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TOEPLITZ–PLUS–HANKEL BEZOUTIANS AND INVERSES OF TOEPLITZ AND TOEPLITZ–PLUS–HANKEL MATRICES

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Abstract. In the present paper Bezoutian-type formulas for the inverses of Toeplitz-plus-Hankel (T+H) matrices are presented which involve bases of kernels of associated rectangular T+H matrices. Special Bezoutians of this type yield inverses of symmetric or skewsymmetric Toeplitz matrices and vice versa. In the skewsymmetric case these formulas lead directly to splitting formulas for inverses of centro-skewsymmetric T+H matrices.

1. Introduction

It is well known that the inverse of a (nonsingular) Toeplitz matrix

$$T_n = [\,a_{i-j}\,]_{i,j=1}^{n}$$

is the Toeplitz Bezoutian of two vectors $\mathbf{u}, \mathbf{v}$. In [7] it was shown that these vectors form a basis of the kernel of the (non-square) Toeplitz matrix ∂T_n which is obtained from T_n by cancelling the first row and adding a column to the right. Conversely, the Toeplitz Bezoutian built from any basis of the kernel of this matrix ∂T_n is up to a constant factor equal to the inverse of T_n. Similar observations can be made for Hankel matrices $H_n = [\,b_{i+j}\,]$.

In the present paper we mainly consider matrices R_n which are the sum of a Toeplitz and a Hankel matrix, $R_n = T_n + H_n$. We call these matrices briefly T+H matrices. In [8] it was discovered that inverses of such matrices possess a generalized Bezoutian structure. These Bezoutians will be referred to as T+H-Bezoutians. Now the question arises if it is possible to define for a nonsingular T+H matrix R_n again connected matrices so that a basis of their kernels yield all parameters needed to construct the inverse R_n^{-1}. The starting point for dealing with this problem was given in [8]. In the present paper we discuss which linear combinations of the vectors of any bases result in the vectors involved in the Bezoutian formula for R_n^{-1}.

To explain the content of the present paper, let us first present some definitions and simple facts. Throughout the paper we consider matrices with entries from a given field

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$\mathbb{F}$ with a characteristic not equal to 2. In all what follows, let J_n stand for the $n \times n$ matrix of the counteridentity

$$J_n = \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix}.$$

For a vector $\mathbf{u} \in \mathbb{F}^n$ we denote

$$\mathbf{u}^J = J_n \mathbf{u}\,, \tag{1.1}$$

for an $m \times n$ matrix A,

$$A^J = J_m A J_n\,.$$

A vector $\mathbf{u}$ is called *symmetric* if $\mathbf{u} = \mathbf{u}^J$, it is called *skewsymmetric* if $\mathbf{u}^J = -\mathbf{u}^J$. A matrix A is called *centrosymmetric* if $A = A^J$ and is called *centro-skewsymmetric* if $A = -A^J$. A square Toeplitz matrix is centrosymmetric if and only if it is symmetric, and it is centro-skewsymmetric if and only if it is skewsymmetric.

We adopt some further notation in Section 2. Section 3 is dedicated to the Toeplitz case. After recalling basic inversion formulas in Subsection 3.1 we consider in Subsections 3.2 and 3.3 inverses of symmetric and skewsymmetric Toeplitz matrices. In both cases we present splitting formulas which involve special T+H-Bezoutians. These kinds of Bezoutians were introduced in [14] and have the nice property that all their rows and columns are either symmetric or skewsymmetric vectors. In particular, the splitting representation of Subsection 3.3 seems to be new.

Section 4 is dedicated to the Toeplitz-plus-Hankel case. In Subsection 4.1 we give the definition of a T+H-Bezoutian and discuss the uniqueness of this representation. In the first part of Subsection 4.2 we show how to get the vectors needed in the Bezoutian formula for the inverse of a T+H matrix from the bases of two associated $(n-2)\times(n+2)$ T+H matrices. Here we present a perhaps new and systematic solution of this problem.

How to compute these bases efficiently is beyond the scope of the present paper. For this question we refer to [18], [3], [15] and the references therein. Note that the idea to take advantage from splitting the vectors into their symmetric and skewsymmetric parts goes back to Delsarte and Genin [1], [2] (see also [17], [5]).

The last part of Subsection 4.2 is concentrated on the special cases of centrosymmetric and centro-skewsymmetric T+H matrices. Here we show how the splitting formulas for symmetric and skewsymmetric Toeplitz matrix inverses designed in Subsections 3.2 and 3.3 directly lead to splitting formulas for inverses of centrosymmetric and centro-skewsymmetric T+H matrices. Such splitting formulas have already been presented in [14], [15]. But, in particular, the proof in the centro-skewsymmetric case was more complicated than our proof presented here at the end of Subsection 4.2.

A justification to design inversion formulas for centrosymmetric and centro-skewsymmetric T or T+H matrices with the help of some special, simply structured T+H Bezoutians is that if $\mathbb{F}$ is the field of real or complex numbers, then these Bezoutians have matrix representations that include only discrete Fourier matrices or matrices of other trigonometric transformations and diagonal matrices. This allows to carry out matrix-vector multiplication with computational complexity $O(n \log n)$ (see [9], [10], [11], [12]).

2. Notation

The Bezoutian concept is conveniently introduced in polynomial language. Let $\ell_n(t)$, $t \in \mathbb{F}$, be the vector

$$\ell_n(t) = (t^j)_{j=0}^{n-1} .$$

First we introduce “polynomial language" for vectors. For $\mathbf{x} = (x_i)_{i=1}^n \in \mathbb{F}^n$, we consider the polynomial

$$\mathbf{x}(t) = \ell_n(t)^T \mathbf{x} = \sum_{k=1}^n x_k t^{k-1} \in \mathbb{F}^n(t)$$

and call it the *generating polynomial of* $\mathbf{x}$. Polynomial language for matrices means that we introduce the *generating polynomial of an* $m \times n$ *matrix* $A = [\, a_{ij} \,]_{i=1,j=1}^{m \quad n} \in \mathbb{F}^{m\times n}$ as the bivariate polynomial

$$A(t,s) = \ell_m(t)^T A \ell_n(s) = \sum_{i=1}^m \sum_{j=1}^n a_{ij}\, t^{i-1} s^{j-1}.$$

The Hankel Bezoutian (shortly H-Bezoutian) of $\mathbf{p}, \mathbf{q} \in \mathbb{F}^{n+1}$ is, by definition, the $n \times n$ matrix B with generating function

$$B(t,s) = \frac{\mathbf{p}(t)\mathbf{q}(s) - \mathbf{q}(t)\mathbf{p}(s)}{t-s} . \tag{2.1}$$

We write $B = \mathrm{Bez}_H(\mathbf{p},\mathbf{q})$. An $n \times n$ matrix B is called the Toeplitz Bezoutian (T-Bezoutian) of $\mathbf{p}, \mathbf{q} \in \mathbb{F}^{n+1}$, $B = \mathrm{Bez}_T(\mathbf{p},\mathbf{q})$, if its generating function is of the form

$$B(t,s) = \frac{\mathbf{p}(t)\mathbf{q}(s^{-1})s^n - \mathbf{q}(t)\mathbf{p}(s^{-1})s^n}{1-ts} . \tag{2.2}$$

Notice that

$$(J_{n+1}\mathbf{p})(t) = \mathbf{p}(t^{-1})t^n.$$

It is well known that a nonsingular matrix is the inverse of a Toeplitz matrix (Hankel matrix) if and only if it is a T-Bezoutian (H-Bezoutian).

We introduce the transformation $\nabla_T : \mathbb{F}^{n\times n} \to \mathbb{F}^{(n+1)\times(n+1)}$ in the language of generating polynomials as follows:

$$(\nabla_T B)(t,s) = (1-ts)B(t,s).$$

Then a nonsingular matrix B is the inverse of a Toeplitz matrix if and only if $\mathrm{rank} \nabla_T B = 2$. Let S_n denote the (forward) shift in $\mathbb{F}^n$,

$$S_n = \begin{bmatrix} 0 & 0 & & 0 \\ 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix} . \tag{2.3}$$

For a matrix A of order n the matrix $\nabla_T A$ can be written in the form

$$\nabla_T A = \begin{bmatrix} A - S_n A S_n^T & * \\ * & * \end{bmatrix} = \begin{bmatrix} * & * \\ * & S_n^T A S_n - A \end{bmatrix}, \tag{2.4}$$

and it is easy to see that A is a Toeplitz matrix if and only if the matrix of order $n-2$ in the center of $\nabla_T A$ is the zero matrix.

Analogously, in the Hankel case one can consider the transformation $\nabla_H : \mathbb{F}^{n\times n} \to \mathbb{F}^{(n+1)\times(n+1)}$ defined by

$$(\nabla_H B)(t,s) = (t-s)B(t,s).$$

Hereafter we denote by $\mathbb{F}^n_+, \mathbb{F}^n_-$ the subspaces of all symmetric, skewsymmetric vectors of $\mathbb{F}^n$, respectively. Let $P_\pm$ be the matrices

$$P_\pm = \frac{1}{2}(I_n \pm J_n)\,. \tag{2.5}$$

These matrices are projections onto $\mathbb{F}^n_\pm$ and

$$P_+ + P_- = I_n, \quad P_+ - P_- = J_n\,.$$

It is easy to see that a centrosymmetric matrix $A \in \mathbb{F}^{n\times n}$ maps $\mathbb{F}^n_\pm$ to $\mathbb{F}^n_\pm$, $AP_\pm = P_\pm A P_\pm$, whereas a centro-skewsymmetric matrix A maps $\mathbb{F}^n_\pm$ to $\mathbb{F}^n_\mp$, $AP_\pm = P_\mp A P_\pm$.

3. The Toeplitz case

3.1. Inverses of Toeplitz matrices

Recall that the inverse of a Toeplitz matrix

$$T_n = [\,a_{i-j}\,]_{i,j=1}^n$$

is a (nonsingular) T-Bezoutian $B = \mathrm{Bez}_T(\mathbf{u},\mathbf{v})$ (and vice versa). The question is how to obtain the involved vectors $\mathbf{u}$ and $\mathbf{v}$ of $\mathbb{F}^{n+1}$. Starting with the Gohberg-Semencul formula [4], where these vectors are given from the first and the last column of T_n^{-1}, there are further possibilities discussed in the literature.

