0) = \left\{\varphi(\lambda) \in L_2^\sigma[0,+\infty) : \int_0^{+\infty} \varphi(\lambda)d\sigma(\lambda) = 0\right\}. \tag{5.8}$$

DEFINITION 5.1. A scalar Herglotz-Nevanlinna function $V(z)$ is called *Stieltjes like function* if it has an integral representation (4.1) with an arbitrary (not necessarily non-negative) constant γ.

We are going to introduce a new class of realizable scalar Stieltjes like functions whose structure is similar to that of $S_0(R)$ of section 4.

DEFINITION 5.2. A Stieltjes like function $V(z)$ is said to be a member of the *class* $SL_0(R)$ if it admits an integral representation

$$V(z) = \gamma + \int_0^\infty \frac{d\sigma(t)}{t-z}, \qquad \big(\gamma \in (-\infty,+\infty)\big), \tag{5.9}$$

where non-decreasing function $\sigma(t)$ satisfies the following conditions

$$\int_0^\infty d\sigma(t) = \infty, \qquad \int_0^\infty \frac{d\sigma(t)}{1+t} < \infty. \tag{5.10}$$

Consider the following subclasses of $SL_0(R)$.

DEFINITION 5.3. A function $V(z) \in SL_0(R)$ belongs to the *class* $SL_0^K(R)$ if

$$\int_0^\infty \frac{d\sigma(t)}{t} = \infty. \tag{5.11}$$

DEFINITION 5.4. A function $V(z) \in SL_0(R)$ belongs to the *class* $SL_{01}^K(R)$ if

$$\int_0^\infty \frac{d\sigma(t)}{t} < \infty. \tag{5.12}$$

The following theorem describes the realization of the class $SL_0(R)$.

THEOREM 5.5. *Let $V(z) \in SL_0(R)$ and the function $\sigma(t)$ be the distribution function of an operator pair $\tilde{B}_\theta$ of the form* (5.4) *and $\tilde{B}$ of the form* (5.5)*. Then there exist unique Schrödinger operator T_h ($\operatorname{Im} h > 0$) of the form* (5.1)*, operator $\mathbb{A}$ given by* (2.6)*, operator K as in* (3.14)*, and the rigged canonical system of the Livsic type*

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathscr{H}_+ \subset L_2[a,+\infty) \subset \mathscr{H}_- & & \mathbb{C} \end{pmatrix}, \tag{5.13}$$

of the form (3.15) *so that $V(z)$ is realized by Θ.*

Proof. Since $\sigma(t)$ is the distribution function of the positive self-adjoint operator, then (see [23]) we can completely restore the operator $\tilde{B}_\theta$ of the form (5.5) as well as a symmetric operator $\tilde{B}$ of the form (5.4). It follows from the definition of the distribution function above that there is operator U defined in (5.6) establishing one-to-one isometric correspondence between $L_2^\sigma[0,+\infty)$ and $L_2[a,+\infty)$ while providing for the unitary equivalence between the operator $\tilde{B}_\theta$ and operator of multiplication by independent variable Λ_σ of the form (5.7). Taking this into account, we realize (see [11]) a Herglotz-Nevanlinna function $V(z)$ with a rigged canonical system

$$\Theta_\Lambda = \begin{pmatrix} \Lambda & K^\sigma & 1 \\ \mathscr{H}_+^\sigma \subset L_2^\sigma[0,+\infty) \subset \mathscr{H}_-^\sigma & & \mathbb{C} \end{pmatrix}.$$

Following the steps for construction of the model system described in [11], we note that

$$\Lambda = \operatorname{Re}\Lambda + iK^\sigma(K^\sigma)^*$$

is a correct ($*$)-extension of an operator T^σ such that $\Lambda \supset T^\sigma \supset \Lambda_\sigma^0$ where Λ_σ^0 is defined in (5.8). The real part $\operatorname{Re}\Lambda$ is a self-adjoint bi-extension of Λ_σ^0 that has a quasi-kernel Λ_σ of the form (5.7). The operator K^σ in the above system is defined by

$$K^\sigma c = c\cdot\alpha, \quad c \in \mathbb{C},\ \alpha \in \mathscr{H}_-^\sigma.$$

In addition we can observe that the function $\eta(\lambda) \equiv 1$ belongs to the space $\mathscr{H}_-^\sigma$. To confirm this we need to show that $(x,1)$ defines a continuous linear functional for every $x \in \mathscr{H}_+^\sigma$. It was shown in [11], [12] that

$$\mathscr{H}_+^\sigma = \mathscr{D}(\Lambda_\sigma^0) \dot{+} \left\{\frac{c_1}{1+t^2}\right\} \dot{+} \left\{\frac{c_2 t}{1+t^2}\right\}, \quad c_1, c_2 \in \mathbb{C}.$$

Consequently, every vector $x \in \mathscr{H}_+^\sigma$ has three components $x = x_1 + x_2 + x_3$ according to the decomposition above. Obviously, $(x_1,1)$ and $(x_2,1)$ yield convergent integrals while $(x_3,1)$ boils down to

$$\int_0^\infty \frac{t}{1+t^2}\,d\sigma(t).$$

To see the convergence of the above integral we notice that

$$\frac{t}{1+t^2} = \frac{t-1}{(1+t^2)(t+1)} + \frac{1}{1+t} \leqslant \frac{1}{1+t^2} + \frac{1}{1+t}.$$

The integrals taken of the last two expressions on the right side converge due to (1.2) and (5.10), and hence so does the integral of the left side. Thus, $(x,1)$ defines a continuous linear functional for every $x \in \mathscr{H}_+^\sigma$, and hence $1 \in \mathscr{H}_-^\sigma$.

