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A NOTE ON ANISOTROPIC POTENTIALS ASSOCIATED WITH THE LAPLACE–BESSEL DIFFERENTIAL OPERATOR

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Abstract. In this note the anisotropic maximal operator and anisotropic Riesz potentials generated by the generalized shift operator are investigated in the anisotropic B-Morrey space $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. We prove that the anisotropic B-maximal operator M_γ is bounded on the anisotropic B-Morrey space $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. Also the anisotropic B-Riesz potential R^α_γ is bounded from the anisotropic B-Morrey spaces $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ if and only if $1/p-1/q=\alpha/(|a|+(a,\gamma)-\lambda)$ and $1<p<(|a|+(a,\gamma)-\lambda)/\alpha$, and its modified version $\widetilde{R}^\alpha_\gamma$ is bounded from the anisotropic B-Morrey space to the anisotropic B-BMO space. Furthermore, we obtain some imbedding relations between the space $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ and the anisotropic B-Stummel-Kato class $S_{p,\theta,\gamma}(\mathbb{R}^n_{k,+})$.

Introduction

The classical maximal operator and Riesz potentials are important technical tools in harmonic analysis, theory of functions and partial differential equations. The maximal operator, singular integrals, Riesz potential and related topics associated with the Laplace-Bessel differential operator

$$\Delta_B=\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}+\sum_{i=1}^{k}\frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i},\quad \gamma_i>0,\ \ i=1,\dots,k$$

have been investigated by many researchers, such as B. Muckenhoupt and E. Stein [20], I. Kipriyanov [17], I. Kipriyanov and M. Klyuchantsev [18], K. Trimeche [25], L. Lyakhov [16], K. Stempak [24], A. D. Gadjiev and I. A. Aliev [8, 2], I. A. Aliev and S. Bayrakci [3], A. Serbetci and I. Ekincioglu [23], V. S. Guliyev [10]–[12], V. S. Guliyev and J. J. Hasanov [13], [9] and others.

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In translations from Russian D. D. Gasanov and J. J. Hasanov are names of the same person.

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Morrey spaces play an important role in the theory of partial differential equations. The classical Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ was introduced by C. B. Morrey in [19] to study the local behavior of solutions to second order elliptic partial differential equations. In [6] F. Chiarenza and M. Frasca proved the boundedness of maximal operator M, and in [1] D. R. Adams proved the boundedness of the Riesz potential I^α on the classical Morrey spaces. The Morrey space $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ (B-Morrey space) associated with the Laplace-Bessel differential operator was studied by V. S. Guliyev in [11]. In [13] V. S. Guliyev and J. J. Hasanov proved the boundedness of the B-maximal operator M_γ and B-Riesz potential I^α_γ on the B-Morrey spaces $L_{p,\lambda,\gamma}(\mathbb{R}^n_+)$ in the isotropic case.

In this paper we deal with the anisotropic B-maximal operator M_γ and anisotropic B-Riesz potentials R^α_γ on the anisotropic B-Morrey spaces $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. If we take $a_i = 1$, $i = \overline{1,n}$ in the results obtained here we get the same results for the isotropic case. We prove that the anisotropic M_γ is bounded on the anisotropic $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $1 \leqslant p < \infty$. We obtain necessary and sufficient conditions for the operator R^α_γ to be bounded from anisotropic $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ and from anisotropic $L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to weak anisotropic B-Morrey space $WL_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. Also we prove that the operator $\widetilde{R}^\alpha_\gamma$ is bounded in the anisotropic $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to anisotropic B-BMO spaces. Furthermore, we establish some imbedding relations between the spaces anisotropic $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ and the anisotropic B-Stummel-Kato class $S_{p,\theta,\gamma}(\mathbb{R}^n_{k,+})$. We prove that $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ is contained in $S_{p,\theta,\gamma}(\mathbb{R}^n_{k,+})$ if $|a|+(a,\gamma)-\theta < \lambda < |a|+(a,\gamma)$, while, in the case $\varphi(t) \sim t^\delta$ the belonging of f to $S_{p,\theta,\gamma}(\mathbb{R}^n_{k,+})$ is equivalent to $f \in L_{p,|a|+(a,\gamma)-\theta+\delta,\gamma}(\mathbb{R}^n_{k,+})$.

1. Preliminaries

Let $\mathbb{R}^n_{k,+} = \{x = (x_1,...,x_n) \in \mathbb{R}^n : x_1 > 0,\dots,x_k > 0,\ 1 \leqslant k \leqslant n\}$, and $\gamma = (\gamma_1,\dots,\gamma_k)$, $\gamma_1 > 0,\dots,\gamma_k > 0$, $a = (a_1,\dots,a_n) \in (0,\infty)^n$, $|a| = \sum_{i=1}^n a_i$, $(a,\gamma) = \sum_{i=1}^k a_i\gamma_i$. For $x \neq 0$, let $|x|_a$ be a positive solution to the equation $\sum_{j=1}^n x_j^2 |x|_a^{-2a_j} = 1$. Note that $|x|_a$ is equivalent to $\sum_{j=1}^n |x_j|^{\frac{1}{a_j}}$, i.e.,

$$c_1|x|_a \leqslant \sum_{j=1}^n |x_j|^{\frac{1}{a_j}} \leqslant c_2|x|_a$$

for certain positive c_1 and c_2 (see [4]).

For a measurable set $E \subset \mathbb{R}^n_{k,+}$ let $E(x,r) = \{y \in \mathbb{R}^n_{k,+} : |x-y|_a < r\}$, $E(0,r) = E_r$, and $|E_r|_\gamma = \int_{E_r}(x')^\gamma dx$, then $|E_r|_\gamma = \omega(n,k,\gamma) r^{|a|+(a,\gamma)}$, where

$$\omega(n,k,\gamma) = \int_{E_1}(x')^\gamma dx = 2^{-k}\pi^{\frac{n-k}{2}}\,\Gamma^{-1}\left(\frac{|a|+(a,\gamma)+2}{2}\right)\prod_{i=1}^k \Gamma\left(\frac{\gamma_i+1}{2}\right).$$

The generalized shift operator T^y is defined by (see, for example [15, 17])

$$T^y f(x) = C_{\gamma,k}\int_0^\pi \dots \int_0^\pi f\left((x',y')_\beta, x''-y''\right)\, d\nu(\beta),$$

where $d\nu(\beta) = \prod_{i=1}^{k} \sin^{\gamma_i - 1}\beta_i \quad d\beta_1 \dots d\beta_k$, $x' = (x_1, \dots, x_k) \in \mathbb{R}^k$, $x'' = (x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-k}$, $(x_i, y_i)_{\beta_i} = (x_i^2 - 2x_i y_i \cos\beta_i + y_i^2)^{\frac{1}{2}}$, $1 \leqslant i \leqslant k$, $(x', y')_\beta = \big((x_1, y_1)_{\beta_1}, \dots, (x_k, y_k)_{\beta_k}\big)$ and $C_{\gamma,k} = \pi^{-\frac{k}{2}} \prod_{i=1}^{k} \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}$.

