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SPECTRAL PERTURBATION BOUNDS FOR SELFADJOINT OPERATORS I

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(*communicated by M. Omladič*)

Abstract. We give general spectral and eigenvalue perturbation bounds for a selfadjoint operator perturbed in the sense of the pseudo-Friedrichs extension. We also give several generalisations of the aforementioned extension. The spectral bounds for finite eigenvalues are obtained by using analyticity and monotonicity properties (rather than variational principles) and they are general enough to include eigenvalues in gaps of the essential spectrum.

1. Introduction

The main purpose of this paper is to derive spectral and eigenvalue bounds for selfadjoint operators. If a selfadjoint operator H in a Hilbert space $\mathscr{H}$ is perturbed into

$$T = H + A \tag{1}$$

with, say, a bounded A then the well-known spectral inclusion holds

$$\sigma(T) \subseteq \{\lambda : \ \mathrm{dist}(\lambda, \sigma(H)) \leqslant \|A\|\}. \tag{2}$$

Here σ denotes the spectrum of a linear operator. (Whenever not otherwise stated we shall follow the notation and the terminology of [3].)

If H, A, T are finite Hermitian matrices then (1) implies

$$|\mu_k - \lambda_k| \leqslant \|A\|, \tag{3}$$

where μ_k, λ_k are the non-increasingly ordered eigenvalues of T, H, respectively. (Here and henceforth we count the eigenvalues together with their multiplicities.)

Whereas (2) may be called an upper semicontinuity bound the estimate (3) contains an *existence* statement: each of the intervals $[\lambda_k - \|A\|, \ \lambda_k + \|A\|]$ contains 'its own' μ_k.

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Colloquially, bounds like (2) may be called 'one-sided' and those like (3) 'two-sided'. As it is well-known (3) can be refined to another two-sided bound

$$\min \sigma(A) \leqslant \mu_k - \lambda_k \leqslant \max \sigma(A). \tag{4}$$

In [9] the following 'relative' two-sided bound was derived

$$|\mu_k - \lambda_k| \leqslant b|\lambda_k|, \tag{5}$$

provided that

$$|(A\psi, \psi)| \leqslant b(|H|\psi, \psi), \quad b < 1.$$

This bound was found to be relevant for numerical computations. Combining (3) and (5) we obtain

$$|\mu_k - \lambda_k| \leqslant a + b|\lambda_k|, \tag{6}$$

or, equivalently,

$$\lambda_k - a - b|\lambda_k| \leqslant \mu_k \leqslant \lambda_k + a + b|\lambda_k|, \tag{7}$$

provided that

$$|(A\psi, \psi)| \leqslant a\|\psi\|^2 + b(|H|\psi, \psi), \quad b < 1. \tag{8}$$

One of our goals is to extend the bound (6) to general selfadjoint operators. Since these may be unbounded we have to make precise what we mean by the sum (1). Now, the condition (8) is exactly the one which guarantees the existence and the uniqueness of a closed extension T of $H + A$, if, say, $\mathscr{D}(A) \supseteq \mathscr{D}(|H|^{1/2})$. The operator T is called *the pseudo-Friedrichs extension* of $H + A$, see [3], Ch. VI. Th. 3.11. Further generalisations of this construction are contained in [2, 6, 5]. All they allow A to be merely a quadratic form, so (1) is understood as the form sum; note that the estimate (8) concerns just forms. Particularly striking by its simplicity is the construction made in [5] for the so-called quasidefinite operators (finite matrices with this property have been studied in [8], cf. also the references given there). Let H, A be bounded and, in the intuitive matrix notation,

$$H = \begin{bmatrix} H_+ & 0 \\ 0 & -H_- \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}, \tag{9}$$

with $H_\pm$ positive definite. Then

$$T = \begin{bmatrix} 1 & 0 \\ B^* H_+^{-1} & 1 \end{bmatrix} \begin{bmatrix} H_+ & 0 \\ 0 & -H_- - B^* H_+^{-1} B \end{bmatrix} \begin{bmatrix} 1 & H_+^{-1} B \\ 0 & 1 \end{bmatrix} \tag{10}$$

with an obvious bounded inverse. This is immediately transferable to unbounded H, A provided that $F = H_+^{-1/2} B H_-^{-1/2}$ is bounded. Indeed, then (10) can be rewritten as

$$T = |H|^{1/2} \begin{bmatrix} 1 & 0 \\ F^* & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 - F^* F \end{bmatrix} \begin{bmatrix} 1 & F \\ 0 & 1 \end{bmatrix} |H|^{1/2} \tag{11}$$

which is selfadjoint as a product of factors which have bounded inverses. Note that in (8) we have $a = 0$ and $b = \|F\|$ and the latter need not be less than one!

In fact, our first task will be to derive further constructions of operators defined as form sums. One of them takes in (9)

$$A = \begin{bmatrix} A_+ & B \\ B^* & A_- \end{bmatrix},$$

where $A_\pm$ are $H_\pm$-bounded as in (8). So, we require $b < 1$ only for 'diagonal blocks'. Another one exhibits 'off-diagonal dominance' inasmuch as $H_\pm$ in (9) are a sort of B-bounded. All these constructions as well as those from [3, 2, 6, 5] are shown to be contained in a general abstract theorem which also helps to get a unified view of the material scattered in the literature. This is done in Sect. 2.

As a rule each such construction will also contain a spectral inclusion like (2). In Sect. 3 we will give some more inclusion theorems under the condition (8) as an immediate preparation for eigenvalue estimates. In the proofs the quasidefinite structure will be repeatedly used. Moreover, the decomposition (10) and the corresponding invertibility property will be carried over to the Calkin algebra, thus allowing tight control of the spectral movement including the monotonicity in gaps both for the total and the essential spectra.

In Sect. 4 we consider two-sided bounds for finite eigenvalues. They are obtained by using analyticity and monotonicity properties.[1] In order to do this we must be able

(i) to count the eigenvalues (note that we may be in a gap of the essential spectrum) and

(ii) to keep the essential spectrum away from the considered region.

The condition (i) is achieved by requiring that at least one end of the considered interval be free from spectrum during the perturbation (we speak od 'impenetrability'). This will be guaranteed by one of the spectral inclusion theorems mentioned above. Similarly, (ii) is guaranteed by analogous inclusions for the essential spectrum. Based on this we first prove a monotonicity result for a general class of selfadjoint holomorphic families and then establish the bound (6) as well as an analogous relative bound generalising (4) which includes the monotonicity of eigenvalues in spectral gaps. Another result, perhaps even more important in practice, is the one in which the form A is perturbed into B with $B - A$ small with respect to A (this corresponds to relatively small perturbations of the potential in quantum mechanical applications). In this case the necessary impenetrability is obtained by a continuation argument which assumes the knowledge of the whole family $H + \eta A$ instead of the mere unperturbed operator $H + A$. All our eigenvalue bounds are sharp.

The corresponding eigenvector bounds as well as systematic study of applications to various particular classes of operators will be treated in forthcoming papers.

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[1] Another possible approach to the monotonicity could be to use variational principles valid also in spectral gaps, see e.g. [4] or [1] but we found the analyticity more elegant.

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2. Construction of operators

Here we will give various constructions of selfadjoint operators by means of forms (cf. [3, 2, 6, 5]). Sometimes our results will generalise the aforementioned ones only slightly, but we will still give the proofs because their ingredients will be used in the later work. We shall include non-symmetric perturbations whenever the proofs naturally allow such possibility.

DEFINITION 2.1. We say that the open interval (λ_-, λ_+) is a *spectral gap* of a selfadjoint operator H, if this interval belongs to the resolvent set $\rho(H)$ and its ends, if finite, belong to the spectrum $\sigma(H)$. The *essential spectral gap* is defined analogously.

DEFINITION 2.2. We say that a sesquilinear form τ, defined in a Hilbert space $\mathscr{H}$ on a dense domain $\mathscr{D}$ *represents* an operator T, if

$$T \text{ is closed and densely defined,} \tag{12}$$

$$\mathscr{D}(T), \mathscr{D}(T^*) \subseteq \mathscr{D} \tag{13}$$

$$(T\psi, \phi) = \tau(\psi, \phi), \quad \psi \in \mathscr{D}(T),\ \phi \in \mathscr{D}, \tag{14}$$

$$(\psi, T^*\phi) = \tau(\psi, \phi), \quad \psi \in \mathscr{D},\ \phi \in \mathscr{D}(T^*). \tag{15}$$

PROPOSITION 2.3. *A closed, densely defined operator T is uniquely defined by (12)–(15). If τ is symmetric then T is selfadjoint.*

Proof. Suppose that T_1 satisfies (12)–(15). Then

$$(T\psi, \phi) = \tau(\psi, \phi) = (\psi, T_1^*\phi), \quad \psi \in \mathscr{D}(T),\ \phi \in \mathscr{D}(T_1^*),$$

$$(T_1\psi, \phi) = \tau(\psi, \phi) = (\psi, T^*\phi), \quad \psi \in \mathscr{D}(T_1),\ \phi \in \mathscr{D}(T^*).$$

The first relation implies $T_1 \supseteq T$ and the second $T \supseteq T_1$. If τ is symmetric then by (14) and (15) both T and T^* are symmetric and therefore equal. □

Let H be selfadjoint in a Hilbert space $\mathscr{H}$ and let $\alpha(\cdot,\cdot)$ be a sesquilinear form defined on $\mathscr{D}$ such that

$$|\alpha(\psi, \phi)| \leqslant \|H_1^{1/2}\psi\|\|H_1^{1/2}\phi\| \quad \psi, \phi \in \mathscr{D} \tag{16}$$

where $\mathscr{D}$ is a core for $|H|^{1/2}$ and

$$H_1 = a + b|H|, \quad a, b \text{ real},\ b \geqslant 0, \quad H_1 \text{ positive definite.} \tag{17}$$

Then the formula

$$(C\psi, \phi) = \alpha(H_1^{-1/2}\psi, H_1^{-1/2}\phi), \quad \psi, \phi \in \mathscr{D}, \tag{18}$$

defines a $C \in \mathscr{B}(\mathscr{H})$ with

$$\|C\| \leqslant 1 \tag{19}$$

(note that $H_1^{1/2}\mathscr{D}$ is dense in $\mathscr{H}$). The form α can obviously be extended to the form $\alpha_{\mathscr{Q}}$, defined on the subspace

$$\mathscr{Q} = \mathscr{D}(|H|^{1/2}) = \mathscr{D}(H_1^{1/2}) \tag{20}$$

by the formula

$$\alpha_{\mathscr{Q}}(\psi,\phi) = \lim_{n,m\to\infty} \alpha(\psi_n,\phi_m) \tag{21}$$

for any sequence $\psi_n \to \psi$, $\phi_m \to \phi$, $H_1^{1/2}\psi_n \to H_1^{1/2}\psi$, $H_1^{1/2}\phi_m \to H_1^{1/2}\phi$. Then (16) holds for $\alpha_{\mathscr{Q}}$ on $\mathscr{Q}$ and

$$(C\psi,\phi) = \alpha_{\mathscr{Q}}(H_1^{-1/2}\psi, H_1^{-1/2}\phi), \quad \psi,\phi \in \mathscr{H}. \tag{22}$$

The sesquilinear form for H is defined on $\mathscr{Q}$ as

$$h(\psi,\phi) = (J|H|^{1/2}\psi, |H|^{1/2}\phi) \tag{23}$$

with

$$J = \operatorname{sign} H. \tag{24}$$

In general there may be several different sign functions J of H with $J^2 = 1$. The form h does not depend on the choice of J. Note that h above represents the operator H in the sense of Definition 2.2.

THEOREM 2.4. *Let H, α, C, $\mathscr{D}$ $\mathscr{Q}$ be as above and such that*

$$C_\zeta = (H-\zeta)H_1^{-1} + C \tag{25}$$

is invertible in $\mathscr{B}(\mathscr{H})$ for some $\zeta \in \mathbf{C}$. Then the form

$$\tau = h + \alpha_{\mathscr{Q}} \tag{26}$$

represents a unique closed densely defined operator T whose domain is a core for $|H|^{1/2}$ and which is given by

$$T - \zeta = H_1^{1/2} C_\zeta H_1^{1/2}, \tag{27}$$

$$T^* - \overline{\zeta} = H_1^{1/2} C_\zeta^* H_1^{1/2}, \quad \zeta \in \mathbf{C} \tag{28}$$

and, whenever $C_\zeta^{-1} \in \mathscr{B}(\mathscr{H})$,

$$(T-\zeta)^{-1} = H_1^{-1/2} C_\zeta^{-1} H_1^{-1/2} \in \mathscr{B}(\mathscr{H}), \tag{29}$$

$$(T^*-\overline{\zeta})^{-1} = H_1^{-1/2} C_\zeta^{-*} H_1^{-1/2} \in \mathscr{B}(\mathscr{H}). \tag{30}$$

We call T the form sum of H and α and write

$$T = H + \alpha. \tag{31}$$

If α is symmetric then T is selfadjoint.