The state space of the system Θ_Λ is $\mathscr{H}_+^\sigma \subset L_2^\sigma[0,+\infty) \subset \mathscr{H}_-^\sigma$, where $\mathscr{H}_+^\sigma = \mathscr{D}\big((\Lambda_\sigma^0)^*\big)$. By the realization theorem [11] we have that $V(z) = V_{\Theta_\Lambda}(z)$.

We can also show that the system Θ_Λ is a prime system. In order to do so we need to show that

$$\underset{\lambda\neq\bar\lambda,\,\lambda\in\rho(T^\sigma)}{c.l.s.} \mathfrak{N}_\lambda = L_2^\sigma[0,+\infty), \tag{5.14}$$

where $\mathfrak{N}_\lambda$ are defect subspaces of the symmetric operator Λ_σ^0. It is known (see [11]) that Λ_σ^0 is a prime operator. Hence we can follow the reasoning of the proof of theorem 4.5 and only confirm that operator T^σ has regular points in the upper and lower half-planes. To see this we first note that non-negative operator Λ_σ^0 has no kernel spectrum [1] on the left real half-axis. Then we apply Theorem 1 of [1] (see page 149 of vol. 2 of [1]) that gives the complete description of the spectrum of T^σ. This theorem implies that there are regular points of T^σ on the left real half-axis. Since $\rho(T^\sigma)$ is an open set we confirm the presence of non-real regular points of T^σ in both half-planes. Thus (5.14) holds and Θ_Λ is a prime system.

Applying theorem 3.2 on unitary equivalence to the isometry U defined in (5.6) we obtain a triplet of isometric operators U_+, U, and U_-, where

$$U_+ = U\big|_{\mathscr{H}_+^\sigma}, \quad U_-^* = U_+^*.$$

This triplet of isometric operators will map the rigged space $\mathscr{H}_+^\sigma \subset L_2^\sigma[0,+\infty) \subset \mathscr{H}_-^\sigma$ into another rigged Hilbert space $\mathscr{H}_+ \subset L_2^\sigma[a,+\infty) \subset \mathscr{H}_-$. Moreover, U_+ is an isometry from $\mathscr{H}_+^\sigma = \mathscr{D}(\Lambda_\sigma^{0*})$ onto $\mathscr{H}_+ = \mathscr{D}(\tilde B^*)$, and $U_-^* = U_+^*$ is an isometry from $\mathscr{H}_+^\sigma$ onto $\mathscr{H}_-$. This is true since the operator U provides the unitary equivalence between the symmetric operators $\tilde B$ and Λ_σ^0.

Now we construct a system

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathscr{H}_+ \subset L_2[a,+\infty) \subset \mathscr{H}_- & & \mathbb{C} \end{pmatrix}$$

where $K = U_- K^\sigma$ and $\mathbb{A} = U_- \Lambda U_+^{-1}$ is a correct $(*)$-extension of operator $T = UT^\sigma U^{-1}$ such that $\mathbb{A} \supset T \supset \tilde B$. The real part $\operatorname{Re}\mathbb{A}$ contains the quasi-kernel $\tilde B_\theta$. This construction of $\mathbb{A}$ is unique due to the theorem on the uniqueness of a $(*)$-extension for a given quasi-kernel (see [27]). On the other hand, all $(*)$-extensions based on a pair $\tilde B$, $\tilde B_\theta$ must take form (2.6) for some values of parameters h and μ. Consequently, our function $V(z)$ is realized by the system Θ of the form (5.13) and

$$V(z) = V_{\Theta_\Lambda}(z) = V_\Theta(z).$$

□

REMARK 5.6. Applying corollary 3.3 to the mapping U defined by (5.6) we obtain that the operator U in the above theorem is unique. The uniqueness of the operator U leads to an interesting observation. Let $u_k(x,\lambda)$, $(k = 1,2)$ be the solutions of the following Cauchy problems:

$$\begin{cases} l(u_1)=\lambda u_1 \\ u_1(a,\lambda)=0 \\ u_1'(a,\lambda)=1 \end{cases}, \qquad \begin{cases} l(u_2)=\lambda u_2 \\ u_2(a,\lambda)=1 \\ u_2'(a,\lambda)=0 \end{cases}.$$

Traditionally, (see [23]) a non-decreasing function $\sigma(\lambda)$ defined on $[0,+\infty)$ is called the *distribution function of a self-adjoint operator* $\tilde{B}_\theta$ of the form (5.5) if the formulas

$$\begin{aligned} \varphi(\lambda) &= Uf(x) = \int_a^{+\infty} f(x)u(x,\lambda)\,dx, \\ f(x) &= U^{-1}\varphi(\lambda) = \int_0^{+\infty} \varphi(\lambda)u(x,\lambda)\,d\sigma(\lambda), \end{aligned} \tag{5.15}$$

where $u(x,\lambda)=u_1(x,\lambda)+\theta u_2(x,\lambda)$, establish one-to-one isometric correspondence U between $L_2^\sigma[0,+\infty)$ and $L_2[a,+\infty)$ such that the operator $\tilde{B}_\theta$ in (5.5) is unitarily equivalent to the operator Λ_σ in (5.7). It is easily seen that if the mapping U in (5.15) has a property that symmetric operators $\widetilde{B}$ in (5.4) and Λ_σ^0 in (5.8) are also unitarily equivalent w.r.t. U, then the unitary operator appearing in the proof of Theorem 5.5 coincides with the one defined by the formulas (5.15). Indeed, assuming that there is another mapping $\tilde{U}$ provided by Theorem 3.2 on unitary equivalence for the systems Θ_Λ and Θ we would violate the uniqueness of the operator U, and thus $\tilde{U}=U$.