Let $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ be the space of measurable functions on $\mathbb{R}^n_{k,+}$ with finite norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})} = \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx\right)^{1/p}, \quad 1 \leqslant p < \infty.$$

For $p = \infty$ the space $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ is defined by

$$\|f\|_{L_{\infty,\gamma}} = \|f\|_{L_\infty} = \operatorname*{esssup}_{x \in \mathbb{R}^n_{k,+}} |f(x)|.$$

Let $1 \leqslant p < \infty$. By $WL_{p,\gamma}(\mathbb{R}^n_{k,+})$ we denote the weak $L_{p,\gamma}$ spaces defined as the set of locally integrable functions f with the finite norms

$$\|f\|_{WL_{p,\gamma}} = \sup_{r>0} r \left|\left\{x \in \mathbb{R}^n_{k,+} \ : \ |f(x)| > r\right\}\right|_\gamma^{1/p}.$$

It is well known that T^y is closely related to the Laplace-Bessel differential operator $\Delta_B = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \sum_{i=1}^{k} \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$. Furthermore, T^y generates the corresponding B-convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y)\, T^y g(x) (y')^\gamma dy$$

for which the Young inequality

$$\|f \otimes g\|_{r,\gamma} \leqslant \|f\|_{p,\gamma} \|g\|_{q,\gamma}, \quad 1 \leqslant p, q, r \leqslant \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

holds.

LEMMA 1. *Let $1 \leqslant p \leqslant \infty$. Then for all $y \in \mathbb{R}^n_{k,+}$*

$$\|T^y f(\cdot)\|_{L_{p,\gamma}} \leqslant \|f\|_{L_{p,\gamma}}.$$

Proof. Firstly, we prove the lemma in the case $p = \infty$:

$$|T^y f(x)| \leqslant C_\gamma \int_0^\pi \left| f\left(x' - y', \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\alpha}\right)\right| \sin^{\gamma-1}\alpha d\alpha$$

$$\leqslant \|f\|_{L_\infty\left(\mathbb{R}^n_{k,+}\right)} C_\gamma \int_0^\pi \sin^{\gamma-1}\alpha d\alpha = \|f\|_{L_\infty\left(\mathbb{R}^n_{k,+}\right)}.$$

Then

$$\|T^y f\|_{L_{\infty,\gamma}\left(\mathbb{R}^n_{k,+}\right)} = \|T^y f\|_{L_{\infty}\left(\mathbb{R}^n_{k,+}\right)} \leqslant \|f\|_{L_{\infty}\left(\mathbb{R}^n_{k,+}\right)} .$$

Secondly,let $p = 1$. Then we have

$$\|T^y f\|_{L_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right)} = \int\limits_{\mathbb{R}^n_{k,+}} |T^y f(x)|\,(x')^\gamma dx \leqslant \int\limits_{\mathbb{R}^n_{k,+}} T^y\,|f(x)|\,(x')^\gamma dx$$

$$= \int\limits_{\mathbb{R}^n_{k,+}} |f(x)|\,(x')^\gamma dx = \|f\|_{L_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right)} .$$

Applying the Riesz-Thorin theorem, for all $y \in \mathbb{R}^n_{k,+}$ we get

$$\|T^y f(\cdot)\|_{L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)} \leqslant \|f\|_{L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)} .$$

□

LEMMA 2. *For all $x \in \mathbb{R}^n_{k,+}$ the following equality is valid*

$$\int_{E(x,t)} g(y)(y')^\gamma dy = C^{-1}_{\gamma,k} \int_{B((x,0),t)} g\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z''\right) d\mu(z,\overline{z'}),$$

where $B((x,0),t) = \{(z,\overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k \,:\, |(x_1 - \sqrt{z_1^2+\overline{z}_1^2},\ldots,x_k - \sqrt{z_k^2+\overline{z}_k^2}, x'' - z'')|_a < t\}$, $d\mu(z,\overline{z'}) = (\overline{z'})^{\gamma-1} dz d\overline{z'}$, $d\overline{z'} = d\overline{z}_1 \cdots d\overline{z}_k$, $(\overline{z'})^{\gamma-1} = (\overline{z}_1)^{\gamma_1 - 1}\cdots(\overline{z}_k)^{\gamma_k-1}$.

LEMMA 3. *For all $x \in \mathbb{R}^n_{k,+}$ the following equality is valid*

$$\int_{E_t} T^y g(x)(y')^\gamma dy = \int_{E((x,0),t)} g\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z''\right) d\mu(z,\overline{z'}),$$

where $E((x,0),t) = \{(z,\overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k \,:\, \left|\left(x - z, \overline{z'}\right)\right|_a < t\}$.

The proof of Lemmas 2, 3 is straightforward via the following substitutions

$$z'' = x'', z_i = x_i \cos\alpha_i,\;\; \overline{z_i} = x_i \sin\alpha_i,\;\; 0 \leqslant \alpha_i < \pi,\; i = 1,\ldots,k,$$
$$x \in \mathbb{R}^n_{k,+},\; \overline{z'} = (\overline{z}_1,\ldots,\overline{z}_k),\; (z,\overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k,\; 1 \leqslant k \leqslant n.$$

LEMMA 4. *Let $0 < \alpha < |a| + (a,\gamma)$. Then for $2|x|_a \leqslant |y|_a$ the following inequality is valid*

$$\left|T^y|x|_a^{\alpha-|a|-(a,\gamma)} - |y|_a^{\alpha-|a|-(a,\gamma)}\right| \leqslant 2^{|a|+(a,\gamma)-\alpha+1}|y|_a^{\alpha-|a|-(a,\gamma)-1}|x|_a. \qquad (1)$$

Proof. We will show that

$$\left|T^y|x|_a^{\alpha-|a|-(a,\gamma)} - |y|_a^{\alpha-|a|-(a,\gamma)}\right|$$
$$\leqslant C_\gamma \int_0^\pi \left|\left|\left((x',y')_\beta, x''-y''\right)\right|_a^{\alpha-|a|-(a,\gamma)} - |y|_a^{\alpha-|a|-(a,\gamma)}\right|\, d\nu(\beta).$$

First estimate

$$\left| \left| \left((x',y')_\beta, x''-y'' \right) \right|_a^{\alpha-|a|-(a,\gamma)} - |y|_a^{\alpha-|a|-(a,\gamma)} \right|.$$

