

SELF-ADJOINT EXTENSIONS OF RESTRICTIONS

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Abstract. We provide a simple recipe for obtaining all self-adjoint extensions, together with their resolvent, of the symmetric operator S obtained by restricting the self-adjoint operator $A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ to the dense, closed with respect to the graph norm, subspace $\mathcal{N} \subset \mathcal{D}(A)$. Neither the knowledge of S^* nor of the deficiency spaces of S is required. Typically A is a differential operator and \mathcal{N} is the kernel of some trace (restriction) operator along a null subset. We parametrise the extensions by the bundle $p : \mathbf{E}(\mathfrak{h}) \rightarrow \mathbf{P}(\mathfrak{h})$, where $\mathbf{P}(\mathfrak{h})$ denotes the set of orthogonal projections in the Hilbert space $\mathfrak{h} \simeq \mathcal{D}(A)/\mathcal{N}$ and $p^{-1}(\Pi)$ is the set of self-adjoint operators in the range of Π . The set of self-adjoint operators in \mathfrak{h} , i.e. $p^{-1}(1)$, parametrises the relatively prime extensions. Any $(\Pi, \Theta) \in \mathbf{E}(\mathfrak{h})$ determines a boundary condition in the domain of the corresponding extension $A_{\Pi, \Theta}$ and explicitly appears in the formula for the resolvent $(-A_{\Pi, \Theta} + z)^{-1}$. The connection with both von Neumann's and Boundary Triples theories of self-adjoint extensions is explained. Some examples related to quantum graphs, to Schrödinger operators with point interactions and to elliptic boundary value problems are given.

1. Introduction

On the Hilbert space \mathcal{H} with scalar product $\langle \cdot, \cdot \rangle$ we consider the self-adjoint operator

$$A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}.$$

We denote by \mathcal{H}_A the Hilbert space given by the operator domain $\mathcal{D}(A)$ endowed with the graph inner product

$$\langle \phi, \psi \rangle_A := \langle A\phi, A\psi \rangle + \langle \phi, \psi \rangle.$$

Given a closed subspace $\mathcal{N} \subset \mathcal{H}_A$ which is dense in \mathcal{H} , we denote by S the closed, densely defined, symmetric operator obtained by restricting A to \mathcal{N} . Here our aim is to find all self-adjoint extensions of S and to provide their resolvent.

Since \mathcal{N} is closed we have $\mathcal{H}_A = \mathcal{N} \oplus \mathcal{N}^\perp$ and thus \mathcal{N} coincides with the kernel of the orthogonal projection onto \mathcal{N}^\perp . Since $\mathcal{N}^\perp \simeq \mathcal{H}_A/\mathcal{N}$ is a Hilbert

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space, without loss of generality we can suppose that \mathcal{N} coincides with the kernel of a surjective bounded linear operator

$$\tau : \mathcal{H}_A \rightarrow \mathfrak{h}$$

with \mathfrak{h} a Hilbert space. This choice has some advantages in practical applications, where usually τ is given in advance. Indeed typically A is a differential operator and τ is some trace (restriction) operator along a null subset.

In Section 2, by using the results in [25], we construct, by explicitly giving their resolvents (see Theorem 2.1), a family of self-adjoint extensions of the symmetric operator S . Such extensions are parametrised by couples (Π, Θ) , where Π is a orthogonal projection in \mathfrak{h} and Θ is a self-adjoint operator in the range of Π . The resolvent Kreĭn-like formula (2.7) we provide (see [18], [19], [31] for the original Kreĭn's formula; also see [11] and references therein) resembles the one obtained, by Boundary Triple theory, in [22], Corollary 5.6, for the case of a dual pair of operators, and in [2], [24] for the case of a single symmetric operator. There a resolvent formula is given in terms of a couple (B_1, B_2) of bounded linear operators which satisfy a commutativity hypothesis and a non-degeneracy one (see (4.1) and (4.2) in Remark 4.4), while here we impose no further conditions on the couple (Π, Θ) .

Then, by using the results in [27], we give (see Theorem 2.4) an alternative description which shows how the couple (Π, Θ) induces a boundary condition in the operator domain of the corresponding extension. Again this has connections with [22], [2] and [24]. We conclude Section 2 by giving (see Theorem 2.5) an additive representation for the self-adjoint extensions of the symmetric operator S . The analogous result in the case of relatively prime extensions was obtained in [27].

In Section 3 we explore the connection with von Neumann's theory of self-adjoint extensions [23]. By extending the results in [27], Section 4, we explicitly provide (see Theorem 3.1) a bijection from unitary operators $U : \mathcal{H}_+ \rightarrow \mathcal{H}_-$ (\mathcal{H}_\pm denoting the defect spaces of S) to couples (Π, Θ) in such a way that $A_U = A_{\Pi, \Theta}$, where A_U is the extension given by von Neumann's Theory and $A_{\Pi, \Theta}$ is the extension given in Theorem 2.1. This show, as a byproduct, that our construction provides all the self-adjoint extensions of S (see Corollary 3.2). Thus the whole set of self-adjoint extensions of the symmetric operator S is parametrised by the bundle $p : \mathbb{E}(\mathfrak{h}) \rightarrow \mathbb{P}(\mathfrak{h})$, where $\mathbb{P}(\mathfrak{h})$ denotes the set of orthogonal projections in \mathfrak{h} and $p^{-1}(\Pi)$ is the set of self-adjoint operators in the range of Π . This kind of parametrisation is compatible with the one obtained, in the case A is injective with a bounded inverse, in [13], Theorem II 2.1 (also see [32], Section 1). We refer to Example 5.5 in Section 5 below for more details in the case of applications to elliptic boundary value problems.