THEOREM 5.7. *Let $V(z)\in SL_0(R)$ satisfy the conditions of theorem 5.5. Then the operator T_h in the conclusion of the theorem 5.5 is accretive if and only if*

$$\gamma^2+\gamma\int_0^\infty \frac{d\sigma(t)}{t}+1\geqslant 0. \tag{5.16}$$

The operator T_h is α-sectorial for some $\alpha\in(0,\pi/2)$ if and only if the inequality (5.16) is strict. In this case the exact value of angle α can be calculated by the formula

$$\tan\alpha = \frac{\int_0^\infty \frac{d\sigma(t)}{t}}{\gamma^2+\gamma\int_0^\infty \frac{d\sigma(t)}{t}+1}. \tag{5.17}$$

Proof. It was shown in [26] that for the system Θ in (5.13) described in the previous theorem the operator T_h is accretive if and only if the function

$$\begin{aligned} V_h(z) &= -i[W_\Theta^{-1}(-1)W_\Theta(z)+I]^{-1}[W_\Theta^{-1}(-1)W_\Theta(z)-I] \\ &= -i\frac{1-[(m_\infty(z)+\bar{h})/(m_\infty(z)+h)][(m_\infty(-1)+h)/(m_\infty(-1)+\bar{h})]}{1+[(m_\infty(z)+\bar{h})/(m_\infty(z)+h)][(m_\infty(-1)+h)/(m_\infty(-1)+\bar{h})]}, \end{aligned} \tag{5.18}$$

is holomorphic in Ext $[0,+\infty)$ and satisfies the following inequality

$$1+V_h(0)\,V_h(-\infty)\geqslant 0. \tag{5.19}$$

Here $W_\Theta(z)$ is the transfer function of (5.13). It is also shown in [26] that the operator T_h is α-sectorial for some $\alpha\in(0,\pi/2)$ if and only if the inequality (5.19) is strict while the exact value of angle α can be calculated by the formula

$$\cot\alpha = \frac{1+V_h(0)\,V_h(-\infty)}{|V_h(-\infty)-V_h(0)|}. \tag{5.20}$$

According to theorem 5.5 and equation (3.5)

$$W_\Theta(z) = (I - iV(z)J)(I + iV(z)J)^{-1}.$$

By direct calculations one obtains

$$W_\Theta(-1) = \frac{1 - i\left[\gamma + \int_0^\infty \frac{d\sigma(t)}{t+1}\right]}{1 + i\left[\gamma + \int_0^\infty \frac{d\sigma(t)}{t+1}\right]}, \qquad W_\Theta^{-1}(-1) = \frac{1 + i\left[\gamma + \int_0^\infty \frac{d\sigma(t)}{t+1}\right]}{1 - i\left[\gamma + \int_0^\infty \frac{d\sigma(t)}{t+1}\right]}. \quad (5.21)$$

Using the following notations

$$a = \gamma + \int_0^\infty \frac{d\sigma(t)}{t+1} \qquad \text{and} \qquad b = \gamma + \int_0^\infty \frac{d\sigma(t)}{t},$$

and performing straightforward calculations we obtain

$$V_h(0) = \frac{a-b}{1+ab} \qquad \text{and} \qquad V_h(-\infty) = \frac{a-\gamma}{1+a\gamma}. \quad (5.22)$$

Substituting (5.22) into (5.20) and performing the necessary steps we get

$$\cot\alpha = \frac{1+b\gamma}{b-\gamma} = \frac{\gamma^2 + \gamma\int_0^\infty \frac{d\sigma(t)}{t} + 1}{\int_0^\infty \frac{d\sigma(t)}{t}}. \quad (5.23)$$

Taking into account that $b - \gamma > 0$ we combine (5.19), (5.20) with (5.23) and this completes the proof of the theorem. □

COROLLARY 5.8. *Let* $V(z) \in SL_0(R)$ *satisfy the conditions of theorem 5.5. Then the operator* T_h *in the conclusion of theorem 5.5 is accretive if and only if*

$$1 + V(0)\,V(-\infty) \geqslant 0. \quad (5.24)$$

The operator T_h *is* α*-sectorial for some* $\alpha \in (0, \pi/2)$ *if and only if the inequality* (5.24) *is strict. In this case the exact value of angle* α *can be calculated by the formula*

$$\tan\alpha = \frac{|V(-\infty) - V(0)|}{1 + V(0)\,V(-\infty)}. \quad (5.25)$$

Proof. Taking into account that

$$V(0) = \gamma + \int_0^\infty \frac{d\sigma(t)}{t},$$

$V(z) = V_\Theta(z)$, and $V_\Theta(-\infty) = \gamma$, we use (5.16) and (5.17) to obtain (5.24) and (5.25). □

THEOREM 5.9. *Let $V(z) \in S_0(R)$ and satisfy the conditions of theorem 5.5. Then the system Θ of the form* (5.13) *is accretive and its symmetric operator A of the form* (2.4) *is such that its Kreĭn-von Neumann extension A_K of the form* (2.8) *does not coincide with its Friedrichs extension A_F of the form* (2.9).

Proof. The proof of the fact that Θ is accretive directly follows from the theorems 4.2 and 5.5. The second part follows from the theorem in [25] that states that a positive symmetric operator A admits a non-self-adjoint accretive extension T if and only if A_F and A_K do not coincide. □

Below we will derive the formulas for calculation of the boundary parameter h in the restored Schrödinger operator T_h of the form (5.1). We consider two major cases.

Case 1. In the first case we assume that $\int_0^\infty \frac{d\sigma(t)}{t} < \infty$. This means that our function $V(z)$ belongs to the class $SL_{01}^K(R)$. In what follows we denote

$$b = \int_0^\infty \frac{d\sigma(t)}{t} \qquad \text{and} \qquad m = m_\infty(-0).$$

Suppose that $b \geqslant 2$. Then the quadratic inequality (5.16) implies that for all γ such that

$$\gamma \in \left(-\infty, \frac{-b-\sqrt{b^2-4}}{2}\right] \cup \left[\frac{-b+\sqrt{b^2-4}}{2}, +\infty\right) \tag{5.26}$$

the restored operator T_h is accretive. Clearly, this operator is extremal accretive if

$$\gamma = \frac{-b \pm \sqrt{b^2-4}}{2}.$$

In particular if $b = 2$ then $\gamma = -1$ and the function

$$V(z) = -1 + \int_0^\infty \frac{d\sigma(t)}{t-z}$$

is realized using an extremal accretive T_h.