By the theorem about mean value we get

$$\begin{aligned} &\left| \left| \left((x',y')_\beta, x''-y'' \right) \right|_a^{\alpha-|a|-(a,\gamma)} - |y|_a^{\alpha-|a|-(a,\gamma)} \right| \\ \leqslant &\left| \left| \left((x',y')_\beta, x''-y'' \right) \right|_a^{\alpha-|a|-(a,\gamma)} - |y|_a \right| \cdot \xi^{\alpha-|a|-(a,\gamma)-1}, \end{aligned}$$

where $\min\left\{ \left| \left((x',y')_\beta, x''-y'' \right) \right|_a, |y|_a \right\} \leqslant \xi \leqslant \max\left\{ \left| \left((x',y')_\beta, x''-y'' \right) \right|_a, |y|_a \right\}.$

Note that

$$\left| \left((x',y')_\beta, x''-y'' \right) \right|_a \leqslant |x|_a + |y|_a \leqslant \frac{3}{2}|y|_a,$$

$$\left| \left((x',y')_\beta, x''-y'' \right) \right|_a \geqslant |x-y|_a \geqslant |y|_a - |x|_a \geqslant \frac{1}{2}|y|_a$$

and

$$\begin{aligned} &\left| \left((x',y')_\beta, x''-y'' \right) \right|_a - |y|_a \leqslant |x|_a + |y|_a - |y|_a \leqslant |x|_a \\ &|y|_a - \left| \left((x',y')_\beta, x''-y'' \right) \right|_a \leqslant |y|_a - |x-y|_a \leqslant |x|_a. \end{aligned}$$

Hence

$$\frac{1}{2}|y|_a \leqslant \left| \left((x',y')_\beta, x''-y'' \right) \right|_a \leqslant \frac{3}{2}|y|_a,$$

$$\left| \left| \left((x',y')_\beta, x''-y'' \right) \right|_a - |y|_a \right| \leqslant |x|_a.$$

Thus we obtain (1). □

We define the anisotropic B-maximal operator (see [10]) as

$$M_\gamma f(x) = \sup_{r>0} |E_r|_\gamma^{-1} \int_{E_r} T^y |f(x)| (y')^\gamma dy.$$

Consider anisotropic B-Riesz potential as

$$R_\gamma^\alpha f(x) = \int_{\mathbb{R}_{k,+}^n} T^y |x|_a^{\alpha-|a|-(a,\gamma)} f(y) (y')^\gamma dy, \quad 0 < \alpha < |a| + (a,\gamma)$$

and isotropic B-Riesz potential as

$$I_\gamma^\alpha f(x) = \int_{\mathbb{R}_{k,+}^n} T^y |x|^{\alpha-n-|\gamma|} f(y) (y')^\gamma dy, \quad 0 < \alpha < n + |\gamma|$$

The modified anisotropic B-Riesz potential

$$\widetilde{R}_\gamma^\alpha f(x) = \int_{\mathbb{R}_{k,+}^n} \left(T^y |x|_a^{\alpha-|a|-(a,\gamma)} - |y|_a^{\alpha-|a|-(a,\gamma)} \chi_{E_1^*}(y) \right) f(y) (y')^\gamma dy,$$

where $E_1^* = \mathbb{R}^n_{k,+} \backslash E_1$.

Let $\Delta_B = \sum\limits_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \sum\limits_{i=1}^{k} \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$, $\gamma_i > 0$, $i = 1, \ldots, k$. The following theorems are true.

THEOREM 1. [18] *If α-is an even non-negative integer, $f(x)$-is a finite, even by the variables $x_1, ..., x_k$ function having $\alpha/2$ continuous derivatives by the variables $x_{k+1}, ..., x_n$ and α are continuous derivatives by $x_1, ..., x_k$, then the potential $I_\gamma^\alpha f(x)$ is a solution of the equation*

$$\triangle_B^{\alpha/2} u(x) = f(x).$$

DEFINITION 1. [10] Let $1 \leqslant p < \infty$, $0 \leqslant \lambda \leqslant |a| + (a,\gamma)$. We denote by $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ the anisotropic B-Morrey spaces as the set of locally integrable functions f with finite norm

$$\|f\|_{L_{p,\lambda,\gamma}} = \sup_{t>0,\, x\in\mathbb{R}^n_{k,+}} \left(t^{-\lambda} \int_{E_t} (T^y |f(x)|)^p (y')^\gamma dy \right)^{1/p}.$$

Note that $L_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\gamma}(\mathbb{R}^n_{k,+})$, $L_{p,|a|+(a,\gamma),\gamma}\left(\mathbb{R}^n_{k,+}\right) = L_\infty\left(\mathbb{R}^n_{k,+}\right)$.

DEFINITION 2. [13] Let $1 \leqslant p < \infty, 0 \leqslant \lambda \leqslant |a| + (a,\gamma)$. We denote by $WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ the weak anisotropic B-Morrey spaces as the set of locally integrable functions f with finite norm

$$\|f\|_{WL_{p,\lambda,\gamma}} = \sup_{r>0} r \sup_{t>0,\, x\in\mathbb{R}^n_{k,+}} \left(t^{-\lambda} \int_{\{y\in E_t:\, T^y|f(x)|>r\}} (y')^\gamma dy \right)^{1/p}.$$

Note that

$$WL_{p,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,0,\gamma}(\mathbb{R}^n_{k,+}), L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \subset WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$$

and

$$\|f\|_{WL_{p,\lambda,\gamma}} \leqslant \|f\|_{L_{p,\lambda,\gamma}}.$$

DEFINITION 3. [10] We denote by $BMO_\gamma(\mathbb{R}^n_{k,+})$, the B-BMO space the set of locally integrable functions f with finite norms

$$\|f\|_{*,\gamma} = \sup_{r>0,\ x\in\mathbb{R}^n_{k,+}} |E_r|_\gamma^{-1} \int_{E_r} |T^y f(x) - f_{E_r}(x)|(y')^\gamma dy < \infty,$$

where $f_{E_r}(x) = |E_r|_\gamma^{-1} \int_{E_r} T^y f(x)(y')^\gamma dy$.

Following [22] ([21]) we define the anisotropic B-Stummel-Kato class:

DEFINITION 4. Let $1 < \theta < |a| + (a,\gamma)$, $1 \leqslant p < \infty$, then the anisotropic B-Stummel-Kato class $S_{p,\theta,\gamma}(\mathbb{R}^n_{k,+})$ is defined by

$$S_{p,\theta,\gamma}(\mathbb{R}^n_{k,+}) = \left\{ f \in L_{1,\gamma}^{loc}(\mathbb{R}^n_{k,+}) :\ \lim_{t\to 0} \varphi(t) = 0 \right\},$$

where $\varphi(t) = \sup\limits_{x\in\mathbb{R}^n_{k,+}} \left(\int_{E_t} \frac{\left(T^y|f(x)|\right)^p}{|y|^{|a|+(a,\gamma)-\theta}} (y')^\gamma dy \right)^{\frac{1}{p}}$.

Note that L_1^{loc} contains so called Stummel-Kato class $S_{1,\theta}$.

2. Statement of main results

The following theorem gives the anisotropic $L_{p,\lambda,\gamma}$-boundedness of the anisotropic B-maximal operator.

