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POLYNOMIAL MATRICES WITH HERMITIAN COEFFICIENTS AND A GENERALIZATION OF THE ENESTRÖM–KAKEYA THEOREM

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Abstract. Polynomial matrices $G(z) = Iz^m - \sum C_i z^i$ with hermitian coefficients C_i are studied. The assumption $\sum |C_i| \leqslant I$ implies that the characteristic values of $G(z)$ lie in the closed unit disc. The characteristic values of modulus one are roots of unity. An extension of the Eneström–Kakeya theorem is proved and a stability criterion for a system of difference equations is given.

1. Introduction

The starting point for this paper is the following theorem, which in part can be traced back to Hurwitz [7]. It deals with a real polynomial and its roots in the unit disc and on the unit circle.

THEOREM 1.1. *Let*

$$g(z) = z^m - (c_{m-1}z^{m-1} + \cdots + c_1 z + c_0) \tag{1.1}$$

be a real polynomial. Suppose $c_0 \neq 0$ *and* $s = \sum_{i=0}^{m-1} |c_i| \leqslant 1$. *Then*

$$\rho(g) = \max\{|\lambda|;\ g(\lambda) = 0\} \leqslant 1.$$

If λ *is a root of* $g(z)$ *with* $|\lambda| = 1$ *then* λ *is a simple root and* $\lambda^d = \pm 1$ *for some* d *with* $d \,|\, m$. *If* $\rho(g) = 1$ *then either* $g(1) = 1$ *and*

$$g(z) = (z^k - 1) f(z^k) \tag{1.2}$$

or $g(1) \neq 1$ *and*

$$g(z) = (z^k + 1) f(z^k), \tag{1.3}$$

and $f(\mu) \neq 0$ *if* $|\mu| = 1$.

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It is the aim of the paper to extend Theorem 1.1 to a complex $n \times n$ polynomial matrix

$$G(z) = Iz^m - (C_{m-1}z^{m-1} + \cdots + C_1 z + C_0) \tag{1.4}$$

with hermitian coefficients. Two applications will be given. The first one is an extension of the Eneström–Kakeya theorem and its sharpness to polynomial matrices with positive semidefinite coefficients. Recall that the following theorem is known as Eneström–Kakeya theorem (see e.g. [10, p. 4], [3, p. 12], [11, p. 255]).

THEOREM 1.2. *Let*

$$h(z) = a_{m-1}z^{m-1} + \cdots + a_1 z + a_0 \tag{1.5}$$

be a real polynomial such that

$$a_{m-1} \geqslant \cdots \geqslant a_1 \geqslant a_0 \geqslant 0, \quad a_{m-1} > 0. \tag{1.6}$$

(i) *Then* $\rho(h) \leqslant 1$.
(ii) *The zeros of* $h(z)$ *lying on the unit circle are simple.*

The second application is a stability criterion for the difference equation

$$x(t+m) = C_{m-1}x(t+m-1) + \cdots + C_1 x(t+1) + C_0 x(t).$$

The following notation will be used. Let $G(z)$ be the polynomial matrix in (1.4). We define

$$\sigma(G) = \{\lambda \in \mathbb{C};\ \det G(\lambda) = 0\}$$

and $\rho(G) = \max\{|\lambda|;\ \lambda \in \sigma(G)\}$. In particular, if $f(z) \in \mathbb{C}^n[z]$ then $\sigma(f)$ shall denote the set of roots of $f(z)$. In accordance with [2, p. 341] the elements of $\sigma(G)$ will be called the characteristic values of $G(z)$. If $v \in \mathbb{C}^n$ satisfies $G(\lambda)v = 0$, $v \neq 0$, then v is said to be an eigenvector corresponding to λ. An r-tuple of vectors $(v_0, v_1, \ldots, v_{r-1})$, $v_i \in \mathbb{C}^n$, $v_0 \neq 0$, is called a Jordan chain (or Keldysh chain [2]) of length r of $G(z)$ if

$$G(\lambda)v_0 = 0, \quad G'(\lambda)v_0 + G(\lambda)v_1 = 0, \cdots,$$
$$\frac{1}{(r-1)!}G^{(r-1)}(\lambda)v_0 + \frac{1}{(r-2)!}G^{(r-2)}(\lambda)v_1 + \cdots + G(\lambda)v_{r-1} = 0.$$

The symbol $\mathbb{D}$ represents the open unit disc. Thus $\partial\mathbb{D}$ is the unit circle and $\overline{\mathbb{D}}$ is the closed unit disc. If $Q, R \in \mathbb{C}^{n\times n}$ are hermitian then we write $Q > 0$ if Q is positive definite, and $R \geqslant 0$ if R is positive semidefinite. The inequality $Q \geqslant R$ means $Q - R \geqslant 0$. If $Q \geqslant 0$ then $Q^{1/2}$ shall denote the positive semidefinite square root of Q. The positive semidefinite part of a hermitian matrix A is given by $|A| = (AA^*)^{1/2} = (A^2)^{1/2}$. Let

$$E_k = \{\zeta \in \mathbb{C};\ \zeta^k = 1\}$$

be the group of k-th roots of unity. If $\zeta \in E_k$ then $\operatorname{ord}\zeta$ will denote the order of ζ, i.e. if $\operatorname{ord}\zeta = s$ then s is the smallest positive divisor of k such that $\zeta^s = 1$. In many instances limits of summation will be omitted. Then $\sum$ shall mean $\sum_{i=0}^{m-1}$.

