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ON THE SPECTRA OF SOME TOEPLITZ AND WIENER–HOPF OPERATORS WITH ALMOST PERIODIC MATRIX SYMBOLS

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(*communicated by L. Rodman*)

Abstract. Constructive invertibilty criteria for Toeplitz and Wiener-Hopf operators with matrix almost periodic symbols in general are not known. Even in the case of 2×2 triangular symbols definite results are available only under some, rather restrictive additional requirements on the entries of those symbols. We show, however, that for certain symbols such additional requirements allow one to go one step further and actually describe the (essential) spectra of the operators in question. This description shows in particular that the number of connected components of the spectrum can be arbitrarily large — a striking difference with the scalar situation.

1. Introduction

First, we fix some notation. For any set X, we denote by $X_{n\times k}$ the set of all n-by-k matrices with entries from X; $X_{n\times 1}$ will be abbreviated to X_n.

For $1 \leqslant p \leqslant \infty$, we denote by $L^p(\mathbb{R})$ the usual Lebesgue space on the real line $\mathbb{R}$ and by H^p the respective Hardy space in the upper half plane. The Riesz projection P of $L^p(\mathbb{R})$ onto H^p is bounded for $1 < p < \infty$, and is extended entrywise on $L^p_n(\mathbb{R})$. For $a \in L^\infty_{n\times n}(\mathbb{R})$, the formula

$$T(a)f = Paf$$

defines the *Toeplitz operator* with the (matrix) symbol a on H^p_n, $1 < p < \infty$.

In its turn, the *convolution operator* with the symbol a is defined formally as

$$W^0(a) := F^{-1}aF.$$

Here F denotes the standard Fourier transform on the real line. The a priori domain of $W^0(a)$ is $L^2_n(\mathbb{R}) \cap L^p_n(\mathbb{R})$; if $W^0(a)$ actually maps this linear manifold into itself and extends continuously to a bounded operator on $L^p_n(\mathbb{R})$, the symbol a is called a *Fourier multiplier* on $L^p_n(\mathbb{R})$. The set of all Fourier multipliers on $L^p(\mathbb{R})$ is denoted M^p. Of course, $M^2 = L^\infty(\mathbb{R})$.

The *Wiener-Hopf operator* with the symbol $a \in M^p_{n\times n}$ acts on $L^p_n(\mathbb{R}_+)$ according to the formula

$$W(a)f := \chi_+ W^0(a)f.$$

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Here χ_+ is the characteristic function of the non-negative half-axis $\mathbb{R}_+ = [0, +\infty)$.

There is a vast amount of literature on the properties of Wiener-Hopf and Toeplitz operators, including their Fredholmness and invertibility, under various additional restrictions on their symbols. We mention here [3] as the most up to date general reference.

In this paper, we will be dealing with *almost periodic* symbols a. To give an explicit description of this class, we start with *almost periodic polynomials*, which by definition are finite linear combinations of the form

$$f = \sum_j c_j e_{\lambda_j}, \quad c_j \in \mathbb{C}, \ \lambda_j \in \mathbb{R}. \tag{1.1}$$

Here and in what follows, the abbreviation e_λ is used to denote the function

$$e_\lambda(x) = e^{i\lambda x}.$$

Almost periodic polynomials form a (non-closed) subalgebra APP of $L^\infty(\mathbb{R})$. The closure of APP in the norm of $L^\infty(\mathbb{R})$ is the classical algebra AP of Bohr almost periodic functions. Every $f \in AP$ has the *Bohr mean*

$$\mathbf{M}(f) := \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} f(t)\, dt.$$

This allows one to define the *Bohr-Fourier coefficients* of f as

$$\hat{f}(\lambda) := \mathbf{M}(e_{-\lambda} f);$$

this definition carries over to $f \in AP_{n\times k}$ entrywise. In its turn, the *Bohr-Fourier spectrum* of $f \in AP_{n\times k}$ is

$$\Omega(f) = \{\lambda \in \mathbb{R} : \hat{f}(\lambda) \neq 0\}.$$

The Bohr-Fourier spectrum of any $f \in AP$ is at most countable.

The algebra AP is *inverse closed* in $L^\infty(\mathbb{R})$, that is, for any $f \in AP$ which is invertible in $L^\infty(\mathbb{R})$ (equivalently, bounded away from 0 on $\mathbb{R}$), $f^{-1} \in AP$. For any such invertible function f the Bohr theorem guarantees the existence of a (obviously, unique) real number $\kappa := \kappa(f)$ such that a branch of $\log(e_{-\kappa} f)$ continuous on $\mathbb{R}$ actually lies in AP. This κ is called the *mean motion* of f.