THEOREM 2. *1) If* $f \in L_{1,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, $0 \leqslant \lambda \leqslant |a| + (a,\gamma)$, *then* $M_\gamma f \in WL_{1,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ *and*

$$\|M_\gamma f\|_{WL_{1,\lambda,\gamma}} \leqslant C\|f\|_{L_{1,\lambda,\gamma}},$$

where C *is independent of* f.

2) If $f \in L_{p,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, $1 < p < \infty, 0 \leqslant \lambda \leqslant |a| + (a,\gamma)$, *then* $M_\gamma f \in L_{p,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ *and*

$$\|M_\gamma f\|_{L_{p,\lambda,\gamma}} \leqslant C_{p,\gamma}\|f\|_{L_{p,\lambda,\gamma}},$$

where $C_{p,\gamma}$ *depends only on* p, γ *and* n.

COROLLARY 1. [9, 11] *1) If* $f \in L_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, *then* $M_\gamma f \in WL_{1,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|M_\gamma f\|_{WL_{1,\gamma}} \leqslant C\|f\|_{L_{1,\gamma}},$$

where C *is independent of* f.

2) If $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p \leqslant \infty$, *then* $M_\gamma f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|M_\gamma f\|_{L_{p,\gamma}} \leqslant C_{p,\gamma}\|f\|_{L_{p,\gamma}},$$

where $C_{p,\gamma}$ *depends only on* p, γ *and* n.

For the anisotropic B-Riesz potentials the following generalized Hardy–Littlewood–Sobolev theorem is valid.

THEOREM 3. *Let* $0 < \alpha < |a| + (a,\gamma)$, $0 \leqslant \lambda \leqslant |a| + (a,\gamma)$.

1) If $1 < p < \frac{|a|+(a,\gamma)-\lambda}{\alpha}$, *then the condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|a|+(a,\gamma)-\lambda}$ *is necessary and sufficient for the boundedness of* R^α_γ *from* $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

2) If $p = 1$, *then the condition* $1 - \frac{1}{q} = \frac{\alpha}{|a|+(a,\gamma)-\lambda}$ *is necessary and sufficient for the boundedness of* R^α_γ *from* $L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ *to* $WL_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

COROLLARY 2. *Let* $0 < \alpha < |a| + (a,\gamma)$.

1) If $1 < p < \frac{|a|+(a,\gamma)-\lambda}{\alpha}$, *then the condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|a|+(a,\gamma)}$ *is necessary and sufficient for the boundedness of* R^α_γ *from* $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{q,\gamma}(\mathbb{R}^n_{k,+})$.

2) If $p = 1$, *then the condition* $1 - \frac{1}{q} = \frac{\alpha}{|a|+(a,\gamma)}$ *is necessary and sufficient for the boundedness of* R^α_γ *from* $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ *to* $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.

REMARK 1. Note that, the sufficiency part of Corollary 2 in the case $n \geqslant 1$, $k = 1$ was proved in [9], (in the isotropic case or for $a_i = 1$, $i = \overline{1,n}$ the sufficiency part of Theorem 3 was proved in [13], [14]) and in the case $n = k \geqslant 1$ in [11].

THEOREM 4. *Let* $0 < \alpha < |a| + (a,\gamma)$, $0 \leqslant \lambda < |a| + (a,\gamma) - \alpha$ *and* $1 < p = \frac{|a|+(a,\gamma)-\lambda}{\alpha}$, *then the operator* $\widetilde{R}^{\alpha}_{\gamma}$ *is bounded from* $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ *to* $BMO_{\gamma}(\mathbb{R}^n_{k,+})$.

Moreover, if for $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $R^{\alpha}_{\gamma}f$ *exists almost everywhere, then* $R^{\alpha}_{\gamma} \in BMO_{\gamma}(\mathbb{R}^n_{k,+})$ *and the following inequality is valid*

$$\|R^{\alpha}_{\gamma}f\|_{BMO_{\gamma}} \leqslant C\|f\|_{L_{p,\lambda,\gamma}},$$

where $C > 0$ *is independent of* f.

In the following two theorems we give some imbedding relations between the space $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ and the anisotropic B-Stummel-Kato class $S_{p,\theta,\gamma}(\mathbb{R}^n_{k,+})$.

THEOREM 5. *Let* $1 < \theta < |a| + (a,\gamma)$, $1 \leqslant p < \infty$, $|a| + (a,\gamma) - \theta < \lambda < |a| + (a,\gamma)$. *If* $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ *then* $f \in S_{p,\theta,\gamma}(\mathbb{R}^n_{k,+})$.

THEOREM 6. *Let* $1 < \theta < |a| + (a,\gamma)$, $1 \leqslant p < \infty$. *If* $f \in S_{p,\theta,\gamma}(\mathbb{R}^n_{k,+})$, $\varphi(t) \sim t^{\delta}$, *then* $f \in L_{p,|a|+(a,\gamma)-\theta+\delta,\gamma}(\mathbb{R}^n_{k,+})$.

From theorem 1 and theorem 3 we have

THEOREM 7. *Let* $0 < \alpha < n + |\gamma|$, $0 \leqslant \lambda \leqslant n + |\gamma|$.

1) If $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \frac{n+|\gamma|-\lambda}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+|\gamma|-\lambda}$. *Then, the following estimation holds:*

$$\|u\|_{L_{q,\lambda,\gamma}} \leqslant C_{p,\lambda} \, \|\Delta_B u\|_{L_{p,\lambda,\gamma}},$$

where $C_{p,\lambda}$ *is independent of* f.

2) If $f \in L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $1 - \frac{1}{q} = \frac{\alpha}{n+|\gamma|-\lambda}$. *Then, the following estimation holds:*

$$\|u\|_{WL_{q,\lambda,\gamma}} \leqslant C_{\lambda} \, \|\Delta_B u\|_{L_{1,\lambda,\gamma}},$$

where C_{λ} *is independent of* f.

3. Proof of Theorems

Proof of Theorem 2. We need to introduce the maximal operator defined on a space of homogeneous type (Y,d,ν). By this we mean a topological space $Y = \mathbb{R}^n \times (0,\infty)^k$ equipped with a continuous pseudometric d and a positive measure ν satisfying

$$\nu(E((x,\overline{x'}),2r)) \leqslant C_1 \nu(E((x,\overline{x'}),r)) \tag{2}$$

with a constant C_1 independent of $(x,\overline{x'})$ and $r > 0$. Here $E((x,\overline{x'}),r) = \{(y,\overline{y'}) \in Y : d(((x,\overline{x'}),(y,\overline{y'})) < r\}$, $d\nu(y,\overline{y'}) = (\overline{y'})^{\gamma-1}dy\,d\overline{y'}$, $(\overline{y'})^{\gamma-1} = (\overline{y_1})^{\gamma_1-1}\cdots(\overline{y_k})^{\gamma_k-1}$, $d((x,\overline{x'}),(y,\overline{y'})) = |(x,\overline{x'}) - (y,\overline{y'})|_a \equiv (|x-y|_a^2 + |\overline{x'} - \overline{y'}|_{a'}^2)^{\frac{1}{2}}$.