In Section 4 we explore, by using the results in [28], the connection with Boundary Triples theory of self-adjoint extensions (see e.g. [4], [16], [12], [10], [5]). In particular we recover Theorem 5.3 in [3] by which any self-adjoint relation in $\mathfrak{h} \oplus \mathfrak{h}$ is of the kind $\mathcal{G}(\Theta) \oplus \mathfrak{h}_0^\perp$, where $\mathfrak{h}_0 \subseteq \mathfrak{h}$ is a closed subspace, Θ is some self-adjoint operator in \mathfrak{h}_0 and $\mathcal{G}(\Theta)$ denotes its graph. The connection with different parametrisations of the set of self-adjoint relations is explicitly given in Theorem 4.5. This provides the bridge between our Kreĭn-like formula (2.7) and the one given in [22], [2] and [24].

Finally, in Section 5, we give some applications by examples related to quantum graphs, to Schrödinger operators with point interactions and to boundary value problems for the Laplace operator on bounded domains.

2. Self-adjoint Extensions

In the following, given a linear operator L we denote by

$$\mathcal{D}(L), \quad \mathcal{K}(L), \quad \mathcal{R}(L), \quad \rho(L)$$

its domain, kernel, range and resolvent set respectively.

Let

$$A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H},$$

$$\tau : \mathcal{H}_A \rightarrow \mathfrak{h}, \quad \mathcal{R}(\tau) = \mathfrak{h}, \quad \overline{\mathcal{K}(\tau)} = \mathcal{H},$$

and

$$S : \mathcal{K}(\tau) \subseteq \mathcal{H} \rightarrow \mathcal{H}$$

be the respectively self-adjoint, bounded and symmetric operators considered in the introduction.

For any z in $\rho(A)$ the linear operator

$$R(z) := (-A + z)^{-1}$$

is bounded on \mathcal{H} onto \mathcal{H}_A . Thus for any $z \in \rho(A)$ we can define the bounded operator

$$G(z) := (\tau R(\bar{z}))^* : \mathfrak{h} \rightarrow \mathcal{H}$$

(here the $*$ denotes the Hilbert space adjoint). The surjectivity of τ makes $G(z)$ injective. By [27], Lemma 2.1, one has that, given the surjectivity hypothesis $\mathcal{R}(\tau) = \mathfrak{h}$, the density one $\overline{\mathcal{K}(\tau)} = \mathcal{H}$ is equivalent to

$$\mathcal{R}(G(z)) \cap \mathcal{D}(A) = \{0\}. \tag{2.1}$$

By the first resolvent identity one easily obtains (see [25], Lemma 2.1), for any z and w in $\rho(A)$,

$$(z - w)R(w)G(z) = G(w) - G(z), \tag{2.2}$$

thus

$$\mathcal{R}(G(w) - G(z)) \subseteq \mathcal{D}(A). \tag{2.3}$$

Let $\Gamma(z) : \mathfrak{h} \rightarrow \mathfrak{h}$, $z \in \rho(A)$, be a family of bounded linear operators such that

$$\Gamma(z) - \Gamma(w) = (z - w)G(\bar{w})^*G(z) \tag{2.4}$$

and

$$\Gamma(z)^* = \Gamma(\bar{z}). \tag{2.5}$$

The class of such families is not void. Indeed by (2.2) and the definition of $\Gamma(z)$ it is easy to check (see [25], Lemma 2.2, for the details) that any of such a family differs by a z -independent bounded self-adjoint operator from the family $\hat{\Gamma}_{(w)}(z)$ defined by

$$\hat{\Gamma}_{(w)}(z) := \tau \left(\frac{G(w) + G(\bar{w})}{2} - G(z) \right), \quad w \in \rho(A). \tag{2.6}$$

Note that $\hat{\Gamma}_{(w)}(z)$ is well defined by (2.3).

Given the orthogonal projection

$$\Pi : \mathfrak{h} \rightarrow \mathfrak{h},$$

pose

$$\mathfrak{h}_0 := \mathcal{R}(\Pi)$$

and let

$$\Theta : \mathcal{D}(\Theta) \subseteq \mathfrak{h}_0 \rightarrow \mathfrak{h}_0$$

be self-adjoint. Then we define the closed operator

$$\Gamma_{\Pi, \Theta}(z) := (\Theta + \Pi \Gamma(z) \Pi) : \mathcal{D}(\Theta) \subseteq \mathfrak{h}_0 \rightarrow \mathfrak{h}_0,$$

and the open set

$$Z_{\Pi, \Theta} := \{z \in \rho(A) : 0 \in \rho(\Gamma_{\Pi, \Theta}(z))\}.$$

The next theorem is nothing but the combination of Theorem 2.1 and Proposition 2.1 in [25] when the bounded and surjective linear map there denoted by τ is given by $\Pi \tau : \mathcal{H}_A \rightarrow \mathfrak{h}_0$. We give a short self-contained proof for the reader's convenience.

THEOREM 2.1. *Let A, τ, S, Π, Θ and $\Gamma_{\Pi, \Theta}$ be as above. Then*

$$\mathbb{C} \setminus \mathbb{R} \subseteq Z_{\Pi, \Theta}$$

and the bounded linear operator

$$R_{\Pi, \Theta}(z) := R(z) + G(z) \Pi \Gamma_{\Pi, \Theta}(z)^{-1} \Pi G(\bar{z})^*, \quad z \in Z_{\Pi, \Theta}, \tag{2.7}$$

is a resolvent of the self-adjoint extension $A_{\Pi, \Theta}$ of S defined by

$$A_{\Pi, \Theta} : \mathcal{D}(A_{\Pi, \Theta}) \subseteq \mathcal{H} \rightarrow \mathcal{H}, \quad (-A_{\Pi, \Theta} + z)\phi := (-A + z)\phi_z,$$

$$\mathcal{D}(A_{\Pi, \Theta}) := \{ \phi \in \mathcal{H} : \phi = \phi_z + G(z) \Pi \Gamma_{\Pi, \Theta}(z)^{-1} \Pi \tau \phi_z, \phi_z \in \mathcal{D}(A) \}.$$

Such a definition is z -independent and the decomposition of ϕ appearing in $\mathcal{D}(A_{\Pi, \Theta})$ is unique.