Now suppose that $0 < b < 2$. For every $\gamma \in (-\infty, +\infty)$ the restored operator T_h will be accretive and α-sectorial for some $\alpha \in (0, \pi/2)$. Consider a function $V(z)$ defined by (5.9). Conducting realizations of $V(z)$ by operators T_h for different values of $\gamma \in (-\infty, +\infty)$ we notice that the operator T_h with the largest angle of sectorialilty occurs when

$$\gamma = -\frac{b}{2}, \tag{5.27}$$

and is found according to the formula

$$\alpha = \arctan \frac{b}{1 - b^2/4}. \tag{5.28}$$

This follows from the formula (5.17), the fact that $\gamma^2 + \gamma b + 1 > 0$ for all γ, and the formula

$$\gamma^2 + \gamma b + 1 = \left(\gamma + \frac{b}{2}\right)^2 + \left(1 - \frac{b^2}{4}\right).$$

Now we will focus on the description of the parameter h in the restored operator T_h.

It was shown in [6] that the quasi-kernel $\hat{A}$ of the realizing system Θ from theorem 5.5 takes a form

$$\begin{cases} \widehat{A}y = -y'' + qy \\ y'(a) = \eta y(a) \end{cases}, \qquad \eta = \frac{\mu \operatorname{Re} h - |h|^2}{\mu - \operatorname{Re} h} \tag{5.29}$$

On the other hand, since $\sigma(t)$ is also the distribution function of the positive self-adjoint operator, we can conclude that $\hat{A}$ equals to the operator $\tilde{B}_\theta$ of the form (5.5). This connection allows us to obtain

$$\theta = \eta = \frac{\mu \operatorname{Re} h - |h|^2}{\mu - \operatorname{Re} h}. \tag{5.30}$$

Assuming that

$$h = x + iy$$

we will use (5.30) to derive the formulas for x and y in terms of γ. First, to eliminate parameter μ, we notice that (3.16) and (3.5) imply

$$W_\Theta(\lambda) = \frac{\mu - h}{\mu - \overline{h}}\,\frac{m_\infty(\lambda) + \overline{h}}{m_\infty(\lambda) + h} = \frac{1 - iV(z)}{1 + iV(z)}. \tag{5.31}$$

Passing to the limit in (5.31) when $\lambda \to -\infty$ and taking into account that $V(-\infty) = \gamma$ and $m_\infty(-\infty) = \infty$ we obtain[2]

$$\frac{\mu - h}{\mu - \overline{h}} = \frac{1 - i\gamma}{1 + i\gamma}.$$

Let us denote

$$a = \frac{1 - i\gamma}{1 + i\gamma}. \tag{5.32}$$

Solving (5.32) for μ yields

$$\mu = \frac{h - a\bar{h}}{1 - a}.$$

Substituting this value into (5.30) after simplification produces

$$\frac{x + iy - a(x - iy)x - (x^2 + y^2)(1 - a)}{x + iy - a(x - iy) - x(1 - a)} = \theta.$$

After straightforward calculations targeting to represent numerator and denominator of the last equation in standard form one obtains the following relation

$$x - \gamma\, y = \theta. \tag{5.33}$$

[2]The fact that $m_\infty(-\infty) = \infty$ (with the assumption $m_\infty(0) < \infty$ considered in this paper for the corresponding nonnegative Schrödinger operator) follows from (2.6), (3.17), (3.12), and (5.23) when $\operatorname{Re} h = -m_\infty(0)$, $\mu = \infty$ as well as from

$$V_{\Theta(\lambda)} = \left((\operatorname{Re}\mathbb{A} - \lambda I)^{-1} g, g\right) = \frac{\operatorname{Im} h}{m_\infty(\lambda) - m_\infty(0)}$$

and the relation $V_\Theta(-\infty) = 0$ established in [6], [15].

It was shown in [26] that the α-sectorialilty of the operator T_h and (5.20) lead to

$$\tan\alpha = \frac{\operatorname{Im} h}{\operatorname{Re} h + m_\infty(-0)} = \frac{y}{x + m_\infty(-0)}. \tag{5.34}$$

Combining (5.33) and (5.34) one obtains

$$x - \gamma(x\tan\alpha + m_\infty(-0)\,\tan\alpha) = \theta,$$

or

$$x = \frac{\theta + \gamma m_\infty(-0)\,\tan\alpha}{1 - \gamma\,\tan\alpha}.$$

But $\tan\alpha$ is also determined by (5.17). Direct substitution of

$$\tan\alpha = \frac{b}{1 + \gamma(\gamma + b)}$$

into the above equation yields

$$x = \theta + \frac{[\theta + m_\infty(-0)]\,b\gamma}{1 + \gamma^2}.$$

Using the short notation and finalizing calculations we get

$$h = x + iy, \qquad x = \theta + \frac{\gamma[\theta + m]b}{1 + \gamma^2}, \qquad y = \frac{[\theta + m]b}{1 + \gamma^2}. \tag{5.35}$$

At this point we can use (5.35) to provide analytical and graphical interpretation of the parameter h in the restored operator T_h. Let

$$c = (\theta + m)b.$$

Again we consider three subcases.

Subcase 1: $b > 2$ Using basic algebra we transform (5.35) into

$$(x - \theta)^2 + \left(y - \frac{c}{2}\right)^2 = \frac{c^2}{4}. \tag{5.36}$$

Since in this case the parameter γ belongs to the interval in (5.26), we can see that h traces the highlighted part of the circle on the figure 1 as γ moves from $-\infty$ towards $+\infty$.