2. Characteristic values in $\overline{\mathbb{D}}$

In this section we are mainly concerned with the location of the characteristic values of $G(z)$. The following observations will be useful.

LEMMA 2.1. *Let $A \in \mathbb{C}^{n\times n}$ be hermitian.*
(i) *If $\eta = \pm 1$ then*

$$|A| \geqslant \eta A. \tag{2.1}$$

(ii) *There exists a unitary matrix U such that*

$$A = |A|\,U = U\,|A| \quad \text{and} \quad \sigma(U) \in \{1,-1\}. \tag{2.2}$$

Proof. Let V be unitary such that $A = V^*\mathrm{diag}\,(\alpha_1,\dots,\alpha_n)V$. Then

$$|A| = V^*\,\mathrm{diag}\,(|\alpha_1|,\dots,|\alpha_n|)\,V.$$

Thus (2.1) is obvious. Set $\eta_i = 1$ if $\alpha_i \geqslant 0$ and $\eta_i = -1$ if $\alpha_i < 0$. Define $U = V^*\mathrm{diag}\,(\eta_1,\dots,\eta_n)V$. Then (2.2) is satisfied. □

THEOREM 2.2. *Let $G(z) = Iz^m - \sum C_i z^i$ be an $n\times n$ polynomial matrix with hermitian coefficients C_i. Set $S = \sum |C_i|$. Suppose $S \leqslant I$. Let $G(\lambda)v = 0$, $v \neq 0$, and $|\lambda| = 1$. Then the following holds.* (i) $\rho(G) \leqslant 1$. (ii) $Sv = v$. (iii) $v^*G(\lambda) = 0$. (iv) *The elementary divisors of $G(z)$ corresponding to λ are linear.*

Proof. Let $G(\lambda)v = 0$ and $v \neq 0$. We can assume $v^*v = 1$. Suppose $\lambda \neq 0$. Then $\lambda^m v = \sum C_i\lambda^i v$ implies

$$1 = \sum \frac{1}{\lambda^{m-i}} v^* C_i v. \tag{2.3}$$

Then (2.1) yields

$$1 \leqslant \sum \frac{1}{|\lambda^{m-i}|}|v^*C_iv| \leqslant \sum \frac{1}{|\lambda^{m-i}|} v^*|C_i|v. \tag{2.4}$$

(i) Set $\mu = \min\{|\lambda|,\dots,|\lambda|^m\}$. Then (2.4) and $S \leqslant I$ imply $1 \leqslant 1/\mu$, that is $|\lambda| \leqslant 1$.

(ii) If $|\lambda| = 1$ then $S \leqslant I$ and (2.4) imply $1 = v^*Iv = v^*Sv$. Hence $(S-I)v = 0$.

(iii) Set

$$\beta_i = \frac{1}{\lambda^{m-i}} v^*C_iv, \ \ i = 0,\dots,m-1. \tag{2.5}$$

From (2.4) follows $1 = \left|\sum \beta_i\right| = \sum |\beta_i|$. Hence $\beta_i = \omega\alpha_i$, $i = 0,\dots,m-1$, with $\alpha_i \in \mathbb{R}$, $\alpha_i \geqslant 0$, $\omega \in \mathbb{C}$, $|\omega| = 1$. From (2.3) we obtain $1 = \omega\sum\alpha_i$. Therefore $\omega = 1$, and $\beta_i \in \mathbb{R}$, $\beta_i \geqslant 0$. Define

$$I(v) = \{i;\ 0 \leqslant i \leqslant m-1,\ v^*C_iv \neq 0\}. \tag{2.6}$$

Then (2.3) implies $I(v) \neq \emptyset$. Since C_i is hermitian we have $v^*C_iv \in \mathbb{R}$. Therefore, if $i \in I(v)$ then $\lambda^{m-i} = \pm 1$ in (2.5). Using (2.1) we obtain

$$1 = \sum_{i \in I(v)} \frac{1}{\lambda^{m-i}} v^* C_i v \leqslant \sum_{i \in I(v)} v^*|C_i|v \leqslant \sum_{0 \leqslant i \leqslant m-1} v^*|C_i|v = 1. \tag{2.7}$$

Hence, if $i \notin I(v)$ then we have $v^*|C_i|v = 0$, or equivalently $|C_i|v = C_iv = 0$. Therefore

$$I(v) = \{i;\ 0 \leqslant i \leqslant m-1,\ C_iv \neq 0\}. \tag{2.8}$$

From (2.7) follows

$$\sum_{i \in I(v)} v^*\Big(|C_i| - \frac{1}{\lambda^{m-i}} C_i\Big) v = 0.$$

Hence $|C_i|v = \frac{1}{\lambda^{m-i}} C_i v = \lambda^{m-i} C_i v$ if $i \in I(v)$. Thus we have shown that

$$\lambda^{m-i} C_i v = |C_i|v, \quad i = 0, \dots, m-1. \tag{2.9}$$

From $v = \sum |C_i|v$ and (2.9) follows $v = \sum \lambda^{m-i} C_i v$. Because of $\bar{\lambda} = \lambda^{-1}$ and $C_i^* = C_i$ this is equivalent to $v^*\lambda^m = v^* \sum \lambda^i C_i$, i.e. to $v^*G(\lambda) = 0$.