Proof. By (2.4) and (2.5), denoting by (\cdot, \cdot) the scalar product in \mathfrak{h} and by $\|\cdot\|$ the norm in \mathcal{H} , one has

$$|(\zeta_0, \Gamma_{\Pi, \Theta}(z)\zeta_0)|^2 \geq \text{Im}(z)^2 \|G(z)\zeta_0\|^4$$

for any $\zeta_0 \in \mathfrak{h}_0$. Thus $\Gamma_{\Pi, \Theta}(z)$ is injective for all $z \in \mathbb{C} \setminus \mathbb{R}$ by the injectivity of $G(z)$. Since

$$\mathcal{R}(\Gamma_{\Pi, \Theta}(z))^\perp = \mathcal{H}(\Gamma_{\Pi, \Theta}(z)^*) = \mathcal{H}(\Gamma_{\Pi, \Theta}(\bar{z})) = \{0\},$$

the range of $\Gamma_{\Pi,\Theta}(z)$ is dense. Since $G(z)\Pi = (\Pi\tau R(\bar{z}))^* : \mathfrak{h}_0 \rightarrow \mathcal{H}$, the range of $G(z)\Pi$ is closed by the surjectivity of both Π , τ , $R(\bar{z})$ and by the closed range theorem. Thus

$$\gamma := \inf_{\zeta_0 \in \mathfrak{h}_0 \setminus \{0\}} \frac{\|G(z)\zeta_0\|}{\|\zeta_0\|} > 0.$$

Therefore

$$\inf_{\zeta_0 \in \mathfrak{h}_0 \setminus \{0\}} \frac{\|\Gamma_{\Pi,\Theta}(z)\zeta_0\|}{\|\zeta_0\|} \geq \inf_{\zeta_0 \in \mathfrak{h}_0 \setminus \{0\}} \frac{|(\zeta_0, \Gamma_{\Pi,\Theta}(z)\zeta_0)|}{\|\zeta_0\|^2} \geq |\operatorname{Im}(z)|\gamma^2 > 0$$

and the range of $\Gamma_{\Pi,\Theta}(z)$ is closed. Thus for any $z \in \mathbb{C} \setminus \mathbb{R}$ the closed operator $\Gamma_{\Pi,\Theta}(z)$ is both injective and surjective. By the inverse mapping theorem $\mathbb{C} \setminus \mathbb{R} \subseteq Z_{\Pi,\Theta}$.

By using (2.4), a simple computation (see [25], page 115) shows that $R_{\Pi,\Theta}(z)$ satisfies the resolvent identity

$$(z - w)R_{\Pi,\Theta}(w)R_{\Pi,\Theta}(z) = R_{\Pi,\Theta}(w) - R_{\Pi,\Theta}(z) \tag{2.8}$$

and, by (2.5),

$$R_{\Pi,\Theta}(z)^* = R_{\Pi,\Theta}(\bar{z}). \tag{2.9}$$

Moreover, by (2.1), $R_{\Pi,\Theta}(z)$ is injective. Thus

$$A_{\Pi,\Theta} := z - \overline{R_{\Pi,\Theta}(z)^{-1}}$$

is well-defined on

$$\mathcal{D}(A_{\Pi,\Theta}) := \mathcal{R}(R_{\Pi,\Theta}(z)),$$

is z -independent by (2.8) and is symmetric by (2.9). It is self-adjoint since $\mathcal{R}(-A_{\Pi,\Theta} \pm i) = \mathcal{H}$ by construction. \square

REMARK 2.2. By the successive results in Section 4 and by [10], Propositions 1 and 2, Section 2, one has that

$$\lambda \in \sigma_p(A_{\Pi,\Theta}) \cap \rho(A) \iff 0 \in \sigma_p(\Gamma_{\Pi,\Theta}(\lambda)),$$

where $\sigma_p(\cdot)$ denotes point spectrum. An analogous result holds for the continuous spectrum. Regarding the eventual eigenvectors and their multiplicity, by [28], Theorem 3.4, one has that

$$G(\lambda) : \mathcal{K}(\Gamma_{\Pi,\Theta}(\lambda)) \rightarrow \mathcal{K}(-A_{\Pi,\Theta} + \lambda)$$

is a bijection for any $\lambda \in \sigma_p(A_{\Pi,\Theta}) \cap \rho(A)$.

Now we provide an alternative description of the self-adjoint extensions obtained in the previous theorem. This result will show how the couple (Π, Θ) induces a boundary conditions on the elements in the operator domain of the corresponding extension.

Since

$$\Gamma(z) - \hat{\Gamma}(z) = \hat{\Theta}, \quad \hat{\Gamma}(z) := \hat{\Gamma}_{(i)}(z),$$

where $\hat{\Theta}$ is a z -independent bounded symmetric operator in \mathfrak{h} , one has

$$\Gamma_{\Pi,\Theta}(z) = (\Theta + \Pi\hat{\Theta}\Pi) + \Pi\hat{\Gamma}(z)\Pi.$$

Since Θ is an arbitrary self-adjoint operator in \mathfrak{h}_0 , from now on we will take, without loss of generality,

$$\Gamma_{\Pi,\Theta}(z) = \hat{\Gamma}_{\Pi,\Theta}(z) := \Theta + \Pi \hat{\Gamma}(z) \Pi. \tag{2.10}$$