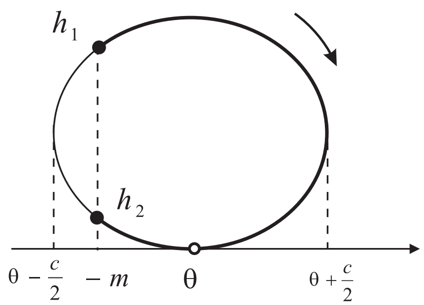

Figure 1. $b > 2$

We also notice that the removed point $(\theta, 0)$ corresponds to the value of $\gamma = \pm\infty$ while the points h_1 and h_2 correspond to the values $\gamma_1 = \frac{-b-\sqrt{b^2-4}}{2}$ and $\gamma_2 = \frac{-b+\sqrt{b^2-4}}{2}$, respectively (see figure 2.).

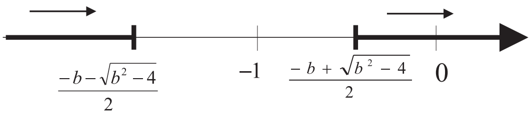

Figure 2. γ interval

Subcase 2: $b < 2$ For every $\gamma \in (-\infty, +\infty)$ the restored operator T_h will be accretive and α-sectorial for some $\alpha \in (0, \pi/2)$. As we have mentioned above, the operator T_h achieves the largest angle of sectorialilty when $\gamma = -\frac{b}{2}$. In this particular case (5.35) becomes

$$h = x + iy, \qquad x = \frac{\theta(4-b^2) - 2b^2 m}{4+b^2}, \qquad y = \frac{4(\theta+m)b}{4+b^2}. \tag{5.37}$$

The value of h from (5.37) is marked on the figure 3.

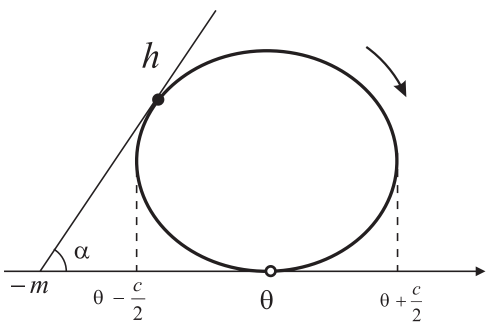

Figure 3. $b < 2$

Subcase 3: $b = 2$ The behavior of parameter h in this case is depicted on the figure 4. It shows that in this case the function $V(z)$ can be realized using an extremal accretive T_h when $\gamma = -1$. The value of the parameter h according to (5.35) then becomes

$$h = x + iy, \qquad x = -m, \qquad y = \theta + m. \tag{5.38}$$

Clockwise direction of the circle again corresponds to the change of γ from $-\infty$ to $+\infty$ and the marked value of h occurs when $\gamma = -1$.

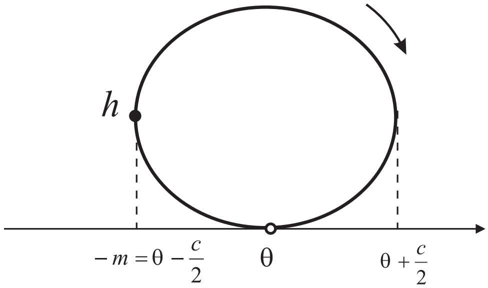

Figure 4. $b=2$

Now we consider the second case.

Case 2. Here we assume that $\int_0^\infty \frac{d\sigma(t)}{t} = \infty$. This means that our function $V(z)$ belongs to the class $SL_0^K(R)$ and $b=\infty$. According to theorem 5.7 and formulas (5.16) and (5.17), the restored operator T_h is accretive if and only if

$$\gamma \geqslant 0,$$

and α-sectorial if and only if $\gamma > 0$. It directly follows from (5.17) that the exact value of the angle α is then found from

$$\tan\alpha = \frac{1}{\gamma}. \tag{5.39}$$

The latter implies that the restored operator T_h is extremal if $\gamma = 0$. This means that a function $V(z) \in SL_0^K(R)$ is realized by a system with an extremal operator T_h if and only if

$$V(z) = \int_0^\infty \frac{d\sigma(t)}{t-z}. \tag{5.40}$$

On the other hand since $\gamma \geqslant 0$ the function $V(z)$ is a Stieltjes function of the class $S_0(R)$. Applying realization theorems from [15] we conclude that $V(z)$ admits realization by an accretive system Θ of the form (3.1) with $\mathbb{A}_R$ containing the Krein-von Neumann extension A_K as a quasi-kernel. Here A_K is defined by (2.8). This yields

$$\theta = -m_\infty(-0) = -m. \tag{5.41}$$

As in the beginning of the previous case we derive the formulas for x and y, where $h = x+iy$. Using (5.30) and (5.33) leads to

$$\begin{cases} \theta = \frac{\mu x-(x^2+y^2)}{\mu - x}, \\ x = \theta + \gamma y. \end{cases} \tag{5.42}$$

Solving this system for x and y leads to

$$x = \frac{\theta+\mu\gamma^2}{1+\gamma^2}, \qquad y = \frac{(\mu-\theta)\gamma}{1+\gamma^2}. \tag{5.43}$$

Combining (5.42) and (5.43) gives

$$x = \frac{-m+\mu\gamma^2}{1+\gamma^2}, \qquad y = \frac{(m+\mu)\gamma}{1+\gamma^2}. \tag{5.44}$$