We recall now an approach which contains these possibilities as special cases (compare [7]). To that aim we introduce the $(n-1)\times(n+1)$ Toeplitz matrix ∂T_n obtained from $T_n = [\,a_{i-j}\,]_{i,j=1}^n$ after deleting the first row and adding another column to the right by preserving the Toeplitz structure,

$$\partial T_n = \begin{bmatrix} a_1 & a_0 & \dots & a_{2-n} & a_{1-n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & \dots & a_0 & a_{-1} \end{bmatrix}. \tag{3.1}$$

If T_n is nonsingular, then ∂T_n has a two-dimensional nullspace. Each basis of this subspace is called a *fundamental system for* T_n. The reason for this notion is that any fundamental system for T_n delivers the desired vectors $\mathbf{u},\mathbf{v}$. To explain this we first mention the following fact. If $\operatorname{span}\{\mathbf{u},\mathbf{v}\} = \operatorname{span}\{\mathbf{u}_1,\mathbf{v}_1\}$, in other words, if there is a nonsingular 2×2 matrix φ such that

$$[\,\mathbf{u}_1 \;\; \mathbf{v}_1\,] = [\,\mathbf{u} \;\; \mathbf{v}\,]\,\varphi\,, \tag{3.2}$$

then

$$\mathbf{u}_1(t)\mathbf{v}_1^J(s) - \mathbf{v}_1(t)\mathbf{u}_1^J(s) = (\det\varphi)\,(\mathbf{u}(t)\mathbf{v}^J(s) - \mathbf{v}(t)\mathbf{u}^J(s))\,.$$

Thus,

$$\mathrm{Bez}_T\,(\mathbf{u}_1,\mathbf{v}_1) = (\det\varphi)\;\mathrm{Bez}_T\,(\mathbf{u},\mathbf{v})\,.$$

In particular, the (nontrivial) T-Bezoutians $\mathrm{Bez}_T(\mathbf{u},\mathbf{v})$ and $\mathrm{Bez}_T(\mathbf{u}_1,\mathbf{v}_1)$ coincide if and only if the vectors $\mathbf{u},\mathbf{v}$ and $\mathbf{u}_1$, $\mathbf{v}_1$ are related via (3.2) with $\det\varphi = 1$.

Now we recall the following basic inversion formula (see [7], Theorem 1.1 and (1.47)). Denote by $\mathbf{f}$ the vector $\mathbf{f} = (a_{-i})_{i=1}^n$ whith a_{-n} arbitrary and by $\{\mathbf{e}_k\}_{k=1}^n$ the canonical basis of $\mathbb{F}^n$.

THEOREM 3.1. *Let the equations*

$$T_n\mathbf{y} = \mathbf{e}_1 \quad \textit{and} \quad T_n\mathbf{z} = \mathbf{f}^J$$

be solvable. Then T_n is nonsingular, and

$$T_n^{-1} = \mathrm{Bez}_T(\mathbf{u},\mathbf{v})\,,$$

where

$$\mathbf{u} = \begin{bmatrix} \mathbf{y} \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -\mathbf{z} \\ 1 \end{bmatrix}. \tag{3.3}$$

It is easy to see that $\{\mathbf{u},\mathbf{v}\}$ with $\mathbf{u},\mathbf{v}$ defined in (3.3) is a fundamental system for T_n which satisfies the following normalization condition. If F is the $2\times(n+1)$ matrix

$$F = \begin{bmatrix} a_0 & \dots & a_{1-n} & a_{-n} \\ 0 & \dots & 0 & 1 \end{bmatrix},$$

then

$$F[\,\mathbf{u} \;\; \mathbf{v}\,] = I_2\,. \tag{3.4}$$

A fundamental system $\{\mathbf{u},\mathbf{v}\}$ which satisfies (3.4) is called *canonical*. For every fixed a_{-n} the canonical fundamental system is unique.

Consider now a general fundamental system $\{\mathbf{u},\mathbf{v}\}$. The matrix $\varphi = F[\,\mathbf{u} \;\; \mathbf{v}\,]$ is nonsingular. Indeed, suppose it is singular. Then there is a nontrivial linear combination $\mathbf{w}(t)$ of $\mathbf{u}(t)$ and $\mathbf{v}(t)$ such that $F\mathbf{w} = \mathbf{0}$. In particular, the highest order coefficient vanishes, i.e. $\mathbf{w}\in\mathbb{F}^n$. Since $\mathbf{w}\in\ker\partial T_n$ we conclude that $T_n\mathbf{w} = \mathbf{0}$, which means that T_n is singular. Now the columns of $[\,\mathbf{u} \;\; \mathbf{v}\,]\varphi^{-1}$ form a canonical fundamental system, and we conclude the following.

THEOREM 3.2. *Let $\{\mathbf{u},\mathbf{v}\}$ be a fundamental system for T_n. Then*

$$T_n^{-1} = \frac{1}{\det\varphi}\,\mathrm{Bez}_T(\mathbf{u},\mathbf{v})\,,$$

where $\varphi = F[\,\mathbf{u} \;\; \mathbf{v}\,]$.

Now we present a modification of the latter result which is especially interesting since the Levinson algorithm computes recursively the vectors under consideration henceforth. Let $T_{n+1} = [\,a_{i-j}\,]_{i,j=1}^{n+1}$ be a nonsingular Toeplitz extension of T_n, $\mathbf{x}_{n+1}^{-}$ the first and $\mathbf{x}_{n+1}^{+}$ the last column of T_{n+1}^{-1}. Then $\{\mathbf{x}_{n+1}^{-}, \mathbf{x}_{n+1}^{+}\}$ is a fundamental systems for T_n and

$$F\left[\begin{array}{cc} \mathbf{x}_{n+1}^{-} & \mathbf{x}_{n+1}^{+} \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ * & \xi_{n+1} \end{array}\right], \tag{3.5}$$

where $\xi_{n+1} = \mathbf{e}_{n+1}^T \mathbf{x}_{n+1}^{+} \neq 0$.

COROLLARY 3.3. *The inverse of the Toeplitz matrix T_n is given by*

$$T_n^{-1} = \frac{1}{\xi_{n+1}} \operatorname{Bez}_T(\mathbf{x}_{n+1}^{-}, \mathbf{x}_{n+1}^{+})\,.$$

3.2. Inverses of symmetric Toeplitz matrices

We discuss now the case of a nonsingular, symmetric Toeplitz matrix T_n. Since T_n^J is equal to the transpose of T_n, a Toeplitz matrix is symmetric if and only if it is centrosymmetric. Let

$$T_{n+1} = [\,a_{|i-j|}\,]_{i,j=1}^{n+1}$$

be a nonsingular, symmetric Toeplitz extension of T_n. (Note that T_{n+1} is nonsingular with the exception of at most two values of a_n.)

Since $\mathbf{x}_{n+1} := \mathbf{x}_{n+1}^{-} = (\mathbf{x}_{n+1}^{+})^J$ we have

$$T_n^{-1} = \frac{1}{\xi_{n+1}} \operatorname{Bez}_T(\mathbf{x}_{n+1}, \mathbf{x}_{n+1}^J)\,.$$

Now we consider the vectors $\mathbf{w}_{n+1}^{\pm} = \mathbf{x}_{n+1} \pm \mathbf{x}_{n+1}^J \in \mathbb{F}_{\pm}^{n+1}$ which are the solutions of

$$T_{n+1}\mathbf{w}_{n+1}^{\pm} = \mathbf{e}_1 \pm \mathbf{e}_{n+1}\,.$$

Obviously, they form a fundamental system of T_n, and

$$F\left[\begin{array}{cc} \mathbf{w}_{n+1}^{+} & \mathbf{w}_{n+1}^{-} \end{array}\right] = \left[\begin{array}{cc} 1 & 1 \\ \mathbf{w}_{n+1}^{+}(0) & -\mathbf{w}_{n+1}^{-}(0) \end{array}\right].$$

PROPOSITION 3.4. *The inverse of a symmetric Toeplitz matrix T_n can be represented as a T-Bezoutian of a symmetric and a skewsymmetric vector,*

$$T_n^{-1} = \frac{1}{\gamma} \operatorname{Bez}_T(\mathbf{w}_{n+1}^{-}, \mathbf{w}_{n+1}^{+})\,,$$

where $\gamma = \mathbf{w}_{n+1}^{+}(0) + \mathbf{w}_{n+1}^{-}(0)$.

Finally let us present an inversion formula for a symmetric Toeplitz matrix T_n which involves a new kind of Bezoutians introduced in [14]. First we recall the definition.

Let $\mathbf{p}, \mathbf{q} \in \mathbb{F}^{n+2}$ be either both symmetric or both skewsymmetric. Then

$$B_{\text{split}}(t,s) = \frac{\mathbf{p}(t)\mathbf{q}(s) - \mathbf{q}(t)\mathbf{p}(s)}{(t-s)(1-ts)}$$

is a polynomial in t and s. The $n \times n$ matrix with the generating polynomial $B_{\text{split}}(t,s)$ will be called the *split Bezoutian* of $\mathbf{p}(t)$ and $\mathbf{q}(t)$ and denoted by $\text{Bez}_{\text{split}}(\mathbf{p}, \mathbf{q})$. As we will see in the next section, split Bezoutians are special Toeplitz-plus-Hankel Bezoutians. It is easy to see that $\text{Bez}_{\text{split}}(\mathbf{p}, \mathbf{q})$ is a symmetric and centrosymmetric matrix. If $\mathbf{p}$ and $\mathbf{q}$ are symmetric, then the columns and rows of the split Bezoutian are symmetric, and we speak of a *split Bezoutian of (+)-type*. If $\mathbf{p}$ and $\mathbf{q}$ are skewsymmetric then the columns and rows of the split Bezoutian are skewsymmetric, and we then refer to a *split Bezoutian of (−)-type*. Hereafter instead of B_{split} we will often write B_+ or B_-, respectivly. Matrices with a generating polynomial of the form

$$B_\pm(t,s)\frac{1 \mp s}{1 \pm s}$$

occur in the skewsymmetric case for even n. They have the property that all rows are in $\mathbb{F}^n_\pm$ and all columns are in $\mathbb{F}^n_\mp$.

Now let $\mathbf{u}_+ \in \mathbb{F}^{n+1}_+, \mathbf{v}_- \in \mathbb{F}^{n+1}_-$ form a fundamental system for the symmetric Toeplitz matrix T_n, and let B be the (symmetric) T-Bezoutian $B = \text{Bez}_T(\mathbf{u}_+, \mathbf{v}_-)$.