Let us define

$$\begin{aligned} R &\equiv R_+ := R(i), \quad R_- := R(-i), \\ G_+ &:= G(i), \quad G \equiv G_- := G(-i), \\ G_* &:= \frac{1}{2} (G_- + G_+). \end{aligned}$$

Thus

$$\hat{\Gamma}(z) = \tau(G_* - G(z)).$$

REMARK 2.3. The choice $w = i$ in the above definitions is not essential. Any different $w \in \mathbb{C} \setminus \mathbb{R}$ would lead to analogous results. Whereas the behaviour of a single extension depends on the choice of the family $\Gamma(z)$, and hence depends on the choice of w , the whole family of extensions does not. In any case one can easily connect any two parametrisations provided by different families $\Gamma_1(z)$ and $\Gamma_2(z)$: it suffices to use the substitution

$$\Theta \leftrightarrow \Theta + \Pi(\Gamma_1(z) - \Gamma_2(z))\Pi.$$

By choosing $w \in \mathbb{C} \setminus \mathbb{R}$ (in particular $w = i$) we can treat the case of an arbitrary self-adjoint extension without making w $Z_{\Pi,\Theta}$ -dependent. In the case one works with a specific operator, different (more appealing) choices are possible. Indeed we can interpret Section 5 below (where A is the Laplacian) as a proof, by examples, of a different version of next Theorem 2.4 valid in the case $w = 0$. In examples 5.1, 5.2 and 5.5 one has $w = 0 \in \rho(A)$ whereas in examples 5.3 and 5.4 $w = 0 \in \sigma(A) \setminus \sigma_p(A)$, thus showing that there are situations in which w it not even required to be in $\rho(A)$ (see [26] for a study of the case $w = 0 \in \sigma(A) \setminus \sigma_p(A)$ in a general setting. By Remark 3.3 in [29] the hypotheses on τ required in [26] can be relaxed, thus allowing wider applications).

The next theorem is nothing but Theorem in [27] (also see [26], Corollary 3.2) when one uses the bounded and surjective map (there denoted by τ) given by $\Pi\tau : \mathcal{H}_A \rightarrow \mathfrak{h}_0$ and notes that, by (2.2),

$$G_* = -iRG + G.$$

Note that in the definition of $\mathcal{D}(A)$ now appears the boundary condition

$$\Pi\tau\phi_* = \Theta\zeta_\phi.$$

THEOREM 2.4. *The self-adjoint extension $A_{\Pi,\Theta}$ can be re-written as*

$$\begin{aligned} A_{\Pi,\Theta} : \mathcal{D}(A_{\Pi,\Theta}) \subseteq \mathcal{H} &\rightarrow \mathcal{H}, \quad A_{\Pi,\Theta}\phi = A\phi_* + RG\zeta_\phi, \\ \mathcal{D}(A_{\Pi,\Theta}) &= \{ \phi \in \mathcal{H} : \phi = \phi_* + G_*\zeta_\phi, \\ \phi_* \in \mathcal{D}(A), \zeta_\phi \in \mathcal{D}(\Theta), \Pi\tau\phi_* &= \Theta\zeta_\phi \}. \end{aligned}$$

We conclude this section by giving an additive representation for the self-adjoint extensions of the symmetric operator S . Let us denote by \mathcal{H}_A^* the Hilbert space obtained by completing \mathcal{H} with respect to the scalar product

$$[\phi, \psi]_A := \langle \phi, (A^2 + 1)^{-1} \psi \rangle \equiv \langle R\phi, R\psi \rangle .$$

Let

$$\bar{A} : \mathcal{H} \rightarrow \mathcal{H}_A^*$$

be the bounded extension of

$$A : \mathcal{H}_A \subseteq \mathcal{H} \rightarrow \mathcal{H}_A^*$$

and let us denote by

$$(\cdot, \cdot)_A : \mathcal{H}_A \times \mathcal{H}_A^* \rightarrow \mathbb{C}$$

the pairing obtained by extending the scalar product in \mathcal{H}

$$\langle \cdot, \cdot \rangle : \mathcal{H}_A \times \mathcal{H} \subseteq \mathcal{H}_A \times \mathcal{H}_A^* \rightarrow \mathbb{C} .$$

We define then

$$\tau^* : \mathfrak{h} \rightarrow \mathcal{H}_A^*$$

by

$$\forall \phi \in \mathcal{H}_A, \quad (\phi, \tau^* \zeta)_A = (\tau \phi, \zeta) ,$$

where (\cdot, \cdot) denotes the scalar product in \mathfrak{h} .

The next theorem is nothing but Theorem 3.4 in [27] when one uses the bounded and surjective map (there denoted by τ) given by $\Pi \tau : \mathcal{H}_A \rightarrow \mathfrak{h}_0$.

THEOREM 2.5. *When restricted to $\mathcal{D}(A_{\Pi, \Theta})$ the linear operator*

$$\tilde{A} : \mathcal{D}(\tilde{A}) \subseteq \mathcal{H} \rightarrow \mathcal{H}_A^*, \quad \tilde{A}\phi := \bar{A}\phi + \tau^* \zeta_\phi ,$$

$$\mathcal{D}(\tilde{A}) = \{ \phi \in \mathcal{H} : \phi = \phi_* + G_* \zeta_\phi, \phi_* \in \mathcal{D}(A), \zeta_\phi \in \mathfrak{h} \} ,$$

is \mathcal{H} -valued and coincides with $A_{\Pi, \Theta}$.

3. The connection with von Neumann's Theory

In this section we explore the connection between the results given in Section 2 and von Neumann's theory of self-adjoint extensions. As a byproduct we will obtain that our construction provides all self-adjoint extensions of the symmetric operator S .