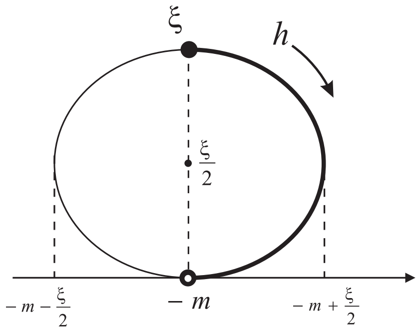

Figure 5. $b = \infty$

To proceed, we first notice that our function $V(z)$ satisfies the conditions of theorem 4.8 of [6]. Indeed, the inequality

$$\mu \geqslant \frac{(\mathrm{Im}\, h)^2}{m_\infty(-0) + \mathrm{Re}\, h} + \mathrm{Re}\, h,$$

turns into

$$\mu = \frac{y^2}{x-m} + x,$$

if you use $\theta = -m$ and the first equation in (5.42). Applying theorem 4.8 of [6] yields

$$\int\limits_0^\infty \frac{d\sigma(t)}{1+t^2} = \frac{\mathrm{Im}\, h}{|\mu - h|^2}\left(\sup_{y\in D(A_K)} \frac{|\mu y(a) - y'(a)|}{\left(\int\limits_a^\infty (|y(x)|^2 + |l(y)|^2)\, dx\right)^{\frac{1}{2}}}\right)^2. \tag{5.45}$$

Taking into account that

$$\mu y(a) - y'(a) = (\mu + m) y(a)$$

and setting

$$c^{1/2} = \sup_{y\in D(A_K)} \frac{|y(a)|}{\left(\int\limits_a^\infty (|y(x)|^2 + |l(y)|^2)\, dx\right)^{\frac{1}{2}}}, \tag{5.46}$$

we obtain

$$\frac{\mathrm{Im}\, h}{|\mu - h|^2}(\mu+m)^2\, c = \int\limits_0^\infty \frac{d\sigma(t)}{1+t^2}. \tag{5.47}$$

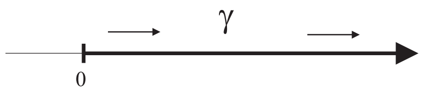

Figure 6. $\gamma \geqslant 0$

Considering that $\operatorname{Im} h = y$ and combining (5.47) with (5.44) we use straightforward calculations to get

$$\mu = -m + \left(\frac{1}{\gamma c}\right) \int\limits_0^\infty \frac{d\sigma(t)}{1+t^2}.$$

Let

$$\xi = \frac{1}{c} \int_0^\infty \frac{d\sigma(t)}{1+t^2}. \tag{5.48}$$

Then the last equation becomes

$$\mu = -m + \frac{\xi}{\gamma}. \tag{5.49}$$

Applying (5.49) on (5.44) yields

$$x = -m + \frac{\gamma \xi}{1+\gamma^2}, \qquad y = \frac{\xi}{1+\gamma^2}, \qquad \gamma \geqslant 0. \tag{5.50}$$

Following the previous case approach we transform (5.50) into

$$(x+m)^2 + \left(y - \frac{\xi}{2}\right)^2 = \frac{\xi^2}{4}. \tag{5.51}$$

The connection between the parameters γ and h in the accretive restored operator T_h is depicted in figures 5. and 6. As we can see h traces the highlighted part of the circle clockwise on the figure 5 as γ moves from 0 towards $+\infty$.

As we mentioned earlier the restored operator T_h is extremal if $\gamma = 0$. In this case formulas (5.50) become

$$x = -m, \qquad y = \xi, \qquad \gamma = 0, \tag{5.52}$$

where ξ is defined by (5.48).

6. Realizing systems with Schrödinger operator

Now once we described all the possible outcomes for the restored accretive operator T_h, we can concentrate on the main operator $\mathbb{A}$ of the system (5.13). We recall that $\mathbb{A}$ is defined by formulas (2.6) and beside the parameter h above contains also parameter μ. We will obtain the behavior of μ in terms of the components of our function $V(z)$ the same way we treated the parameter h. As before we consider two major cases dividing them into subcases when necessary.

Case 1. Assume that $b=\int_0^\infty \frac{d\sigma(t)}{t}<\infty$. In this case our function $V(z)$ belongs to the class $SL_{01}^K(R)$. First we will obtain the representation of μ in terms of x and y, where $h=x+iy$. We recall that

$$\mu=\frac{h-a\bar{h}}{1-a},$$

where a is defined by (5.32). By direct computations we derive that

$$a=\frac{1-\gamma^2}{1+\gamma^2}-\frac{2\gamma}{1+\gamma^2}i,\qquad 1-a=\frac{2\gamma^2}{1+\gamma^2}+\frac{2\gamma}{1+\gamma^2}i,$$

and

$$h-a\bar{h}=\left(\frac{2\gamma^2}{1+\gamma^2}x+\frac{2\gamma}{1+\gamma^2}y\right)+\left(\frac{2}{1+\gamma^2}y+\frac{2\gamma}{1+\gamma^2}x\right)i.$$

Plugging the last two equations into the formula for μ above and simplifying we obtain

$$\mu=x+\frac{1}{\gamma}y. \tag{6.1}$$

We recall that during the present case x and y parts of h are described by the formulas (5.35).

Once again we elaborate in three subcases.

Subcase 1: $b>2$ As we have shown this above, the formulas (5.35) can be transformed into equation of the circle (5.36). In this case the parameter γ belongs to the interval in (5.26), the accretive operator T_h corresponds to the values of h shown in the bold part of the circle on the figure 1 as γ moves from $-\infty$ towards $+\infty$.

Substituting the expressions for x and y from (5.35) into (6.1) and simplifying we get

$$\mu=\theta+\frac{(\theta+m)b}{\gamma}. \tag{6.2}$$

The connection between values of γ and μ is depicted on the figure 7.