PROPOSITION 3.5. *The matrices $B_\pm = BP_\pm$ are split Bezoutians of (±)-type,*

$$B_\pm = \pm\frac{1}{2}\,\text{Bez}_{\text{split}}(\mathbf{p}_\pm, \mathbf{q}_\pm)\,,$$

where $\mathbf{p}_+$ and $\mathbf{q}_+$ are symmetric, and $\mathbf{p}_-$ and $\mathbf{q}_-$ are skewsymmetric vectors given by

$$\mathbf{p}_\pm(t) = (t \pm 1)\mathbf{u}_+(t) \quad \textit{and} \quad \mathbf{q}_\pm(t) = (t \mp 1)\mathbf{v}_-(t)\,. \tag{3.6}$$

Proof. We compute the generating polynomial of $B_\pm = BP_\pm$,

$$\begin{aligned} B_\pm(t,s) &= \frac{1}{2}\left(B(t,s) \pm B(t,s^{-1})s^{n-1}\right) \\ &= \frac{1}{2}\left(-\frac{\mathbf{u}_+(t)\mathbf{v}_-(s) + \mathbf{v}_-(t)\mathbf{u}_+(s)}{1-ts} \pm \frac{\mathbf{u}_+(t)\mathbf{v}_-(s) - \mathbf{v}_-(t)\mathbf{u}_+(s)}{t-s}\right) \\ &= \pm\frac{1}{2}\,\frac{(t\pm 1)\mathbf{u}_+(t)(s\mp 1)\mathbf{v}_-(s) - (t\mp 1)\mathbf{v}_-(t)(s\pm 1)\mathbf{u}_+(s)}{(t-s)(1-ts)} \\ &= \pm\frac{1}{2}\,\left(\text{Bez}_{\text{split}}(\mathbf{p}_\pm, \mathbf{q}_\pm)\right)\,(t,s). \end{aligned}$$

(Note that in the proof we do not use the nonsingularity of B.) $\square$

Clearly, $B = BP_+ + BP_-$, and the considerations above lead to the following splitting of the inverse of T_n into a sum of two split Bezoutians of different type.

COROLLARY 3.6. *The inverse of a symmetric Toeplitz matrix* T_n *can be represented in the form*

$$T_n^{-1} = \frac{1}{2\gamma}\left(\mathrm{Bez}_{\mathrm{split}}(\mathbf{p}_+,\mathbf{q}_+) - \mathrm{Bez}_{\mathrm{split}}(\mathbf{p}_-,\mathbf{q}_-)\right),$$

where $\mathbf{p}_\pm, \mathbf{q}_\pm$ *are defined in (3.6) with* $\mathbf{u}_+ = \mathbf{w}_{n+1}^+, \mathbf{v}_- = \mathbf{w}_{n+1}^-$ *and* γ *is defined in Proposition 3.4.*

3.3. Inverses of skewsymmetric Toeplitz matrices.

In the case of a nonsingular, skewsymmetric Toeplitz matrix T_n, $n = 2m$, a Levinson-type algorithm can be used to compute vectors spanning the nullspace of T_{2k-1} for $k = 1,\ldots,m$ (compare [13], [16]). So it is reasonable to ask for a fundamental system $\{\mathbf{u},\mathbf{v}\}$ consisting of vectors of this kind.

Let $\mathbf{x}$ be any vector spanning the nullspace of T_{n-1}. From the relation $T_{n-1}^J = -T_{n-1}$ it follows that also the vector $\mathbf{x}^J$ belongs to the nullspace of T_{n-1}. Thus $\mathbf{x}$ is either symmetric or skewsymmetric. In [13] (compare also [14]) it was shown that the latter is not possible: The vector $\mathbf{x}$ is symmetric. Now it is immediately clear that

$$\mathbf{u} = \begin{bmatrix} 0 \\ \mathbf{x} \\ 0 \end{bmatrix} \in \ker \partial T_n\,.$$

Furthermore, let T_{n+1} be any $(n+1)\times(n+1)$ skewsymmetric Toeplitz extension of T_n and $\mathbf{v}$ a (symmetric) vector spanning the nullspace of T_{n+1}. Since T_n is nonsingular, we may assume that $\mathbf{v}$ is monic, i.e. $\mathbf{e}_1^T\mathbf{v} = 1$. Now $\{\mathbf{u},\mathbf{v}\}$ is a fundamental system consisting of two symmetric vectors, and

$$F\begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} = \begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix}, \quad \gamma = [\,a_1 \ldots a_{n-1}\,]\mathbf{x} \neq 0\,.$$

Thus the inverse of the nonsingular, skewsymmetric Toeplitz matrix T_n is given by

$$T_n^{-1} = \frac{1}{\gamma}\,\mathrm{Bez}_T(\mathbf{u},\mathbf{v})\,.$$

In analogy to the symmetric case we define symmetric and skewsymmetric vectors by

$$\mathbf{p}_\pm(t) = (t\pm 1)\mathbf{u}(t) \quad \text{and} \quad \mathbf{q}_\pm(t) = (t\pm 1)\mathbf{v}(t)\,. \tag{3.7}$$

Now we observe that the generating polynomial of the matrices $B_{(\pm)} = \mathrm{Bez}_T(\mathbf{u},\mathbf{v})P_\pm$ can be represented in the form

$$\begin{aligned} B_{(\pm)}(t,s) &= \frac{1}{2}\left(\frac{\mathbf{u}(t)\mathbf{v}(s)-\mathbf{v}(t)\mathbf{u}(s)}{1-ts} \pm \frac{\mathbf{u}(t)\mathbf{v}(s)-\mathbf{v}(t)\mathbf{u}(s)}{s-t}\right) \\ &= \pm\frac{1}{2}\,\frac{\mathbf{p}_\mp(t)\mathbf{q}_\pm(s)-\mathbf{q}_\mp(t)\mathbf{p}_\pm(s)}{(t-s)(1-ts)}. \end{aligned}$$

Since $\mathbf{u}(t) = \frac{\mathbf{p}_+(t)}{t+1}$ we have $\mathbf{p}_-(t) = \frac{t-1}{t+1}\mathbf{p}_+(t)$, analogously, $\mathbf{q}_-(t) = \frac{t-1}{t+1}\mathbf{q}_+(t)$. By putting these into $B_{(\pm)}(t,s)$ we obtain

$$B_{(\pm)}(t,s) = \pm\frac{1}{2}\,B_{\mp}(t,s)\frac{s\pm 1}{s\mp 1} = \pm\frac{1}{2}\,\frac{t\mp 1}{t\pm 1}B_{\pm}(t,s)\,,$$

which leads to the following splittings of T_n^{-1}.

PROPOSITION 3.7. *Let $B_\pm = \mathrm{Bez}_{\mathrm{split}}(\mathbf{p}_\pm, \mathbf{q}_\pm)$ with $\mathbf{p}_\pm, \mathbf{q}_\pm$ being defined in (3.7). Then*

$$\begin{aligned} T_n^{-1}(t,s) &= \frac{1}{2\gamma}\left(B_+(t,s)\frac{1-s}{1+s} - B_-(t,s)\frac{1+s}{1-s}\right) \\ &= \frac{1}{2\gamma}\left(\frac{1+t}{1-t}B_-(t,s) - \frac{1-t}{1+t}B_+(t,s)\right). \end{aligned} \tag{3.8}$$

4. The Toeplitz-plus Hankel case

4.1. Toeplitz-plus-Hankel Bezoutians

Now we want to generalize results for Toeplitz or Hankel matrices to matrices which are the sum of such structured matrices. In particular, we recall the fact known from [8] that the inverse of a (nonsingular) matrix which is the sum of a Toeplitz plus a Hankel matrix possesses again a Bezoutian structure, though in a generalized sense.

An $n\times n$ matrix B is called a *Toeplitz-plus-Hankel Bezoutian*, briefly *T+H-Bezoutian,* if there are eight polynomials $\mathbf{g}_i(t)$, $\mathbf{f}_i(t)$ $(i = 1,2,3,4)$ of $\mathbb{F}^{n+2}(t)$ such that

$$B(t,s) = \frac{\sum_{i=1}^{4}\mathbf{g}_i(t)\mathbf{f}_i(s)}{(t-s)(1-ts)}\,. \tag{4.1}$$

In analogy to the Hankel or Toeplitz case we here use the notation $B = \mathrm{Bez}_{T+H}((\mathbf{g}_i,\mathbf{f}_i)_1^4)$. Clearly, H-Bezoutians or T-Bezoutians are also T+H-Bezoutians. Moreover, since the generating polynomials of the flip matrix J_n and of the shift matrix S_n are

$$J_n(t,s) = \frac{t^n - s^n}{t-s} \quad\text{and}\quad S_n(t,s) = \frac{t - t^n s^{n-1}}{1-ts}\,,$$

J_n is an H- and S_n is a T-Bezoutian. The sum $S_n + J_n$ is a T+H-Bezoutian,

$$(S_n + J_n)(t,s) = \frac{(t^n + t^2) - t^{n+1}(s + s^{n-1}) + (t^n - 1)s^n + t(s^{n+1} - s)}{(t-s)(1-ts)}\,.$$

However, in general the sum of a T- and an H-Bezoutian $\mathrm{Bez}_H(\mathbf{u},\mathbf{v}) + \mathrm{Bez}_T(\mathbf{g},\mathbf{f})$ is no T+H-Bezoutian, since the rank of the matrix with the generating polynomial

$$(1-ts)(\mathbf{u}(t)\mathbf{v}(s) - \mathbf{v}(t)\mathbf{u}(s)) + (t-s)(\mathbf{g}(t)\mathbf{h}^J(s) - \mathbf{h}(t)\mathbf{g}^J(s))$$

is not expected to be less than or equal to 4.

The T+H analogue of the transformations ∇_H or ∇_T is the transformation ∇_{T+H} mapping a matrix $A = [\,a_{ij}\,]_{i,j=1}^n$ of order n to a matrix of order $n+2$ according to

$$\nabla_{T+H}A = [\,a_{i-1,j} - a_{i,j-1} + a_{i-1,j-2} - a_{i-2,j-1}\,]_{i,j=1}^{n+2}\ .$$

Here we put $a_{ij} = 0$ if $i \notin \{1,2,\dots,n\}$ or $j \notin \{1,2,\dots,n\}$. Denoting $W_n = S_n + S_n^T$ we have

$$\nabla_{T+H}A = \begin{bmatrix} 0 & -\mathbf{e}_1^T A & 0 \\ A\mathbf{e}_1 & AW_n - W_nA & A\mathbf{e}_n \\ 0 & -\mathbf{e}_n^T A & 0 \end{bmatrix}. \tag{4.2}$$

The generating polynomial of $\nabla_{T+H}A$ is

$$(\nabla_{T+H}A)(t,s) = (t-s)(1-ts)A(t,s)\,. \tag{4.3}$$

Hence a matrix B is a T+H-Bezoutian if and only if

$$\mathrm{rank}\nabla_{T+H}B \leqslant 4\,.$$

Note that also T+H matrices can be characterized with this transformation: An $n \times n$ matrix R_n is a T+H matrix if and only if the matrix of order $n-2$ in the center of $\nabla_{T+H}R_n$ is the zero matrix.

Let us discuss the question under which conditions different vector systems $\{\mathbf{g}_i, \mathbf{f}_i\}_{i=1}^4$, $\{\widetilde{\mathbf{g}}_i, \widetilde{\mathbf{f}}_i\}_{i=1}^4$ produce the same T+H-Bezoutian. Clearly, $B = \mathrm{Bez}_{T+H}((\mathbf{g}_i, \mathbf{f}_i)_1^4)$ is equal to $\widetilde{B} = \mathrm{Bez}_{T+H}((\widetilde{\mathbf{g}}_i, \widetilde{\mathbf{f}}_i)_1^4)$ if and only if $\nabla_{T+H}B = \nabla_{T+H}\widetilde{B}$. Thus we can use the following lemma to answer the question.