By defining the deficiency spaces

$$\mathcal{K}_\pm := \mathcal{K}(-S^* \pm i)$$

and posing

$$\mathcal{N} := \mathcal{K}(\tau) ,$$

von Neumann's theory says that

$$\mathcal{D}(S^*) = \mathcal{N} \oplus \mathcal{K}_+ \oplus \mathcal{K}_- , \quad S^*(\phi_0 + \phi_+ + \phi_-) = A\phi_0 + i\phi_+ - i\phi_- ,$$

the direct sum decomposition being orthogonal with respect to the graph inner product of S^* ; any self-adjoint extension A_U of S is then obtained by restricting S^* to a subspace of the kind $\mathcal{N} \oplus \mathcal{G}(U)$, where $U : \mathcal{K}_+ \rightarrow \mathcal{K}_-$ is unitary and $\mathcal{G}(U)$ denotes its graph.

In the next theorem we pose $\hat{\Gamma} := \hat{\Gamma}(i) \equiv \frac{1}{2} \tau(G_- - G_+)$, thus $\hat{\Gamma}^* = -\hat{\Gamma}$. Moreover in 1) we use the decomposition $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_0^\perp$.

THEOREM 3.1. 1) *The linear operators*

$$G_\pm : \mathfrak{h} \rightarrow \mathcal{K}_\pm$$

are continuous bijection and the linear operator

$$U : \mathcal{K}_+ \rightarrow \mathcal{K}_-, \quad U := -G_-((1 + 2(\Theta - \Pi \hat{\Gamma} \Pi)^{-1} \Pi \hat{\Gamma} \Pi) \oplus 1) G_+^{-1},$$

which can be alternatively re-written, in the case Θ is bounded, as

$$U = -G_-((\Theta - \Pi \hat{\Gamma} \Pi)^{-1}(\Theta + \Pi \hat{\Gamma} \Pi) \oplus 1) G_+^{-1},$$

is unitary. The corresponding extension A_U given by von Neumann's theory coincides with the self-adjoint operator $A_{\Pi, \Theta}$.

2) *Let A_U be the self-adjoint extension of S corresponding, by von Neumann's theory, to the unitary operator $U : \mathcal{K}_+ \rightarrow \mathcal{K}_-$ and let $\mathcal{M} \subseteq \mathcal{H}_A$ be the closed subspace $\mathcal{M} := \mathcal{D}(A_U) \cap \mathcal{D}(A)$. Then there exists a closed subspace \mathfrak{h}_0 (see (3.1) for the precise definition),*

$$\mathfrak{h}_0 \subseteq (\tau[\mathcal{M} \cap \mathcal{N}^\perp])^\perp \subseteq \mathfrak{h},$$

such that, denoting by Π the orthogonal projection onto \mathfrak{h}_0 and by $U_A := (-A + i)(-A - i)^{-1}$ the Cayley transform of A , the linear operator $\Theta : \mathcal{D}(\Theta) \subseteq \mathfrak{h}_0 \rightarrow \mathfrak{h}_0$ defined by

$$\Theta := i\Pi G^*(U - U_A)(U + U_A)^{-1} G \Pi$$

is self-adjoint. The corresponding self-adjoint extension $A_{\Pi, \Theta}$ coincides with A_U .

Proof. 1) The first half of the theorem is consequence of Theorem 4.1 in [27] when one uses the bounded and surjective map (there denoted by τ) given by $\Pi \tau : \mathcal{H}_A \rightarrow \mathfrak{h}_0$. 2) Since $\mathcal{N} \subseteq \mathcal{M}$ we have the orthogonal decomposition

$$\mathcal{H}_A = \mathcal{N} \oplus (\mathcal{M} \cap \mathcal{N}^\perp) \oplus \mathcal{M}^\perp,$$

so that, if $\phi = \phi_0 + \phi_1 + \phi_2$,

$$\tau \phi = \tau \phi_1 + \tau \phi_2.$$

If $\tau \phi_1 = \tau \phi_2$ then $\phi_1 - \phi_2 \in \mathcal{N}$ which gives $\phi_1 = \phi_2 = 0$ since both ϕ_1 and ϕ_2 are in \mathcal{N}^\perp and $\phi_1 \perp \phi_2$. Thus

$$\mathfrak{h} = \tau[\mathcal{M} \cap \mathcal{N}^\perp] + \tau[\mathcal{M}^\perp], \quad \tau[\mathcal{M} \cap \mathcal{N}^\perp] \cap \tau[\mathcal{M}^\perp] = \{0\}.$$

Since τ is continuous and surjective, by the open mapping theorem both $\tau[\mathcal{M} \cap \mathcal{N}^\perp]$ and $\tau[\mathcal{M}^\perp]$ are closed and therefore there exists a unique continuous projection P in \mathfrak{h} such that

$$\mathcal{K}(P) = \mathfrak{h}_1 := \tau[\mathcal{M} \cap \mathcal{N}^\perp], \quad \mathcal{R}(P) = \mathfrak{h}_2 := \tau[\mathcal{M}^\perp].$$

Moreover

$$P\tau : \mathcal{H}_A \rightarrow \mathfrak{h}_2$$

is a continuous surjection with $\mathcal{H}(P\tau) = \mathcal{M}$ and we can use the results in [27] when the map there denoted by τ is given by $P\tau$. In particular by Theorem 4.3 in [27] the linear operator in \mathfrak{h}_2

$$\Sigma := iPG^*(U - U_A)(U + U_A)^{-1}GP^*,$$

is densely defined and self-adjoint. Moreover

$$\begin{aligned} \hat{A} : \mathcal{D}(\hat{A}) \subseteq \mathcal{H} &\rightarrow \mathcal{H}, & \hat{A}\phi &= A\phi_* + RGP^*\xi_\phi, \\ \mathcal{D}(\hat{A}) &= \{ \phi \in \mathcal{H} : \phi = \phi_* + G_*P^*\xi_\phi, \\ & \phi_* \in \mathcal{D}(A), \xi_\phi \in \mathcal{D}(\Sigma), P\tau\phi_* = \Sigma\xi_\phi \} \end{aligned}$$