Proof. In view of what was said above we may obviously suppose that $\mathscr{D}$ is already equal to $\mathscr{Q}$.[2] We first prove that $\mathscr{D}(H_1^{1/2}C_\zeta H_1^{1/2})$ is independent of ζ and is dense in $\mathscr{H}$. Indeed, for $\zeta, \zeta' \in \mathbf{C}$ and $\psi \in \mathscr{D}(H_1^{1/2}C_\zeta H_1^{1/2})$ we have $\psi \in \mathscr{Q}$ and

$$\begin{aligned}\mathscr{Q} \ni C_\zeta H_1^{1/2}\psi &= (H-\zeta)H_1^{-1}H_1^{1/2}\psi + CH_1^{1/2}\psi \\ &= (H-\zeta')H_1^{-1/2}\psi + CH_1^{1/2}\psi + (\zeta'-\zeta)H_1^{-1/2}\psi \\ &= C_{\zeta'}H_1^{1/2}\psi + (\zeta'-\zeta)H_1^{-1/2}\psi.\end{aligned}$$

Thus, by $(\zeta'-\zeta)H_1^{-1/2}\psi \in \mathscr{Q}$ we have $C_{\zeta'}H_1^{1/2}\psi \in \mathscr{Q}$, hence $\psi \in \mathscr{D}(H_1^{1/2}C_{\zeta'}H_1^{1/2})$. Since ζ, ζ' are arbitrary $\mathscr{D}(H_1^{1/2}C_\zeta H_1^{1/2})$ is indeed independent of ζ and (27) holds. Now take ζ with $C_\zeta^{-1} \in \mathscr{B}(\mathscr{H})$. Then the three factors on the right hand side of (27) have bounded, everywhere defined inverses, so (29) holds as well and T is closed. We now prove that $\mathscr{D}(T)$ is a core for $|H|^{1/2}$ or, equivalently, for $H_1^{1/2}$. That is, $H_1^{1/2}\mathscr{D}(T)$ must be dense in $\mathscr{H}$ (see [3] III, Exercise 51.9). By taking ζ with $C_\zeta^{-1} \in \mathscr{B}(\mathscr{H})$ we have

$$H_1^{1/2}\mathscr{D}(T) = H_1^{1/2}\mathscr{D}(T-\zeta) = H_1^{1/2}\left\{\psi \in \mathscr{Q}:\ C_\zeta H_1^{1/2}\psi \in \mathscr{Q}\right\} = C_\zeta^{-1}\mathscr{Q}$$

and this is dense because C_ζ maps bicontinuously $\mathscr{H}$ onto itself. In particular, $\mathscr{D}(T)$ is dense in $\mathscr{H}$. By

$$C_\zeta^* = (H-\overline{\zeta})H_1^{-1} + C^*$$

all properties derived above are seen to hold for T^* as well. The identities (14), (15) follow immediately from (27) by using the obvious identity

$$\tau(\psi,\phi) - \zeta(\psi,\phi) = (C_\zeta H_1^{1/2}\psi, H_1^{1/2}\phi), \tag{32}$$

valid for any $\psi, \phi \in \mathscr{Q}$, $\zeta \in \mathbf{C}$. Finally, if α is symmetric then T, T^* is also symmetric and therefore selfadjoint. □

COROLLARY 2.5. *Let H, α, τ, T be as in Theorem 2.4. Then*

$$\tau(\psi,\phi) = (\chi,\phi)$$

for some $\psi \in \mathscr{Q}$, $\chi \in \mathscr{H}$ and all $\phi \in \mathscr{Q}$ is equivalent to

$$\psi \in \mathscr{D}(T), \quad T\psi = \chi.$$

Proof. We use the identity (32):

$$(C_0H_1^{1/2}\psi, H_1^{1/2}\phi) = (\chi,\phi)$$

for all $\phi \in \mathscr{Q}$ and the selfadjointness of $H_1^{1/2}$ imply $C_0H_1^{1/2}\psi \in \mathscr{Q} = \mathscr{D}(|H|^{1/2})$ i.e. $\psi \in \mathscr{D}(|H|^{1/2}C_0H_1^{1/2}) = \mathscr{D}(T)$ and $|H|^{1/2}C_0H_1^{1/2}\psi = T\psi = \chi$. □

[2] This assumption will be made throughout the rest of the paper, if not stated otherwise.

REMARK 2.6. Although fairly general, Theorem 2.4 does not cover all relevant form representations. If $T = H + \alpha$ and α_1 is any bounded form, then $\tau_1 = \tau + \alpha_1$ obviously generates a T_1 in the sense of Def. 2.2 – we again write $T = H + \alpha + \alpha_1$ – while H, $\alpha + \alpha_1$ need not satisfy the conditions of Theorem 2.4.

REMARK 2.7. If α is symmetric then (16) is equivalent to

$$|\alpha(\psi,\psi)| \leqslant \|H_1\psi\|^2. \tag{33}$$

In general, (33) implies (16) but with b replaced by $2b$.

REMARK 2.8. By Proposition 2.3 the operator $T = H + \alpha$ does not depend on the choice of a, b in the operator H_1 from (17). Moreover, in the construction (27) H_1 may be replaced by any selfadjoint $H_2 = f(H)$ where f is a real positive-valued function and

$$0 < m \leqslant \frac{a + b|\lambda|}{f(\lambda)} \leqslant M < \infty.$$

Then

$$(T - \zeta)^{-1} = H_2^{-1/2} D_\zeta^{-1} H_2^{-1/2}, \tag{34}$$

where

$$D_\zeta = (H - \zeta)H_2^{-1} + D,$$
$$D = H_1^{1/2} f(H)^{-1/2} C H_1^{1/2} f(H)^{-1/2}$$

and

$$D_\zeta = H_1^{1/2} f(H)^{-1/2} C_\zeta H_1^{1/2} f(H)^{-1/2}$$

is invertible in $\mathscr{B}(\mathscr{H})$, if and only if C_ζ is such.

COROLLARY 2.9. *Let H, $H_1 = a + |H|$, $\mathscr{Q}$, C $\alpha = \alpha_{\mathscr{Q}}$, h and J be as in (20)–(24) such that $J + C$ is invertible in $\mathscr{B}(\mathscr{H})$. Then the form $\tau = h + \alpha$ represents a unique closed densely defined operator $T = H + \alpha$ in the sense of Remark 2.6. Moreover, $\mathscr{D}(T)$ is a core for $|H|^{1/2}$ and*

$$T + aJ = H_1^{1/2}(J + C)H_1^{1/2} \tag{35}$$

(and similarly for T^*).

Note that the preceding construction – in contrast to the related one in Theorem 2.4 does not give an immediate representation of the resolvent, except, if $a = 0$.

In the following theorem we will use the well known formulae

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}, \tag{36}$$

$$(\lambda - BA)^{-1} = \frac{1}{\lambda} + \frac{B(\lambda - AB)^{-1}A}{\lambda} \tag{37}$$

$$Bf(AB) = f(BA)B, \tag{38}$$

where $A, B \in \mathscr{B}(\mathscr{H})$ and f is analytic on $\sigma(AB) \cup \{0\}$.

THEOREM 2.10. *Let H, α, $\mathscr{Q}$, C satisfy (16)–(22). Let, in addition,*

$$C = Z_2^* Z_1,\ Z_{1,2} \in \mathscr{B}(\mathscr{H}). \tag{39}$$

Then C_ζ from (25) is invertible in $\mathscr{B}(\mathscr{H})$, if and only if

$$F_\zeta = 1 + Z_1 H_1 (H-\zeta)^{-1} Z_2^*. \tag{40}$$

is such. In this case Theorem 2.4 holds and

$$(T-\zeta)^{-1} = (H-\zeta)^{-1} - H_1^{1/2}(H-\zeta)^{-1} Z_2^* F_\zeta^{-1} Z_1 H_1^{1/2} (H-\zeta)^{-1}. \tag{41}$$

Proof. C_ζ is invertible in $\mathscr{B}(\mathscr{H})$, if and only if

$$1 + H_1(H-\zeta)^{-1} C = 1 + H_1 (H-\zeta)^{-1} Z_2^* Z_1$$

is invertible in $\mathscr{B}(\mathscr{H})$. Now,

$$\sigma(H_1(H-\zeta)^{-1} Z_2^* Z_1) \setminus \{0\} = \sigma(Z_1 H_1 (H-\zeta)^{-1} Z_2^*) \setminus \{0\}$$

Hence F_ζ is invertible in $\mathscr{B}(\mathscr{H})$ if an only if C_ζ is such. In this case (29) gives

$$\begin{aligned}
(T-\zeta)^{-1} &= H_1^{-1/2}(1 + H_1(H-\zeta)^{-1} Z_2^* Z_1)^{-1} H_1^{1/2} (H-\zeta)^{-1} \\
&= H_1^{-1/2}\left(1 - H_1(H-\zeta)^{-1} Z_2^* Z_1 (1 + H_1 (H-\zeta)^{-1} Z_2^* Z_1)^{-1}\right) H_1^{1/2} (H-\zeta)^{-1} \\
&= (H-\zeta)^{-1} - H_1^{1/2}(H-\zeta)^{-1} Z_2^* F_\zeta^{-1} Z_1 H_1^{1/2} (H-\zeta)^{-1}.
\end{aligned}$$

□

We now apply Theorem 2.4 to further cases in which the key operator C_ζ from (25) is invertible in $\mathscr{B}(\mathscr{H})$.

THEOREM 2.11. *Let H be selfadjoint and let α satisfy (16) with $b < 1$ and (17). Then the conditions of Theorem 2.4 are satisfied and $\zeta = \lambda + i\eta \in \rho(T)$ whenever*

$$|\eta| > \frac{a + |\lambda| b.}{\sqrt{1-b^2}} \tag{42}$$

Proof. To prove $C_\zeta^{-1} = \left((H-\zeta)H_1^{-1} + C\right)^{-1} \in \mathscr{B}(\mathscr{H})$ it is enough to show

$$\|H_1(H-\zeta)^{-1}\| < 1. \tag{43}$$

for some $\zeta = \lambda + i\eta$. Now,

$$\|(H-\zeta)^{-1} H_1\| \leqslant \sup_{\xi \in \mathbf{R}} \psi(\xi, a, b, \lambda, \eta), \quad \psi(\xi, a, b, \lambda, \eta) = \frac{b|\xi| + a}{\sqrt{(\xi-\lambda)^2 + \eta^2}}$$

A straightforward, if a bit tedious, calculation (see Appendix) shows

$$\max_\xi \psi = \frac{1}{|\eta|}\sqrt{(a + |\lambda| b)^2 + b^2 \eta^2} \tag{44}$$

Hence (42) implies (43). □

Another similar criterion for the validity of Theorem 2.4 — oft independent of that of Theorem 2.11 — is given by the following

COROLLARY 2.12. *Let H, α, C satisfy (16)–(18) and let*

$$\|Z_1 H_1 (H-\zeta)^{-1} Z_2^*\| < 1. \tag{45}$$

for some $\zeta \in \rho(H)$ and with $Z_{1,2}$ from (39). Then Theorem 2.10 applies.

Typically we will have

$$\alpha(\psi,\phi) = (V_1\psi, V_2\phi), \tag{46}$$

where $V_{1,2}$ are linear operators defined on $\mathscr{Q}$ such that

$$Z_{1,2} = V_{1,2} H_1^{1/2} \in \mathscr{B}(\mathscr{H}). \tag{47}$$

In this case the formula (45) can be given a more familiar, if not always rigorous, form (cf. [6])

$$\|V_1(H-\zeta)^{-1} V_2^*\| < 1.$$

REMARK 2.13. If $\alpha(\psi,\phi) = (A\psi,\phi)$, where A is a linear operator defined on $\mathscr{D} \subseteq \mathscr{D}(H)$, $\mathscr{D}$ a core for $|H|^{1/2}$ then Theorem 2.11 applies and, by construction, the obtained operator coincides with the one in [3] VI. Th. 3.11. The uniqueness of T as an extension of $H+A$, proved in [3] makes no sense in our more general, situation. Our notion of form uniqueness (which was used by [6] in the symmetric case) will be appropriate in applications to both Quantum and Continuum Mechanics. Thus, our Theorem 2.11 can be seen as a slight generalisation of [3]. On the other side, our proof of Theorem 2.4 closely follows the one from [3].

Cor. 2.9 and Th. 2.10 are essentially Theorems. 2.1, 2.2 in [6] except for the following: (i) our α need not be symmetric, (ii) we use a more general factorisability (39) instead of (46) which is supposed in [6] and finally, (iii) we need no relative compactness argument to establish Theorem 2.10. The fact that the mentioned results from [6] are covered by our theory will facilitate to handle perturbations of the form α which are not easily accessible, if α is factorised as in (46). The spectral inclusion formula (42) seems to be new.