The algebra APP can also be supplied with the *Wiener norm*

$$\|f\|_W := \sum_j |c_j|,$$

where f is a (still finite) sum (1.1). The closure APW of APP with respect to this norm consists of all functions of the form (1.1) for which the series of the coefficients c_j converges absolutely. Of course, APW is a subalgebra of AP, dense in the uniform norm. This algebra also is inverse closed in $L^\infty(\mathbb{R})$ (and in AP). Moreover, $APW \subset M^p$ for all $p \in (1, \infty)$.

For any subset X of the real line, we let

$$\Pi_X f = \sum_{\lambda \in X} \hat{f}(\lambda) e_\lambda. \tag{1.2}$$

Of course, formula (1.2) defines a projection with norm one on APW.

Scalar ($n=1$) Wiener-Hopf and Toeplitz operators with almost periodic symbols were treated by Coburn-Douglas and Gohberg-Feldman about forty years ago, see [2] for the exact references and the history of the subject. According to their results, the invertibility of $T(a)$ with $a \in AP$ on H^p for any $p \in (1,\infty)$ is equivalent to its Fredholmness, and it occurs if and only if a is invertible with the zero mean motion. The same statement holds for $W(a)$ provided that $a \in AP \cap M^p$, in particular if $a \in APW$.

From here it instantly follows that the spectrum and the essential spectrum of the operators $T(a)$ and $W(a)$ with the scalar symbol in AP (respectively, APW) do not depend on $p \in (1,\infty)$ and coincide with the union of $\mathscr{R}(a)$ (the closure of the range of a) and

$$\{\lambda \in \mathbb{C}\colon \kappa(f-\lambda) \neq 0\}. \tag{1.3}$$

The mean motion of $f-\lambda$ as a function of λ is constant on the connected components of $\mathbb{C} \setminus \mathscr{R}(a)$, so that the set (1.3) is the union of some of these components.

In the matrix ($n>1$) case, the invertibility/Fredhomness criterion can be extracted from [2, Chapters 18,19], and it reads as follows.

THEOREM 1.1. *Let $a \in APW_{n\times n}$. Then the following conditions are equivalent:*

1. *the operator $T(a)$ is invertible on H^p_n for at least one/all $p \in (1,\infty)$,*
2. *the operator $T(a)$ is Fredholm on H^p_n for at least one/all $p \in (1,\infty)$,*
3. *the operator $W(a)$ is invertible on $L^p_n(\mathbb{R}_+)$ for at least one/all $p \in (1,\infty)$,*
4. *the operator $W(a)$ is Fredholm on $L^p_n(\mathbb{R}_+)$ for at least one/all $p \in (1,\infty)$,*
5. *the symbol a admits a representation*

$$a = a_- a_+ \tag{1.4}$$

in which $a_+^{\pm 1} \in APW^+_{n\times n}$ and $a_-^{\pm 1} \in APW^-_{n\times n}$.

Here

$$APW^{\pm} = \{f \in APW\colon \Omega(f) \subset \mathbb{R}_{\pm}\},$$

with $\mathbb{R}_- = (-\infty, 0]$.

According to Theorem 1.1 the spectrum $\Sigma(a)$ of the operator $T(a)$ on H^p_n coincides with its essential spectrum and does not depend on p. Moreover, it is the same as the (essential) spectrum of $W(a)$ on any of the spaces $L^p_n(\mathbb{R}_+)$, $p \in (1,\infty)$. In this respect, the situation does not change when passing from the scalar to the matrix setting.

On the other hand, condition (5) of Theorem 1.1 is much more subtle than its scalar counterpart. It is of course necessary for (1.4) to hold that a is invertible and that $\kappa(\det a) = 0$ but the converse is not true. As a matter of fact, a constructive equivalent of (1.4) is not known in general, or even for a seemingly very simple class of triangular 2×2 matrix functions of the form

$$G_f = \begin{bmatrix} e_\lambda & f \\ 0 & e_{-\lambda} \end{bmatrix} \tag{1.5}$$

with $\lambda > 0$ and $f \in APW$.

Consequently, there is no explicit description of $\Sigma(G_f)$, let alone $\Sigma(a)$ for general $a \in APW_{n\times n}$. The purpose of our paper is to give such a description of $\Sigma(G_f)$ under various additional conditions imposed on the structure of f.