LEMMA 4.1. *Let $G_j, F_j\ (j = 1,2)$ be full rank matrices of order $m \times r, n \times r$, respectively, $r = \mathrm{rank}\, G_j = \mathrm{rank}\, F_j$. Then*

$$G_1F_1^T = G_2F_2^T \tag{4.4}$$

if and only if there is a nonsingular $r \times r$ matrix φ such that

$$G_2 = G_1\varphi\,,\ F_1 = F_2\varphi^T\,. \tag{4.5}$$

Let $B, \widetilde{B}$ be $n \times n$ T+H-Bezoutians and let $\nabla_{\mathrm{T+H}}B$ and $\nabla_{\mathrm{T+H}}\widetilde{B}$ allow the rank decompositions

$$\nabla_{\mathrm{T+H}}B = GF^T\,,\ \nabla_{\mathrm{T+H}}\widetilde{B} = \widetilde{G}\widetilde{F}^T\,,$$

where $G, \widetilde{G}, F, \widetilde{F}$ are full rank matrices with

$$r = \mathrm{rank}\, G = \mathrm{rank}\, F \leqslant 4\,,\ \widetilde{r} = \mathrm{rank}\, \widetilde{G} = \mathrm{rank}\, \widetilde{F} \leqslant 4\,.$$

The T+H-Bezoutians B and $\widetilde{B}$ coincide if and only if $r = \widetilde{r}$ and if there is a nonsingular $r \times r$ matrix φ so that

$$\widetilde{G} = G\varphi\,,\ F = \widetilde{F}\varphi^T\,.$$

To specify this for the nonsingular case we recall from [8] that if B is an $n \times n$ matrix $(n \geqslant 2)$ with rank $\nabla_{\mathrm{T+H}}B < 4$, then B is a singular matrix. In particular, if rank $\nabla_{\mathrm{T+H}} B < 4$ then the first and the last rows (or the first and the last columns) of B are linearly dependent. For T-(or H-)Bezoutians B, the condition rank $\nabla_T B < 2$ (or rank $\nabla_H B < 2$) leads to $B \equiv 0$. But in the T+H case nontrivial T+H-Bezoutians B with rank $\nabla_{\mathrm{T+H}} B < 4$ exist. Examples are $B = I_n + J_n$ and the split Bezoutians introduced in Subsection 3.2.

Now we present the result for the nonsingular case.

PROPOSITION 4.2. *The nonsingular T+H-Bezoutians*

$$B = \mathrm{Bez}_{\mathrm{T+H}}((\mathbf{g}_i, \mathbf{f}_i)_1^4) \text{ and } \widetilde{B} = \mathrm{Bez}_{\mathrm{T+H}}((\widetilde{\mathbf{g}}_i, \widetilde{\mathbf{f}}_i)_1^4)$$

coincide if and only if there is a nonsingular 4×4 *matrix* φ *such that*

$$[\, \mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3\ \mathbf{g}_4 \,]\, \varphi = [\, \widetilde{\mathbf{g}}_1\ \widetilde{\mathbf{g}}_2\ \widetilde{\mathbf{g}}_3\ \widetilde{\mathbf{g}}_4 \,]$$

and

$$[\, \widetilde{\mathbf{f}}_1\ \widetilde{\mathbf{f}}_2\ \widetilde{\mathbf{f}}_3\ \widetilde{\mathbf{f}}_4 \,]\, \varphi^T = [\, \mathbf{f}_1\ \mathbf{f}_2\ \mathbf{f}_3\ \mathbf{f}_4 \,].$$

In [8] it was shown that, in analogy to the Toeplitz and Hankel cases, a nonsingular matrix is an T+H-Bezoutian if and only if it is the inverse of a T+H matrix. To be self-contained we recall the proof and start with the following part of this assertion.

THEOREM 4.3. *Let B be a nonsingular T+H-Bezoutian. Then B^{-1} is a T+H matrix.*

Proof. We have rank $\nabla_{T+H} B = 4$, and a rank decomposition of $\nabla_{T+H} B$ is of the form

$$\nabla_{\mathrm{T+H}} B = \begin{bmatrix} 0 \\ B\mathbf{e}_1 \\ 0 \end{bmatrix} [\,1 * 0\,] + \begin{bmatrix} 0 \\ B\mathbf{e}_n \\ 0 \end{bmatrix} [\,0 * 1\,] - \begin{bmatrix} 1 \\ * \\ 0 \end{bmatrix} [\,0\ \mathbf{e}_1^T B\ 0\,] - \begin{bmatrix} 0 \\ * \\ 1 \end{bmatrix} [\,0\ \mathbf{e}_n^T B\ 0\,], \tag{4.6}$$

where $*$ stands for some vector of $\mathbb{F}^n$. In particular, this means that there are vectors $\mathbf{z}_i \in \mathbb{F}^n, i = 1,2,3,4$ such that

$$BW_n - W_n B = B\mathbf{e}_1\, \mathbf{z}_1^T + B\mathbf{e}_n\, \mathbf{z}_2^T + \mathbf{z}_3\, \mathbf{e}_1^T B + \mathbf{z}_4\, \mathbf{e}_n^T B.$$

Applying B^{-1} to both sides of the last equality leads to

$$B^{-1} W_n - W_n B^{-1} = -(\mathbf{e}_1\, \mathbf{z}_1^T B^{-1} + \mathbf{e}_n\, \mathbf{z}_2^T B^{-1} + B^{-1}\mathbf{z}_3\, \mathbf{e}_1^T + B^{-1}\mathbf{z}_4\, \mathbf{e}_n^T).$$

Thus, the matrix of order $n-2$ in the center of $\nabla_{T+H}(B^{-1})$ is the zero matrix. This proves that B^{-1} is a T+H matrix. $\square$

In the next subsection we will show that the converse is also true, i.e., the inverse of a (nonsingular) T+H matrix is a T+H-Bezoutian.

4.2. Inverses of T+H-matrices

We now consider $n \times n$ matrices R_n which are the sum of a Toeplitz matrix $T_n = T_n(\mathbf{a})$, $\mathbf{a} = (a_i)_{i=1-n}^{n-1}$ and a Hankel matrix $H_n = T_n(\mathbf{b}) J_n$, $\mathbf{b} = (b_i)_{i=1-n}^{n-1}$,

$$R_n = T_n(\mathbf{a}) + T_n(\mathbf{b}) J_n = \begin{bmatrix} a_0 & \dots & a_{1-n} \\ \vdots & \ddots & \vdots \\ a_{n-1} & \dots & a_0 \end{bmatrix} + \begin{bmatrix} b_{1-n} & \dots & b_0 \\ \vdots & \iddots & \vdots \\ b_0 & \dots & b_{n-1} \end{bmatrix}. \tag{4.7}$$

Note that the *chess-board matrices*,

$$B=\begin{bmatrix} c & b & c & \cdots \\ b & c & b & \cdots \\ c & b & c & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \qquad (c,b\in\mathbb{F}) \tag{4.8}$$

are both Toeplitz and Hankel matrices. Thus the representation (4.7) is not unique. We want to prove that the inverse of a T+H matrix R_n is a T+H-Bezoutian and even more, we want to present inversion formulas

$$R_n^{-1}=\mathrm{Bez}_{T+H}((\mathbf{g}_i,\mathbf{f}_i)_1^4)\,.$$

Thus, we have to answer the question how to obtain the vectors $\mathbf{g}_i,\mathbf{f}_i$, $i=1,2,3,4$.

1. Fundamental systems. Besides the nonsingular T+H matrix R_n of (4.7) we consider the $(n-2)\times(n+2)$ T+H matrices $\partial_{T+H}R_n,\partial_{T+H}R_n^T$ obtained from R_n,R_n^T after deleting the first and last rows and adding one column to the right and one to the left by preserving the T+H structure,

$$\partial_{T+H}R_n=\begin{bmatrix} a_2 & a_1 & \dots & a_{2-n} & a_{1-n} \\ a_3 & a_2 & \dots & a_{3-n} & a_{2-n} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-1} & a_{n-2} & \dots & a_{-1} & a_{-2} \end{bmatrix}+\begin{bmatrix} b_{1-n} & b_{2-n} & \dots & b_1 & b_2 \\ b_{2-n} & b_{3-n} & \dots & b_2 & b_3 \\ \vdots & \vdots & & \vdots & \vdots \\ b_{-2} & b_{-1} & \dots & b_{n-2} & b_{n-1} \end{bmatrix}, \tag{4.9}$$

$$\partial_{T+H}R_n^T=\begin{bmatrix} a_{-2} & a_{-1} & \dots & a_{n-2} & a_{n-1} \\ a_{-3} & a_{-2} & \dots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{1-n} & a_{2-n} & \dots & a_1 & a_2 \end{bmatrix}+\begin{bmatrix} b_{1-n} & b_{2-n} & \dots & b_1 & b_2 \\ b_{2-n} & b_{3-n} & \dots & b_2 & b_3 \\ \vdots & \vdots & & \vdots & \vdots \\ b_{-2} & b_{-1} & \dots & b_{n-2} & b_{n-1} \end{bmatrix}. \tag{4.10}$$

Since R_n is nonsingular both matrices $\partial_{T+H}R_n$ and $\partial_{T+H}R_n^T$ are of full rank, which means

$$\dim\ker\partial_{T+H}R_n=\dim\ker\partial_{T+H}R_n^T=4\,.$$

In contrast to the Toeplitz case, where $\{\mathbf{u},\mathbf{v}\}$ is a basis of $\ker\partial T_n$ if and only if $\{\mathbf{u}^J,\mathbf{v}^J\}$ is a basis of $\ker\partial T_n^T$, a connection between the kernels of $\partial_{T+H}R_n$ and $\partial_{T+H}R_n^T$ is not transparent.

Any system of eight vectors $\{\mathbf{u}_i\}_{i=1}^4,\{\mathbf{v}_i\}_{i=1}^4$, where $\{\mathbf{u}_i\}_{i=1}^4$ is a basis of $\ker\partial_{T+H}R_n$ and $\{\mathbf{v}_i\}_{i=1}^4$ is a basis of $\ker\partial_{T+H}R_n^T$, is called a *fundamental system for* R_n . (Note that the notion of a fundamental system for T+H matrices was also introduced in [6], but in a different way.) The reason for our definition here is that these vectors completely determine the inverse R_n^{-1} . In order to show this we consider first a special fundamental system.