Thus, our Theorem 2.4 seems to cover essentially all known constructions thus far.[3]

Next we give some results on the invariance of the essential spectrum.

THEOREM 2.14. *Let H, h, α, C, $\mathscr{D}$ $\mathscr{Q}$ satisfy (16)–(24) with α symmetric.*
(i) If the operator C is compact then Theorem 2.11 holds and $\sigma_{ess}(T) = \sigma_{ess}(H)$.
(ii) If Theorem 2.4 holds and $H_1^{-1}C$ is compact then again $\sigma_{ess}(T) = \sigma_{ess}(H)$.

[3] There are two obvious extensions: (i) adding a bounded form (Remark 2.6) and (ii) multiplying T by a bicontinuous operator. An example of the latter is $T = H + \alpha$ described in Cor. 2.9.

Proof. In any of the cases (i), (ii) we can find a ζ for which $C_\zeta^{-1} \in \mathscr{B}(\mathscr{H})$ (in the case (i) this follows from the known argument that for a compact C the estimate (16) will hold with arbitrarily small b) so Theorem 2.4 holds anyway. By (29) we have

$$\begin{aligned}(T-\zeta)^{-1}-(H-\zeta)^{-1} &= H_1^{-1/2}\left(((H-\zeta)H_1^{-1}+C)^{-1}-H_1(H-\zeta)^{-1}\right)H_1^{-1/2}\\ &= H_1^{-1/2}\left((1+A)^{-1}-1\right)H_1^{1/2}(H-\zeta)^{-1}\end{aligned}$$

where $H_1^{-1}A = (H-\zeta)^{-1}C$ is compact and by $C_\zeta^{-1} \in \mathscr{B}(\mathscr{H})$ also $(1+A)^{-1} \in \mathscr{B}(\mathscr{H})$. Hence

$$(T-\zeta)^{-1}-(H-\zeta)^{-1} = -H_1^{-1/2}A(1+A)^{-1}H_1^{1/2}(H-\zeta)^{-1}$$

is compact and the Weyl theorem applies. □

Finally we borrow from [6] the following result which will be of interest for Dirac operators with strong Coulomb potentials.

THEOREM 2.15. *Let H, α, $\mathscr{Q}$, C, $V_{1,2}$, $Z_{1,2}$, T be as in Theorem 2.10. Let, in addition, C_ζ from Theorem 2.4 be invertible in $\mathscr{B}(\mathscr{H})$ and*

1. H have a bounded inverse,

2. the operator $Z_2^*(H-\zeta)^{-1}Z_1$ be compact for some (and then all) $\zeta \in \rho(H)$.

Then $\sigma_{ess}(T) \subseteq \sigma_{ess}(H)$.

The key invertibility of the operator C_ζ can be achieved in replacing the requirement $b < 1$ in (16) by some condition on the structure of the perturbation. One such structure is given, at least symbolically, by the matrix

$$\begin{bmatrix} W_+ & B \\ B & -W_- \end{bmatrix}, \tag{48}$$

where $W_\pm$ are accretive. Such operator matrices appear in various applications (Stokes operator, Dirac operator, especially on a manifold ([5], [10]) and the like). Even more general cases could be of interest, namely those where $b < 1$ in (16) is required to hold only on the "diagonal blocks" of the perturbation α. We have

THEOREM 2.16. *Let H, $\alpha = \alpha_{\mathscr{Q}}$, C, h satisfy (16)–(24) such that H has a spectral gap (λ_-, λ_+) containing zero. Suppose*

$$\pm\,\Re\alpha(\psi,\psi) \leqslant a_\pm\|\psi\|^2 + b_\pm\||H-d|^{1/2}\psi\|^2, \quad \psi \in P_\pm\mathscr{Q}, \tag{49}$$

$$a_\pm > 0, \quad 0 < b_\pm < 1, \tag{50}$$

$$\alpha(\psi,\phi) = \overline{\alpha(\phi,\psi)}, \quad \psi \in P_+\mathscr{Q}, \quad \phi \in P_-\mathscr{Q}. \tag{51}$$

where $P_\pm = (1 \pm J)/2$. Finally, suppose

$$\widehat{\lambda}_- = \lambda_- + b_-|\lambda_-| < \widehat{\lambda}_+ = \lambda_+ - b_+|\lambda_+|. \tag{52}$$

Then $\tau = h + \alpha$ generates a closed, densely defined operator T with $\mathscr{D}(T)$ a core for $|H|^{1/2}$ and

$$(\widehat{\lambda}_-, \widehat{\lambda}_+) + i\mathbf{R} \subseteq \rho(T). \tag{53}$$

The operator T is selfadjoint, if α is symmetric.

Proof. We split the perturbation α into two parts

$$\alpha = \chi + \chi'$$

where χ is the 'symmetric diagonal part' of α, that is,

$$\alpha_d(\psi,\phi) = \alpha(P_+\psi, P_+\phi) + \alpha(P_-\psi, P_-\phi),$$
$$\chi(\psi,\phi) = \frac{1}{2}\left(\alpha_d(\psi,\phi) + \overline{\alpha_d(\phi,\psi)}\right).$$

Symbolically,[4]

$$\chi = \begin{bmatrix} \chi_+ & 0 \\ 0 & -\chi_- \end{bmatrix}, \qquad h = \begin{bmatrix} h_+ & 0 \\ 0 & -h_- \end{bmatrix}.$$

Now (49) and the standard perturbation result for closed symmetric forms ([3] Ch. VI, Th. 3.6) implies $\widetilde{h}_\pm = h_\pm + \chi_\pm$ is symmetric, bounded from below by

$$\pm\lambda_\pm - b_\pm|\lambda_\pm| - a_\pm$$

and closed on $\mathscr{Q}$. The thus generated selfadjoint operator $\widetilde{H}_\pm$ has $\mathscr{D}(|\widetilde{H}_\pm|^{1/2}) = P_\pm\mathscr{Q}$. Now,

$$\tau = h + \alpha = \widetilde{h} + \chi', \qquad \widetilde{h} = \begin{bmatrix} \widetilde{h}_+ & 0 \\ 0 & -\widetilde{h}_- \end{bmatrix}.$$

We write

$$\tau = h + \alpha = \widetilde{h} + \chi', \qquad \widetilde{h} = \begin{bmatrix} \widetilde{h}_+ & 0 \\ 0 & -\widetilde{h}_- \end{bmatrix}, \qquad \widetilde{H} = \begin{bmatrix} \widetilde{H}_+ & 0 \\ 0 & -\widetilde{H}_- \end{bmatrix},$$

where $\widetilde{H}$ has a spectral gap contained in $(\widetilde{\lambda}_-, \widetilde{\lambda}_+)$ and

$$J = \operatorname{sign}(\widetilde{H} - d) = \operatorname{sign} H, \qquad \widetilde{\lambda}_- < d < \widetilde{\lambda}_+.$$

We will apply Theorem 2.4 to $\widetilde{H}$, χ'. We have first to prove that $\widetilde{H}, \chi'$ satisfy the conditions (16), (17) (possibly with different constants a, b). By (49) we have

$$0 \leqslant h_\pm \leqslant \frac{a_\pm}{1 - b_\pm} + \frac{\widetilde{h}_\pm}{1 - b_\pm}.$$

Hence

$$|H| \leqslant c|\widetilde{H} - d|$$

for any $d \in (\widetilde{\lambda}_-, \widetilde{\lambda}_+)$ and some $c = c(d)$. So, $\widetilde{H}, \chi'$ satisfy (16), (17) with $|H|$ replaced by $|\widetilde{H} - d|$. We take $\zeta = d + i\eta$ and set

$$\widetilde{T} - \zeta = |\widetilde{H} - d|^{1/2} D_\zeta |\widetilde{H} - d|^{1/2} \tag{54}$$

$$D_\zeta = J - \zeta|\widetilde{H} - d|^{-1} + D,$$

[4] Throughout this paper we will freely use matrix notation for bounded operators as well as for unbounded ones or forms whenever the latter are unambigously defined. The matrix partition refers to the orthogonal decomposition $\mathscr{H} = P_+\mathscr{H} \oplus P_-\mathscr{H}$.

$$(D\psi, \phi) = \chi'(|\widetilde{H} - d|^{-1/2}\psi, |\widetilde{H} - d|^{-1/2}\phi),$$
$$D = \begin{bmatrix} D_+ & F \\ F^* & -D_- \end{bmatrix}.$$

By the construction we have

$$\Re\chi'(P_\pm\psi, P_\pm\psi) = 0. \tag{55}$$

Hence $D_\pm$ are skew Hermitian and

$$D_\zeta = \begin{bmatrix} 1 - i\eta(\widetilde{H}_+ - d)^{-1} + D_+ & F \\ F^* & -1 - i(d - \widetilde{H}_-)^{-1} - D_- \end{bmatrix}$$

where the first diagonal block is uniformly accretive and the second uniformly dissipative, so $D_\zeta^{-1} \in \mathscr{B}(\mathscr{H})$ by virtue of the factorisation (10) which obviously holds in this case, too. Thus,

$$(T - \zeta)^{-1} = |\widetilde{H} - d|^{-1/2}\widetilde{D}_\zeta^{-1}|\widetilde{H} - d|^{1/2} \in \mathscr{B}(\mathscr{H})$$

and Theorem 2.4 applies. Note also that $|H|^{1/2}$ and $|\widetilde{H}|^{1/2}$ have the same set of cores. □

COROLLARY 2.17. *If in the preceding theorem we drop the condition (52) or even the existence of the spectral gap of H we still have $T = H + \alpha$ but without the spectral inclusion (53).*

Proof. We first apply the preceding theorem to $\widehat{T} = \widehat{H} + \alpha$ with $\widehat{H} = H + \delta J$ and $\delta > 0$ large enough to insure that (52) holds. Then set $T = \widehat{T} - \delta J$. □

REMARK 2.18. Theorem 2.16 becomes particularly elegant, if we set $a_\pm, b_\pm = 0$. If, in addition, α is taken as symmetric then we have a 'quasidefinite form' τ as was mentioned in Sect. 1. In this case we require only the condition (16) with no restriction on the size of a, b (for H, α non-negative this is a well-known fact).

There is an alternative proof of Theorem 2.16 which we now illustrate (we assume for simplicity that $a_\pm = 0$). Instead of the pair H, α consider $JH = |H|$, $J\alpha$ where $J\alpha$ is the 'product form' naturally defined by

$$J\alpha(\psi, \phi) = \alpha(\psi, J\phi)$$

As one immediately sees the new form

$$J\tau = Jh + J\alpha$$

is sectorial and its symmetric part Jh is closed non-negative, so by the standard theory ([3] Ch. VI. §3) $J\tau$ generates a closed sectorial operator which we denote by JT. Symbolically,

$$JT = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} A_+ & B \\ B^* & -A_- \end{bmatrix} = \begin{bmatrix} A_+ & B \\ -B^* & A_- \end{bmatrix}.$$

The reason why we still stick at our previous proof is its constructivity (here we have no direct access to the resolvent) as well as its 'symmetry', (here even for a symmetric α a detour through non-symmetric objects is made).

Another case in which Theorem 2.4 can be applied is the one in which (48) is 'off-diagonally dominant' (cf. [10]). We set

$$H = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix} \tag{56}$$

where B is a closed, densely defined operator between the Hilbert spaces $\mathscr{H}_-$ and $\mathscr{H}_+$. It is easy to see that H is selfadjoint on $\mathscr{D}(B^*) \oplus \mathscr{D}(B)$ (see [7], Lemma 5.3). Denote by

$$B = U\sqrt{B^*B} \tag{57}$$

the corresponding polar decomposition (see [3], Ch. VI, §2.7) and suppose that U is an isometry from $\mathscr{H}_-$ onto $\mathscr{H}_+$. Then

$$H = \begin{bmatrix} 0 & U\sqrt{B^*B} \\ U^*\sqrt{BB^*} & 0 \end{bmatrix} = \begin{bmatrix} 0 & U \\ U^* & 0 \end{bmatrix} \begin{bmatrix} \sqrt{BB^*} & \\ 0 & \sqrt{B^*B} \end{bmatrix} = J|H|,$$

$$J^2 = I.$$

The form α is defined as follows. Denoting

$$\psi = \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}, \quad \psi_+ \in \mathscr{D}(B^*),\ \psi_- \in \mathscr{D}(B)$$

we set

$$\alpha(\psi, \phi) = \alpha_+(\psi_+, \phi_+) - \alpha_-(\psi_-, \phi_-) \tag{58}$$

where $\alpha_\pm$, defined on $\mathscr{D}(B^*)$, $\mathscr{D}(B)$, respectively, are symmetric and non-negative.