Let (Y,d,ν) be a space of homogeneous type. Define

$$M_{\nu}\overline{f}(x,\overline{x'}) = \sup_{r>0} \nu(E((x,\overline{x'}),r))^{-1} \int_{E((x,\overline{x'}),r)} \left|\overline{f}(y,\overline{y'})\right| d\nu(y),$$

where $\overline{f}(x,\overline{x'}) = f\left(\sqrt{x_1^2 + \overline{x}_1^2}, \ldots, \sqrt{x_k^2 + \overline{x}_k^2}, x''\right)$.

It is well known that the maximal operator M_ν is of weak type $(1,\ 1)$ and is bounded on $L_p(Y,d\nu)$ for $1<p<\infty$ (see [5]). Here we concern with the maximal operator defined by using the measure $d\nu(y,\overline{y'})=(\overline{y'})^{\gamma-1}dy\,d\overline{y'}$. It is clear that this measure satisfies the (2) doubling condition.

It can be proved that

$$M_\gamma f\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z''\right)=M_\nu\overline{f}\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\right), \quad (3)$$

and

$$M_\gamma f(x)=M_\nu\overline{f}(x,0). \quad (4)$$

Indeed, by the Lemma 3 the equalities

$$\int_{E_r}T^y\left|f\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z''\right)\right|(y')^\gamma\,dy$$
$$=\int\limits_{E\left(\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\right),r\right)}\left|\overline{f}(y,\overline{y'})\right|\,d\nu(y,\overline{y'})$$

and

$$|E_r|_\gamma=\nu E\left(\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\right),r\right)$$

imply (3). Furthermore, taking $\overline{z}_k=0$ in (3) we get (4).

Using Lemma 3 and equality (3) we obtain

$$\int_{E_r}\left(T^yM_\gamma f(x)\right)^p(y')^\gamma\,dy$$
$$=\int_{E((x,0),r)}\left(M_\gamma f\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z''\right)\right)^p\,d\nu(z,\overline{z'})$$
$$=\int_{E((x,0),r)}\left(M_\nu\overline{f}\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\right)\right)^p\,d\nu(z,\overline{z'}).$$

In [7] it was proved that the analogue of the Fefferman-Stein theorem for the maximal operator defined on a space of homogeneous type is valid, if condition (2) is satisfied. Therefore

$$\int_{E((x,\overline{x'}),r)}\left(M_\nu\varphi(y,\overline{y'})\right)^p\psi(y,\overline{y'})d\nu(y,\overline{y'})$$
$$\leqslant C_2\int_{E((x,\overline{x'}),r)}|\varphi(y,\overline{y'})|^pM_\nu\psi(y,\overline{y'})\,d\nu(y,\overline{y'}). \quad (5)$$

Then taking $\varphi(y,\overline{y'})=\overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)$ and $\psi(y,\overline{y'})\equiv 1$ we obtain from inequality (5) and Lemma 3 that

$$\int_{E_r}\left(T^yM_\gamma f(x)\right)^p(y')^\gamma dy$$

$$= \int_{E((x,0),r)} \left(M_\nu \overline{f} \left(\sqrt{y_1^2+\overline{y}_1^2}, \ldots, \sqrt{y_k^2+\overline{y}_k^2}, y'', 0 \right) \right)^p \, d\nu(y,\overline{y'})$$
$$\leqslant C_3 \int_{E((x,0),r)} \left| \overline{f} \left(\sqrt{y_1^2+\overline{y}_1^2}, \ldots, \sqrt{y_k^2+\overline{y}_k^2}, y'', 0 \right) \right|^p \, d\nu(y,\overline{y'})$$
$$= C_3 \int_{E((x,0),r)} \left| f \left(\sqrt{y_1^2+\overline{y}_1^2}, \ldots, \sqrt{y_k^2+\overline{y}_k^2}, y'' \right) \right|^p \, d\nu(y,\overline{y'})$$
$$= C_3 \int_{E_r} (T^y |f(x)|)^p (y')^\gamma dy \leqslant C_3 \, r^\lambda \, \|f\|_{L_{p,\lambda,\gamma}}^p . \qquad \square$$

Proof of Theorem 3. 1) *Sufficiency*: Let $f \in L_{p,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, then we can write

$$\begin{aligned} R^\alpha_\gamma f(x) &= \left(\int_{E_t} + \int_{\mathbb{R}^n_{k,+} \setminus E_t} \right) T^y f(x) |y|_a^{\alpha-|a|-(a,\gamma)} (y')^\gamma dy \\ &\equiv F_1(x,t) + F_2(x,t). \end{aligned} \tag{6}$$

Firstly, we estimate $F_1(x,t)$:

$$\begin{aligned} |F_1(x,t)| &\leqslant \int_{E_t} T^y |f(x)| \, |y|_a^{\alpha-|a|-(a,\gamma)} (y')^\gamma dy \\ &\leqslant \sum_{k=-\infty}^{-1} \left(2^k t\right)^{\alpha-|a|-(a,\gamma)} \int_{E_{2^{k+1}t} \setminus E_{2^k t}} T^y |f(x)| \, (y')^\gamma dy \leqslant C_4 t^\alpha M_\gamma f(x). \end{aligned}$$

We find the following inequality

$$|F_1(x,t)| \leqslant C_4 t^\alpha M_\gamma f(x). \tag{7}$$

To estimate $F_2(x,t)$ we use Hölder's inequality and get the following inequality

$$\begin{aligned} |F_2(x,t)| &\leqslant \left(\int_{\mathbb{R}^n_{k,+} \setminus E_t} |y|_a^{-\beta} (T^y |f(x)|)^p (y')^\gamma dy \right)^{1/p} \\ &\times \left(\int_{\mathbb{R}^n_{k,+} \setminus E_t} |y|_a^{\left(\frac{\beta}{p}+\alpha-|a|-(a,\gamma)\right)p'} (y')^\gamma dy \right)^{1/p'} = J_1 \cdot J_2. \end{aligned}$$

Let $\lambda < \beta < |a| + (a,\gamma) - \alpha p$. For J_1 we have

$$\begin{aligned} J_1 &= \left(\sum_{j=0}^{\infty} \int_{E_{2^{j+1}t} \setminus E_{2^j t}} (T^y |f(x)|)^p |y|_a^{-\beta} (y')^\gamma dy \right)^{1/p} \\ &\leqslant 2^{\frac{\lambda}{p}} t^{\frac{\lambda-\beta}{p}} \|f\|_{L_{p,\lambda,\gamma}} \left(\sum_{j=0}^{\infty} 2^{(\lambda-\beta)j} \right)^{1/p} = C_5 t^{\frac{\lambda-\beta}{p}} \|f\|_{L_{p,\lambda,\gamma}} . \end{aligned} \tag{8}$$