Hereafter we use the following notation. For a given vector $\mathbf{a} = (a_j)_{j=1-n}^{n-1}$ we define

$$\mathbf{a}_{(\pm)} = (a_{\pm j})_{j=1}^{n} , \tag{4.11}$$

where $a_{\pm n}$ can be chosen arbitrarily. The $n \times n$ matrix in the center of $\nabla_{T+H} R_n$ (compare (4.2))

$$\nabla(R_n) = R_n W_n - W_n R_n$$

allows a rank decomposition of the form

$$\nabla(R_n) = -(\mathbf{a}_{(+)}+\mathbf{b}_{(-)}^J)\mathbf{e}_1^T-(\mathbf{a}_{(-)}^J+\mathbf{b}_{(+)})\mathbf{e}_n^T+\mathbf{e}_1(\mathbf{a}_{(-)}+\mathbf{b}_{(-)}^J)^T+\mathbf{e}_n(\mathbf{a}_{(+)}^J+\mathbf{b}_{(+)})^T . \tag{4.12}$$

Multiplying (4.12) from both sides by R_n^{-1} we obtain a rank decomposition of $\nabla(R_n^{-1})$.

PROPOSITION 4.4. *We have*

$$\nabla\left(R_n^{-1}\right) = \mathbf{x}_1\mathbf{y}_1^T + \mathbf{x}_2\mathbf{y}_2^T - \mathbf{x}_3\mathbf{y}_3^T - \mathbf{x}_4\mathbf{y}_4^T , \tag{4.13}$$

where $\mathbf{x}_i$ $(i = 1,2,3,4)$ *are the solutions of*

$$R_n\mathbf{x}_1 = \mathbf{a}_{(+)} + \mathbf{b}_{(-)}^J , \quad R_n\mathbf{x}_2 = \mathbf{a}_{(-)}^J + \mathbf{b}_{(+)} , \quad R_n\mathbf{x}_3 = \mathbf{e}_1 , \quad R_n\mathbf{x}_4 = \mathbf{e}_n , \tag{4.14}$$

and $\mathbf{y}_i$ $(i = 1,2,3,4)$ *are the solutions of*

$$R_n^T\mathbf{y}_1 = \mathbf{e}_1 , \quad R_n^T\mathbf{y}_2 = \mathbf{e}_n , \quad R_n^T\mathbf{y}_3 = \mathbf{a}_{(-)} + \mathbf{b}_{(-)}^J , \quad R_n^T\mathbf{y}_4 = \mathbf{a}_{(+)}^J + \mathbf{b}_{(+)} . \tag{4.15}$$

According to (4.9), (4.10) we obtain the following fundamental system for R_n.

PROPOSITION 4.5. *Let* $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{F}^n$ *be defined by (4.14), (4.15). The vector system*

$$\left\{ \mathbf{u}_1 = \begin{bmatrix} 1 \\ -\mathbf{x}_1 \\ 0 \end{bmatrix} , \ \mathbf{u}_2 = \begin{bmatrix} 0 \\ -\mathbf{x}_2 \\ 1 \end{bmatrix} , \ \mathbf{u}_3 = \begin{bmatrix} 0 \\ \mathbf{x}_3 \\ 0 \end{bmatrix} , \ \mathbf{u}_4 = \begin{bmatrix} 0 \\ \mathbf{x}_4 \\ 0 \end{bmatrix} \right\} \tag{4.16}$$

is a basis of $\ker \partial_{T+H} R_n$, *and the vector system*

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} 0 \\ \mathbf{y}_1 \\ 0 \end{bmatrix} , \ \mathbf{v}_2 = \begin{bmatrix} 0 \\ \mathbf{y}_2 \\ 0 \end{bmatrix} , \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ -\mathbf{y}_3 \\ 0 \end{bmatrix} , \ \mathbf{v}_4 = \begin{bmatrix} 0 \\ -\mathbf{y}_4 \\ 1 \end{bmatrix} \right\} \tag{4.17}$$

is a basis of $\ker \partial_{T+H} R_n^T$.

2. Inversion. The special fundamental system of Proposition 4.5 delivers the parameters needed in a Bezoutian formula for R_n^{-1}. This basic inversion formula is the initial point for our further considerations.

THEOREM 4.6. *[8] Let* R_n *be the nonsingular T+H matrix (4.7) and* $\{\mathbf{u}_i\}_{i=1}^4$, $\{\mathbf{v}_i\}_{i=1}^4$ *be the fundamental system for* R_n *given by (4.16), (4.14), (4.17), (4.15). Then* R_n^{-1} *is the T+H-Bezoutian with the generating polynomial*

$$R_n^{-1}(t,s) = \frac{\mathbf{u}_3(t)\mathbf{v}_3(s) + \mathbf{u}_4(t)\mathbf{v}_4(s) - \mathbf{u}_1(t)\mathbf{v}_1(s) - \mathbf{u}_2(t)\mathbf{v}_2(s)}{(t-s)(1-ts)} . \tag{4.18}$$

Proof. Since $\mathbf{x}_3$ is the first, $\mathbf{x}_4$ the last column, $\mathbf{y}_1^T$ is the first, $\mathbf{y}_2^T$ the last row of R_n^{-1} we conclude from (4.2) that

$$\nabla_{T+H} R_n^{-1} = \begin{bmatrix} 0 & -\mathbf{y}_1^T & 0 \\ \mathbf{x}_3 & \nabla(R_n^{-1}) & \mathbf{x}_4 \\ 0 & -\mathbf{y}_2^T & 0 \end{bmatrix}.$$

Taking (4.13) into account this leads to

$$\nabla_{T+H} R_n^{-1} = [\,-\mathbf{u}_1 \;\; -\mathbf{u}_2 \;\; \mathbf{u}_3 \;\; \mathbf{u}_4\,]\,[\,\mathbf{v}_1\;\mathbf{v}_2\;\mathbf{v}_3\;\mathbf{v}_4\,]^T,$$

where the vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ are defined in (4.16), (4.17). Formula (4.18) follows now from (4.3). $\square$

In particular, this theorem shows that if we want to use the vectors of any fundamental system for R_n in a Bezoutian formula for the inverse R_n^{-1}, then a "normalization" of them is necessary. For this purpose we introduce the following $(n+2)\times 4$ matrices:

$$F = [\,\mathbf{e}_1\;\mathbf{e}_{n+2}\;\mathbf{f}_1\;\mathbf{f}_2\,], \quad G = [\,\mathbf{g}_1\;\mathbf{g}_2\;\mathbf{e}_1\;\mathbf{e}_{n+2}\,],$$

where

$$\mathbf{f}_1 = (a_{1-i} + b_{i-n})_{i=0}^{n+1}, \;\; \mathbf{f}_2 = (a_{n-i} + b_{i-1})_{i=0}^{n+1},$$
$$\mathbf{g}_1 = (a_{i-1} + b_{i-n})_{i=0}^{n+1}, \;\; \mathbf{g}_2 = (a_{i-n} + b_{i-1})_{i=0}^{n+1}$$

with $a_{\pm n}, b_{\pm n}$ arbitrarily chosen. We call a fundamental system $\{\mathbf{u}_i\}_{i=1}^4, \{\mathbf{v}_i\}_{i=1}^4$ for R_n *canonical* if

$$F^T\,[\,\mathbf{u}_1\;\mathbf{u}_2\;\mathbf{u}_3\;\mathbf{u}_4\,] = G^T\,[\,\mathbf{v}_1\;\mathbf{v}_2\;\mathbf{v}_3\;\mathbf{v}_4\,] = I_4\,. \tag{4.19}$$

PROPOSITION 4.7. *A fundamental system $\{\mathbf{u}_i\}_{i=1}^4, \{\mathbf{v}_i\}_{i=1}^4$ for R_n is canonical if and only if the $\mathbf{u}_i$ are of the form (4.16), (4.14), and the $\mathbf{v}_i$ are of the form (4.17), (4.15) for $i = 1,2,3,4$.*

Proof. If $\{\mathbf{u}_i\}_{i=1}^4$ and $\{\mathbf{v}_i\}_{i=1}^4$ form a canonical system then (4.19) means, in particular, that the first component of $\mathbf{u}_1$ and $\mathbf{v}_3$ as well as the last component of $\mathbf{u}_2$ and $\mathbf{v}_4$ are one. The first and last components of the other vectors are zero. Hence there are vectors $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{F}^n$ such that $\mathbf{u}_i, \mathbf{v}_i$ are of the form (4.16), (4.17). Now by (4.19) we have

$$[\,I_{+-}\mathbf{f}_1\; I_{+-}\mathbf{f}_2\,]^T[\,\mathbf{x}_3\;\mathbf{x}_4\,] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.20}$$

where for a given vector $\mathbf{h} = (h_i)_{i=0}^{n+1} \in \mathbb{F}^{n+2}$ the vector $I_{+-}\mathbf{h} \in \mathbb{F}^n$ is defined by

$$I_{+-}\mathbf{h} = (h_i)_{i=1}^n. \tag{4.21}$$

Since

$$(I_{+-}\mathbf{f}_1)^T = e_1^T R_n\,, \;\; (I_{+-}\mathbf{f}_2)^T = e_n^T R_n\,,$$

and since $\begin{bmatrix} 0 \\ \mathbf{x}_3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathbf{x}_4 \\ 0 \end{bmatrix}$ are in $\ker \partial_{T+H} R_n$, equality (4.20) leads to

$$R_n \mathbf{x}_3 = e_1\,,\; R_n \mathbf{x}_4 = e_n\,.$$

Moreover, $\begin{bmatrix} 1 \\ -\mathbf{x}_1 \\ 0 \end{bmatrix} \in \ker \partial_{T+H} R_n$ means that $R_n \mathbf{x}_1 = \mathbf{a}_{(+)} + \mathbf{b}^J_{(-)}$ and $\begin{bmatrix} 0 \\ -\mathbf{x}_2 \\ 1 \end{bmatrix} \in \ker \partial_{T+H} R_n$ means that $R_n \mathbf{x}_2 = \mathbf{a}^J_{(-)} + \mathbf{b}_{(+)}$. Similar arguments show that $\mathbf{y}_i$, $i = 1,2,3,4$, are the solutions of (4.15), and the necessity part of the proof is complete.