THEOREM 2.19. *Let H, α, B, U be as above. Let*

$$\alpha_+(\psi, \psi) \leqslant a_+\|\psi\|^2 + b_+\left((BB^*)^{1/2}\psi, \psi\right), \quad \psi \in \mathscr{D}(B^*), \tag{59}$$

$$\alpha_-(\psi, \psi) \leqslant a_-\|\psi\|^2 + b_-\left((B^*B)^{1/2}\psi, \psi\right), \quad \psi \in \mathscr{D}(B), \tag{60}$$

for some $a_\pm, b_\pm > 0$. Then $\tau = h + \alpha$ generates a unique $T = H + \alpha$ in the sense of Definition 2.2.

Proof. Since α is defined on

$$\mathscr{D}(B^*) \oplus \mathscr{D}(B) = \mathscr{D}(\sqrt{BB^*}) \oplus \mathscr{D}(\sqrt{B^*B})$$

which is obviously a core for $|H|^{1/2}$ we can use (21) to extend α to $\mathscr{Q}$ still keeping the estimates (59), (60) and similarly with $\alpha_\pm$. (For simplicity we denote the extended forms again by $\alpha, \alpha_\pm$, respectively.)

We first consider the special case, in which B has an inverse in $\mathscr{B}(\mathscr{H}_+, \mathscr{H}_-)$. Then we can obviously assume that $a_\pm = 0$ (by increasing the size of $b_\pm$, if necessary, note that now both BB^*, B^*B are positive definite). Clearly, $H^{-1} \in \mathscr{B}(\mathscr{H})$, so we may use the representation $T = |H|^{1/2}(J+D)|H|^{1/2}$ with

$$J = \operatorname{sign} H = \begin{bmatrix} 0 & U \\ U^* & 0 \end{bmatrix}$$

and

$$D = \begin{bmatrix} D_+ & 0 \\ 0 & -D_- \end{bmatrix}$$

where $D_\pm$ are bounded symmetric non-negative. Now, we have to prove the bounded invertibility of

$$J + D = \begin{bmatrix} 1 & 0 \\ -D_-U^* & 1 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & U^* + D_-U^*D_+ \end{bmatrix} \begin{bmatrix} U^*D_+ & 1 \\ 1 & 0 \end{bmatrix} \tag{61}$$

which, in turn, depends on the bounded invertibility of

$$U^* + D_-U^*D_+$$

or, equivalently, of $1 + UD_-U^*D_+$. The latter is true because the spectrum of the product of two bounded symmetric non-negative operators is known to be real and non-negative.

In general we first apply Theorem 2.19 to $\tau_1 = h_1 + \alpha$ where h_1 belongs to

$$\begin{bmatrix} 0 & B_1 \\ B_1^* & 0 \end{bmatrix}$$

and

$$B_1 = (\sqrt{BB^*} + \delta)U, \ \delta > 0.$$

Indeed, by $B_1^*B_1 = B^*B + \delta$ and $B_1B_1^* = UB_1B_1^*U^* + \delta = BB^* + \delta$ (here we have used the assumed isomorphy property of U) the inequalities (58) are valid for B_1 as well. Thus, τ_1 generates T_1 and $\tau = \tau_1 + (\tau - \tau_1)$ generates T the difference $\tau - \tau_1$ being bounded. □

REMARK 2.20. If in the preceding theorem we have $B^{-1} \in \mathscr{B}(\mathscr{H}_+, \mathscr{H}_-)$ then we can take $a_\pm = 0$ and $T^{-1} \in \mathscr{B}(\mathscr{H})$ follows. This is immediately seen from the factorisation (61).

REMARK 2.21. The property of off-diagonal dominance was used in [10] for a special Dirac operator with a bounded form α including the decomposition (61). This decomposition has a similar disadvantage as the one described in Remark 2.18: it is not symmetric i.e. it has not the form of a congruence like e.g. (10), but we know of no better as yet.

If in (48) we drop the positive definiteness of, say, H_- we still may have a positive definite Schur complement. This gives one more possibility of constructing selfadjoint operators.

THEOREM 2.22. *Let τ be a symmetric sesquilinear form defined on a dense subspace $\mathscr{Q} \subseteq \mathscr{H}$. Let P_+, P_- be an orthogonal decomposition of the identity such that*

(i) $P_\pm \mathscr{Q} \subseteq \mathscr{Q}$,
(ii) τ, *restricted to* $P_+\mathscr{Q}$ *is closed and positive definite,*
(iii)

$$\sup_{\psi \in P_-\mathscr{Q}, \phi \in P_+\mathscr{H},\ \psi,\phi \neq 0} \frac{|\tau(\psi, H_+^{-1/2}\phi)|}{\|\psi\|\|\phi\|} < \infty,$$

where H_+ *is the operator generated by* τ *in* $P_+\mathscr{H}$,
(iv) *denoting by* $N \in \mathscr{B}(P_-\mathscr{H}, P_+\mathscr{H})$ *the operator, defined by* $(N\psi, \phi) = \tau(\psi, H_+^{-1/2}\phi)$, *the form*

$$P_-\mathscr{Q} \ni \psi, \phi \mapsto -\tau(\psi, \phi) + (N\psi, N\phi) \tag{62}$$

is closed and positive definite.

Then there exists a unique selfadjoint operator T *such that*
(a) $\mathscr{D}(T) \subseteq \mathscr{Q}$,
(b) $\tau(\psi, \phi) = (T\psi, \phi), \quad \psi \in \mathscr{D}(T), \phi \in \mathscr{Q}$.

The operator T *is given by the formulae*

$$T = WH_1W^*, \tag{63}$$

$$W = \begin{bmatrix} 1 & 0 \\ NH_+^{-1/2} & 1 \end{bmatrix} \in \mathscr{B}(\mathscr{H}), \tag{64}$$

$$H_1 = \begin{bmatrix} H_+ & 0 \\ 0 & -\widetilde{H}_- \end{bmatrix}, \tag{65}$$

where $\widetilde{H}_-$ *is generated by the form (62).*

Proof. Obviously

$$T^{-1} = W^{-*}H_1^{-1}W^{-1} \in \mathscr{B}(\mathscr{H}),$$

where every factor is bounded. Also

$$W^*\mathscr{Q} \subseteq \mathscr{Q}, \quad W^{-*}\mathscr{Q} \subseteq \mathscr{Q},$$

$$\mathscr{D}(T) \subseteq \mathscr{Q} = \mathscr{D}(|T_1|^{1/2}).$$

Now take

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \in \mathscr{D}(T), \quad \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \in \mathscr{Q}.$$

Then

$$\begin{aligned} (T\psi, \phi) &= (H_1W^*\psi, W^*\phi) \\ &= \left(H_1 \begin{bmatrix} \psi_1 + H_+^{-1/2}N\psi_2 \\ \psi_2 \end{bmatrix}, \begin{bmatrix} \phi_1 + H_+^{-1/2}N\phi_2 \\ \phi_2 \end{bmatrix}\right) \\ &= \left(\begin{bmatrix} H_+^{1/2}\psi_1 + N\psi_2 \\ -\widetilde{H}_-^{1/2}\psi_2 \end{bmatrix}, \begin{bmatrix} H_+^{1/2}\phi_1 + N\phi_2 \\ \widetilde{H}_-^{1/2}\phi_2 \end{bmatrix}\right) \\ &= \tau(\psi_1, \phi_1) + (H_+^{1/2}\psi_1, N\phi_2) + (N\psi_2, H_+^{1/2}\phi_1) \\ &\quad +(N\psi_2, N\phi_2) + \tau(\psi_2, \phi_2) - (N\psi_2, N\phi_2). \end{aligned}$$

Now by $(N\psi_2, H_+^{1/2}\phi_1) = \tau(\psi_2, \psi_1)$ we obtain

$$(T\psi, \phi) = \tau(\psi, \phi)$$

whereas the uniqueness follows from Proposition 2.3. □

3. More spectral inclusions

Some spectral inclusion results are already contained in the construction Theorems 2.11 and 2.16. They control the spectral gap at zero. In the sequel we produce additional results valid for general spectral gaps. We restrict ourselves here and in the following to symmetric forms α and therefore to selfadjoint operators $T = H + \alpha$.

THEOREM 3.1. *Let* (λ_-, λ_+) *be an open interval, contained in* $\rho(H)$ *such that* $\lambda_\pm \in \sigma(H)$ *(we allow* $\lambda_\pm = \pm\infty$*) and let* $T = H + \alpha$ *satisfy Theorem 2.11. Let, in addition, the open interval*

$$\mathscr{I} = (\lambda_- + (a + b|\lambda_-|),\ \lambda_+ - (a + b|\lambda_+|)) \tag{66}$$

be non-void. Then $\mathscr{I} \subseteq \rho(T)$.

Proof. Without loss of generality we may take $\lambda_+ > 0$ (otherwise consider $-H, -T$). We supose first that both λ_- and λ_+ are finite. For $d \in (\lambda_-, \lambda_+)$ we will have

$$(T - d)^{-1} = H_1^{-1/2}((H - d)H_1^{-1} + C)^{-1}H^{-1/2} \in \mathscr{B}(\mathscr{H}),$$

if

$$\|(H - d)^{-1}H_1\| < 1.$$

Now,

$$\|(H - d)^{-1}H_1\| = \sup_{\lambda \notin (\lambda_-, \lambda_+)} f(\lambda),$$

$$f(\lambda) = \frac{b|\lambda| + a}{|\lambda - d|}.$$

We now compute the supremum above.

Case 1: $\lambda_- > 0$. Then $d > 0$.

$$\lambda \geqslant \lambda_+ : \quad \left(\frac{b\lambda + a}{\lambda - d}\right)' = \frac{b(\lambda - d) - (b\lambda + a)}{(\lambda - d)^2} = \frac{-db - a}{(\lambda - d)^2}, \tag{67}$$

$$\max_{\lambda \geqslant \lambda_+} f(\lambda) = \frac{b\lambda_+ + a}{\lambda_+ - d} > b;$$

$$0 \leqslant \lambda \leqslant \lambda_- : \quad \left(\frac{b\lambda + a}{d - \lambda}\right)' = \frac{b(d - \lambda) + (b\lambda + a)}{(\lambda - d)^2} = \frac{db + a}{(\lambda - d)^2}, \tag{68}$$

$$\max_{0 \leqslant \lambda \leqslant \lambda_-} f(\lambda) = \frac{b\lambda_- + a}{d - \lambda_-} > \frac{a}{d};$$

$$\lambda \leqslant 0: \quad \left(\frac{-b\lambda+a}{d-\lambda}\right)' = \frac{-b(d-\lambda)+(-b\lambda+a)}{(\lambda-d)^2} = \frac{-db+a}{(\lambda-d)^2}, \tag{69}$$

$$\sup_{\lambda\leqslant 0} f(\lambda) = \begin{cases} a/d, & a > db \\ b, & a \leqslant db \end{cases}$$

Altogether,

$$\max_{\lambda\notin(\lambda_-,\lambda_+)} f(\lambda) = \max\left\{\frac{b\lambda_+ + a}{\lambda_+ - d}, \frac{b\lambda_- + a}{d-\lambda_-}\right\}$$

and this is obviously less than one, if $d \in \mathscr{I}$.