is self-adjoint and coincides with A_U . Since P^* is the unique continuous projection in \mathfrak{h} such that

$$\mathcal{H}(P^*) = \mathfrak{h}_2^\perp, \quad \mathcal{R}(P^*) = \mathfrak{h}_1^\perp,$$

denoting by Π_2 the orthogonal projection onto \mathfrak{h}_2 , the linear map

$$Q := P^*\Pi_2 : \mathfrak{h}_2 \rightarrow \mathfrak{h}_0 := \mathcal{R}(P^*\Pi_2) \tag{3.1}$$

is a continuous bijection. Thus for any $\xi_\phi \in \mathfrak{h}_2$ there exists an unique $\zeta_\phi \in \mathfrak{h}_0$ such that $P^*\xi_\phi = \zeta_\phi$ and

$$P\tau\phi_* = \Sigma\xi_\phi \iff (Q^*)^{-1}P\tau\phi_* = (Q^*)^{-1}\Sigma Q^{-1}\zeta_\phi$$

Thus $\hat{A} \equiv A_{\Pi, \Theta}$ with

$$\Pi := (Q^*)^{-1}P, \quad \Theta := (Q^*)^{-1}\Sigma Q^{-1}.$$

□

The previous theorem shows that the self-adjoint extensions we provided in Theorem 2.1 exhaust the set of all self-adjoint extensions of S (the case of relatively prime extensions was already contained in [27]). Thus we have the following

COROLLARY 3.2. *The set of operators provided by Theorem 2.1 coincides with the set of all self-adjoint extensions of the symmetric operator S . Such a set is parametrised by the bundle $p : \mathbf{E}(\mathfrak{h}) \rightarrow \mathbf{P}(\mathfrak{h})$, where $\mathbf{P}(\mathfrak{h})$ denotes the set of orthogonal projections in \mathfrak{h} and $p^{-1}(\Pi)$ is the set of self-adjoint operators in the range of Π . The set of self-adjoint operators in \mathfrak{h} , i.e. $p^{-1}(1)$, parametrises the extensions for which $\mathcal{D}(A_{1, \Theta}) \cap \mathcal{D}(A) = \mathcal{N}$ i.e. parametrises all relatively prime extensions of S .*

4. The connection with Boundary Triples Theory

In this section we explore the connection between the results given in Section 2 and Boundary Triples Theory.

A triple $\{\mathfrak{h}, \beta_1, \beta_2\}$, where \mathfrak{h} is a Hilbert space with inner product (\cdot, \cdot) and

$$\beta_1 : \mathcal{D}(S^*) \rightarrow \mathfrak{h}, \quad \beta_2 : \mathcal{D}(S^*) \rightarrow \mathfrak{h},$$

are two linear surjective maps, is said to be a boundary triple for S^* if

$$\langle \phi, S^* \psi \rangle - \langle S^* \phi, \psi \rangle = (\beta_1 \phi, \beta_2 \psi) - (\beta_2 \phi, \beta_1 \psi).$$

A closed subspace $\Lambda \subset \mathfrak{h} \oplus \mathfrak{h}$ is said to be a symmetric closed relation if

$$\forall ((\zeta_1, \zeta_2), (\xi_1, \xi_2)) \in \Lambda \oplus \Lambda, \quad (\zeta_1, \xi_2) = (\zeta_2, \xi_1).$$

Then Λ is said to be a self-adjoint relation if it is maximal symmetric, i.e. if it does not exist a closed symmetric relation $\tilde{\Lambda}$ such that $\Lambda \subsetneq \tilde{\Lambda}$. Of course the graph of a self-adjoint operator is a particular case of self-adjoint relation.

One of the main results of boundary triples theory (see e.g. [12], Theorem 1.6, Chapter 3) is the following

THEOREM 4.1. *The self-adjoint extensions of S are parametrised by the set of self-adjoint relations in $\mathfrak{h} \oplus \mathfrak{h}$. Any self-adjoint extension of S is obtained by restricting S^* to the subspace*

$$\{\phi \in \mathcal{D}(S^*) : (\beta_1 \phi, \beta_2 \phi) \in \Lambda\},$$

where Λ is some self-adjoint relation and $\{\mathfrak{h}, \beta_1, \beta_2\}$ is a boundary triple for S^* .

Now let us take A , τ and S as in section 1. In [28], Theorem 3.1, the following result (with slight different notations and definitions) was obtained:

THEOREM 4.2. *The adjoint of S is given by*

$$S^* : \mathcal{D}(S^*) \subseteq \mathcal{H} \rightarrow \mathcal{H}, \quad S^* \phi = A\phi_* + RG\zeta_\phi,$$

$$D(S^*) = \{\phi \in \mathcal{H} : \phi = \phi_* + G_*\zeta_\phi, \phi_* \in D(A), \zeta_\phi \in \mathfrak{h}\}.$$

The triple $\{\mathfrak{h}, \hat{\beta}_1, \hat{\beta}_2\}$, where

$$\hat{\beta}_1 : \mathcal{D}(S^*) \rightarrow \mathfrak{h}, \quad \hat{\beta}_1 \phi := \zeta_\phi,$$

$$\hat{\beta}_2 : \mathcal{D}(S^*) \rightarrow \mathfrak{h}, \quad \hat{\beta}_2 \phi := \tau\phi_*,$$

is a boundary triple for S^* .