The general approach to this problem is discussed in Section 2. Sections 3–5 deal with the particular cases, the nature of which is clear from their headings. The results of these sections are used in final Section 6 to address the question about the possible number of connected components of the (essential) spectra of Toeplitz operators with matrix symbols.

2. General remarks

For convenience of references recall that (1.4) is nothing but the *canonical right AP factorization* of a. The results regarding its existence and construction can be found in [2], and some more recent development, relevant to the task at hand, in [1] and [5]. We note that some of the literature is written in the language of the *left AP* factorization, the difference being in the order of multiples $a_\pm$. Of course, the results carry from one type of factorization to the other by taking transposes. In what follows, this simple remark is used silently.

For a given $f \in APW$ and $\mu \in \mathbb{C}$, let

$$f_\mu = \sum_k \hat{f}(k\lambda)\mu^{|k|} + \sum_j f_{(j)} \left(e_{-j\lambda}\mu^{|j|} + e_{-(j+1)\lambda}\mu^{|j+1|}\right). \tag{2.1}$$

Here $f_{(j)}$ is an abbreviated notation for $\Pi_{(j\lambda,(j+1)\lambda)}f$ and the summation is with respect to integer j,k. Formula (2.1) defines an element of APW whenever $0<|\mu|<1$, and

$$\Omega(f_\mu) \subset (-\lambda,\lambda).$$

For $\mu = 0$ it is natural by continuity considerations to set

$$f_0 = \Pi_{(-\lambda,\lambda)}f.$$

If the function f itself is such that

$$\Omega(f) \subset (-\lambda,\lambda), \tag{2.2}$$

then the formulas for f_μ simplify to

$$f_\mu = f + \mu\left(e_\lambda f_{(-1)} + e_{-\lambda} f_{(0)}\right). \tag{2.3}$$

Finally, let $\mathbb{M}(G_f)$ denote the set of all such $\mu \in \mathbb{D}$ that G_{f_μ} does not admit a canonical right AP factorization. As usual, $\mathbb{T}$ stands for the unit circle and $\mathbb{D}$ for the (open) unit disk.

THEOREM 2.1. *For any matrix function G_f of the form (1.5),*

$$\Sigma(G_f) = \mathbb{T} \cup \mathbb{M}(G_f). \tag{2.4}$$

Proof. According to Theorem 1.1, $\mu \in \Sigma(G_f)$ if and only if the matrix function

$$G_f - \mu I = \begin{bmatrix} e_\lambda - \mu & f \\ 0 & e_{-\lambda} - \mu \end{bmatrix}$$

does not admit a canonical AP factorization. This is indeed the case for any $\mu \in \mathbb{T}$ because then $G_f - \mu I$ is not invertible. If $|\mu| > 1$, then we will make use of the identity

$$\begin{bmatrix} e_\lambda - \mu & f \\ 0 & e_{-\lambda} - \mu \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e_{-\lambda} - \mu \end{bmatrix} \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_\lambda - \mu & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.5}$$

The middle factor in the right hand side of (2.5) admits a canonical factorization, as does any non-singular triangular matrix function with constant diagonal. The left and right factors, being invertible in $APW^{\mp}_{2\times 2}$, respectively, do not affect the (right) factorability. So, for all such μ the matrix $G_f - \mu I$ admits a canonical AP factorization, and therefore $\Sigma(G_f) \subset \mathbb{T} \cup \mathbb{D}$.

It remains to show that

$$\Sigma(G_f) \cap \mathbb{D} = \mathbb{M}(G_f).$$

To this end, notice that $G_f - \mu I$ can also be represented as

$$\begin{bmatrix} 1 - \mu e_{-\lambda} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_\lambda & f(1 - \mu e_{-\lambda})^{-1}(1 - \mu e_\lambda)^{-1} \\ 0 & e_{-\lambda} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 - \mu e_\lambda \end{bmatrix}. \tag{2.6}$$

For $|\mu| < 1$, the left and right multiples in (2.6) are invertible in $APW^{\mp}_{2\times 2}$, respectively, so that $G_f - \mu I$ admits a canonical AP factorization only simultaneously with the middle factor in (2.6). The right upper entry of the latter, in its turn, equals

$$f \sum_{j,k=0}^{\infty} \mu^{j+k} e_{\lambda(j-k)} = f \sum_{l=-\infty}^{+\infty} e_{\lambda l} \sum_{k=\max(0,-l)}^{\infty} \mu^{l+2k} = \frac{1}{1-\mu^2} f \sum_{l=-\infty}^{+\infty} e_{\lambda l} \mu^{|l|} = \frac{1}{1-\mu^2} g,$$