For J_2 we have

$$J_2 = \left(\int_{\mathbb{S}^{n-1}_{k,+}} \sum_{i=1}^{n} a_i \xi_i^2 (\xi')^{\gamma} d\sigma(\xi) \int_t^{\infty} r^{|a|+(a,\gamma)-1+\left(\frac{\beta}{p}+\alpha-|a|-(a,\gamma)\right)p'} dr \right)^{\frac{1}{p'}}$$
$$= C_6 t^{\frac{\beta}{p}+\alpha-\frac{|a|+(a,\gamma)}{p}}.$$

Then

$$|F_2(x,t)| \leqslant C_6 t^{\alpha-\frac{|a|+(a,\gamma)-\lambda}{p}} \|f\|_{L_{p,\lambda,\gamma}}. \tag{9}$$

From (7) and (9) we have

$$\left|R_\gamma^\alpha f(x)\right| \leqslant C_4 t^\alpha M_\gamma f(x) + C_6 t^{\alpha-\frac{|a|+(a,\gamma)-\lambda}{p}} \|f\|_{L_{p,\lambda,\gamma,a}}.$$

Minimizing with respect to t, at $t = \left[\left(M_\gamma f(x)\right)^{-1} \|f\|_{L_{p,\lambda,\gamma}}\right]^{p/(|a|+(a,\gamma)-\lambda)}$ we obtain

$$\left|R_\gamma^\alpha f\ (x)\right| \leqslant C_7 \left(M_\gamma f(x)\right)^{p/q} \|f\|_{L_{p,\lambda,\gamma}}^{1-p/q}.$$

Hence, by Theorem 2 we find

$$\int_{E_t} \left(T^y \left|R_\gamma^\alpha f(x)\right|\right)^q (y')^\gamma dy \leqslant C_8 \|f\|_{L_{p,\lambda,\gamma}}^{q-p} \int_{E_t} \left(T^y M_\gamma f(y)\right)^p (y')^\gamma dy$$
$$\leqslant C_9 t^\lambda \|f\|_{L_{p,\lambda,\gamma}}^{q-p} \|f\|_{L_{p,\lambda,\gamma}}^{p} \leqslant C_9 t^\lambda \|f\|_{L_{p,\lambda,\gamma}}^{q}.$$

Necessity: Let $1 < p < \frac{|a|+(a,\gamma)-\lambda}{\alpha}$ and R_γ^α be bounded from $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Define $f_t(x) =: f(tx)$. Then

$$\|f_t\|_{L_{p,\lambda,\gamma}} = t^{-\frac{|a|+(a,\gamma)}{p}} \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left(r^{-\lambda} \int_{E_{tr}} (T^y|f(tx)|)^p (y')^\gamma dy\right)^{1/p}$$
$$= t^{-\frac{|a|+(a,\gamma)-\lambda}{p}} \|f\|_{L_{p,\lambda,\gamma}}$$

and

$$R_\gamma^\alpha f_t(x) = t^{-\alpha} R_\gamma^\alpha f(tx),$$
$$\left\|R_\gamma^\alpha f_t\right\|_{L_{q,\lambda,\gamma}} = t^{-\alpha} \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left(r^{-\lambda} \int_{E_r} \left(T^{ty} \left|R_\gamma^\alpha f(tx)\right|\right)^q (y')^\gamma dy\right)^{1/q}$$
$$= t^{-\alpha-\frac{|a|+(a,\gamma)}{q}} \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left(r^{-\lambda} \int_{E_{tr}} \left(T^y \left|R_\gamma^\alpha f(x)\right|\right)^q (y')^\gamma dy\right)^{1/q}$$
$$= t^{-\alpha-\frac{|a|+(a,\gamma)-\lambda}{q}} \left\|R_\gamma^\alpha f\right\|_{L_{q,\lambda,\gamma}}.$$

By the boundedness R_γ^α from $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$

$$\left\|R_\gamma^\alpha f\right\|_{L_{q,\lambda,\gamma}} \leqslant C_{p,q,\lambda,\gamma} t^{\alpha+\frac{|a|+(a,\gamma)-\lambda}{q}-\frac{|a|+(a,\gamma)-\lambda}{p}} \|f\|_{L_{p,\lambda,\gamma}},$$

where $C_{p,q,\lambda,\gamma}$ depends only on p, q, λ, γ, k and n.

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{|a|+(a,\gamma)-\lambda}$, then for all $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, we obtain $\left\|R_\gamma^\alpha f\right\|_{L_{q,\lambda,\gamma}} = 0$ as $t \to 0$.

If $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{|a|+(a,\gamma)-\lambda}$, then for all $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ we obtain $\left\|R_\gamma^\alpha f\right\|_{L_{q,\lambda,\gamma}} = 0$ as $t \to \infty$.

Therefore we get $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{|a|+(a,\gamma)-\lambda}$.

2) *Sufficiency*: Let $f \in L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, then we have

$$\begin{aligned}\left|\left\{y \in E_t \,:\, T^y \left|R_\gamma^\alpha f(x)\right| > 2\beta\right\}\right|_\gamma &\leqslant |\{y \in E_t \,:\, T^y |F_1(x,t)| > \beta\}|_\gamma \\ &\quad + |\{y \in E_t \,:\, T^y |F_2(x,t)| > \beta\}|_\gamma .\end{aligned}$$

Taking into account inequality (7) and Theorem 2, we have

$$\begin{aligned}|\{y \in E_t \,:\, T^y |F_1(x,t)| > \beta\}|_\gamma &\leqslant \left|\left\{y \in E_t \,:\, T^y \left(M_\gamma f(x)\right) > \frac{\beta}{C_3 t^\alpha}\right\}\right|_\gamma \\ &\leqslant \frac{C_{10} t^\alpha}{\beta} \cdot t^\lambda \|f\|_{L_{1,\lambda,\gamma}} .\end{aligned}$$

Thus if we choose $C_6 t^{-\frac{|a|+(a,\gamma)-\lambda}{q}} \|f\|_{L_{1,\lambda,\gamma}} = \beta$, then $|F_2(x,t)| \leqslant \beta$ and consequently we get $|\{y \in E_t \,:\, T^y |F_2(x,t)| > \beta\}|_\gamma = 0$.