(iv) According to [2, p. 342]) the degree of elementary divisors is related to the length of Jordan chains. Hence we have to show that the eigenvector v can not be extended to a Jordan chain of length greater than 1. Suppose there exists a vector $w \in \mathbb{C}^n$ such that $G'(\lambda)v + G(\lambda)w = 0$. Then $v^*G(\lambda) = 0$ implies

$$0 = v^*\big[G(\lambda)w + G'(\lambda)v\big] = v^*G'(\lambda)v = v^*\big(m\lambda^{m-1} - \sum iC_i\lambda^{i-1}\big)v.$$

Thus we would obtain $mv^*v \leqslant \sum_{i=0}^{m-1} iv^*|C_i|v$, in contradiction to $v^*v = \sum v^*|C_i|v$. □

Hermitian polynomial matrices $G(z)$ with positive semidefinite coefficients C_i have been studied in [13]. In the present paper we no longer assume $C_i \geqslant 0$. This will require a more elaborate approach.

3. Characteristic values on the unit circle

We continue to assume $S = \sum |C_i| \leqslant I$. In this section the focus is on characteristic values of $G(z)$ on the unit circle. To a vector $v \in \mathbb{C}^n$, $v \neq 0$, we associate the set

$$M(v) = \{\lambda \in \mathbb{C};\ |\lambda| = 1,\ G(\lambda)v = 0\}.$$

If $M(v) \neq \emptyset$ then it follows from (2.6) and (2.8) that $v^*C_iv \neq 0$ if and only if $C_iv \neq 0$. We define $t = \min\{i;\ i \in I(v)\}$ and $\epsilon = \operatorname{sign} v^*C_tv$. Then $C_tv \neq 0$, and $C_iv = 0$ if $i < t$. It will be shown in Lemma 3.2 below that all elements of $M(v)$ are roots of unity. In Theorem 3.6 and Theorem 3.8 it will be proved that either $M(v) = E_d$ or $M(v) = \{\lambda;\ \lambda^d = -1\}$ for some divisor d of $m - t$.

LEMMA 3.1. *We have*

$$G(\lambda)v = 0,\ v \neq 0, \quad \textit{and} \quad |\lambda| = 1 \tag{3.1}$$

if and only if

$$Sv = v,\ \ v \neq 0, \quad \text{and} \tag{3.2a}$$

$$C_i \lambda^{m-i} v = |C_i|\, v, \quad i = 0, \ldots, m-1. \tag{3.2b}$$

Proof. In Theorem 2.2 and its proof we have seen that (3.1) implies (3.2). Conversely, let (3.2) be satisfied. Then (3.2b) yields

$$v^*|C_i|v = |\lambda^{m-i}||v^* C_i v| \leqslant |\lambda^{m-i}|v^*|C_i|v. \tag{3.2}$$

Suppose $|\lambda| < 1$. But then (3.2) would imply $v^*|C_i|v = 0$, i.e. $|C_i|v = 0$, $i = 0, \ldots, m-1$. We would obtain $Sv = 0$, which is incompatible with (3.2a). It follows that $|\lambda| \geqslant 1$. Hence $\rho(G) \leqslant 1$ implies $|\lambda| = 1$. To prove $G(\lambda)v = 0$ we recall Theorem 2.2(iii) and note that $G(\bar{\lambda})v = 0$ is equivalent to $G(\lambda)v = 0$ if $|\lambda| = 1$. Using $\bar{\lambda} = \lambda^{-1}$ and (3.2) we obtain

$$\begin{aligned} G(\bar{\lambda})v &= \left[\bar{\lambda}^m I - \sum C_i \bar{\lambda}^i\right] v = \bar{\lambda}^m \left[I - \sum \lambda^{m-i} C_i\right] v \\ &= \bar{\lambda}^m \left[I - \sum |C_i|\right] v = \bar{\lambda}^m (I - S)v = 0, \end{aligned}$$

which completes the proof. □

LEMMA 3.2. *For all $\lambda \in M(v)$ we have $\lambda^{m-t} = \epsilon$ and $\lambda^{2(m-t)} = 1$. If $1 \in M(v)$ then $\epsilon = 1$ and $M(v) \subseteq E_{m-t}$.*

Proof. From (3.2b) and (2.2) we obtain

$$C_t \lambda^{m-t} v = U_t \lambda^{m-t}\ |C_t|\, v = |C_t|\, v.$$

Hence $(U_t \lambda^{m-t} - I)\,|C_t|\, v = 0$. From $C_t v \neq 0$ follows $\lambda^{t-m} \in \sigma(U_t)$. Thus $\lambda^{m-t} \in \{1, -1\}$. Then $v^* C_t v\, \lambda^{m-t} = v^*|C_t|v > 0$ yields $\lambda^{m-t} = \epsilon$. □

LEMMA 3.3. *Let λ be an element of $M(v)$ of order k.*
(i) *If k is odd then*

$$C_{t+j}\, v \neq 0 \quad \text{only if} \quad j \in k\mathbb{Z}. \tag{3.3}$$

Moreover $k \mid (m-t)$ and $\epsilon = 1$, and

$$C_{t+\nu k} v = |C_{t+\nu k}| v, \quad \nu = 0, 1, \ldots, \ell - 1, \quad \ell = \tfrac{m-t}{k}. \tag{3.4}$$