where

$$g = f \sum_{l=-\infty}^{+\infty} e_{\lambda l} \mu^{|l|}.$$

Consequently, $G_f - \mu I$ admits a canonical AP factorization only simultaneously with

$$\begin{bmatrix} e_\lambda & \frac{1}{1-\mu^2} g \\ 0 & e_{-\lambda} \end{bmatrix} = \begin{bmatrix} \frac{1}{1-\mu^2} & 0 \\ 0 & 1 \end{bmatrix} G_g \begin{bmatrix} 1-\mu^2 & 0 \\ 0 & 1 \end{bmatrix},$$

and thus only simultaneously with G_g. According to [2, Proposition 13.4], the matrices G_g and $G_{\Pi_{(-\lambda,\lambda)} g}$ have the same AP factorization properties. It remains to observe that

$$\Pi_{(-\lambda,\lambda)} g = \Pi_{(-\lambda,\lambda)} \left(\sum_{k,l} \hat{f}(k\lambda) \mu^{|l|} e_{\lambda(k+l)} + \sum_{j,l} f_{(j)} \mu^{|l|} e_{\lambda l} \right) = f_\mu.$$

□

In the rest of the paper, we consider some cases in which $\mathbb{M}(G_f)$, and therefore $\Sigma(G_f)$, can be explicitly described.

3. Commensurable distances

Let the Bohr-Fourier spectrum of the element f in (1.5) be such that the distances between any two of its points are commensurable, in other words,

$$\Omega(f) \subset \Gamma \tag{3.1}$$

for some affine transformation Γ of $\mathbb{Z}$: $\Gamma = -\nu + h\mathbb{Z}$. The factorability properties of (1.5) in this situation were described in [6, 8], see also [2, Chapter 14].

According to (2.1), the commensurability of distances between the points of the Bohr-Fourier spectrum in general is not inherited by the function f_μ. However, this becomes the case if λ/h is rational. The description of $\Sigma(G_f)$ can then in principle be extracted from the above mentioned results on AP factorability of G_f with f satisfying (3.1).

We give concrete statements for the situation when f also satisfies (2.2). Refining the step h if necessary, we may then without loss of generality even suppose that

$$\lambda/h = N \in \mathbb{N}. \tag{3.2}$$

Then of course

$$\Omega(f_\mu) \subset \Gamma$$

simultaneously with (3.1).

The following result can be extracted from [2, Theorem 14.14].

THEOREM 3.1. *Let*

$$\Omega(f) \cap (-\lambda, \lambda) \subset -\nu + h\mathbb{Z} := \Gamma$$

for some $\nu, h \in \mathbb{R}$ *such that (3.2) holds. Denote* $c_j = M_{\tau+jh}(f)$, *where* τ *is the smallest non-negative element of* Γ, *and let*

$$T = T_N(f) = (c_{j-i})_{i,j=1}^N = \begin{bmatrix} c_0 & c_1 & \cdots & c_{N-1} \\ c_{-1} & c_0 & \cdots & c_{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1-N} & c_{2-N} & \cdots & c_0 \end{bmatrix},$$

$$\Delta = \Delta_N(f) = (c_{j-i-1})_{i,j=1}^N = \begin{bmatrix} c_{-1} & c_0 & \cdots & c_{N-2} \\ c_{-2} & c_{-1} & \cdots & c_{N-3} \\ \vdots & \vdots & \ddots & \vdots \\ c_{-N} & c_{1-N} & \cdots & c_{-1} \end{bmatrix}.$$

Then the matrix function (1.5) admits a canonical right AP factorization if and only if one of the following two conditions holds:

(a) $0 \in \Gamma$, *and* $\det T \neq 0$*;*

(b) $0 \notin \Gamma$ *and* $\det T \det \Delta \neq 0$.

Applying Theorem 3.1 to G_{f_μ} in place of G_f, with f_μ given by (2.3), we arrive to the following conclusion.

THEOREM 3.2. *Let us keep the setting and notation of Theorem 3.1. Along with T and Δ, introduce the following (also Toeplitz) matrices:*

$$\begin{aligned} B &= \begin{bmatrix} 0 & c_{1-N} & \cdots & c_{-1} \\ c_{N-1} & 0 & \cdots & c_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \cdots & 0 \end{bmatrix}, \\ C &= c_{-N}I + B, \\ D &= \begin{bmatrix} c_{N-1} & c_{-N} & \cdots & c_{-2} \\ c_{N-2} & c_{N-1} & \cdots & c_{-3} \\ \vdots & \vdots & \ddots & \vdots \\ c_0 & c_1 & \cdots & c_{N-1} \end{bmatrix}. \end{aligned}$$

Then $\Sigma(G_f)$ is given by (2.4) with $\mathbb{M}(G_f)$ consisting of the roots (in $\mathbb{D}$) of the linear pencil $T+\mu B$ if $0 \in \Gamma$ and of two linear pencils, $T+\mu C$ and $\Delta+\mu D$, if $0 \notin \Gamma$.