Finally, we obtain

$$\left|\left\{y \in E_t \,:\, T^y \left|R_\gamma^\alpha f(x)\right| > 2\beta\right\}\right|_\gamma$$

$$\leqslant C_{10} t^{\alpha+\lambda} \frac{\|f\|_{L_{1,\lambda,\gamma}}}{\beta} = C_{11} t^\lambda \left(\frac{\|f\|_{L_{1,\lambda,\gamma}}}{\beta}\right)^{\frac{q}{|a|+(a,\gamma)-\lambda}+1} = C_{11} t^\lambda \left(\frac{\|f\|_{L_{1,\lambda,\gamma}}}{\beta}\right)^q .$$

Necessity: Let R_γ^α be bounded from $L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. Then we have

$$\begin{aligned}\left\|R_\gamma^\alpha f_t\right\|_{WL_{q,\lambda,\gamma}} &= \sup_{r>0} r \sup_{\tau>0,\, x\in\mathbb{R}^n_{k,+}} \left(\tau^{-\lambda} \int_{\{y\in E_\tau \,:\, T^y|R_\gamma^\alpha f_t(x)|>r\}} (y')^\gamma dy\right)^{1/q} \\ &= t^{-\alpha} \sup_{r>0} r t^\alpha \sup_{\tau>0,\, x\in\mathbb{R}^n_{k,+}} \left(\tau^{-\lambda} \int_{\{y\in E_\tau \,:\, T^{ty}|R_\gamma^\alpha f(tx)|>rt^\alpha\}} (y')^\gamma dy\right)^{1/q} \\ &= t^{-\alpha-\frac{|a|+(a,\gamma)}{q}} \sup_{r>0} r t^\alpha \sup_{\tau>0,\, x\in\mathbb{R}^n_{k,+}} \left(t^\lambda (t\tau)^{-\lambda} \int_{\{y\in E_{t\tau}: T^y|R_\gamma^\alpha f(x)|>rt^\alpha\}} (y')^\gamma dy\right)^{1/q} \\ &= t^{-\alpha-\frac{|a|+(a,\gamma)-\lambda}{q}} \left\|R_\gamma^\alpha f\right\|_{WL_{q,\lambda,\gamma}} .\end{aligned}$$

From the boundedness of R_γ^α from $L_{1,\lambda,\gamma}(\mathbb{R}_{k,+}^n)$ to $WL_{q,\lambda,\gamma}(\mathbb{R}_{k,+}^n)$ we obtain

$$\left\|R_\gamma^\alpha f\right\|_{WL_{q,\lambda,\gamma}} \leqslant C_{1,q,\lambda,\gamma} t^{\alpha+\frac{|a|+(a,\gamma)-\lambda}{q}-(|a|+(a,\gamma)-\lambda)} \|f\|_{L_{1,\lambda,\gamma}},$$

where $C_{1,q,\lambda,\gamma}$ depends only on q, λ, γ, k and n.

If $1 < \frac{1}{q} + \frac{\alpha}{|a|+(a,\gamma)-\lambda}$, then for all $f \in L_{1,\lambda,\gamma}(\mathbb{R}_{k,+}^n)$ we obtain $\left\|R_\gamma^\alpha f\right\|_{WL_{q,\lambda,\gamma}} = 0$ as $t \to 0$.

Similarly, if $1 > \frac{1}{q} + \frac{\alpha}{|a|+(a,\gamma)-\lambda}$, then for all $f \in L_{1,\lambda,\gamma}(\mathbb{R}_{k,+}^n)$ we obtain $\left\|R_\gamma^\alpha f\right\|_{WL_{q,\lambda,\gamma}} = 0$ as $t \to \infty$.

Therefore we get $1 = \frac{1}{q} + \frac{\alpha}{|a|+(a,\gamma)-\lambda}$. $\square$

Proof of Theorem 4. $f \in L_{p,\lambda,\gamma}\left(\mathbb{R}_{k,+}^n\right)$, $1 < p = \frac{|a|+(a,\gamma)-\lambda}{\alpha}$. For any $t > 0$ we denote

$$f_1(z) = f(z)\chi_{E_{2t}}(z), \quad f_2(z) = f(z) - f_1(z), \tag{10}$$

where $\chi_{E_{2t}}$ is the characteristic function of the set E_{2t}. Then we have

$$\widetilde{R}_\gamma^\alpha f(z) = \widetilde{R}_\gamma^\alpha f_1(z) + \widetilde{R}_\gamma^\alpha f_2(z) = F_1(z) + F_2(z),$$

where

$$F_1(z) = \int_{E_{2t}} \left(T^y |z|_a^{\alpha-|a|-(a,\gamma)} - |y|_a^{\alpha-|a|-(a,\gamma)} \chi_{E_1^*}(y)\right) f(y)(y')^\gamma dy,$$

$$F_2(z) = \int_{\mathbb{R}_{k,+}^n \setminus E_{2t}} \left(T^y |z|_a^{\alpha-|a|-(a,\gamma)} - |y|_a^{\alpha-|a|-(a,\gamma)} \chi_{E_1^*}(y)\right) f(y)(y')^\gamma dy.$$

Note that the function f_1 has compact (bounded) support and therefore

$$a_1 = -\int_{E_{2t} \setminus E_{\min\{1,2t\}}} |y|_a^{\alpha-|a|-(a,\gamma)} f(y)(y')^\gamma dy$$

is finite.

Note also that

$$\begin{aligned} F_1(z) - a_1 &= \int_{E_{2t}} T^y |z|_a^{\alpha-|a|-(a,\gamma)} f(y)(y')^\gamma dy \\ &\quad - \int_{E_{2t} \setminus E_{\min\{1,2t\}}} |y|_a^{\alpha-|a|-(a,\gamma)} f(y)(y')^\gamma dy + \int_{E_{2t} \setminus E_{\min\{1,2t\}}} |y|_a^{\alpha-|a|-(a,\gamma)} f(y)(y')^\gamma dy \\ &= \int_{\mathbb{R}_{k,+}^n} T^y |z|_a^{\alpha-|a|-(a,\gamma)} f_1(y)(y')^\gamma dy = R_\gamma^\alpha f_1(z). \end{aligned}$$

Therefore

$$|F_1(z) - a_1| \leqslant \int\limits_{\mathbb{R}^n_{k,+}} |y|_a^{\alpha-|a|-(a,\gamma)} |T^y f_1(z)| (y')^\gamma dy$$

$$= \int\limits_{\{y\in\mathbb{R}^n_{k,+} : T^y|z|_a<2t\}} |y|_a^{\alpha-|a|-(a,\gamma)} |T^y f(z)| (y')^\gamma dy.$$

Further, for $z \in E_t$, $T^y|z|_a < 2t$ we have

$$|y|_a \leqslant |z|_a + |z-y|_a \leqslant |z|_a + T^y|z|_a < 3t.$$

Consequently

$$|F_1(z) - a_1| \leqslant \int_{E_{3t}} |y|_a^{\alpha-|a|-(a,\gamma)} T^y |f(z)| (y')^\gamma dy, \tag{11}$$

if $z \in E_t$.