Thus by Corollary 3.2 and Theorem 4.1 we have the following

THEOREM 4.3. *1) Any self-adjoint relation in $\mathfrak{h} \oplus \mathfrak{h}$ can be written as*

$$\Lambda_{(\Pi, \Theta)} := \{(\zeta_1, \zeta_2) \in \mathfrak{h} \oplus \mathfrak{h} : \zeta_1 \in \mathcal{D}(\Theta), \quad \Theta\zeta_1 = \Pi\zeta_2\},$$

for some $(\Pi, \Theta) \in \mathbf{E}(\mathfrak{h})$.

2) Any self-adjoint extension of S is given by restricting S^* to a subspace of the kind

$$\{\phi \in \mathcal{D}(S^*) : \hat{\beta}_1 \phi \in \mathcal{D}(\Theta), \quad \Theta\hat{\beta}_1 \phi = \Pi\hat{\beta}_2 \phi\}$$

for some $(\Pi, \Theta) \in \mathbf{E}(\mathfrak{h})$.

REMARK 4.4. By

$$\mathfrak{h} = \mathcal{R}(\Pi) \oplus \mathcal{K}(\Pi),$$

by

$$\begin{aligned} \mathfrak{h} \oplus \mathfrak{h} &\simeq \mathcal{R}(\Pi) \oplus \mathcal{R}(\Pi) \oplus \mathcal{K}(\Pi) \oplus \mathcal{K}(\Pi), \\ \mathcal{K}(\Pi) &\simeq \mathcal{K}(\Pi) \oplus \{0\} \subset \mathfrak{h} \oplus \mathfrak{h} \end{aligned}$$

and denoting by

$$\mathcal{G}(\Theta) \subset \mathcal{R}(\Pi) \oplus \mathcal{R}(\Pi)$$

the graph of Θ , Theorem 4.3 gives

$$\Lambda_{(\Pi, \Theta)} \simeq \mathcal{G}(\Theta) \oplus \mathcal{K}(\Pi).$$

This reproduces Theorem 5.3 in [3].

By [24], Proposition 4 and Lemma 5, any self-adjoint relation in $\mathfrak{h} \oplus \mathfrak{h}$ can be written as

$$\Lambda^{(B_1, B_2)} := \{(\zeta_1, \zeta_2) : B_1 \zeta_1 = B_2 \zeta_2\} = \{(B_2^* \zeta, B_1^* \zeta), \zeta \in \mathfrak{h}\},$$

where B_1 and B_2 are bounded linear operators in \mathfrak{h} such that

$$B_1 B_2^* = B_2 B_1^* \tag{4.1}$$

and

$$0 \in \rho \left(M^{(B_1, B_2)} \right), \quad M^{(B_1, B_2)} := \begin{pmatrix} B_1 & -B_2 \\ B_2 & B_1 \end{pmatrix}. \tag{4.2}$$

In the case \mathfrak{h} is finite dimensional the condition (4.2) is equivalent either to

$$\mathcal{K}(B_1^*) \cap \mathcal{K}(B_2^*) = \{0\} \tag{4.3}$$

or to

$$\det(B_1 B_1^* + B_2 B_2^*) \neq 0. \tag{4.4}$$

Conditions (4.1) and (4.4) were obtained in [30]. Their infinite dimensional analogue, as other equivalent conditions, are given in [9], Section 3.2.

The connection between the representation of self-adjoint relations in terms of $\Lambda_{(\Pi, \Theta)}$ and the one in terms of $\Lambda^{(B_1, B_2)}$ is provided by the following

THEOREM 4.5. *Given $(\Pi, \Theta) \in \mathbf{E}(\mathfrak{h})$ and posing $\mathfrak{h} = \mathcal{R}(\Pi) \oplus \mathcal{K}(\Pi)$, let us define*

$$B_1 := \Theta(-\Theta + i)^{-1} \oplus 1, \quad B_2 := (-\Theta + i)^{-1} \oplus 0. \tag{4.5}$$

Conversely, given (B_1, B_2) satisfying (4.1) and (4.2), let Π be the orthogonal projection onto $\mathcal{K}(B_2)^\perp$ and let Θ be the self-adjoint operator in $\mathcal{K}(B_2)^\perp$ defined by

$$\Theta : \mathcal{R}(B_2^*) \subseteq \mathcal{K}(B_2)^\perp \rightarrow \mathcal{K}(B_2)^\perp, \quad \Theta := \Pi B_1^* (B_2^* \tilde{\Pi})^{-1} \Pi, \tag{4.6}$$

where $\tilde{\Pi}$ is the orthogonal projection onto $\mathcal{K}(B_2^)^\perp$.*

Then

$$\Lambda_{(\Pi, \Theta)} = \Lambda^{(B_1, B_2)}.$$

Proof. Checking that $\Lambda^{(B_1, B_2)} \simeq \mathcal{G}(\Theta) \oplus \mathcal{K}(\Pi)$, where (B_1, B_2) is defined by (4.5) is straightforward.

Conversely, by

$$\mathfrak{h} = \mathcal{K}(B_2)^\perp \oplus \mathcal{K}(B_2) = \mathcal{K}(B_2^*)^\perp \oplus \mathcal{K}(B_2^*),$$

one has

$$\begin{aligned} & \{(B_2^* \zeta, B_1^* \zeta), \zeta \in \mathfrak{h}\} \\ & = \{(\zeta_0, B_1^*(B_2^* \tilde{\Pi})^{-1} \zeta_0 + B_1^* \zeta_1), \zeta_0 \in \mathcal{R}(B_2^*) \subseteq \mathcal{K}(B_2)^\perp, \zeta_1 \in \mathcal{K}(B_2^*)\}. \end{aligned}$$

By (4.1)

$$\zeta_1 \in \mathcal{K}(B_2^*) \implies B_1^* \zeta_1 \in \mathcal{K}(B_1).$$

Thus

$$\Lambda^{(B_1, B_2)} \simeq \mathcal{G}(\Theta) \oplus \mathcal{R}((1 - \Pi)B_1^*).$$