4. Low number of terms

We start with the simplest case of a *monomial* f.

THEOREM 4.1. *Let*

$$f = ce_\alpha, \quad c \neq 0. \tag{4.1}$$

Then $\Sigma(G_f) = \mathbb{T}$ if $\alpha = 0$ and $\Sigma(G_f) = \mathbb{T} \cup \{0\}$ otherwise.

Proof. A canonical right AP factorization of G_f with f given by (4.1) exists if and only if $\alpha = 0$. This takes care of the statement concerning zero.

For $\mu \neq 0$ formula (2.1) implies that f_μ is a non-zero constant (if $\alpha/\lambda \in \mathbb{Z}$) or a binomial with the distance between its exponents exactly equal λ (if $\alpha/\lambda \notin \mathbb{Z}$). Either way, G_{f_μ} admits a canonical AP factorization according to [2, Section 14.3]. It remains to invoke Theorem 2.1. □

The case of a binomial f with the distance between exponents divisible by λ can be tackled along the same lines. We skip the details, and turn to certain *trinomials* instead. This course will eventually allow us to cover a more interesting case of arbitrary binomials satisfying (2.2).

So, let

$$f = c_{-1}e_{-\nu} + c_0 + c_1e_\alpha, \text{ where } \alpha, \nu > 0 \text{ and } \alpha + \nu = \lambda. \tag{4.2}$$

Factorability of matrix functions G_f with such f was treated in [7]. Of course, the case of rational ν/α can also be extracted from Theorem 3.1. For the readers' convenience, we state the result.

THEOREM 4.2. *Let f be of the form (4.2). Then the existence of a canonical right AP factorization of the matrix function (1.5) is equivalent to*

$$c_0^{m+n} \neq (-1)^{m+n} c_1^m c_{-1}^n$$

if ν/α is rational and m/n is its representation in lowest terms, and to

$$|c_0|^\lambda \neq |c_1|^\nu \, |c_{-1}|^\alpha$$

if ν/α is irrational.

Observe that for f given by (4.2), the function f_μ will have its Bohr-Fourier spectrum contained in the same set $\{-\nu, 0, \alpha\}$, the difference being only in the coefficients:

$$f_\mu = (c_{-1} + \mu c_1)e_{-\nu} + c_0 + (c_1 + \mu c_{-1})e_\alpha.$$

The description of $\Sigma(G_f)$ therefore follows immediately.

THEOREM 4.3. *Let G_f satisfy (4.2). Then $\Sigma(G_f)$ is given by (2.4) with $\mathbb{M}(G_f)$ equal the set of $\mu \in \mathbb{D}$ for which*

$$(c_1 + \mu c_{-1})^m (c_{-1} + \mu c_1)^n = (-c_0)^{m+n} \tag{4.3}$$

if $\frac{\nu}{\alpha}$ $\left(= \frac{m}{n}\right)$ is rational, and

$$|c_1 + \mu c_{-1}|^\nu \, |c_{-1} + \mu c_1|^\alpha = |c_0|^\lambda$$

otherwise.

Of course, if two out of three coefficients in (4.2) vanish, the results of Theorem 4.3 agree with those of Theorem 4.1. Letting just one coefficient vanish, we arrive at the following statements regarding binomial f.

COROLLARY 4.4. *Let*

$$f = ae_\alpha + be_\beta, \ \textit{with } \alpha \in (0, \lambda), \ \beta = \alpha - \lambda, \ ab \neq 0.$$

Then $\Sigma(G_f) = \mathbb{T} \cup \{z\}$, where

$$z = \begin{cases} -a/b & \textit{if } |a| \leqslant |b| \\ -b/a & \textit{otherwise.} \end{cases}$$

COROLLARY 4.5. *Let*

$$f = a + be_\beta, \ \textit{with } 0 < |\beta| < \lambda, \ a \neq 0.$$

Then $\Sigma(G_f)$ equals $\mathbb{T}$ if $|a| \geqslant |b|$ and $\mathbb{T} \cup \mathbb{S}$ if $|a| < |b|$. Here $\mathbb{S}$ is the circle centered at the origin with the radius $\left|\frac{a}{b}\right|^{\lambda/|\beta|}$ if β/λ is irrational, or a finite subset of the points

$$\left(-\frac{a}{b}\right)^{\lambda/|\beta|}$$

on this circle otherwise.