Case 2: $\lambda_- \leqslant 0$. Then d may be negative. By (67),

$$\sup_{\lambda\geqslant\lambda_+} f(\lambda) = \begin{cases} \dfrac{b\lambda_+ + a}{\lambda_+ - d} & a+db \geqslant 0 \\ b, & a+db \leqslant 0 \end{cases} \tag{70}$$

By (69),

$$\sup_{\lambda\leqslant\lambda_-} f(\lambda) = \begin{cases} \dfrac{-b\lambda_- + a}{d-\lambda_-} & a > db \\ b, & a \leqslant db \end{cases} \tag{71}$$

Again, both suprema are less than one, if $d \in \mathscr{I}$. If one of $\lambda_\pm$ is infinite the proof goes along the same lines and is simpler still. □

Tighter bounds can be obtained, if more is known on the perturbation α. If α is, say, non-negative then

$$\alpha = \alpha_0 + e_0, \quad e_0 = \inf_\psi \frac{\alpha(\psi,\psi)}{(\psi,\psi)}$$

and both α_0 and e_0 are non-negative. Now for

$$T = H + \alpha = H_1^{1/2}(HH_1^{-1} + C)H_1^{1/2}$$

we have

$$T - e_0 = H + \alpha_0 = H_1^{1/2}(HH_1^{-1} + C_0)H_1^{1/2}$$

where

$$C_0 = C - e_0 H_1^{-1}$$

is again non-negative bur smaller than C, in particular,

$$\min \sigma(C_0) = 0.$$

Indeed,

$$\frac{(C_0\psi,\psi)}{(\psi,\psi)} = \frac{\alpha_0(\phi,\phi)}{(\phi,\phi)}\frac{(\phi,\phi)}{\|H_1^{1/2}\phi\|^2}, \quad \psi = H_1^{1/2}\phi.$$

where

$$\inf_\phi \frac{\alpha_0(\phi,\phi)}{(\phi,\phi)} = 0, \tag{72}$$

$$\sup_{\psi} \frac{(\phi,\phi)}{\|H_1^{1/2}\phi\|^2} < \infty.$$

In this way we can always extract away the trivial scalar part e_0 of the perturbation α (and similarly for a non-positive α). In the following theorem we will therefore suppose that

$$\inf_{\phi} \frac{\alpha_0(\phi,\phi)}{(\phi,\phi)} = 0, \text{ if } \alpha \text{ is non-negative,} \tag{73}$$

$$\sup_{\phi} \frac{\alpha_0(\phi,\phi)}{(\phi,\phi)} = 0, \text{ if } \alpha \text{ is non-positive.} \tag{74}$$

Then

$$\min \sigma(C) = 0, \text{ if } \alpha \text{ is non-negative,} \tag{75}$$

$$\max \sigma(C) = 0, \text{ if } \alpha \text{ is non-positive.} \tag{76}$$

THEOREM 3.2. *Let* (λ_-, λ_+), H, α, T, C *be as in the previuos theorem and let* α *satisfy (73, 74) above. If the interval*

$$\mathscr{I} = (\lambda_- + c_+(a + b|\lambda_-|),\ \lambda_+ + c_-(a + b|\lambda_+|)) \tag{77}$$

where

$$c_- = \min(\sigma(C)) = \inf_{\psi} \frac{\alpha(\psi,\psi)}{\|H_1^{1/2}\psi\|^2}, \quad c_+ = \max(\sigma(C)) = \sup_{\psi} \frac{\alpha(\psi,\psi)}{\|H_1^{1/2}\psi\|^2}, \tag{78}$$

is not void then it is contained in $\rho(T)$.

Proof. We supose first that the interval (λ_-, λ_+) is finite. Then by virtue of (75) or (76) this interval must contain $\mathscr{I}$.

For every $d \in \mathscr{I}$ the complementary projections

$$P_{\pm} = \frac{1}{2}(\pm \text{sign}\,(H-d) + 1))$$

obviously do not depend on d. In the corresponding matrix representation we have

$$(H-d) = \begin{bmatrix} (H-d)_+ & 0 \\ 0 & -(H-d)_- \end{bmatrix},$$

$$(H-d)H_1^{-1} = \begin{bmatrix} (H-d)_+(a+bH_+)^{-1} & 0 \\ 0 & -(H-d)_-(a+bH_-)^{-1} \end{bmatrix},$$

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{12}^* & C_{22} \end{bmatrix},$$

$$T - d = H_1^{1/2} Z H_1^{1/2}$$

with

$$Z = \begin{bmatrix} A & C_{12} \\ C_{12}^* & -B \end{bmatrix},$$

$$A = (H-d)_+(a+bH_+)^{-1} + C_{11}, \quad B = (H-d)_-(a+bH_-)^{-1} - C_{22}$$

so that $Z^{-1} \in \mathscr{B}(\mathscr{H})$ implies $d \in \rho(T)$. By the obvious identity

$$Z = \begin{bmatrix} 1 & 0 \\ C_{12}^* A^{-1} & 1 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & -B - C_{12}^* A^{-1} C_{12} \end{bmatrix} \begin{bmatrix} 1 & A^{-1}C_{12} \\ 0 & 1 \end{bmatrix}$$

we see that $Z^{-1} \in \mathscr{B}(\mathscr{H})$ follows, if both operators A, B are positive definite, in particular, if both

$$(H-d)_+ H_{1+}^{-1} + c_- \text{ and } (H-d)_- H_{1-}^{-1} - c_+$$

are positive definite. This, in turn, is equivalent to

$$1 + c_- \sup_{\lambda \geqslant \lambda_+} \frac{a+b|\lambda|}{\lambda - d} > 0 \tag{79}$$

and

$$1 - c_+ \sup_{\lambda \leqslant \lambda_-} \frac{a+b|\lambda|}{d - \lambda} > 0. \tag{80}$$

Noting that (70) is valid for any possible value of λ_- we may rewrite (79) as

$$\lambda_+ - d + c_-(a + b\lambda_+) > 0 \ \& \ 1 + c_- b > 0.$$

Here the second inequality is fullfilled by $0 \leqslant b < 1$, $|c_-| \leqslant 1$ whereas the first is implied by $d \in \mathscr{I}$. Now for (79). If $\lambda_- > 0$ then by (68) and (69) we have

$$\sup_{\lambda \leqslant \lambda_-} \frac{a+b|\lambda|}{d-\lambda} = \max\left\{\frac{b\lambda_- + a}{d - \lambda_-},\ b\right\}$$

and (80) can be written as

$$1 > c_+ \max\left\{\frac{b\lambda_- + a}{d - \lambda_-},\ b\right\}$$

which is again guaranteed by $d \in \mathscr{I}$. Here, too, the proof is even simpler, if one of $\lambda_\pm$ is infinite. □

REMARK 3.3. (i) Neither of the above two theorems appears to be stronger or weaker than the other – in spite of the fact that the interval $\mathscr{I}$ from Theorem 3.1 is smaller than the one from Theorem 3.2. This lack of elegance is due to the fact that relative bounds are not shift-invariant.

(ii) Both theorems can be understood as upper-semicontinuity spectral bounds. According to Theorem 3.1 a boundary spectral point λ cannot move further than $\pm|\lambda|(a+b|\lambda|)$. Similarly, by Theorem 3.2 λ can move as far as $\lambda + c_\pm(a+b|\lambda|)$. In particular, *the spectrum moves monotonically* even in spectral gaps: for, say, α non-negative,

$$\mathscr{I} = (\lambda_- + c_+(a+b|\lambda_-|),\ \lambda_+). \tag{81}$$

(iii) If $T = H + A$, A bounded then

$$\mathscr{I} = (\lambda_- + \max\sigma(A),\ \lambda_+ + \min\sigma(A)).$$

Bounds for the essential spectra. The proofs of the preceding two Theorems have enough of algebraic structure to be transferable to the Calkin quotient $\mathscr{C}^*$ algebra $\mathscr{B}(\mathscr{H})/\mathscr{C}(\mathscr{H})$, where $\mathscr{C}(\mathscr{H})$ is the ideal of all compact operators. Using this we will now derive analogous bounds for the essential spectra.

We first list some simple facts which will be used. Let $\mathscr{A}$ be a semisimple $\mathscr{C}^*$ algebra with the identity e. If $p \in \mathscr{A}$, $p \neq e$, is a projection then the subalgebra

$$\mathscr{A}_p = \{b \in \mathscr{A} : \ bp = pb = b\}$$

is again semisimple with the unit p. An element $b \in \mathscr{A}$ is invertible in $\mathscr{A}_p$, if and only if in $\mathscr{A}$ its spectrum has zero as an isolated point and the corresponding projection is $q = e - p$. If $\mathscr{A} = \mathscr{B}(\mathscr{H})$ then $\mathscr{A}_p$ is naturally identified with $\mathscr{B}(p\mathscr{H})$. An element $b \in \mathscr{A}$ is called positive, if its spectrum is non-negative. A sum of two positive elements, one of which is invertible, is itself positive and invertible.

PROPOSITION 3.4. *Let $a = a^* \in \mathscr{A}$ be invertible and let $p, q \neq 0$ be the projections belonging to the positive and the negative part of $\sigma(a)$, respectively. Let $b = b^* \in \mathscr{A}$ and $pbp = qbq = 0$. Then $a + b$ is invertible.*

Proof. The elements ap, aq are invertible with the inverses $a^{(p)}, a^{(q)}$ in $\mathscr{A}_p, \mathscr{A}_q$, respectively. Moreover both $a^{(p)}$ and $-a^{(q)}$ are positive. The fundamental identity (the Schur-complement decomposition)

$$a + b = za_0z^*$$

with

$$z = e + qba^{(p)}, \quad z^{-1} = e - qba^{(p)},$$
$$a_0 = ap + aq - qba^{(p)}bq$$

is readily verified. Thus, we have to prove the invertibility of a_0 in $\mathscr{A}$. Obviously $\widetilde{a} = aq - qba^{(p)}bq$ is invertible in $\mathscr{A}_q$ (being a sum of negative elements one of which is invertible). Denoting by $\widetilde{a}^{(q)}$ its inverse in $\mathscr{A}_q$ we have

$$a_0^{-1} = a^{(p)} + \widetilde{a}^{(q)}.$$

Indeed,

$$(a^{(p)} + \widetilde{a}^{(q)})(ap + \widetilde{a}) = (ap + \widetilde{a})(a^{(p)} + \widetilde{a}^{(q)}) = p + q = e.$$

□

We now prove an analog of Theorem 3.2 for the essential spectrum.

THEOREM 3.5. *Let $(\lambda_-, \lambda_+) \cap \sigma_{ess}(H) = \emptyset$, $\lambda_\pm \in \sigma_{ess}(H)$ and let $T = H + \alpha$ satisfy Theorem 2.11 as well as (73,74), respectively. If the interval*

$$\mathscr{I} = (\lambda_- + c_+(a + b|\lambda_-|),\ \lambda_+ + c_-(a + b|\lambda_+|)) \tag{82}$$

with

$$c_- = \min(\sigma_{ess}(C)), \quad c_+ = \max(\sigma_{ess}(C)) \tag{83}$$

is not void then $\mathscr{I} \cap \sigma_{ess}(T) = \emptyset$.

Proof. Obviously, $\sigma_{ess}(C) = \sigma(\widehat{C})$, where

$$\widehat{\ } \ : \mathscr{B}(\mathscr{H}) \to \mathscr{A}$$

is the Calkin homomorphism. Whenever C_ζ is invertible in $\mathscr{B}(\mathscr{H})$ and in particular for $\zeta = i\eta$, $|\eta|$ large (27) yields

$$r(\zeta) = \widehat{(T-\zeta)^{-1}} = \widehat{H_1^{-1/2}}(\widehat{D} + \widehat{C})^{-1}\widehat{H_1^{-1/2}}, \tag{84}$$

$$D = (H - \zeta)H_1^{-1}.$$

By the spectral mapping principle $r(\zeta)$ is analytically continued onto the complement of $\sigma_{ess}(T)$. This complement contains all real $\zeta = d \in (\lambda_-, \lambda_+)$ for which $(\widehat{D} + \widehat{C})^{-1} \in \mathscr{A}$. Obviously

$$\sigma(\widehat{D}) = f(\sigma_{ess}(H)), \quad f(\lambda) = \frac{\lambda - d}{a + b|\lambda|}$$

and $\widehat{D}^{-1} \in \mathscr{A}$. Let $p, q \in \mathscr{A}$ be the projections corresponding to the positive and the negative part of the spectrum of $\widehat{D}$, respectively. As in Theorem 3.2 one proves that

$$a_p = p\widehat{D} + p\widehat{C}p, \ -a_q = -q\widehat{D} - q\widehat{C}q$$

are positive and invertible in $\mathscr{A}_p$, $\mathscr{A}_q$, respectively. Now apply Proposition 3.4 to

$$a = a_p + a_q, \ b = p\widehat{C}q + q\widehat{C}p$$

thus obtaining the invertibility in $\mathscr{A}$ of $a + b = \widehat{D} + \widehat{C}$. □

In particular, *the essential spectrum depends monotonically on* α. Of course, if C is compact then $c_\pm = 0$ and we have $(\lambda_-, \lambda_+) \cap \sigma_{ess}(T) = \emptyset$ as was known from Theorem 2.14.

There is an essential-spectrum analog of Theorem 3.1 as well:

THEOREM 3.6. *Let* $(\lambda_-, \lambda_+) \cap \sigma_{ess}(H) = \emptyset$, $\lambda_\pm \in \sigma_{ess}(H)$ *and let* $T = H + \alpha$ *satisfy Theorem 2.11. If the interval*

$$\mathscr{I} = (\lambda_- + (a + b|\lambda_-|), \ \lambda_+ - (a + b|\lambda_+|)) \tag{85}$$

is not void then $\mathscr{I} \cap \sigma_{ess}(T) = \emptyset$.

The proof is similar as above and is omitted.

4. Finite eigenvalues

All forms in this section will be symmetric. The following theorem is a necessary tool from the analytic perturbations which will be repeatedly used later on.