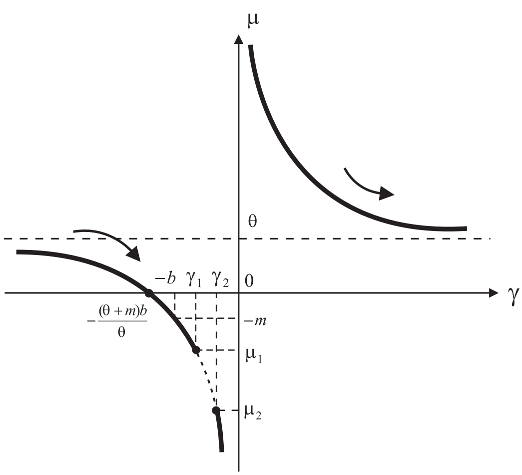

Figure 7. $b>2$

We note that $\mu = 0$ when $\gamma = -\frac{(\theta+m)b}{\theta}$. Also, the endpoints

$$\gamma_1 = \frac{-b-\sqrt{b^2-4}}{2} \qquad \text{and} \qquad \gamma_2 = \frac{-b+\sqrt{b^2-4}}{2}$$

of γ-interval (5.26) are responsible for the μ-values

$$\mu_1 = \theta + \frac{(\theta+m)b}{\gamma_1} \qquad \text{and} \qquad \mu_2 = \theta + \frac{(\theta+m)b}{\gamma_2}.$$

The values of μ that are acceptable parameters of operator $\mathbb{A}$ of the restored system make the bold part of the hyperbola on the figure 7. It follows from theorem 4.2 that the operator $\mathbb{A}$ of the form (2.6) is accretive if and only if $\gamma \geqslant 0$ and thus μ sweeps the right branch on the hyperbola. We note that figure 7 shows the case when $-m < 0$, $\theta > 0$, and $\theta > -m$. Other possible cases, such as $(-m < 0,\ \theta < 0,\ \theta > -m)$, $(-m < 0,\ \theta = 0)$, and $(m = 0,\ \theta > 0)$ require corresponding adjustments to the graph shown in the picture 7.

Subcase 2: $b < 2$ For every $\gamma \in (-\infty, +\infty)$ the restored operator T_h will be accretive and α-sectorial for some $\alpha \in (0, \pi/2)$. As we have mentioned above, the operator T_h achieves the largest angle of sectorialilty when $\gamma = -\frac{b}{2}$. In this particular case (5.35) becomes

$$h = x + iy, \qquad x = \frac{\theta(4-b^2) - 2b^2 m}{4+b^2}, \qquad y = \frac{4(\theta+m)b}{4+b^2}.$$

Substituting $\gamma = b/2$ into (6.1) we obtain

$$\mu = -(\theta + 2m). \tag{6.3}$$

This value of μ from (6.3) is marked on the figure 8. The corresponding operator $\mathbb{A}$ of the realizing system is based on these values of parameters h and μ.

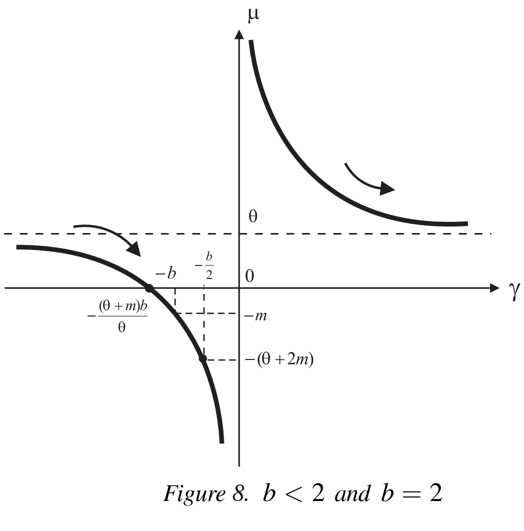

Figure 8. $b < 2$ *and* $b = 2$

Subcase 3: $b=2$ The behavior of parameter μ in this case is also shown on the figure 8. It was shown above that in this case the function $V(z)$ can be realized using an extremal accretive T_h when $\gamma=-1$. The values of the parameters h and μ then become

$$h=x+iy, \qquad x=-m, \qquad y=\theta+m, \qquad \mu=-(\theta+2m).$$

The value of μ above is marked on the left branch of the hyperbola and occurs when $\gamma=-1=-b/2$.

Case 2. Again we assume that $\int_0^\infty \frac{d\sigma(t)}{t}=\infty$. Hence $V(z)\in SL_0^K(R)$ and $b=\infty$. As we mentioned above the restored operator T_h is accretive if and only if $\gamma\geqslant 0$ and α-sectorial if and only if $\gamma>0$. It is extremal if $\gamma=0$. The values of x, y, and μ were already calculated and are given in (5.50) and (5.49), respectively. That is

$$x=-m+\frac{\gamma\,\xi}{1+\gamma^2}, \quad y=\frac{\xi}{1+\gamma^2}, \quad \mu=-m+\frac{\xi}{\gamma}, \qquad \gamma\geqslant 0.$$

where ξ is defined in (5.48). Figure 9. gives graphical representation of this case. Only the right bold branch of hyperbola shows the values of μ in the case $b=\infty$. If $m=0$ then

$$\mu=\frac{\xi}{\gamma}$$

and the graph should be adjusted accordingly.

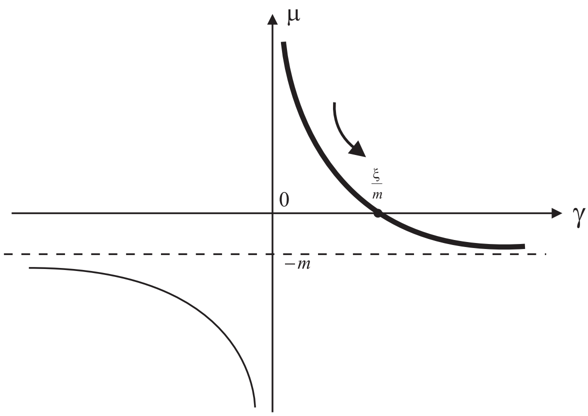

Figure 9. $b=\infty$

In the case when $\gamma=0$ and T_h is extremal we have

$$x=-m, \qquad y=\xi, \qquad \mu=\infty, \qquad h=-m+i\xi, \tag{6.4}$$

and according to (2.6) we have

$$\mathbb{A}y=-y''+q(x)y+[(-m+i\xi)y(a)-y'(a)]\delta(x-a), \tag{6.5}$$

that is the main operator of the realizing system.