When all three terms in (4.2) are actually present, the shape of $\Sigma(G_f)$ becomes more involved. The figures below illustrate this point.

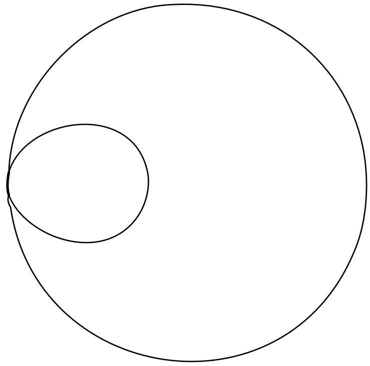

Figure 1. $c_{-1} = 2,\ c_0 = 1,\ c_1 = 1,\ \alpha = .5,\ \lambda = 1$

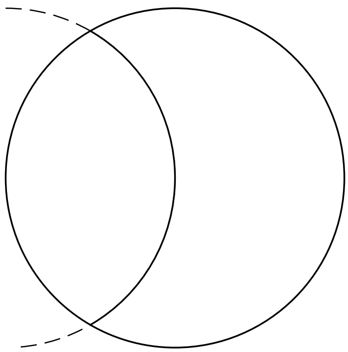

Figure 2. $c_{-1} = c_0 = 1,\ c_1 = 1,\ \alpha = .3,\ \lambda = 1$

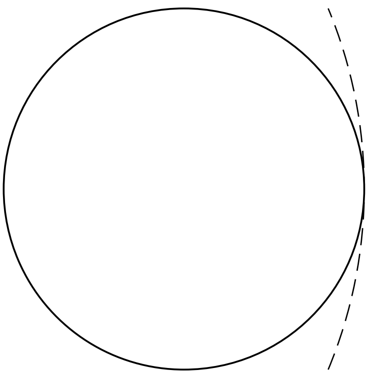

Figure 3. $c_{-1} = c_0 = 1,\ c_1 = 2,\ \alpha = .3,\ \lambda = 1$

Recently [1] the canonical AP factorization criterion was established for matrices (1.5) with a quadrinomial f of the form

$$f = c_{-2}e_{\beta-\lambda} + c_{-1}e_{\alpha-\lambda} + c_2e_\beta + c_1e_\alpha \text{ with } \lambda > \alpha > \beta > 0. \tag{4.4}$$

The results in [1] are stated in terms of the invertibility of a certain difference operator acting on $L^2[0,1]$; when transplanted to our setup, they can be reformulated as follows. The notation $\lfloor x \rfloor$ and $\lceil x \rceil$ stand for the best integer approximation of $x \in \mathbb{R}$ from below and above, respectively.

THEOREM 4.6. *Let f be of the form (4.4). Then the existence of a canonical right AP factorization of the matrix function (1.5) is equivalent to*

$$(-1)^n c_1^{n-m-k} c_{-1}^{m+k} \neq c_2^{n-k} c_{-2}^{k} \qquad (k = \lfloor n\beta/\lambda \rfloor, \lceil n\beta/\lambda \rceil) \tag{4.5}$$

if $\frac{\alpha-\beta}{\lambda}$ is rational, with the lowest terms representation $\frac{m}{n}$, and to

$$|c_1|^{\lambda-\alpha}\, |c_{-1}|^{\alpha} \neq |c_2|^{\lambda-\beta}\, |c_{-2}|^{\beta}$$

otherwise.

Observe that (4.5) contains just one condition if $n\beta/\lambda$ is an integer.

Similarly to the situation of Theorem 4.2, the case of rational $\frac{\alpha-\beta}{\lambda}$ in the setting of Theorem 4.6 can be tackled with the use of Theorem 3.1. On the other hand, cases when some coefficients in (4.4) vanish are covered by earlier results, see [10] and [2, Sections 14.2 and 15.4].

As in the trinomial case (4.2), the spectral structure of f given by (4.4) is inherited by f_μ:

$$f_\mu = (c_{-2} + \mu c_2) e_{\beta-\lambda} + (c_{-1} + \mu c_1) e_{\alpha-\lambda} + (c_2 + \mu c_{-2}) e_\beta + (c_1 + \mu c_{-1}) e_\alpha.$$

Therefore, Theorem 4.6 yields the following spectral picture.