THEOREM 4.1. *Let* H, $\alpha = \alpha_\varepsilon$ *for* ε *from an open interval* $\mathscr{I}$ *satisfy the conditions of Theorem 2.4 and such that* α_ε *is symmetric and* $C = C_\varepsilon$ *from (22) is real analytic in* $\varepsilon \in \mathscr{I}$ *and*

$$C_{\zeta,\varepsilon} = (H - \zeta)H_1^{-1} + C_\varepsilon$$

is invertible in $\mathscr{B}(\mathscr{H})$ *for all* ζ *from an open set* $\mathscr{O} \subseteq \mathscr{C}$ *and all* $\varepsilon \in \mathscr{I}$. *Then the operator family* $T_\varepsilon = T + \alpha_\varepsilon$ *is holomorphic in the sense of [3], Ch. VII, 1. Moreover, the derivative of an isolated holomorphic eigenvalue* $\lambda(\varepsilon)$ *of* T_ε *with finite multiplicity is given by*

$$\lambda'(\varepsilon) = \frac{1}{m} Tr\left((H_1^{1/2} P_\varepsilon)^* C'_\varepsilon H_1^{1/2} P_\varepsilon\right). \tag{86}$$

Here m, P_ε *denotes the multiplicity and the spectral projection on the (total) eigenspace for* $\lambda(\varepsilon)$, *respectively.*

Proof. The formula (86) is plausible being akin to known analogous expressions from the analytic perturbation theory ([3], Ch. VII). For completeness we provide a proof in this more general situation.[5]

Let $\varepsilon_0 \in \mathscr{I}$ and let Γ be a closed Jordan curve separating $\lambda(\varepsilon_0)$ from the rest of $\sigma(T_{\varepsilon_0})$. Let $\Gamma_1 \subseteq \rho(T_{\varepsilon_0})$ be another curve connecting $\mathscr{O}$ and Γ. Take any connected neighbourhood $\mathscr{O}_0$ of $\Gamma \cup \Gamma_1$ with $\mathscr{O}_0 \subseteq \rho(T_{\varepsilon_0})$. According to [3], Ch. VII Th. 1.7 there exists a complex neighbourhood $\mathscr{U}_0$ of ε_0 such that $(\lambda - T_\varepsilon)^{-1}$ is holomorhic in $\mathscr{O}_0 \times \mathscr{U}_0$.

For $\lambda \in \mathscr{O}$ and $\varepsilon \in \mathscr{U}_0$ we have

$$(\lambda - T_\varepsilon)^{-1} = -H_1^{-1/2} C_{\lambda,\varepsilon}^{-1} H_1^{-1/2},$$

$$\frac{\partial}{\partial \varepsilon}(\lambda - T_\varepsilon)^{-1} = H_1^{-1/2} C_{\lambda,\varepsilon}^{-1} C'_\varepsilon C_{\lambda,\varepsilon}^{-1} H_1^{-1/2}.$$

Note that $\mathscr{R}(P_\varepsilon) \subseteq \mathscr{Q}$ and hence

$$H_1^{-1/2} C_{\lambda,\varepsilon}^{-1} H_1^{-1/2} P_\varepsilon = \frac{1}{\lambda - \lambda(\varepsilon)} P_\varepsilon.$$

By $H_1^{1/2} P_\varepsilon \in \mathscr{B}(\mathscr{H})$ we have

$$C_{\lambda,\varepsilon}^{-1} H_1^{-1/2} P_\varepsilon = \frac{1}{\lambda - \lambda(\varepsilon)} H_1^{1/2} P_\varepsilon,$$

$$P_\varepsilon \frac{\partial}{\partial \varepsilon}(\lambda - T_\varepsilon)^{-1} P_\varepsilon = \frac{1}{(\lambda - \lambda(\varepsilon))^2} (H_1^{1/2} P_\varepsilon)^* C'_\varepsilon H_1^{1/2} P_\varepsilon. \tag{87}$$

On the other hand (see [3])

$$P_\varepsilon = \frac{1}{2\pi i} \int_\Gamma (\lambda - T_\varepsilon)^{-1} d\lambda,$$

$$T_\varepsilon P_\varepsilon = \frac{1}{2\pi i} \int_\Gamma \lambda (\lambda - T_\varepsilon)^{-1} d\lambda = \frac{1}{2\pi i} \int_\Gamma \lambda P_\varepsilon (\lambda - T_\varepsilon)^{-1} P_\varepsilon d\lambda,$$

$$\lambda(\varepsilon) = \frac{1}{m} Tr(T_\varepsilon P_\varepsilon).$$

[5] Our case is close to the holomorphic family of type (C) from [3], Ch. VII, §5.1 where no such details are elaborated.

Using $P_\varepsilon^2 = P_\varepsilon$ and $P'_\varepsilon P_\varepsilon = 0$ we have

$$\lambda'(\varepsilon) = \frac{1}{2\pi i m} Tr\left(\int_\Gamma \lambda P_\varepsilon \frac{\partial}{\partial\varepsilon}(\lambda - T_\varepsilon)^{-1} P_\varepsilon d\lambda\right) \tag{88}$$

(here the integration over λ and the differentiation over ε obviously commute). The formula (87) can be analytically continued in $\lambda \in \mathscr{O}_0$ and inserted into (88). By using the obvious identity

$$\frac{1}{2\pi i}\int_\Gamma \frac{\lambda d\lambda}{(\lambda - \lambda(\varepsilon))^2} = 1$$

and taking trace we obtain (86). □

The preceding theorem is not general enough to cover all situations of interest:

THEOREM 4.2. *Let $T_\varepsilon = H + \alpha_\varepsilon$ be as in Theorem 4.1 above and let $B(\varepsilon)$ be a bounded symmetric family, analytic in ε. Let, in addition the set $\mathscr{O}$ contain a full vertical half-line. Then the assertions of Theorem 4.1 hold true for $T_\varepsilon + B(\varepsilon)$ but instead of (86) we have*

$$\lambda'(\varepsilon) = \frac{1}{m} Tr\left((H_1^{1/2}P_\varepsilon)^* C'_\varepsilon H_1^{1/2} P_\varepsilon + P_\varepsilon B'(\varepsilon) P_\varepsilon\right). \tag{89}$$

Proof. We proceed as in the proof of Theorem 4.1 keeping in mind that the formula (29) is not immediately applicable to $T_\varepsilon + B(\varepsilon)$. We can take Γ_1 so as to contain a point $\lambda_0 \in \mathscr{O}$ such that $\|B(\varepsilon_0)(\lambda - T_{\varepsilon_0})^{-1}\| < 1$. This insures $\lambda \in \rho(T_\varepsilon + B(\varepsilon))$ for $\lambda \in \mathscr{O}_1 \subseteq \mathscr{O}$, $\varepsilon \in \mathscr{U}_0$. Then

$$\begin{aligned}
\frac{\partial}{\partial\varepsilon}(\lambda - T_\varepsilon - B(\varepsilon))^{-1} &= \frac{\partial}{\partial\varepsilon}\left[(\lambda - T_\varepsilon)^{-1}\left(1 - B(\varepsilon)(\lambda - T_\varepsilon)^{-1}\right)^{-1}\right] \\
&= \frac{\partial}{\partial\varepsilon}(\lambda - T_\varepsilon)^{-1}\left(1 - B(\varepsilon)(\lambda - T_\varepsilon)^{-1}\right)^{-1} \\
&\quad + (\lambda - T_\varepsilon)^{-1}\left(1 - B(\varepsilon)(\lambda - T_\varepsilon)^{-1}\right)^{-1} \\
&\quad \times \left(B'(\varepsilon)(\lambda - T_\varepsilon)^{-1} + B(\varepsilon)\frac{\partial}{\partial\varepsilon}(\lambda - T_\varepsilon)^{-1}\right) \\
&\quad \times \left(1 - (\lambda - T_\varepsilon)^{-1}B(\varepsilon)\right)^{-1} \\
&= \left(1 - B(\varepsilon)(\lambda - T_\varepsilon)^{-1}\right)^{-1}\frac{\partial}{\partial\varepsilon}(\lambda - T_\varepsilon)^{-1}\left(1 - B(\varepsilon)(\lambda - T_\varepsilon)^{-1}\right)^{-1} \\
&\quad + (\lambda - T_\varepsilon - B(\varepsilon))^{-1} B'(\varepsilon)(\lambda - T_\varepsilon - B(\varepsilon))^{-1}
\end{aligned}$$

Then using (87)

$$P_\varepsilon \frac{\partial}{\partial\varepsilon}(\lambda - T_\varepsilon - B(\varepsilon))^{-1} P_\varepsilon = \frac{1}{(\lambda - \lambda(\varepsilon))^2}\left[(H_1^{1/2}P_\varepsilon)^* C'_\varepsilon H_1^{1/2} P_\varepsilon + P_\varepsilon B'(\varepsilon) P_\varepsilon\right]$$

which leads to (89) as in the theorem above. □

The first application of Theorems 4.1, 4.2 will be a result on monotonicity. We have to assume that the spectrum under consideration is sufficiently protected from unwanted spectral points. We say that a real point d is *impenetrable* (*essentially impenetrable*) for a selfadjoint family T_γ, γ from any set of indices, if $d \notin \sigma(T_\gamma)$ ($d \notin \sigma_{ess}(T_\gamma)$).

THEOREM 4.3. *Let $T_\varepsilon = H + \alpha_\varepsilon$ be analytic in $\varepsilon \in [\varepsilon_0, \varepsilon_1]$ in the sense of Theorem 4.1.*[6] *Let α_ε be non-decreasing in ε, let an open interval (d, d_1) be essentially impenetrable and one of its ends, say, d be impenetrable for T_ε. Let*

$$\lambda_1^1 \leqslant \lambda_2^1 \leqslant \cdots$$

be the eigenvalues in (d, d_1) of T_{ε_1}. Then the spectrum of T_{ε_0} in (d, d_1) consists of the eigenvalues which can be ordered as

$$\lambda_1^0 \leqslant \lambda_2^0 \leqslant \cdots$$

and they satisfy

$$\lambda_k^0 \leqslant \lambda_k^1, \ k = 1, 2, \ldots \tag{90}$$

Proof. For a fixed n let $\lambda_1^1, \lambda_2^1 \leqslant \cdots \leqslant \lambda_n^1$ be the smallest n eigenvalues of T_{ε_1}. Any of them can be analytically continued to a neighbourhood of $\varepsilon = \varepsilon_1$. By the assumed monotonicity (we use Theorem 4.1 with $C'_\varepsilon \geqslant 0$ as well as the assumed impenetrabilities) this analytic continuation covers the whole of $[\varepsilon_0, \varepsilon_1]$ i.e. we obtain analytic non-decreasing functions

$$d < \lambda_1(\varepsilon), \lambda_2(\varepsilon), \ldots \lambda_n(\varepsilon) < d_1$$

as eigenvalues of T_ε. By a permutation, piecewise constant in ε, we obtain

$$d < \widehat{\lambda}_1(\varepsilon) \leqslant \widehat{\lambda}_2(\varepsilon) \leqslant \cdots \leqslant \widehat{\lambda}_n(\varepsilon) < d_1$$

which are continuous, piecewise analytic,[7] still non-decreasing in ε and satisfy $\widehat{\lambda}_k(\varepsilon_1) = \lambda_k^1$. By setting $\varepsilon = \varepsilon_0$ we obtain n eigenvalues of T_{ε_0}

$$d < \widehat{\lambda}_1(\varepsilon_0) \leqslant \widehat{\lambda}_2(\varepsilon_0) \leqslant \cdots \leqslant \widehat{\lambda}_n(\varepsilon_0) < d_1$$

which obviously satisfy

$$\widehat{\lambda}_k(\varepsilon_0) \leqslant \lambda_k^1, \ k = 1, 2, \ldots n.$$

Then *a fortiori*

$$\lambda_k^0 \leqslant \widehat{\lambda}_k(\varepsilon_0) \leqslant \lambda_k^1, \ k = 1, 2, \ldots n. \tag{91}$$

and n is arbitrary. The fact that there exists the smallest eigenvalue λ_1^0 of T_{ε_0} is due to the impenetrability of the point d. □

REMARK 4.4. Note that, in fact, the theorem above asserts *the existence* of at least that much eigenvalues of T_{ε_0} as the λ_k^1. Obviously, if we assume that both interval ends are impenetrable, then Theorem 4.3 applies in both directions and the eigenvalues of T_{ε_0} and T_{ε_1} have the same cardinality which is finite.

[6] Analyticity in a closed interval means the same in a complex neighbourhood of that interval.

[7] A real function f is called piecewise analytic on an open interval $\mathscr{J}$, if it is real-analytic on $\mathscr{J} \setminus \mathscr{S}$ where $\mathscr{S}$ is a discrete set and f has analytic continuation from each side of any point from $\mathscr{S}$, but the two continuatons need not to coincide.