(ii) *If k is even, $k = 2s$, then*

$$C_{t+j}\, v \neq 0 \quad \text{only if} \quad j \in s\mathbb{Z}. \tag{3.5}$$

Moreover $s \mid (m-t)$ and

$$C_{t+\nu s} v = \epsilon(-1)^\nu |C_{t+\nu s}| v, \quad \nu = 0, \ldots, (\ell - 1), \quad \ell = \tfrac{m-t}{s}. \tag{3.6}$$

Proof. From $C_i v = |C_i|\lambda^{-(m-i)} v$, $i = 0, \dots, m-1$, and $\lambda^{m-t} = \epsilon$ we obtain

$$C_{t+j} v = |C_{t+j}|\lambda^{-(m-t)}\lambda^j v = \epsilon\lambda^j |C_{t+j}|\, v, \tag{3.7}$$

$j = 0, \dots, m-1-t$. Let j be such that $C_{t+j} v \neq 0$. Then $v^* C_{t+j} v \in \mathbb{R} \setminus \{0\}$, and therefore (3.7) yields $\lambda^j = \pm 1$.

(i) If $\operatorname{ord} \lambda = k$ is odd then $\lambda^i \neq -1$ for all i. Hence $\epsilon = \lambda^{m-t} = 1$. Moreover, if $C_{t+j} v \neq 0$ then $\lambda^j = 1$, that is $j \in k\mathbb{Z}$, which proves (3.3). By Lemma 3.2 we have $\lambda^k = 1 = \lambda^{2(m-t)}$. Hence $k \mid (m-t)$. We obtain (3.4) if we take $j = \nu k$ in (3.7).

(ii) If $\operatorname{ord} \lambda = k = 2s$ then $\lambda^s = -1$. Therefore $\lambda^j = \pm 1$ is equivalent to $j \in s\mathbb{Z}$. Hence, if $C_{t+j} v \neq 0$ then $j \in s\mathbb{Z}$, and we have (3.5). From $\lambda^k = 1 = \lambda^{2(m-t)}$ and $k = 2s$ follows $s | (m-t)$. The assertion (3.6) is an immediate consequence of (3.7) with $j = \nu s$. □

With (3.3) in mind we make the following definition. Let $D(v)$ be the set of positive integers such that $d \in D(v)$ if and only if $d \mid (m-t)$ and d is a common divisor of the numbers $\{j;\ 0 \leqslant j < m-t, \text{ s.th. } C_{t+j} v \neq 0\}$. Thus $d \in D(v)$ is equivalent to

$$d | (m-t) \quad \text{and} \quad C_{t+j} v = 0 \ \text{ if } \ j \notin d\mathbb{Z}, \tag{3.8}$$

and also to

$$G(z)v = z^t \Big\{ I z^{d\ell} - \Big[C_{d(t+\ell-1)} z^{d(\ell-1)} + \cdots + C_{t+d} z^d + C_t \Big] \Big\} v, \quad m-t = d\ell. \tag{3.9}$$

If $\lambda \in M(v)$ then (3.9) implies $\{\mu;\ \mu^d = \lambda^d\} \subseteq M(v)$.

The subsequent lemmas prepare the ground for the description of $M(v)$. It will make an essential difference whether $M(v)$ contains elements of odd order or not.

LEMMA 3.4. *Let $M(v) \neq \emptyset$. Then the following statements are equivalent.*
(i) *The set $M(v)$ contains an element λ of odd order.*
(ii) *We have*

$$C_i v = |C_i| v, \quad i = 0, \dots, m-1. \tag{3.10}$$

(iii) $1 \in M(v)$.

Proof. (i) ⇒ (ii) If the order of $\lambda \in M(v)$ is odd then (3.3) and (3.4) imply (3.10). (ii) ⇒ (iii) Because of $M(v) \neq \emptyset$ we have (3.2) for some $\lambda \in \partial\mathbb{D}$. Then (3.10) implies that (3.2b) holds for $\lambda = 1$. Hence Lemma 3.1 yields $G(1)v = 0$. The implication (iii) ⇒ (i) is obvious because of $\operatorname{ord} 1 = 1$. □

LEMMA 3.5. *If $1 \in M(v)$ then the following statements are equivalent.*
(i) $d \in D(v)$.
(ii) *We have*

$$G(z)v = z^t (z^d - 1) f(z^d), \quad f(z) \in \mathbb{C}^n[z]. \tag{3.11}$$

(iii) $E_d \subseteq M(v)$.
(iv) *There exists an element $\lambda \in M(v)$ such that* $\operatorname{ord} \lambda = d$.

Proof. If $G(1)v = 0$ then (3.9) implies that $G(\mu)v = 0$ for all $\mu \in E_d$. Hence $d \in D(v)$ is equivalent to (3.11) and also to $E_d \subseteq M(v)$. Suppose (iv) holds. If d is odd then (3.8) follows immediately from Lemma 3.3(i). If d is even, $d = 2s$, then $\epsilon = 1 = \lambda^{m-t}$ implies $d|(m-t)$. Moreover, (3.6) yields $C_{t+\nu s}v = 0$ when ν is odd. Hence (3.8) is valid also when d is even. □

For the polynomial $g(z)$ in (1.1) the condition $1 \in M(v)$ amounts to $g(1) = 1$. Thus the identity (3.14) below generalizes the factorization (1.2) of Theorem 1.1.