If $\{\mathbf{u}_i\}_{i=1}^4, \{\mathbf{v}_i\}_{i=1}^4$ are of the form (4.16), (4.14), and (4.17), (4.15) then, obviously, (4.19) is satisfied. $\square$

Given an arbitrary fundamental system $\{\widetilde{\mathbf{u}}_i\}_{i=1}^4, \{\widetilde{\mathbf{v}}_i\}_{i=1}^4$ we define two 4×4 nonsingular matrices Γ_F, Γ_G,

$$F^T [\, \widetilde{\mathbf{u}}_1 \; \widetilde{\mathbf{u}}_2 \; \widetilde{\mathbf{u}}_3 \; \widetilde{\mathbf{u}}_4 \,] = \Gamma_F \,, \quad G^T [\, \widetilde{\mathbf{v}}_1 \; \widetilde{\mathbf{v}}_2 \; \widetilde{\mathbf{v}}_3 \; \widetilde{\mathbf{v}}_4 \,] = \Gamma_G \,.$$

We conclude that by

$$[\, \mathbf{u}_1 \; \mathbf{u}_2 \; \mathbf{u}_3 \; \mathbf{u}_4 \,] = [\, \widetilde{\mathbf{u}}_1 \; \widetilde{\mathbf{u}}_2 \; \widetilde{\mathbf{u}}_3 \; \widetilde{\mathbf{u}}_4 \,] \, \Gamma_F^{-1} \tag{4.22}$$

and

$$[\, \mathbf{v}_1 \; \mathbf{v}_2 \; \mathbf{v}_3 \; \mathbf{v}_4 \,] = [\, \widetilde{\mathbf{v}}_1 \; \widetilde{\mathbf{v}}_2 \; \widetilde{\mathbf{v}}_3 \; \widetilde{\mathbf{v}}_4 \,] \, \Gamma_G^{-1} \tag{4.23}$$

a canonical fundamental system $\{\mathbf{u}_i\}_{i=1}^4, \{\mathbf{v}_i\}_{i=1}^4$ is given. Note that for fixed $a_{\pm n}, b_{\pm n}$ the canonical fundamental system is unique. The following becomes clear.

THEOREM 4.8. *Let R_n be the nonsingular T+H matrix (4.7) and $\{\widetilde{\mathbf{u}}_i\}_{i=1}^4, \{\widetilde{\mathbf{v}}_i\}_{i=1}^4$ be a fundamental system for R_n. Then the inverse R_n^{-1} is the T+H-Bezoutian (4.18), where $\{\mathbf{u}_i\}_{i=1}^4, \{\mathbf{v}_i\}_{i=1}^4$ are given by (4.22), (4.23).*

Let R_n be given by (4.7). Hereafter we also use a representation of R_n which involves the projections $P_\pm = \frac{1}{2}(I_n \pm J_n)$ onto $\mathbb{F}^n_\pm$ and the vectors

$$\mathbf{c} = (c_j)_{j=1-n}^{n-1} = \mathbf{a} + \mathbf{b} \,, \; \mathbf{d} = (d_j)_{j=1-n}^{n-1} = \mathbf{a} - \mathbf{b} \,,$$

namely

$$R_n = T_n(\mathbf{c}) P_+ + T_n(\mathbf{d}) P_- \,. \tag{4.24}$$

Instead of the solutions $\mathbf{x}_i$ of (4.14) and the solutions $\mathbf{y}_i$ of (4.15) we consider now the solutions of the following equations the right hand sides of which depend on $\mathbf{c}, \mathbf{d}$ and $\widetilde{\mathbf{c}} = \mathbf{a}^J + \mathbf{b}$, $\widetilde{\mathbf{d}} = \mathbf{a}^J - \mathbf{b}$:

$$R_n \mathbf{w}_1 = \frac{1}{2}(\mathbf{c}_{(+)} + \mathbf{c}^J_{(-)}) \,, \quad R_n \mathbf{w}_2 = \frac{1}{2}(\mathbf{d}_{(+)} - \mathbf{d}^J_{(-)}) \,,$$

$$R_n \mathbf{w}_3 = P_+ \mathbf{e}_1 \,, \quad R_n \mathbf{w}_4 = P_- \mathbf{e}_1 \tag{4.25}$$

and

$$R_n^T \mathbf{z}_1 = P_+ \mathbf{e}_1 \,, \quad R_n^T \mathbf{z}_2 = P_- \mathbf{e}_1 \,, \quad R_n^T \mathbf{z}_3 = \frac{1}{2}(\widetilde{\mathbf{c}}_{(+)} + \widetilde{\mathbf{c}}^J_{(-)}) \,,$$

$$R_n^T \mathbf{z}_4 = \frac{1}{2}(\widetilde{\mathbf{d}}_{(+)} - \widetilde{\mathbf{d}}^J_{(-)}) \,, \tag{4.26}$$

where we use the notation (4.11). We introduce the vectors

$$\breve{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ -2\mathbf{w}_1 \\ 1 \end{bmatrix}, \quad \breve{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ -2\mathbf{w}_2 \\ -1 \end{bmatrix}, \quad \breve{\mathbf{u}}_3 = \begin{bmatrix} 0 \\ \mathbf{w}_3 \\ 0 \end{bmatrix}, \quad \breve{\mathbf{u}}_4 = \begin{bmatrix} 0 \\ \mathbf{w}_4 \\ 0 \end{bmatrix},$$
$$\breve{\mathbf{v}}_1 = \begin{bmatrix} 0 \\ \mathbf{z}_1 \\ 0 \end{bmatrix}, \quad \breve{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ \mathbf{z}_2 \\ 0 \end{bmatrix}, \quad \breve{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ -2\mathbf{z}_3 \\ 1 \end{bmatrix}, \quad \breve{\mathbf{v}}_4 = \begin{bmatrix} 1 \\ -2\mathbf{z}_4 \\ -1 \end{bmatrix}. \tag{4.27}$$

Now an inversion formula which involves these vectors follows from formula (4.18).

PROPOSITION 4.9. *Let R_n be the nonsingular T+H matrix (4.24). Then the inverse R_n^{-1} is given by*

$$R_n^{-1}(t,s) = \frac{\breve{\mathbf{u}}_3(t)\breve{\mathbf{v}}_3(s) + \breve{\mathbf{u}}_4(t)\breve{\mathbf{v}}_4(s) - \breve{\mathbf{u}}_1(t)\breve{\mathbf{v}}_1(s) - \breve{\mathbf{u}}_2(t)\breve{\mathbf{v}}_2(s)}{(t-s)(1-ts)}, \tag{4.28}$$

where $\{\breve{\mathbf{u}}_i\}_{i=1}^4$, $\{\breve{\mathbf{v}}_i\}_{i=1}^4$ are defined in (4.27).

Proof. Since

$$[\,\breve{\mathbf{u}}_1\ \breve{\mathbf{u}}_2\ \breve{\mathbf{u}}_3\ \breve{\mathbf{u}}_4\,] = [\,\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3\ \mathbf{u}_4\,]\,\varphi$$

and

$$[\,\breve{\mathbf{v}}_1\ \breve{\mathbf{v}}_2\ \breve{\mathbf{v}}_3\ \breve{\mathbf{v}}_4\,] = [\,\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\,]\,\varphi^{-1},$$

where φ is the block diagonal matrix

$$\varphi = \operatorname{diag}\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right),$$

the proposition follows from Proposition 4.2 and (4.18). □

3. Inversion of symmetric T+H matrices. Now we consider the inversion of symmetric T+H matrices. It is easy to see that a T+H matrix is symmetric if and only if the Toeplitz part has this property. Let R_n be a nonsingular, symmetric T+H matrix (4.7). Then the solutions of (4.14) and (4.15) coincide,

$$\mathbf{y}_1 = \mathbf{x}_3\,,\ \mathbf{y}_2 = \mathbf{x}_4\,,\ \mathbf{y}_3 = \mathbf{x}_1\,,\ \mathbf{y}_4 = \mathbf{x}_2\,.$$

Using the inversion formula (4.18), R_n^{-1} is given by the vectors $\{\mathbf{u}_i\}_{i=1}^4$ of (4.16),

$$R_n^{-1}(t,s) = \frac{\mathbf{u}_3(t)\mathbf{u}_1(s) - \mathbf{u}_1(t)\mathbf{u}_3(s) + \mathbf{u}_4(t)\mathbf{u}_2(s) - \mathbf{u}_2(t)\mathbf{u}_4(s)}{(t-s)(1-ts)}. \tag{4.29}$$

Since $\mathbf{a} = \mathbf{a}^J$ we have $\mathbf{c} = \widetilde{\mathbf{c}}$, $\mathbf{d} = \widetilde{\mathbf{d}}$, and the inversion formula (4.28) can be simplified as well,

$$R_n^{-1}(t,s) = \frac{\breve{\mathbf{u}}_3(t)\breve{\mathbf{u}}_1(s) - \breve{\mathbf{u}}_1(t)\breve{\mathbf{u}}_3(s) + \breve{\mathbf{u}}_4(t)\breve{\mathbf{u}}_2(s) - \breve{\mathbf{u}}_2(t)\breve{\mathbf{u}}_4(s)}{(t-s)(1-ts)}. \tag{4.30}$$

If we have any basis $\{\widetilde{\mathbf{u}}_i\}_{i-1}^4$ of $\ker \partial_{T+H} R_n$, it remains to compute Γ_F, and $\{\mathbf{u}_i\}_{i=1}^4$ is given by (4.22).

4. Inversion of centrosymmetric T+H matrices. If R_n from (4.7) is centrosymmetric, i.e. $R_n^J = R_n$, then in view of $T_n(a)^J = T_n(a^J)$,

$$R_n = \frac{1}{2}(R_n + R_n^J) = T_n\left(\frac{1}{2}(\mathbf{a}+\mathbf{a}^J)\right) + T_n\left(\frac{1}{2}(\mathbf{b}+\mathbf{b}^J)\right) J_n .$$

We conclude the following (compare [14]).

PROPOSITION 4.10. *Let R_n be an $n\times n$ T+H matrix. Then the following assertions are equivalent.*

1. *R_n is centrosymmetric.*
2. *In the representation (4.7) (resp. (4.24)) the Toeplitz matrices $T_n(\mathbf{a})$ and $T_n(\mathbf{b})$ (resp. $T_n(\mathbf{c})$ and $T_n(\mathbf{d})$) are symmetric.*
3. *In the representation (4.7) (resp. 4.24)) $\mathbf{a}$ and $\mathbf{b}$ (resp. $\mathbf{c}$ and $\mathbf{d}$) are symmetric vectors.*

COROLLARY 4.11. *A centrosymmetric T+H matrix R_n is also symmetric.*

Moreover, in the centrosymmetric case the representation (4.24) can be written in the form

$$R_n = P_+ T_n(\mathbf{c}) P_+ + P_- T_n(\mathbf{d}) P_- . \tag{4.31}$$

Now we specify the results for general T+H matrices to centrosymmetric T+H matrices R_n. Since R_n is symmetric we can use the simplifications of the previous subsection. Furthermore, we observe that the right hand sides of the first and the third equations of (4.25) are symmetric and that those of the second and the fourth equations are skewsymmetric if we choose

$$c_n = c_{-n},\ d_n = d_{-n} .$$

Since centrosymmetric matrices map symmetric (skewsymmetric) vectors into symmetric (skewsymmetric) vectors, we conclude that the solutions $\mathbf{w}_1, \mathbf{w}_3$ of (4.25) as well as their extensions $\breve{\mathbf{u}}_1, \breve{\mathbf{u}}_3$ of (4.27) are symmetric, whereas $\mathbf{w}_2, \mathbf{w}_4$ and $\breve{\mathbf{u}}_2, \breve{\mathbf{u}}_4$ are skewsymmetric vectors. This leads to further simplications of the inversion formula (4.30). But before presenting the result let us introduce a more unified notation, where the subscript $+$ designates symmetric and the subscript $-$ skewsymmetric vectors in the fundamental system,

$$\mathbf{u}_+ = \begin{bmatrix} 0 \\ \mathbf{w}_3 \\ 0 \end{bmatrix},\ \mathbf{u}_- = \begin{bmatrix} 0 \\ \mathbf{w}_4 \\ 0 \end{bmatrix},\ \mathbf{v}_+ = \begin{bmatrix} 1 \\ -2\mathbf{w}_1 \\ 1 \end{bmatrix},\ \mathbf{v}_- = \begin{bmatrix} 1 \\ -2\mathbf{w}_2 \\ -1 \end{bmatrix}. \tag{4.32}$$