REMARK 4.5. Theorem 4.3 also holds under the conditions of Theorem 4.2, if we assume that the form $\alpha_\varepsilon + B(\varepsilon)$ is non-decreasing.

COROLLARY 4.6. *Let in Theorem 4.3* $\alpha_\varepsilon = \alpha_0 + \varepsilon\alpha_1$ *and let in (90) the equality hold for some* k. *Then there is* $\psi \neq 0$ *with*

$$T_{\varepsilon_1}\psi = T_{\varepsilon_0}\psi = \lambda_k^1\psi, \tag{92}$$

Proof. By the assumption, and using the inequalities (91) from the proof of Theorem 4.3 we obtain

$$\lambda_k = \widehat{\lambda}_k(\varepsilon) = \widehat{\lambda}_k(\varepsilon_0) = \widehat{\lambda}_k(\varepsilon_1), \ \text{ for all } \varepsilon \in [\varepsilon_0, \varepsilon_1].$$

Thus, $\widehat{\lambda}_k(\varepsilon)$ is constant in $\varepsilon \in [\varepsilon_0, \varepsilon_1]$. Now (86) yields

$$Tr(H_1^{1/2}P_\varepsilon)^* C_1 H_1^{1/2} P_\varepsilon = 0 \tag{93}$$

for ε from a neighbourhood of ε_1, where P_ε is the (total) projection belonging to the spectral point $\lambda_k(\varepsilon)$, $\varepsilon < \varepsilon_1$ and

$$C_\varepsilon = C_0 + \varepsilon C_1$$

with $C'_\varepsilon = C_1$ non-negative. Thus, (93) implies

$$C_1 H_1^{1/2} P_\varepsilon = 0$$

and, in particular,

$$\alpha_1(\psi, \phi) = 0, \ \text{ for all } \phi \in \mathscr{Q},$$

where $T_\varepsilon\psi = \lambda_k^1\psi$ for all ε, in particular, $T_{\varepsilon_0}\psi = T_{\varepsilon_1}\psi = \lambda_k^1\psi = \lambda_k^0\psi$. □

The existence of an impenetrable point d was crucial in Theorem 4.3. It can be guaranteed by one of the spectral inclusions, contained in Theorems 2.16, 3.1, 3.2; each of them contains some restrictions on the size of α in comparison to H. Deeper reaching criteria will compare an 'unknown' α with a known α_0, which has desired properties:

DEFINITION 4.7. let H, $\alpha = \alpha_0 \leqslant 0$, $\mathscr{Q}$ be as in (16), (17), (20).[8] Set

$$\mathscr{A} = \{\alpha : \ \mathscr{D}(\alpha) \supseteq \mathscr{Q}, \ |\alpha| \leqslant c\alpha_0, \ c < 1\}. \tag{94}$$

We call α_0 *H-regular*, if the following four conditions are fulfilled:

1. Each $\alpha \in \mathscr{A}$ satisfies the conditions of Theorem 2.4,
2. $\sigma_{ess}(H+\alpha) \subseteq \sigma_{ess}(H) = (-\infty, -m] \cup [m, \infty)$ for some $m > 0$ and all $\alpha \in \mathscr{A}$,
3. For some $\delta > 0$ and all η with $0 \leqslant \eta < 1$

$$(-m, -m+\delta] \subseteq \rho(H + \eta\alpha_0) \tag{95}$$

4. $\max \sigma(C_0) = 0$, where C_0, C are generated by (18) and α_0, α, respectively.

[8]Of course, $\alpha_0 \geqslant 0$ would do as well. Our definition of the regularity is, in fact, modeled after a standard situation in the applications: the Dirac operator with the attractive Coulomb potential.

THEOREM 4.8. *Let* α_0 *be* H*-regular and* $\alpha \in \mathscr{A}$, $\alpha \leqslant 0$. *Then*

$$(-m, -m+\delta] \subseteq \rho(H+\eta\alpha), \quad 0 \leqslant \eta \leqslant 1$$

with m, δ *from Definition 4.7.*

Proof. Take $\eta_0 \in (0,1]$ such that for a, b from (16) we have $\eta_0 b < 1$ and

$$-m+\delta < m + \eta_0 c_-^0 (a+bm)$$

where $c_-^0 = \min \sigma(C_0)$. Then the conditions of Theorems 2.11, 3.2 hold for $H + \alpha_{\varepsilon,\eta}$ with

$$\alpha_{\varepsilon,\eta} = (1-\varepsilon)\eta c\alpha_0 + \varepsilon\eta\alpha,$$

(c from Def. 4.7) uniformly in $0 \leqslant \eta \leqslant \eta_0$, $0 \leqslant \varepsilon \leqslant 1$; this follows from

$$c\alpha_0 \leqslant \alpha_{\varepsilon,\eta} \leqslant 0.$$

Obviously, $\alpha_{\varepsilon,\eta}$ belongs to $\mathscr{A}$ and is non-decreasing in ε and non-increasing in η. By Theorem 3.2 we have

$$(-m, -m+\delta] \subseteq (-m, m + \eta c_-^0(a+bm)) \subseteq \rho(H + \alpha_{\varepsilon,\eta}),$$

$0 \leqslant \eta \leqslant \eta_0$, $0 \leqslant \varepsilon \leqslant 1$. By

$$H + \alpha_{\varepsilon,\eta} - \zeta = H_1^{1/2}((H-\zeta)H_1^{-1} + (1-\varepsilon)\eta C + \varepsilon\eta c C_0)H_1^{1/2}$$

we see that $H + \alpha_{\varepsilon,\eta}$ is continuous in the sense of the uniform resolvent topology jointly in $\varepsilon, \eta \in [0,1]$ and the same is true for $\sigma(H + \alpha_{\varepsilon,\eta})$ (see [3] Ch. V. Th. 4.10). Thus, the set

$$\mathscr{S} = \{\eta \in [0,1] : \ (-m, -m+\delta] \subseteq \rho(H + \alpha_{\varepsilon,\eta}) \ \text{for all } \varepsilon \in [0,1]\}$$

is open in $[0,1]$ and it obviously contains $[0,\eta_0]$. We will prove that the component of $\mathscr{S}$ containing $[0,\eta_0]$ is equal to $[0,1]$. If this were not so then this component would read $[0,\eta_1)$, $\eta_0 \leqslant \eta_1 < 1$. In this case there would exist an ε_1 such that

$$\sigma(H + \alpha_{\varepsilon_1,\eta_1}) \cap (-m, -m+\delta] \neq \emptyset \tag{96}$$

whereas

$$(-m, -m+\delta] \subseteq \rho(H + \alpha_{\varepsilon,\eta}) \tag{97}$$

for all $\eta < \eta_1$ and all $\varepsilon \in [0,1]$.

Now, by the mentioned spectral continuity we would still have $(-m, -m+\delta) \subseteq \rho(H + \alpha_{\varepsilon,\eta_1})$ for all $\varepsilon \in [0,1]$, more precisely, $-m+\delta = \lambda_1(\varepsilon_1, \eta_1)$, where $\lambda_1(\varepsilon, \eta)$ denotes the lowest eigenvalue of $H + \alpha_{\varepsilon,\eta}$ in $(-m, m)$. Thus, Theorem 4.3 is applicable to the family

$$[0, \varepsilon_1] \ni \varepsilon \mapsto H + \alpha_{\varepsilon,\eta_1}$$

and by (95) we would have

$$-m+\delta < \lambda_1(0,\eta_1) \leqslant \lambda_1(\varepsilon_1,\eta_1) \leqslant -m+\delta \tag{98}$$

– a contradiction. Now take in (97) $\varepsilon = 1$ which gives the statement of our theorem. □

The theorem above can be regarded as an abstract analog of a result of Wüst [11], obtained for the Dirac operator with the Coulomb interaction α_0.

COROLLARY 4.9. *If α_0 is regular then any non-positive $\alpha \in \mathscr{A}$ is regular also.*

COROLLARY 4.10. *Let α_0 be H-regular and $0 \geqslant \beta \geqslant \alpha \in \mathscr{A}$. Then the spectrum of $H+\alpha$, $H+\beta$ in $(-m,m)$ consists of the eigenvalues*

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots$$

$$\mu_1 \leqslant \mu_2 \leqslant \cdots$$

respectively, and

$$\lambda_k \leqslant \mu_k, \quad k = 1, 2 \ldots \tag{99}$$

holds.

Proof. By Theorem 4.8 we have $(-m, -m+\delta) \subseteq \rho(T_\varepsilon)$ where

$$T_\varepsilon = H + (1-\varepsilon)\alpha + \varepsilon\beta, \quad 0 \leqslant \varepsilon \leqslant 1,$$

$$(1-\varepsilon)\alpha + \varepsilon\beta \in \mathscr{A}.$$

Now Theorem 4.3 applies and (99) follows. □

THEOREM 4.11. *Let α_0 be H-regular and let*

$$|\alpha - c\alpha_0| \leqslant -\varepsilon c \alpha_0, \tag{100}$$

$$\varepsilon < \min\{1, \frac{1}{c} - 1\}.$$

Then the spectrum of $H+\alpha$ in $(-m,m)$ consists of the eigenvalues

$$\mu_1 \leqslant \mu_2 \leqslant \cdots$$

and they satisfy

$$\lambda_k((1+\varepsilon)c) \leqslant \mu_k \leqslant \lambda_k((1-\varepsilon)c), \quad k = 1, 2 \ldots \tag{101}$$

where

$$\lambda_1(\eta) \leqslant \lambda_2(\eta) \leqslant \cdots$$

are the eigenvalues of $H + \eta\alpha_0$.

Proof. (100) can be written as

$$(1+\varepsilon)c\alpha_0 \leqslant \alpha \leqslant (1-\varepsilon)c\alpha_0,$$

from which it follows $\alpha \leqslant 0$ and

$$0 \geqslant \alpha \geqslant c_1 \alpha_0$$

with $c_1 = (1+\varepsilon)c < 1$. Thus, $\alpha \in \mathscr{A}$. Now apply Corollary 4.10 to the operators

$$H + (1+\varepsilon)c\alpha_0,\ H+\alpha,\ H+(1-\varepsilon)c\alpha_0$$

and (101) follows. $\square$

REMARK 4.12. The estimates (101) are sharp: by taking the perturbation $\beta = (1 \pm \varepsilon)\alpha$ the equality on the respective side in (101) is obtained. The bound (101) is particularly useful, if the eigenvalues $\lambda(\eta)$ are explicitly known as functions of η as is the case with several important quantum mechanical systems.

Let us now turn to the promised bound (7). We will combine the monotonicity from Theorem 4.3 with one of the spectral inclusion results above to insure the necessary impenetrabilities. There are quite few of the latter, so we will present the most typical cases.

THEOREM 4.13. *Let H, α, T, C be as in Theorem 2.11 and α symmetric. Let $\mathscr{I} = (\lambda_{--}, \lambda_{++})$ be an essential spectral gap for H and λ_{-+} the lowest eigenvalue of H in $\mathscr{I}$ such that the open interval*

$$\mathscr{I}_- = (\lambda_{--} + a + b|\lambda_{--}|,\ \lambda_{-+} - a - b|\lambda_{-+}|) \tag{102}$$

is non-void. Furthermore, let either

(i) λ_{+-} be the highest eigenvalue of H in $\mathscr{I}$ such that the open interval

$$\mathscr{I}_+ = (\lambda_{+-} + a + b|\lambda_{+-}|,\ \lambda_{++} - a - b|\lambda_{++}|) \tag{103}$$

is non-void or

(ii) the form α satisfy the conditions of Theorem 2.14.

By

$$\lambda_1 = \lambda_{-+} \leqslant \lambda_2 \leqslant \cdots$$

denote the (finite or infinite) sequence of the eigenvalues of H in $\mathscr{I}$. Set

$$\widetilde{\lambda} = \begin{cases} \lambda_{++} - a - b|\lambda_{++}|, & \text{in case (i)} \\ \lambda_{++}, & \text{in case (ii)} \end{cases}$$

Then the spectrum of T in $\widetilde{\mathscr{I}} = (\lambda_{--} + a + b|\lambda_-|, \widetilde{\lambda})$ consists of the eigenvalues

$$\mu_1 \leqslant \mu_2 \leqslant \cdots$$

and they satisfy (7) in the following sense: in the case (i) λ 's and μ 's have the same cardinality and (7) holds for all of them whereas in the case (ii) (7) holds as long as $\lambda_k + a + b|\lambda_k| < \lambda_{++}$.