THEOREM 3.6. *Assume $S = \sum |C_i| \leqslant I$ and $1 \in M(v)$. Set*

$$\hat{k} = \gcd\big\{\{m-t\} \cup \{j;\, 0 \leqslant j < m-t,\ C_{t+j}v \neq 0\}\big\}. \tag{3.12}$$

Then $M(v) = E_{\hat{k}}$. If $m - t = \hat{k}\ell$ then

$$G(z)v = z^t \left\{ I z^{\hat{k}\ell} - \sum\nolimits_{\nu=0}^{\ell-1} C_{t+\hat{k}\nu} z^{\hat{k}\nu} \right\} v, \tag{3.13}$$

or equivalently

$$G(z)v = z^t (z^{\hat{k}} - 1) f(z^{\hat{k}}) \tag{3.14}$$

for some $f(z) \in \mathbb{C}^n[z]$ such that

$$\sigma(f) \cap \partial\mathbb{D} = \emptyset. \tag{3.15}$$

Proof. From Lemma 3.2 follows $M(v) \subseteq E_{m-t}$. Let λ_1, λ_2 be elements of $M(v)$ such that $k_i = \operatorname{ord} \lambda_i$, $i = 1, 2$. By Lemma 3.5 we have $k_1, k_2 \in D(v)$. Set $p = \operatorname{lcm}(k_1, k_2)$. Take $d = k_1$ and $d = k_2$ in (3.8). Then we have $C_{t+j}v \neq 0$ only if $j \in k_1\mathbb{Z} \cap k_2\mathbb{Z} = p\mathbb{Z}$. Hence $p \in D(v)$. Therefore $G(z)v = z^t(z^p - 1) f(z^p)$, and from $(\lambda_1\lambda_2)^p = 1$ follows $\lambda_1\lambda_2 \in M(v)$. Hence $M(v)$ is a subgroup of E_{m-t}. Thus $M(v) = E_{\tilde{k}}$ and $\tilde{k} = \max\{\operatorname{ord} \lambda;\, \lambda \in M(v)\}$. We have

$$\tilde{k} = \max\{d; d \in D(v)\} = \hat{k},$$

which implies $M(v) = E_{\hat{k}}$. It remains to show that the polynomial vector $f(z^{\hat{k}})$ satisfies (3.15). Suppose $f(\mu) = 0$ for some $\mu \in \partial\mathbb{D}$. Then $\mu = \eta^{\hat{k}}$ with $\eta \in \partial\mathbb{D}$. Thus (3.14) implies $\eta \in M(v)$, and therefore $\eta \in E_{\hat{k}}$. Hence $G(z)$ would have an elementary divisor $(z-\eta)^r$ with $r \geqslant 2$, in contradiction to Theorem 2.2(iv). □

Suppose all coefficients C_i are positive semidefinite. Then (3.10) is satisfied, and $1 \in M(v)$ if $M(v) \neq \emptyset$. Thus in the case of the polynomial $g(z)$ in (1.1) we have recovered a result which is due to of Ostrowski (see also [1] and [10, p. 3]).

COROLLARY 3.7. [9, p. 92] *Let $g(z) = z^m - \sum_{i=0}^{m-1} c_i z^i$ be a real polynomial with nonnegative coefficients c_i such that $c_0 > 0$ and $\sum_{i=0}^{m-1} c_i = 1$.*

(i) *Then $g(1) = 0$, and the absolute values of the other roots of $g(z)$ do not exceed 1.*

(ii) *Moreover, $\lambda = 1$ is the only root of $g(z)$ on the unit circle if and only if the greatest common divisor of the indices i of all positive coefficients c_i is equal to 1.*

In the next theorem we assume $1 \notin M(v)$ and we obtain a counterpart to Theorem 3.6. The identity (3.18) below yields the factorization (1.3) in Theorem 1.1.

THEOREM 3.8. *Suppose $M(v) \neq \emptyset$ and $1 \notin M(v)$. Then all elements of $M(v)$ have even order. Set*

$$\hat{k} = \operatorname{lcm}\{\tfrac{1}{2}\operatorname{ord}\lambda;\ \lambda \in M(v)\}. \tag{3.16}$$

Then $\hat{k}\,|\,(m-t)$ and

$$M(v) = \{\lambda \in \mathbb{C};\ \lambda^{\hat{k}} + 1 = 0\}. \tag{3.17}$$

If $m - t = \hat{k}\ell$ then

$$G(z)v = z^t(z^{\hat{k}} + 1)f(z^{\hat{k}}) \tag{3.18}$$

for some $f(z) \in \mathbb{C}^n[z]$ with $\sigma(f) \cap \partial\mathbb{D} = \emptyset$.