Example

We conclude this paper with simple illustration. Consider a function

$$V(z) = \frac{i}{\sqrt{z}}. \tag{6.6}$$

A direct check confirms that $V(z)$ is a Stieltjes function. It was shown in [23] (see pp. 140-142) that the inversion formula

$$\sigma(\lambda) = C + \lim_{y\to 0} \frac{1}{\pi} \int_0^{\lambda} \operatorname{Im}\left(\frac{i}{\sqrt{x+iy}}\right) dx \tag{6.7}$$

describes the distribution function for a self-adjoint operator

$$\begin{cases} \tilde{B}_0 y = -y'' \\ y'(0) = 0. \end{cases}$$

The corresponding to $\tilde{B}_0$ symmetric operator is

$$\begin{cases} B_0 y = -y'' \\ y(0) = y'(0) = 0. \end{cases} \tag{6.8}$$

It was also shown in [23] that $\sigma(\lambda) = 0$ for $\lambda \leqslant 0$ and

$$\sigma'(\lambda) = \frac{1}{\pi\sqrt{\lambda}} \qquad \text{for } \lambda > 0. \tag{6.9}$$

By direct calculations one can confirm that

$$V(z) = \int_0^{\infty} \frac{d\sigma(t)}{t-z} = \frac{i}{\sqrt{z}},$$

and that

$$\int_0^{\infty} \frac{d\sigma(t)}{t} = \int_0^{\infty} \frac{dt}{\pi t^{3/2}} = \infty.$$

It is also clear that the constant term in the integral representation (4.1) is zero, i.e. $\gamma = 0$.

Let us assume that $\sigma(t)$ satisfies our definition of spectral distribution function of the pair B_0, $\tilde{B}_0$ given in the section 5. Operating under this assumption, we proceed to restore parameters h and μ and apply formulas (5.50) for the values $\gamma = 0$ and $m = -\theta = 0$. This yields $x = 0$. To obtain y we first find the value of

$$\int_0^{\infty} \frac{d\sigma(t)}{1+t^2} = \frac{1}{\sqrt{2}},$$

and then use formula (5.46) to get the value of c. This yields $c = 1/\sqrt{2}$. Consequently,

$$\xi = \frac{1}{c}\int_0^{\infty} \frac{d\sigma(t)}{1+t^2} = 1,$$

and hence $h = yi = i$. From (5.49) we have that $\mu = \infty$ and (6.5) becomes

$$\mathbb{A}\, y = -y'' + [iy(0) - y'(0)]\delta(x). \tag{6.10}$$

The operator T_h in this case is

$$\begin{cases} T_h y = -y'' \\ y'(0) = iy(0). \end{cases}$$

The channel vector g of the form (3.12) then equals

$$g = \delta(x), \tag{6.11}$$

satisfying

$$\operatorname{Im}\mathbb{A} = \frac{\mathbb{A} - \mathbb{A}^*}{2i} = KK^* = (.,g)g,$$

and channel operator $Kc = cg$, ($c \in \mathbb{C}$) with

$$K^* y = (y, g) = y(0). \tag{6.12}$$

The real part of $\mathbb{A}$

$$\operatorname{Re}\mathbb{A}\, y = -y'' - y'(0)\delta(x)$$

contains the self-adjoint quasi-kernel

$$\begin{cases} \widehat{A} y = -y'' \\ y'(0) = 0. \end{cases}$$

A system of the Livšic type with Schrödinger operator of the form (5.13) that realizes $V(z)$ can now be written as

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathscr{H}_+ \subset L_2[a,+\infty) \subset \mathscr{H}_- & & \mathbb{C} \end{pmatrix}.$$

where $\mathbb{A}$ and K are defined above. Now we can back up our assumption on $\sigma(t)$ to be the spectral distribution function of the pair B_0, $\tilde{B}_0$. Indeed, calculating the function $V_\Theta(z)$ for the system Θ above directly via formula (3.17) with $\mu = \infty$ and comparing the result to $V(z)$ gives the exact value of $h = i$. Using the reasoning of remark 5.6 we confirm that $\sigma(t)$ is the spectral distribution function of the pair B_0, $\tilde{B}_0$.

REMARK 6.1. All the derivations above can be repeated for a Stieltjes like function

$$V(z) = \gamma + \frac{i}{\sqrt{z}}, \qquad -\infty < \gamma < +\infty,\ \gamma \neq 0$$

with very minor changes. In this case the restored values for h and μ are described as follows:

$$h = x + iy, \qquad x = \frac{\gamma}{1+\gamma^2}, \qquad y = \frac{1}{1+\gamma^2}, \qquad \mu = \frac{1}{\gamma}.$$

The dynamics of changing h according to changing γ is depicted on the figure 5 where the circle has the center at the point $i/2$ and radius of $1/2$. The behavior of μ is described by a hyperbola $\mu = 1/\gamma$ (see figure 9 with $\theta = 0$). In the case when $\gamma > 0$ our function becomes Stieltjes and the restored system Θ is accretive. The operators $\mathbb{A}$ and K of the restored system are given according to the formulas (2.6) and (3.14), respectively.

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Sergey Belyi
Department of Mathematics
Troy State University
Troy
AL 36082
USA
e-mail: sbelyi@troy.edu
URL: http://spectrum.troy.edu/∼belyi

Eduard Tsekanovskiĭ
Department of Mathematics
Niagara University
NY 14109
USA
e-mail: tsekanov@niagara.edu