Proof. We introduce an auxiliary family

$$\widetilde{T}_\varepsilon = H + \widetilde{\alpha}_\varepsilon,$$

$$\widetilde{\alpha}_\varepsilon = \varepsilon(a + b\widehat{h}) + (1-\varepsilon)\alpha, \quad 0 \leqslant \varepsilon \leqslant 1,$$

where $\widehat{h}$ is the closed form belonging to the operator $|H|$. This family satisfies the conditions of Theorem 2.11 uniformly in ε:

$$|\widetilde{\alpha}_\varepsilon(\psi,\psi)| \leqslant \varepsilon(a+b\widehat{h})(\psi,\psi) + (1-\varepsilon)(a+b\widehat{h})(\psi,\psi) = (a+b\widehat{h})(\psi,\psi)$$

and the operator $\widetilde{C}_\varepsilon$, constructed from $\widetilde{T}_\varepsilon$ according to (18) is here given by

$$(\widetilde{C}_\varepsilon\psi,\phi) = \widetilde{\alpha}_\varepsilon(H_1^{-1/2}\psi, H_1^{-1/2}\phi) = ((\varepsilon+(1-\varepsilon)C)\psi,\phi),$$

so, $\widetilde{C}_\varepsilon = \varepsilon + (1-\varepsilon)C$ is holomorphic with $\|\widetilde{C}_\varepsilon\| \leqslant 1$ and $\widetilde{C}'_\varepsilon = 1 - C$ is non-negative. In particular, $\widetilde{T}_\varepsilon$ fulfills the conditions of Theorem 4.1 as well as Theorem 3.1, uniformly in $\varepsilon \in [0,1]$. Thus, $\mathscr{I}_-$ is impenetrable for $\widetilde{T}_\varepsilon$. We now show that any open interval (d,d_1) with $d \in \mathscr{I}_-$ and

$$d_1 \begin{cases} \in \mathscr{I}_+, & \text{in case (i)} \\ = \lambda_{++}, & \text{in case (ii)} \end{cases}$$

is essentially impenetrable for $\widetilde{T}_\varepsilon$. To this end we introduce another auxiliary family

$$H + \varepsilon(a+b\widehat{h}) = H + \varepsilon H_1$$

to which both Theorem 2.11 and Theorem 3.1 hold, again uniformly in ε. Therefore its spectrum in $\widetilde{\mathscr{I}}$ consists of the eigenvalues

$$\lambda_1 + \varepsilon(a+b|\lambda_1|) \leqslant \lambda_2 + \varepsilon(a+b|\lambda_2|) \leqslant \cdots$$

In particular, $\widetilde{\mathscr{I}}$ is essentially impenetrable for $H+\varepsilon H_1$. The form sum $\widetilde{T}_\varepsilon = H + \widetilde{\alpha}_\varepsilon$ can obviously be represented as another form sum

$$\widetilde{T}_\varepsilon = (H+\varepsilon H_1) + (1-\varepsilon)\alpha \tag{104}$$

again in the sense of Theorems 2.4 and 2.11. Indeed, using the the functional calculus we obtain the operator inequality (a proof is provided in the Appendix)

$$H_1 \leqslant \frac{a}{1-\varepsilon b} + \frac{b|H+\varepsilon H_1|}{1-\varepsilon b}, \tag{105}$$

hence

$$\begin{aligned}|(1-\varepsilon)\alpha(\psi,\psi)| &\leqslant \frac{1-\varepsilon}{1-\varepsilon b}a\|\psi\|^2 + \frac{(1-\varepsilon)b\||H+\varepsilon H_1|^{1/2}\psi\|^2}{1-\varepsilon b} \\ &\leqslant a\|\psi\|^2 + b\||H+\varepsilon H_1|^{1/2}\psi\|^2.\end{aligned}$$

Furthermore, the form sum (104) satisfies the conditions of Theorem 2.14. As a matter of fact, the operator $\widetilde{C}$, defined by

$$(\widetilde{C}\psi\phi) = \alpha((a+b|H+\varepsilon H_1|)^{-1/2}\psi, (a+b|H+\varepsilon H_1|)^{-1/2}\phi)$$

satisfies

$$(a+b|H+\varepsilon H_1|)^{-1}\widetilde{C} = BH_1^{-1}CB$$

where $H_1^{-1}C$ is known to be compact and by (105)

$$B = H_1^{1/2}(a + b|H + \varepsilon H_1|)^{-1/2}$$

is bounded. Thus, (d, d_1) is essentially impenetrable for $\widetilde{T}_\varepsilon$ in the case (ii). The case (i) is even simpler: due to the impenetrability from both sides for $H + \varepsilon H_1$ the cardinalities of the eigenvalues of H and $H + H_1$ are finite and equal, the same is then true of T and $H + H_1$ now due to the impenetrability from both sides for $\widetilde{T}_\varepsilon$. Now all conditions of Theorem 4.3 are fulfilled for the family $\widetilde{T}_\varepsilon$ for which $\widetilde{T}_0 = T$ and $\widetilde{T}_1 = H + H_1$. Hence the eigenvalues of T in $\widetilde{\mathscr{I}}$ are

$$\mu_1 \leqslant \mu_2 \leqslant \cdots,$$

they are at least as much as those $\lambda_k + a + b|\lambda_k|$ which are smaller than $\widetilde{\lambda}$ and they satisfy the right hand side of (7). To obtain the other we use the form $\widetilde{\alpha}_\varepsilon = -\varepsilon(a + b\widehat{h}) + (1 - \varepsilon)\alpha$. □

REMARK 4.14. (i) In the proof above the right hand side of the inequality (7) had to be proved first because this step guarantees *the existence* of the perturbed eigenvalues. This asymetry is natural and is due to the fact that in general only the left end of the 'window' (d, d_1) is assumed as impenetrable (case (ii)). The other direction is handled by considering $H = -H$. (ii) The restrictive condition that $\lambda_k + a + b|\lambda_k|$ be smaller than $\widetilde{\lambda}$ is trivially fulfilled for all k, if $\lambda_{++} = \infty$.

An analogous result holds under the conditions of Theorem 3.2.

THEOREM 4.15. *Let H, α, T, C be as in Theorem 2.11 and α symmetric and let, in addition, α satisfy (73), (74) with $c_\pm$ from (78). Let $\mathscr{I} = (\lambda_{--}, \lambda_{++})$ be an essential spectral gap for H and λ_{-+} the lowest eigenvalue of H in $\mathscr{I}$ such that the open interval*

$$\mathscr{I}_- = (\lambda_{--} + c_+(a + b|\lambda_{--}|),\ \lambda_{-+} + c_-(a + b|\lambda_{-+}|)) \tag{106}$$

is non-void. Furthermore, let either

(i) λ_{+-} be the highest eigenvalue of H in $\mathscr{I}$ such that the open interval

$$\mathscr{I}_+ = (\lambda_{+-} + c_+(a + b|\lambda_{+-}|),\ \lambda_{++} + c_-(a + b|\lambda_{++}|)) \tag{107}$$

is non-void or

(ii) the form α satisfy the conditions of Theorem 2.14.

By

$$\lambda_1 = \lambda_{-+} \leqslant \lambda_2 \leqslant \cdots$$

denote the (finite or infinite) sequence of the eigenvalues of H in $\mathscr{I}$. Set

$$\widetilde{\lambda} = \begin{cases} \lambda_{++} + c_-(a + b|\lambda_k|), & \text{in case (i)} \\ \lambda_{++}, & \text{in case (ii)} \end{cases}$$

Then the spectrum of T in $\widetilde{\mathscr{I}} = (\lambda_{--} + a + b|\lambda_-|, \widetilde{\lambda})$ consists of the eigenvalues

$$\mu_1 \leqslant \mu_2 \leqslant \cdots$$

and they satisfy

$$\lambda_k + c_-(a + b|\lambda_k|) \leqslant \mu_k \leqslant \lambda_k + c_+(a + b|\lambda_k|). \tag{108}$$

in the following sense: in the case (i) λ *'s and* μ *'s have the same cardinality and (7) holds for all of them whereas in the case (ii) (7) holds as long as* $\lambda_k + c_+(a + b|\lambda_k|) < \lambda_{++}$ *.*

We omit the proof, it follows the lines of the one of Theorem 4.13 above. The only difference in the proof is the form $\widetilde{\alpha}_\varepsilon$ which now reads

$$\widetilde{\alpha}_\varepsilon = c_\pm \varepsilon(a + b\widehat{h}) + (1 - \varepsilon)\alpha.$$

Also, Remark 4.14 applies accordingly.

REMARK 4.16. If in the preceding theorem the form α is non-negative then the bound (108) reads

$$0 \leqslant \mu_k - \lambda_k \leqslant c_+(a + b|\lambda_k|) \leqslant a + b|\lambda_k|. \tag{109}$$

The preceding theorems cover perturbation estimates already known: by setting $a = 0$ the bound (7) was obtained in [9] for finite matrices. Also by setting $b = 0$ we have $T = H + A$, $A \in \mathscr{B}(\mathscr{H})$, $C = A/a$; here (108) gives the mentioned bound (4). Both (7) and (108) are sharp, they obviously become equalities on scalars.

Positioning of an impenetrable point is user dependent; usually a most convenient choice is to take broad spectral gaps. In the most notorious case of a positive definite H with a compact inverse the impenetrability from below is trivially fulfilled.

The proofs of Theorem 4.13 and 4.15 consist of two main ingredients:

1. upper semicontinuity bounds for general spectra from Theorem 3.6, 3.5 and
2. lower semicontinuity bounds for finite eigenvalues, obtained by the construction of monotone holomorphic operator families.

So, we may say that in order to fully control the eigenvalues in a gap by using (7) or (108) have to 'pay a price', that is, the perturbation should be so small as to insure that the impenetrability conditions (102), (103), (106), (107), respectively, are fulfilled. These expressions as well as the estimates in (7) or (108) use the same bound $\pm(a + b|\lambda|)$, so the price is completely adequate. This fact may be seen as a mark of the naturality of the obtained bounds.

5. Appendix

Proof of (44). Obviously the point $\xi = 0$ is a local minimum of $\psi(\cdot, a, b, \lambda, \eta)$. By

$$\psi(-\xi, a, b, \lambda, \eta) = \psi(\xi, a, b, -\lambda, \eta)$$

it is sufficient to take $\lambda \geqslant 0$. We distinguish two cases.

$\xi \geqslant 0$:

$$\psi_\xi = \frac{-\xi(a + \lambda b) + (\lambda^2 + \eta^2)b + \lambda a}{((\xi - \lambda)^2 + \eta^2)^{3/2}}.$$

The maximum is reached at

$$\xi = \xi_0 = \lambda + \frac{\eta^2 b}{a + \lambda b}$$

and it is equal to

$$\psi(\xi_0, a, b, \lambda, \eta) = \frac{1}{|\eta|}\sqrt{(a + \lambda b)^2 + \eta^2 b^2}$$

and this is (44).

$\xi \leqslant 0$:

$$\psi_\xi = \frac{-\xi(a - \lambda b) - (\lambda^2 + \eta^2)b + \lambda a}{((\xi - \lambda)^2 + \eta^2)^{3/2}}.$$

The maximum is reached at

$$\xi = \xi_1 = \lambda - \frac{\eta^2 b}{a - \lambda b}$$

and it is equal to

$$\psi(\xi_1, a, b, \lambda, \eta) = \frac{1}{|\eta|}\sqrt{(a - \lambda b)^2 + \eta^2 b^2},$$

provided that $a > \lambda b$ and

$$\lambda \leqslant \frac{b\eta^2}{a - \lambda b},$$

otherwise the maximum is reached on the boundary $\{-\infty, 0\}$. All three values are obviously less than (44) which is the sought global maximum. □

Proof of (105). For real λ we have

$$|\lambda + \varepsilon(a + b|\lambda|)| = \begin{cases} \lambda + \varepsilon(a + b\lambda), & \lambda \geqslant 0 \\ |\lambda + \varepsilon(a - b\lambda)|, & \lambda \leqslant 0 \end{cases}$$

Thus, for $\lambda \geqslant 0$

$$\lambda = \frac{-\varepsilon a}{1 + b\varepsilon} + \frac{|\lambda + \varepsilon(a + b|\lambda|)|}{1 + b\varepsilon} \leqslant \frac{\varepsilon a}{1 - b\varepsilon} + \frac{|\lambda + \varepsilon(a + b|\lambda|)|}{1 - b\varepsilon}$$

and for $\lambda \leqslant 0$

$$|\lambda + \varepsilon(a + b|\lambda|)| = |\varepsilon a + \lambda(1 - b\varepsilon)| \geqslant -\varepsilon a - \lambda(1 - b\varepsilon)$$

hence

$$-\lambda \leqslant \frac{\varepsilon a}{1 - b\varepsilon} + \frac{|\lambda + \varepsilon(a + b|\lambda|)|}{1 - b\varepsilon}.$$

Altogether

$$|\lambda| \leqslant \frac{\varepsilon a}{1 - b\varepsilon} + \frac{|\lambda + \varepsilon(a + b|\lambda|)|}{1 - b\varepsilon}.$$

Taking corresponding functions of H we obtain (105). □

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