Proof. It follows from Lemma 3.4 that the order of all elements of $M(v)$ is even. Suppose $M(v) = \{\lambda_1, \dots, \lambda_r\}$ and $\operatorname{ord}\lambda_i = k_i = 2s_i$, $i = 1, \dots, r$. Then $s_i \mid (m-t)$. Set $\hat{k} = \operatorname{lcm}(s_1, \dots, s_r)$ and $\ell = (m-t)/\hat{k}$. Then (3.5) implies that $C_{t+j}v \neq 0$ only if $j \in \hat{k}\mathbb{Z}$. Hence

$$G(z)v = z^t\Big[Iz^{\ell\hat{k}} - \sum_{\nu=0}^{\ell-1} C_{t+\nu\hat{k}} z^{\nu\hat{k}}\Big]v. \tag{3.19}$$

It is impossible that $\mu^{\hat{k}} = 1$ for some $\mu \in M(v)$. Otherwise (3.19) would imply $G(\mu)v = G(1)v = 0$ and we would have $1 \in M(v)$. Thus $\lambda_i^{\hat{k}} = -1$, $i = 1, \dots, r$. Hence $M(v) \subseteq \{\lambda;\ \lambda^{\hat{k}} + 1 = 0\}$. On the other hand (3.19) and $G(\lambda_1)v = 0$ yield $G(\lambda)v = 0$ if $\lambda^{\hat{k}} = \lambda_1^{\hat{k}}$. Therefore $\lambda_1^{\hat{k}} = -1$ implies $\{\lambda;\ \lambda^{\hat{k}} + 1 = 0\} \subseteq M(v)$, and we have established (3.17). The factorization (3.18) follows from (3.19) and (3.17). We can use Theorem 2.2(iv) again to show that $f(\mu) \neq 0$ if $|\mu| = 1$. □

If $1 \notin M(v)$ and $\hat{k}$ is given by (3.16) then we conclude from (3.17) that there exists a $\lambda \in M(v)$ with $\operatorname{ord}\lambda = 2\hat{k}$. Thus (3.6) and (3.19) imply

$$G(z)v = z^t\Big[Iz^{\ell\hat{k}} - \sum_{\nu=0}^{\ell-1} \epsilon(-1)^{\nu}|C_{t+\nu\hat{k}}| z^{\nu\hat{k}}\Big]v.$$

On the other hand, if $1 \in M(v)$ and $\hat{k}$ is given by (3.12), such that $\hat{k} = \operatorname{lcm}\{\operatorname{ord}\lambda;\ \lambda \in M(v)\}$, then (3.13) can be written as

$$G(z)v = z^t\Big[Iz^{\ell\hat{k}} - \sum_{\nu=0}^{\ell-1} |C_{t+\nu\hat{k}}| z^{\nu\hat{k}}\Big]v.$$

With these observations we can make Theorem 1.1 more precise.

THEOREM 3.9. *Let $g(z) = z^m - \sum_{i=0}^{m-1} c_i z^i$ be a complex polynomial. Set $s = \sum_{i=0}^{m-1} |c_i|$. Suppose $c_0 \neq 0$ and $s \leqslant 1$. Then $g(z)$ has a root on the unit circle if only if $s = 1$, and either $c_i \geqslant 0$ for all i and*

$$g(z) = z^{k\ell} - \big(|c_0| + |c_k|z^k + \dots + |c_{k(\ell-1)}|z^{(\ell-1)k}\big), \quad m = k\ell,$$

or otherwise $c_j < 0$ *for some* j *and*

$$g(z) = z^{k\ell} - \epsilon\big(|c_0| - |c_k|z^k + |c_{2k}|z^{2k} \mp \cdots + (-1)^{\ell-1}|c_{(\ell-1)k}|z^{(\ell-1)k}\big)$$

with $\epsilon = \operatorname{sign} c_0 = (-1)^{\ell}$.

According to H. Schneider [12] there is a striking similarity between properties of eigenvalues of p-norm contractive maps (see [8]) and of characteristic values of hermitian polynomial matrices $G(z)$ satisfying the condition $S \leqslant I$. For $1 \leqslant p < \infty$ the p-norm on $\mathbb{R}^n$ is defined by

$$|x|_p = \Big(\sum\nolimits_{j=1}^{n} |x_j|^p\Big)^{1/p}, \quad x = (x_1, \ldots, x_n)^T,$$

and the ∞-norm is given by $|x|_\infty = \max\{|x|_j; j = 1, \ldots, n\}$. For a matrix $A \in \mathbb{R}^{n\times n}$ let $\|A\|_p$ denote the corresponding operator norm, and let A be called p-norm contractive if $\|A\|_p \leqslant 1$. We refer to Lemmens and Van Gaans [8] for the following result.

THEOREM 3.10. *Let* $1 \leqslant p \leqslant \infty$ *and* $p \neq 2$. *If* $A \in \mathbb{R}^{n\times n}$ *is* p*-norm contractive then there exists a* $q \in \mathbb{N}$ *such that*

$$q \mid 2(n!) \quad \textit{and} \quad \lambda^q = 1 \quad \textit{for all} \quad \lambda \in \sigma(A) \cap \partial\mathbb{D}.$$

The sequence $(A^{qj})_{j\in\mathbb{N}}$ *is convergent.*

The corresponding result for $G(z)$ is the following.

THEOREM 3.11. *Assume* $S \leqslant I$. *Then there exists a* $q \in \mathbb{N}$ *such that*

$$q \mid 2(m!) \quad \textit{and} \quad \lambda^q = 1 \quad \textit{for all} \quad \lambda \in \sigma(G) \cap \partial\mathbb{D}. \tag{3.20}$$

Proof. If $\lambda \in \sigma(G) \cap \partial\mathbb{D}$ then Lemma 3.3 implies that the order of λ is equal to r or to $2r$ for some $r \in \{1, \ldots, m\}$. Hence

$$q = \operatorname{lcm}\{\operatorname{ord} \lambda \in \sigma(G) \cap \partial\mathbb{D}\} \tag{3.21}$$

satisfies (3.20). □

4. Applications

4.1. The Eneström–Kakeya theorem

According to Anderson, Saff and Varga [1] it is of interest to know when the inequality $\rho(h) \leqslant 1$ in Theorem 1.2 is sharp. We write $h \in \pi^+_{m-1}$ if the polynomial $h(z)$ in (1.5) satisfies (1.6).