THEOREM 4.7. *Let G_f satisfy (4.4) and let m, n be as in Theorem 4.6. Then $\Sigma(G_f)$ is given by (2.4) with $\mathbb{M}(G_f)$ equal the set of $\mu \in \mathbb{D}$ satisfying*

$$(-1)^n (c_1 + \mu c_{-1})^{n-m-k} (c_{-1} + \mu c_1)^{m+k} = (c_2 + \mu c_{-2})^{n-k} (c_{-2} + \mu c_2)^k$$

for $k = \lfloor n\beta/\lambda \rfloor$ or $\lceil n\beta/\lambda \rceil$ if $\frac{\alpha-\beta}{\lambda}$ is rational, and

$$|c_1 + \mu c_{-1}|^{\lambda-\alpha}\, |c_{-1} + \mu c_1|^{\alpha} = |c_2 + \mu c_{-2}|^{\lambda-\beta}\, |c_{-2} + \mu c_2|^{\beta}$$

otherwise.

Setting exactly two of the coefficients of (4.4) at zero allows one to handle all the remaining cases of binomials

$$f = a e_\xi + b e_\eta, \text{ with } ab \neq 0 \tag{4.6}$$

satisfying (2.2). In particular, the choices $c_{-1} = c_{-2} = 0$ and $c_1 = c_2 = 0$ yield the description of $\Sigma(G_f)$ for the binomial f with one sided Bohr-Fourier spectrum.

COROLLARY 4.8. *Let in (4.6) $0 < \xi < \eta < \lambda$ or $-\lambda < \eta < \xi < 0$. Then $\Sigma(G_f)$ equals $\mathbb{T} \cup \{0\}$ if $|a| \geqslant |b|$ and $\mathbb{T} \cup \{0\} \cup \mathbb{S}$ if $|a| < |b|$. Here $\mathbb{S}$ is the circle centered at the origin with the radius $\left|\frac{a}{b}\right|^{\lambda/|\xi-\eta|}$ if $(\xi-\eta)/\lambda$ is irrational, or a finite subset of the points*

$$\left(-\frac{a}{b}\right)^{\lambda/|\xi-\eta|}$$

on this circle otherwise.

The case of binomials (4.6) such that $\xi\eta < 0$ can be handled by applying Theorem 4.7 with $c_{-1} = c_2 = 0$ or $c_{-2} = c_1 = 0$, depending on whether $|\xi - \eta|$ is bigger or smaller than λ. For rational $(\xi - \eta)/\lambda$ the resulting statements are somewhat cumbersome, and therefore we skip them. (Besides, they can be extracted from more general statements of Section 3.) For irrational $(\xi - \eta)/\lambda$ the results are as follows.

COROLLARY 4.9. *Let in (4.6)* $-\lambda < \eta < 0 < \xi < \lambda$ *and* $(\xi - \eta)/\lambda$ *is irrational. Then* $\Sigma(G_f)$ *equals one of three sets:*
$\mathbb{T} \cup \{0\}$ *if* $\xi + \eta = 0,\ |a| \neq |b|$, *or* $\xi + \eta > 0,\ |b| \geqslant |a|$, *or* $\xi + \eta < 0,\ |b| \leqslant |a|$*;*
$\mathbb{T} \cup \{0\} \cup \left\{ \left|\frac{b}{a}\right|^{\lambda/(\xi+\eta)} \right\}$ *if* $\xi + \eta > 0,\ |b| < |a|$ *or* $\xi + \eta < 0,\ |b| > |a|$*;*
the closed unit disk $\mathbb{T} \cup \mathbb{D}$ *if* $\xi + \eta = 0,\ |b| = |a|$.

A few possible shapes of $\Sigma(G_f)$ in case of four non-zero terms in (4.4) are illustrated by the figures 4 and 5 below.

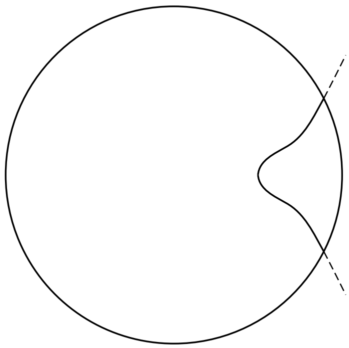

Figure 4. $c_{-2} = .75,\ c_{-1} = -.1,\ c_1 = .8,\ c_2 = 1.5,\ \alpha = \sqrt{.48},\ \beta = .6, \lambda = 1$