Here $\mathbf{w}_i$ are the solutions of (4.25) which turn obviously into pure Toeplitz equations,

$$\begin{aligned} T_n(\mathbf{c})\mathbf{w}_1 &= P_+\mathbf{c}_{(+)},\ T_n(\mathbf{d})\mathbf{w}_2 = P_-\mathbf{d}_{(+)}, \\ T_n(\mathbf{c})\mathbf{w}_3 &= P_+\mathbf{e}_1,\ T_n(\mathbf{d})\mathbf{w}_4 = P_-\mathbf{e}_1 . \end{aligned} \tag{4.33}$$

Note that these equations have unique symmetric or skewsymmetric solutions. Thus,

$$\frac{\mathbf{u}_\pm(t)\mathbf{v}_\pm(s) - \mathbf{v}_\pm(t)\mathbf{u}_\pm(s)}{(t-s)(1-ts)}$$

are polynomials, and the inversion formula (4.30) can be rewritten as a sum of a split Bezoutian of $(+)$-type and a split Bezoutian of $(-)$-type. Thus, we arrive at the following.

THEOREM 4.12. *Let R_n be a nonsingular, centrosymmetric T+H matrix given by (4.24) and $\mathbf{u}_\pm, \mathbf{v}_\pm$ be the vectors of $\mathbb{F}_\pm^{n+2}$ defined in (4.32), where the $\mathbf{w}_i$ are the unique symmetric or skewsymmetric solutions of the Toeplitz equations (4.33). Then*

$$R_n^{-1} = B_+ + B_- ,$$

where $B_\pm$ are the split Bezoutians of $(\pm)$-type given by

$$B_\pm = \mathrm{Bez}_{\mathrm{split}}(\mathbf{u}_\pm, \mathbf{v}_\pm) .$$

Similar ideas as those of Subsection 3.2 lead to a slight modification of the last theorem. We extend the nonsingular centrosymmetric T+H matrix R_n given by (4.24) to a nonsingular centrosymmetric T+H matrix R_{n+2} such that R_n is its central submatrix of order n :

$$R_{n+2} = T_{n+2}(\mathbf{c})P_+ + T_{n+2}(\mathbf{d})P_- . \tag{4.34}$$

Here $\mathbf{c}$ and $\mathbf{d}$ are extensions of the original vectors $\mathbf{c}$ and $\mathbf{d}$ by appropriate components

$$c_{-n} = c_n \,,\; d_{-n} = d_n \text{ and } c_{-n-1} = c_{n+1} \,,\; d_{-n-1} = d_{n+1} \,.$$

Let $\mathbf{x}_{n+2}^\pm, \mathbf{x}_n^\pm$ be the unique symmetric or skewsymmetric solutions of

$$\begin{aligned} T_{n+2}(\mathbf{c})\mathbf{x}_{n+2}^+ &= P_+\mathbf{e}_1 \,, \quad T_n(\mathbf{c})\mathbf{x}_n^+ = P_+\mathbf{e}_1 \,, \\ T_{n+2}(\mathbf{d})\mathbf{x}_{n+2}^- &= P_-\mathbf{e}_1 \,, \quad T_n(\mathbf{d})\mathbf{x}_n^- = P_-\mathbf{e}_1 \,. \end{aligned} \tag{4.35}$$

(Note that $\mathbf{x}_n^+ = \mathbf{w}_3$, $\mathbf{x}_n^- = \mathbf{w}_4$. The solutions $\mathbf{x}_{n+2}^\pm$ are up to a constant factor equal to the vectors $\mathbf{v}_\pm$.)

COROLLARY 4.13. *[15] Let R_{n+2} be a nonsingular, centrosymmetric extension (4.34) of R_n. Then the equations (4.35) have unique symmetric or skewsymmetric solutions and*

$$R_n^{-1}(t,s) = \frac{1}{r_+}B_+(t,s) + \frac{1}{r_-}B_-(t,s) ,$$

where $B_\pm = \mathrm{Bez}_{\mathrm{split}}(\mathbf{x}_{n+2}^\pm, \mathbf{u}_\pm)$, $r_\pm$ are the first components of $\mathbf{x}_{n+2}^\pm$, and $\mathbf{u}_\pm = \begin{bmatrix} 0 \\ \mathbf{x}_n^\pm \\ 0 \end{bmatrix}$.

If $T_n(\mathbf{c})$ and $T_n(\mathbf{d})$ are nonsingular then R_n is nonsingular. Indeed, taking (4.31) into account, the equality $R_n\mathbf{u} = 0$ leads to

$$P_+T_n(\mathbf{c})P_+\mathbf{u} = -P_-T_n(\mathbf{d})P_-\mathbf{u} .$$

Hence $P_+\mathbf{u} = \mathbf{0}$ and $P_-\mathbf{u} = \mathbf{0}$, which means $\mathbf{u} = \mathbf{0}$. The converse is not true. Take, for example, $\mathbf{c} = (1,\ 1,\ 1)$ and $\mathbf{d} = (-1,\ 1, -1)$. Then $T_2(\mathbf{c})$ and $T_2(\mathbf{d})$ are singular, whereas $R_2 = 2I_2$ is nonsingular. One might conjecture that for a nonsingular R_n there is always a representation (4.24) with nonsingular $T_n(\mathbf{c})$ and $T_n(\mathbf{d})$. For $n = 2$ this is true, but this fails to be true for greater n (see [14]).

Let us consider besides $R_n = T_n(\mathbf{a}) + T_n(\mathbf{b})J_n$ the matrix $R_n^- = T(\mathbf{a}) - T(\mathbf{b})J_n$. If R_n is represented in the form (4.31) then the corresponding representation of R_n^- is

$$R_n^- = P_+T_n(\mathbf{d})P_+ + P_-T_n(\mathbf{c})P_- ,$$

which means that the roles of $\mathbf{c}$ and $\mathbf{d}$ are interchanged. We conclude the following.

PROPOSITION 4.14. *The (symmetric) Toeplitz matrices $T_n(\mathbf{c})$ and $T_n(\mathbf{d})$ are nonsingular if and only if both R_n and R_n^- are nonsingular.*

Proof. We have already shown that the nonsingularity of $T_n(\mathbf{c})$ and $T_n(\mathbf{d})$ implies the nonsingularity of R_n. The nonsingularity of R_n^- follows by the same arguments. It remains to show that the singularity of $T_n(\mathbf{c})$ (or $T_n(\mathbf{d})$) leads to the singularity of R_n or R_n^-. Let $\mathbf{u}$ be a nontrivial vector such that $T_n(\mathbf{c})\mathbf{u} = 0$. We split $\mathbf{u}$ into its symmetric and skewsymmetric parts,

$$\mathbf{u} = \mathbf{u}_+ + \mathbf{u}_- \quad (\mathbf{u}_\pm \in \mathbb{F}^n_\pm).$$

Clearly, at least one of the vectors $\mathbf{u}_+$ or $\mathbf{u}_-$ is nonzero, and $T_n(\mathbf{c})\mathbf{u}_+ = T_n(\mathbf{c})\mathbf{u}_- = 0$. Since

$$R_n\mathbf{u}_+ = T_n(\mathbf{c})\mathbf{u}_+ , \quad R_n^-\mathbf{u}_- = T_n(\mathbf{c})\mathbf{u}_-$$

we obtain that R_n or R_n^- is singular. This is also obtained if we assume that $T_n(\mathbf{d})$ is singular. $\square$

5. Inversion of centro-skewsymmetric T+H matrices. Finally we consider the special case of a T+H matrix R_n which is centro-skewsymmetric, $R_n = -R_n^J$. Since $\det A = (-1)^n \det A$ for an $n \times n$ centro-skewsymmetric matrix A all centro-skewsymmetric matrices of odd order are singular. Hence we here consider mainly matrices of even order. The centro-skewsymmetric counterpart of Proposition 4.10 is as follows (see [14]).

PROPOSITION 4.15. *Let R_n be an $n \times n$ T+H matrix. Then the following assertions are equivalent.*

1. *R_n is centro-skewsymmetric.*
2. *There is a representation (4.7) (resp. (4.24)) such that the Toeplitz matrices $T_n(\mathbf{a})$ and $T_n(\mathbf{b})$ (resp. $T_n(\mathbf{c})$ and $T_n(\mathbf{d})$) are skewsymmetric.*
3. *There is a representation (4.7) (resp. (4.24)) such that $\mathbf{a}$ and $\mathbf{b}$ (resp. $\mathbf{c}$ and $\mathbf{d}$) are skewsymmetric vectors.*

In the remaining part of this subsection we only use such representations. In this case (4.24) can be rewritten as

$$R_n = P_-T_n(\mathbf{c})P_+ + P_+T_n(\mathbf{d})P_- .$$

Its transposed matrix is given by

$$R_n^T = -(P_-T_n(\mathbf{d})P_+ + P_+T_n(\mathbf{c})P_-) ,$$

and we have $\widetilde{\mathbf{c}} = -\mathbf{d}$ and $\widetilde{\mathbf{d}} = -\mathbf{c}$ in equations (4.26).