THEOREM 4.1. ([7], [1]) *Let* $h(z) = \sum_{i=0}^{m-1} a_i z^i$ *be a real polynomial and suppose the coefficients* a_i *satisfy*

$$\begin{aligned} 0 < \ & a_0 = a_1 = \cdots = a_{r_1-1} \\ & \qquad < a_{r_1} = a_{r_1+1} = \cdots = a_{r_2-1} < \cdots \\ & \qquad\qquad < a_{r_s} = a_{r_s+1} = \cdots = a_{m-1}. \end{aligned} \tag{4.1}$$

Set $k = \gcd(m, r_1, \dots, r_s)$. *Then* $\rho(h) = 1$ *if and only if* $k > 1$. *In that case*

$$0 < a_0 = \cdots = a_{k-1} \leqslant a_k = \cdots = a_{2k} \leqslant \cdots \leqslant a_{m-k} = \cdots = a_{m-1},$$

and

$$h(z) = (1 + z + \cdots + z^{k-1})p(z^k), \quad p \in \pi^+_{\ell-1}, \quad \sigma(p) \cap \partial\mathbb{D} = \emptyset,$$

with $\ell = m/k$. *We have* $\sigma(h) \cap \partial\mathbb{D} = E_k \setminus \{1\}$.

The Eneström–Kakeya theorem and its refinement in Theorem 4.1 can be extended to polynomial matrices. Note that Furuta and Nakamura [5] generalized Theorem 1.2 (i) to polynomials $H(z) = \sum A_i z^i$ with positive definite operator coefficients A_i. The approach of [5] relies on a power inequality for the numerical radius of a linear operator acting on a Hilbert space. For the subsequent theorem we refer to [4]. In the present paper a different proof is given, which should be more straightforward.

THEOREM 4.2. *Let* $H(z) = A_{m-1}z^{m-1} + \cdots + A_1 z + A_0$ *be a polynomial matrix with hermitian coefficients* A_i *such that*

$$A_{m-1} > 0, \quad A_{m-1} \geqslant A_{m-2} \geqslant \cdots \geqslant A_0 \geqslant 0.$$

(i) *Then* $\rho(H) \leqslant 1$ *and* $1 \notin \sigma(H)$.

(ii) *If* $\lambda \in \sigma(H)$ *and* $|\lambda| = 1$ *then the corresponding elementary divisors of* $H(z)$ *are linear. Moreover,* $\lambda^k = 1$ *for some* k, $0 < k \leqslant m$. *If* $A_0 > 0$ *then* $\lambda^m = 1$.

Proof. From $A_{m-1} > 0$ follows $\sum A_i = H(1) > 0$. Therefore $1 \notin \sigma(H)$. Set $\tilde{A}_i = A_{m-1}^{-1/2} A_i A_{m-1}^{-1/2}$. Then

$$A_{m-1}^{-1/2} H(z) A_{m-1}^{-1/2} = I z^{m-1} + \sum\nolimits_{i=0}^{m-2} \tilde{A}_i z^i,$$

Thus we can assume $A_{m-1} = I$. Put $A_{-1} = 0$. Using the multiplier $(z-1)$ we define $G(z) = (z-1)H(z)$. Then $G(z) = Iz^m - \sum_{i=0}^m C_i z^i$ and

$$C_i = A_i - A_{i-1} \geqslant 0, \quad i = 0, \dots, m-1,$$

and $\sum C_i = I$. Moreover, $\sigma(H) = \sigma(G) \setminus \{1\}$. To complete the proof we apply Theorem 2.2 and Lemma 3.2. □

The following generalization of Theorem 4.1 deals with an eigenvector v of $H(z)$ and the corresponding characteristic values on the unit circle. With regard to (4.1) we make the assumptions $A_0 > 0$ and

$$\begin{aligned} A_0 v = \cdots = A_{r_1-1}v,\ \ A_{r_1-1}v \neq A_{r_1}v, \\ A_{r_1}v = \cdots = A_{r_2-1}v,\ \ A_{r_2-1}v \neq A_{r_2}v, \cdots, \\ A_{r_s-1}v \neq A_{r_s}v,\ \ A_{r_s}v = \cdots = A_{m-1}v. \end{aligned} \tag{4.2}$$

THEOREM 4.3. *Suppose* $H(\lambda)v = 0$, $v \neq 0$, *and* $|\lambda| = 1$. *Let* $r_1, \dots, r_s$, *be given by* (4.2). *Define* $\hat{k} = \gcd\{m, r_1, \dots, r_s\}$. *Then* $\lambda^{\hat{k}} = 1$, *and*

$$H(z)v = z^t(1 + z + \cdots + z^{\hat{k}-1}) f(z^{\hat{k}}) \tag{4.3}$$

where $f(z) \in \mathbb{C}^n[z]$ *and* $\sigma(f) \cap \partial\mathbb{D} = \emptyset$.