By the Theorem 1 and the inequality (11), for $\alpha p = |a| + (a,\gamma) - \lambda$ we obtain

$$|E_t|_\gamma^{-1} \int\limits_{E_t} |T^x F_1(z) - a_1| (z')^\gamma dz$$

$$\leqslant |E_t|_\gamma^{-1} \int\limits_{E_t} T^x \left(\int\limits_{E_{3t}} |y|_a^{\alpha-|a|-(a,\gamma)} T^y |f(z)| (y')^\gamma dy \right) (z')^\gamma dz$$

$$\leqslant \frac{2^{|a|+(a,\gamma)-\alpha} 3^\alpha}{2^\alpha - 1} t^{\alpha-|a|-(a,\gamma)} \cdot t^{(|a|+(a,\gamma))/p'} \left(\int\limits_{E_t} \left(T^x M_\gamma(f(z))\right)^p (z')^\gamma dz \right)^{1/p}$$

$$\leqslant C_{12} \|f\|_{L_{p,\lambda,\gamma}}. \tag{12}$$

Denote

$$a_2 = \int\limits_{E_{\max\{1,2t\}} \setminus E_{2t}} |y|_a^{\alpha-|a|-(a,\gamma)} f(y) (y')^\gamma dy.$$

We estimate $|F_2(z) - a_2|$ for $z \in E_t$.

$$|F_2(z) - a_2| \leqslant \int\limits_{\mathbb{R}^n_{k,+} \setminus E_{2t}} |f(y)| \left| T^y|z|_a^{\alpha-|a|-(a,\gamma)} - |y|_a^{\alpha-|a|-(a,\gamma)} \right| (y')^\gamma dy.$$

Applying Lemma 4 and Hölder's inequality we have

$$
\begin{aligned}
|F_2(z) - a_2| &\leqslant 2^{|a|+(a,\gamma)-\alpha+1}|z|_a \int\limits_{\mathbb{R}^n_{k,+}\setminus E_{2t}} |f(y)||y|_a^{\alpha-|a|-(a,\gamma)-1}(y')^\gamma dy \\
&\leqslant 2^{|a|+(a,\gamma)-\alpha+1}|z|_a \left(\int_{\mathbb{R}^n_{k,+}\setminus E_t} |y|_a^{-\beta} |f(y)|^p (y')^\gamma dy\right)^{1/p} \\
&\quad \times \left(\int_{\mathbb{R}^n_{k,+}\setminus E_t} |y|_a^{\left(\frac{\beta}{p}+\alpha-|a|-(a,\gamma)-1\right)p'} (y')^\gamma dy\right)^{1/p'} \\
&= 2^{|a|+(a,\gamma)-\alpha+1}|z|_a I_1 \cdot I_2.
\end{aligned}
$$

Let $\lambda < \beta < |a| + (a,\gamma) - \alpha p + p$. For I_1 the inequality (8) is valid. For I_2 we obtain

$$
\begin{aligned}
I_2 &= \left(\int_{\mathbb{S}^{n-1}_{k,+}} \sum_{i=1}^n a_i \xi_i^2 (\xi')^\gamma d\sigma(\xi) \int_t^\infty r^{|a|+(a,\gamma)-1+\left(\frac{\beta}{p}+\alpha-|a|-(a,\gamma)-1\right)p'} dr\right)^{\frac{1}{p'}} \\
&= C_{13} t^{\frac{\beta}{p}+\alpha-\frac{|a|+(a,\gamma)}{p}-\frac{1}{p'}}.
\end{aligned}
$$

Then for any $z \in E_t$

$$
|F_2(z) - a_2| \leqslant C_{14}|z|_a t^{\alpha-1-\frac{|a|+(a,\gamma)-\lambda}{p}} \|f\|_{L_{p,\lambda,\gamma}} \leqslant C_{14}|z|_a t^{-1}\|f\|_{L_{p,\lambda,\gamma}} \leqslant C_{14}\|f\|_{L_{p,\lambda,\gamma}}.
$$

Thus for $\alpha p = |a| + (a,\gamma) - \lambda$ and for all $x \in \mathbb{R}^n_{k,+}$, $z \in E_t$ we obtain

$$
|T^x F_2(z) - a_2| \leqslant T^x |F_2(z) - a_2| \leqslant C_{14}\|f\|_{L_{p,\lambda,\gamma}}. \tag{13}
$$

Denote

$$
a_f = a_1 + a_2 = \int\limits_{E_{\max\{1,2t\}}} |y|_a^{\alpha-|a|-(a,\gamma)} f(y)(y')^\gamma dy.
$$

Finally, from (12) and (13) we have

$$
\sup_{x,t} |E_t|_\gamma^{-1} \int\limits_{E_t} \left|T^x \widetilde{R}^\alpha_\gamma f(z) - a_f\right| (z')^\gamma dz \leqslant (C_{12} + C_{14})\|f\|_{L_{p,\lambda,\gamma}}.
$$

Thus

$$
\left\|\widetilde{R}^\alpha_\gamma f\right\|_{BMO_\gamma} \leqslant 2\sup_{x,t} |E_t|_\gamma^{-1} \int\limits_{E_t} \left|T^z \widetilde{R}^\alpha_\gamma f(x) - a_f\right| (z')^\gamma dz \leqslant C_{15}\|f\|_{L_{p,\lambda,\gamma}}. \quad \square
$$

Proof of Theorem 5. Let $f\in L_{p,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$. Then we have

$$\int_{E_t}\frac{(T^y|f(x)|)^p}{|y|_a^{|a|+(a,\gamma)-\theta}}(y')^\gamma dy=\sum_{k=-\infty}^{-1}\int_{E_{2^{k+1}t}\setminus E_{2^k t}}\frac{(T^y|f(x)|)^p}{|y|_a^{|a|+(a,\gamma)-\theta}}(y')^\gamma dy$$
$$\leqslant\sum_{k=-\infty}^{-1}\left(2^k t\right)^{\theta-|a|-(a,\gamma)}\int_{E_{2^{k+1}t}\setminus E_{2^k t}}(T^y|f(x)|)^p(y')^\gamma dy$$
$$\leqslant\|f\|^p_{L_{p,\lambda,\gamma}}\sum_{k=-\infty}^{-1}\left(2^k t\right)^{\theta-|a|-(a,\gamma)+\lambda}$$
$$\leqslant C_{16}t^{\theta-|a|-(a,\gamma)+\lambda}\,\|f\|^p_{L_{p,\lambda,\gamma}}.$$

Thus the proof of the theorem is completed. □

Proof of Theorem 6. Let $f\in S_{p,\theta,\gamma}\left(\mathbb{R}^n_{k,+}\right)$. Then we have

$$\int_{E_t}(T^y|f(x)|)^p(y')^\gamma dy\leqslant t^{|a|+(a,\gamma)-\theta}\int_{E_t}\frac{(T^y|f(x)|)^p}{|y|_a^{|a|+(a,\gamma)-\theta}}(y')^\gamma dy$$
$$\leqslant t^{|a|+(a,\gamma)-\theta}\varphi(t)\leqslant t^{|a|+(a,\gamma)-\theta+\delta}.$$

This completes the proof. □

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