In general, R_n is neither symmetric nor skewsymmetric, and thus a connection between the solutions of (4.25) and (4.26) is not obvious. If we choose $c_n = -c_{-n}$ and $d_n = -d_{-n}$ than $\mathbf{c}_{(-)} = -\mathbf{c}_+$, $\mathbf{d}_{(-)} = -\mathbf{d}_{(+)}$. Hence the right-hand sides of the equations (4.25), (4.26) are either symmetric or skewsymmetric. Since R_n as a centro-skewsymmetric matrix maps $\mathbb{F}^n_\pm$ to $\mathbb{F}^n_\mp$, we obtain that the solutions are also either symmetric or skewsymmetric. Let us indicate these symmetry properties again by denoting

$$\begin{gathered}\mathbf{w}_+ = \mathbf{w}_1\,, \quad \mathbf{w}_- = \mathbf{w}_2\,, \quad \mathbf{x}_- = \mathbf{w}_3\,, \quad \mathbf{x}_+ = \mathbf{w}_4\,, \\ \widetilde{\mathbf{x}}_- = \mathbf{z}_1\,, \quad \widetilde{\mathbf{x}}_+ = \mathbf{z}_2\,, \quad \widetilde{\mathbf{w}}_+ = \mathbf{z}_3\,, \quad \widetilde{\mathbf{w}}_- = \mathbf{z}_4.\end{gathered}$$

Since these symmetries pass to the augmented vectors $\breve{\mathbf{u}}_j$, $\breve{\mathbf{v}}_j$ of (4.27) we set

$$\begin{gathered}\mathbf{v}_+ = \breve{\mathbf{u}}_1\,, \quad \mathbf{v}_- = \breve{\mathbf{u}}_2\,, \quad \mathbf{u}_- = \breve{\mathbf{u}}_3\,, \quad \mathbf{u}_+ = \breve{\mathbf{u}}_4\,, \\ \widetilde{\mathbf{v}}_+ = \breve{\mathbf{v}}_3\,, \quad \widetilde{\mathbf{v}}_- = \breve{\mathbf{v}}_4\,, \quad \widetilde{\mathbf{u}}_- = \breve{\mathbf{v}}_1\,, \quad \widetilde{\mathbf{u}}_+ = \breve{\mathbf{v}}_2\,.\end{gathered} \tag{4.36}$$

The equations (4.25), (4.26) turn into Toeplitz equations,

$$\begin{gathered}T_n(\mathbf{c})\mathbf{x}_+ = P_-\mathbf{e}_1\,, \quad T_n(\mathbf{c})\mathbf{w}_+ = P_-\mathbf{c}_{(+)}\,, \\ T_n(\mathbf{d})\mathbf{x}_- = P_+\mathbf{e}_1\,, \quad T_n(\mathbf{d})\mathbf{w}_- = P_+\mathbf{d}_{(+)}\end{gathered} \tag{4.37}$$

and

$$\begin{gathered}T_n(\mathbf{c})\widetilde{\mathbf{x}}_- = -P_+\mathbf{e}_1\,,\ T_n(\mathbf{c})\widetilde{\mathbf{w}}_- = P_+\mathbf{c}_{(+)}\,, \\ T_n(\mathbf{d})\widetilde{\mathbf{x}}_+ = -P_-\mathbf{e}_1\,,\ T_n(\mathbf{d})\widetilde{\mathbf{w}}_+ = P_-\mathbf{d}_{(+)}\,.\end{gathered} \tag{4.38}$$

According to Proposition 4.9 and (4.3), R_n^{-1} is given by the augmented vectors (4.36) of these solutions via

$$\nabla_{T+H} R_n^{-1} = \mathbf{u}_-\widetilde{\mathbf{v}}_+^T - \mathbf{v}_+\widetilde{\mathbf{u}}_-^T - \mathbf{v}_-\widetilde{\mathbf{u}}_+^T + \mathbf{u}_+\widetilde{\mathbf{v}}_-^T\,. \tag{4.39}$$

Note that for a nonsingular matrix R_n all equations (4.37) and (4.38) are uniquely solvable. Indeed we observe that $\mathbf{x} = \mathbf{x}_+ - \widetilde{\mathbf{x}}_-$ is a solution of $T_n(\mathbf{c})\mathbf{x} = \mathbf{e}_1$ and $\mathbf{w} = \mathbf{w}_+ - \widetilde{\mathbf{w}}_-$ is a solution of $T_n(\mathbf{c})\mathbf{w} = \mathbf{c}^J_{(-)}$. Taking Theorem 3.1 into account we obtain the nonsingularity of $T_n(\mathbf{c})$. Analogously, $T_n(\mathbf{d})$ is nonsingular. This leads to the following conclusion revealing an essential difference between the centrosymmetric and centro-skewsymmetric cases (see [14]).

PROPOSITION 4.16. *For a centro-skewsymmetric T+H matrix*

$$R_n = T(\mathbf{a}) + T(\mathbf{b})J_n = T(\mathbf{c})P_+ + T(\mathbf{d})P_-$$

with skewsymmetric vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, *the following assertions are equivalent.*

1. R_n *is nonsingular.*
2. $R_n^- = T(\mathbf{a}) - T(\mathbf{b})J_n$ *is nonsingular.*
3. $T(\mathbf{c})$ *and* $T(\mathbf{d})$ *are nonsingular.*

COROLLARY 4.17. *Let* R_n *be nonsingular. Then*

$$R_n^{-1} = T_n(\mathbf{c})^{-1}P_- + T_n(\mathbf{d})^{-1}P_+ = P_-T_n(\mathbf{c})^{-1} + P_+T_n(\mathbf{d})^{-1}\,. \tag{4.40}$$

Now we use Proposition 3.7 twice for $T_n = T_n(\mathbf{c})$ and for $T_n = T_n(\mathbf{d})$. All numbers, vectors, and matrices are designated by the upscript $\mathbf{c}$ respective $\mathbf{d}$. In particular, $\mathbf{x}^{\mathbf{c}}$ is any vector spanning the nullspace of $T_{n-1}(\mathbf{c})$, the monic vector $\mathbf{v}^{\mathbf{c}}$ spans the nullspace of $T_{n+1}(\mathbf{c})$, so that $\{\mathbf{u}^{\mathbf{c}}, \mathbf{v}^{\mathbf{c}}\}$ with $\mathbf{u}^{\mathbf{c}}(t) = t\mathbf{x}^{\mathbf{c}}(t)$ is a fundamental system for $T_n(\mathbf{c})$, $\gamma^{\mathbf{c}} = [\, c_1 \dots c_{n-1}\,]\mathbf{x}^{\mathbf{c}}$, $\mathbf{p}^{\mathbf{c}}_{\pm}(t) = (t \pm 1)\mathbf{u}^{\mathbf{c}}(t), \mathbf{q}^{\mathbf{c}}_{\pm}(t) = (t \pm 1)\mathbf{v}^{\mathbf{c}}(t)$. Inversion formula (3.8) looks now as follows:

$$T_n(\mathbf{c})^{-1}(t,s) = \frac{1}{2\gamma^{\mathbf{c}}}\,(B^{\mathbf{c}}_{+}(t,s)\frac{1-s}{1+s} - B^{\mathbf{c}}_{-}(t,s)\frac{1+s}{1-s}),$$

where $B^{\mathbf{c}}_{\pm} = \mathrm{Bez}_{\mathrm{split}}(\mathbf{p}^{\mathbf{c}}_{\pm}, \mathbf{q}^{\mathbf{c}}_{\pm})$. In such a manner also $T_n(\mathbf{d})^{-1}$ can be represented. By putting these representations into (4.40) we obtain the following splitting of R_n^{-1}.

PROPOSITION 4.18.

$$R_n^{-1}(t,s) = \frac{1}{2\gamma^{\mathbf{c}}}\,B^{\mathbf{c}}_{+}(t,s)\frac{1-s}{1+s} - \frac{1}{2\gamma^{\mathbf{d}}}\,B^{\mathbf{d}}_{-}(t,s)\frac{1+s}{1-s}.$$

A splitting formula of this form has been already presented in [14]. But there the proof is based on connections between the solutions of (4.37) and (4.38). Let us recall the results for the augmented vectors:

$$\mathbf{u}_{+}(t) = \frac{1+t}{1-t}\,\widetilde{\mathbf{u}}_{-}(t), \quad \mathbf{v}_{+}(t) = \frac{1+t}{1-t}\,\widetilde{\mathbf{v}}_{-}(t).$$

Replacing $\mathbf{c}$ by $\mathbf{d}$ we obtain

$$\widetilde{\mathbf{u}}_{+}(t) = \frac{1+t}{1-t}\,\mathbf{u}_{-}(t), \quad \widetilde{\mathbf{v}}_{+}(t) = \frac{1+t}{1-t}\,\mathbf{v}_{-}(t).$$

Inserting these into (4.39) leads to the following splitting formula given in [14]:

$$R_n^{-1}(t,s) = B^{+}(t,s)\,\frac{s-1}{s+1} + B^{-}(t,s)\,\frac{s+1}{s-1},$$

where

$$B^{\pm} = \mathrm{Bez}_{\mathrm{split}}(\mathbf{u}_{\pm}, \mathbf{v}_{\pm}).$$

This inversion formula is, obviously, identical with the representation of R_n^{-1} in Proposition 4.18.

To present the skewsymmetric counterpart of Corollary 4.13 let us extend the nonsingular centro-skewsymmetric T+H matrix R_n given by (4.24) to a nonsingular centro-skewsymmetric T+H matrix R_{n+2} such that R_n is its central submatrix of order n:

$$R_{n+2} = T_{n+2}(\mathbf{c})P_{+} + T_{n+2}(\mathbf{d})P_{-}, \tag{4.41}$$

where $\mathbf{c}$ and $\mathbf{d}$ are extensions of the original vectors $\mathbf{c}$ and $\mathbf{d}$ by appropriate components $c_{-j} = -c_j$, $d_{-j} = -d_j$ $(j = n, n+1)$. Let $\mathbf{x}^{\pm}_{n+2}, \mathbf{x}^{\pm}_{n}$ be the unique symmetric or skewsymmetric solutions of

$$\begin{aligned} T_{n+2}(\mathbf{c})\mathbf{x}^{-}_{n+2} &= P_{+}\mathbf{e}_1, \quad T_n(\mathbf{c})\mathbf{x}^{-}_{n} = P_{+}\mathbf{e}_1, \\ T_{n+2}(\mathbf{d})\mathbf{x}^{+}_{n+2} &= P_{-}\mathbf{e}_1, \quad T_n(\mathbf{d})\mathbf{x}^{+}_{n} = P_{-}\mathbf{e}_1. \end{aligned} \tag{4.42}$$

Note that $\mathbf{x}^{\pm}_{n} = -\widetilde{\mathbf{x}}_{\pm}$, and thus $-\mathbf{u}_{\pm}$ are the augmented vectors defined by $\mathbf{u}_{\pm}(t) = t\mathbf{x}^{\pm}_{n}(t)$. The solutions $\mathbf{x}^{\pm}_{n+2}$ are up to a constant factor equal to the vectors $\mathbf{v}_{\pm}$.

COROLLARY 4.19. *[15] Let R_{n+2} be a nonsingular, centro-skewsymmetric extension (4.41) of R_n. Then the equations (4.42) have unique symmetric or skewsymmetric solutions and*

$$R_n^{-1}(t,s) = \frac{1}{r_+} B_+(t,s) \frac{s-1}{s+1} + \frac{1}{r_-} B_-(t,s) \frac{s+1}{s-1},$$

where $B_\pm = \mathrm{Bez}_{\mathrm{split}}(\mathbf{x}_{n+2}^{\pm}, \mathbf{u}_\pm)$ and $r_\pm$ are the first components of $\mathbf{x}_{n+2}^{\pm}$.

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