Proof. Again, it is no loss of generality to assume $A_{m-1} = I$. Because of $C_i = A_i - A_{i-1}$ the condition (4.2) means that $C_j v \neq 0$ only if $j \in \{m, r_1, \dots, r_s\}$. Then (3.14) in Theorem 3.6 yields

$$G(z)v = (z-1)H(z)v = z^t(z^{\hat{k}} - 1)f(z^{\hat{k}}),$$

which implies (4.3). □

4.2. A difference equation

We consider the linear time-invariant equation

$$x(t+m) = C_{m-1}x(t+m-1) + \cdots + C_1x(t+1) + C_0x(t), \tag{4.4a}$$

$$x(0) = x_0, \dots, x(m-1) = x_{m-1}. \tag{4.4b}$$

THEOREM 4.4. *Let $C_i \in \mathbb{C}^{n\times n}$, $i = 0, \dots, m-1$, be hermitian matrices and suppose $S = \sum |C_i| \leqslant I$.*

(i) *Then all solutions $\big(x(t)\big)$ of* (4.4) *are bounded for $t \to \infty$.*

(ii) *There exists a positive integer q such that $q \mid 2(m!)$ and the sequence $\big(x(qj)\big)$ is convergent.*

Proof. (i) It is well known (see e.g. [6]) that the solutions of (4.4) are bounded if and only if all characteristic values of $G(z) = Iz^m - \sum C_i z^i$ are in the closed unit disc and if those on the unit circle have linear elementary divisors. Hence stability of (4.4) follows immediately from Theorem 2.2.

(ii) Let q be given as in (3.21) such that (3.20) holds. If

$$C = \begin{pmatrix} C_{m-1} & \dots & C_2 & C_1 & C_0 \\ I & \dots & 0 & 0 & 0 \\ . & \dots & . & . & . \\ . & \dots & . & . & . \\ . & \dots & I & 0 & 0 \\ 0 & \dots & 0 & I & 0 \end{pmatrix}$$

is the block companion matrix associated with $G(z)$ then $\sigma(C) = \sigma(G)$, and C and $G(z)$ have the same elementary divisors. Set

$$y(t) = \big(x(t+m-1)^T, \dots, x(t+1)^T, x(t)^T\big)^T$$

and define y_0 conforming to (4.4b). Then (4.4) is equivalent to $y(t+1) = Cy(t)$, $y(0) = y_0$. The corresponding equation for $\big(w(j)\big) = \big(x(jq)\big)$ is $w(j+1) = C^q w(j)$. We have $\rho(C) \leqslant 1$ and $\lambda^q = 1$ for all $\lambda \in \sigma(C) \cap \partial\mathbb{D}$. Therefore $\sigma(C^q) \subseteq \{1\} \cup \mathbb{D}$. The matrix C^q is similar to $\operatorname{diag}(I, \hat{C})$ with $\sigma(\hat{C}) \subseteq \mathbb{D}$. Hence $\big(w(j)\big)$ is convergent. □

REFERENCES

[1] N. ANDERSON, E. B. SAFF, AND R. S. VARGA, *On the Eneström–Kakeya theorem and its sharpness*, Linear Algebra Appl. 28, 5–16 (1979).

[2] H. BAUMGÄRTEL, *Analytic Perturbation Theory for Matrices and Operators*, Operator Theory, Advances and Applications, Vol. 15, Birkhäuser, Basel, 1985.

[3] P. BORWEIN AND T. ERDÈLYI, *Polynomials and Polynomial Inequalities*, Springer-Verlag, New York, 1995.

[4] G. DIRR AND H. K. WIMMER, *An Eneström-Kakeya theorem for hermitian polynomial matrices*, IEEE Trans. Automat. Control 52, 2151–2153 (2007).

[5] T. FURUTA AND M. NAKAMURA, *An operator version of the Eneström-Kakeya theorem*, Mathem. Jap. 37, 459–497 (1992).

[6] I. GOHBERG, P. LANCASTER AND L. RODMAN, *Matrix Polynomials*, Academic Press, New York, 1982.

[7] A. HURWITZ, *Über einen Satz des Herrn Kakeya*, Tôhoku Math. J. 4, 89–93 (1913); in: Mathematische Werke von A. Hurwitz, 2. Band, 627–631, Birkhäuser, Basel, 1933.

[8] B. LEMMENS AND O. VAN GAANS, *Iteration of linear p-norm nonexpansive maps*, Linear Algebra Appl. 371, 265–276 (2003).

[9] A. M. OSTROWSKI, *Solutions of Equations in Euclidean and Banach Spaces*, Academic Press, New York, 1973.

[10] V. V. PRASOLOV, *Polynomials, Algorithms and Computation in Mathematics*, Vol. 11, Springer, New York, 2004.

[11] Q. I. RAHMAN AND G. SCHMEISSER, *Analytic Theory of Polynomials*, Oxford University Press, Oxford, 2002.

[12] H. SCHNEIDER, *A remark on a paper of Lemmens and Van Gaans*, private communication, Moscow, 2006.

[13] H. K. WIMMER, *Discrete-time stability of a class of hermitian polynomial matrices with positive semi-definite coefficients*, to appear in Matrix Methods: Theory and Algorithms, V. Olshevsky and E. Tyrtyshnikov, Editors, to be published by World Scientific.

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