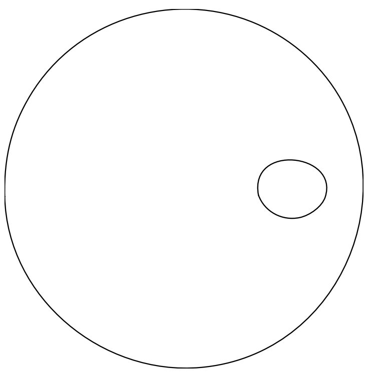

Figure 5. $c_{-2} = .75,\ c_{-1} = -1,\ c_1 = .7,\ c_2 = .5,\ \alpha = \sqrt{.48},\ \beta = .6, \lambda = 1$

5. Karlovich's case

In this section, we consider f of the form

$$f = (a + be_{-\lambda})Q, \text{ where } Q = \prod_{k=1}^{m}(d_k - c_k e_{\beta_k}), \tag{5.1}$$

$\beta_1 + \cdots + \beta_m < \lambda$, $\beta_k > 0$ and β_k/λ are irrational for all k. In particular, such a representation exists for f satisfying

$$\Omega(f) \subset [-\lambda, \lambda),\ \Omega(f) \cap (0, \lambda) \subset \beta\mathbb{Z},\ \Omega(f) \cap (-\lambda, 0) \subset -\lambda + \beta\mathbb{Z}$$

for some $\beta > 0$ provided that β/λ is irrational and that the quotients $\hat{f}(j\beta)/\hat{f}(j\beta - \lambda)$ are the same for all $j = 0, \ldots, \lfloor \lambda/\beta \rfloor$.

Without loss of generality, $c_k \neq 0$ for all k in (5.1); otherwise the k-th multiple in Q is constant and can be dropped (with a, b properly adjusted). From the results of [5] it follows that the matrix (1.5) with f as just described admits a canonical right AP factorization if and only if

$$a \neq 0 \text{ and } |b/a|^{\beta_k} \neq |d_k/c_k|^{\lambda} \text{ for } k = 1, \ldots, m.$$

It follows from (2.1) that for f of the form (5.1) the function f_μ only by a constant multiple of $e_{-\lambda}$ differs from

$$[(a + b\mu) + (b + a\mu)e_{-\lambda}]Q.$$

The latter function has the same structure as (5.1); only the coefficients vary.

Therefore, we have the following spectral picture.

THEOREM 5.1. *Let f satisfy (5.1). Then $\Sigma(G_f)$ is:*
the unit circle $\mathbb{T}$ if $a = \pm b \neq 0$ and $|c_k| \neq |d_k|$ for at least one value of k;
the closed unit disk $\mathbb{T} \cup \mathbb{D}$ if $a = \pm b \neq 0$ and $|c_k| = |d_k|$ for all k;
in all other cases $\Sigma(G_f) = \mathbb{T} \cup (\mathbb{D} \cap \mathbb{S})$, where $\mathbb{S}$ is the union of the images of concentric circles

$$|z| = \left|\frac{d_k}{c_k}\right|^{\lambda/\beta_k}, \quad k = 1, \ldots, m$$

under the fractional-linear transformation $\mu = \frac{az-b}{a-bz}$ and the point $-a/b$.

6. Final remarks

The spectrum and the essential spectrum of scalar Toeplitz operators $T(a)$ on H^2 with arbitrary L^∞ symbols a are both always connected. This beautiful result for the regular spectrum was established by H. Widom [11]. The essential spectrum was treated by R. Douglas, see [4, Theorem 7.45]. Widom actually showed [12] that the connectedness of the regular spectrum persists for the spaces H^p with p different from 2. As conjectured by Böttcher and Silbermann [3, Section 2.35], Douglas' result is also most likely valid in the more general H^p setting.

However, the situation changes in the matrix case: simply by considering diagonal symbols it is easy to see that for $n \times n$ matrix functions a the (essential) spectrum of $T(a)$ may contain anywhere from 1 to n connected components. Using the known Fredholmness criteria [3, 9], it is not difficult to show that the number of connected component stays limited by n for any $n \times n$ continuous or piecewise continuous symbol a. A natural question about the upper bound for the number of such components for arbitrary symbols was raised in [9, Notes to Chapter 7].

The results of Sections 3–5 show that there is no such bound, even for triangular 2×2 matrix functions. Namely, an arbitrarily large number of components may be achieved in the setting of Theorem 3.2, Corollary 4.5 (with all but one component being singletons) or Theorem 5.1 (with all components being circles). On the other hand, in some settings (such as Corollary 4.4) the number of components stays limited by 2.

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