By (4.2) one has

$$\forall \zeta_0 \in \mathcal{K}(B_2) \quad \exists (\zeta_1, \zeta_2) \in \mathfrak{h} \oplus \mathfrak{h} \quad \text{s.t.} \quad (1 - \Pi)(B_1^* \zeta_1 + B_2^* \zeta_2) = \zeta_0.$$

Since $\mathcal{R}(B_2^*) \subseteq \mathcal{K}(B_2)^\perp$, one obtains

$$\mathcal{R}((1 - \Pi)B_1^*) = \mathcal{K}(B_2)$$

and the proof is done. □

5. Examples

For the sake of simplicity in the next examples we take A equal to the Laplace operator. With some more effort these examples could be extended to the case in which A is a variable-coefficients differential operator. Moreover Theorem 2.1 could be applied to not semi-bounded self-adjoint operators (including the case $\sigma(A) = \mathbb{R}$) of the kind $A = iW$ where the skew-adjoint W is associated to some abstract wave equations (see [29]; also see [7] for an application to acoustics).

EXAMPLE 5.1. (The Laplacian on a bounded interval) Let

$$\begin{aligned} A : \mathcal{D}(A) \subseteq L^2(0, a) &\rightarrow L^2(0, a), & A\psi &= \psi'' , \\ \mathcal{D}(A) &= \{\psi \in H^2(0, a) : \psi(0+) = \psi(a-) = 0\}, \\ \tau : H^2(0, a) &\rightarrow \mathbb{C}^2, & \tau\psi &= (\psi'(0+), -\psi'(a-)) . \end{aligned}$$

Here $H^2(0, a) \subset C^1(0, a)$ denotes the usual Sobolev-Hilbert space of square integrable functions with square integrable second order (distributional) derivative. We look for all self-adjoint extensions of the symmetric operator

$$S : \mathcal{D}(S) \subseteq L^2(0, a) \rightarrow L^2(0, a), \quad S\psi = \psi'' ,$$

$$\mathcal{D}(S) \equiv H_0^2(0, a) := \{\psi \in H^2(0, a) : \psi(0+) = \psi'(0+) = \psi(a-) = \psi'(a-) = 0\} .$$

Since

$$\begin{aligned} \left(-\frac{d^2}{dx^2} + z\right)^{-1} \psi(x) &= \frac{\sin(\sqrt{-z}(a-x))}{\sqrt{-z} \sin(\sqrt{-z}a)} \int_0^x \sin(\sqrt{-z}y) \psi(y) dy \\ &\quad + \frac{\sin(\sqrt{-z}x)}{\sqrt{-z} \sin(\sqrt{-z}a)} \int_x^a \sin(\sqrt{-z}(a-y)) \psi(y) dy, \\ z &\neq -\left(\frac{n\pi}{a}\right)^2, \\ \left(-\frac{d^2}{dx^2}\right)^{-1} \psi(x) &= \frac{a-x}{a} \int_0^x y \psi(y) dy + \frac{x}{a} \int_x^a (a-y) \psi(y) dy, \end{aligned}$$

one has

$$\begin{aligned} G(z) : \mathbb{C}^2 &\rightarrow L^2(0, a), \\ [G(z)\zeta](x) &= \begin{cases} \frac{\sin(\sqrt{-z}(a-x))}{\sin(\sqrt{-z}a)} \zeta_1 + \frac{\sin(\sqrt{-z}x)}{\sin(\sqrt{-z}a)} \zeta_2 & z \neq -\left(\frac{n\pi}{a}\right)^2 \\ \frac{a-x}{a} \zeta_1 + \frac{x}{a} \zeta_2 & z = 0, \end{cases} \end{aligned} \tag{5.1}$$

where $\zeta \equiv (\zeta_1, \zeta_2)$ and

$$G(\bar{z})^* : L^2(0, a) \rightarrow \mathbb{C}^2, \quad G(\bar{z})^* \equiv (G(\bar{z})^*_1, G(\bar{z})^*_2) \tag{5.2}$$

$$\begin{aligned} G(\bar{z})^*_1 \psi &= \begin{cases} \int_0^a \frac{\sin(\sqrt{-z}(a-x))}{\sin(\sqrt{-z}a)} \psi(x) dx & z \neq -\left(\frac{n\pi}{a}\right)^2 \\ \int_0^a \frac{a-x}{a} \psi(x) dx & z = 0. \end{cases} \\ G(\bar{z})^*_2 \psi &= \begin{cases} \int_0^a \frac{\sin(\sqrt{-z}x)}{\sin(\sqrt{-z}a)} \psi(x) dx & z \neq -\left(\frac{n\pi}{a}\right)^2 \\ \int_0^a \frac{x}{a} \psi(x) dx & z = 0. \end{cases} \end{aligned}$$

Note that $G(z)\zeta$ solves the Dirichlet boundary value problem

$$\begin{aligned} (G(z)\zeta)'' &= zG(z)\zeta, \\ \rho G(z)\zeta &= \zeta, \end{aligned}$$

where

$$\rho : H^2(0, a) \rightarrow \mathbb{C}^2, \quad \rho\psi := (\psi(0+), \psi(a-)).$$

Thus

$$\mathcal{R}(G(z)) \cap \mathcal{D}(A) = \{0\}.$$

Then one defines

$$\Gamma(z) : \mathbb{C}^2 \rightarrow \mathbb{C}^2, \quad \Gamma(z) := -\tau G(z),$$

i.e

$$\Gamma(z) = \frac{\sqrt{-z}}{\sin(\sqrt{-z}a)} \begin{pmatrix} \cos(\sqrt{-z}a) & -1 \\ -1 & \cos(\sqrt{-z}a) \end{pmatrix}, \quad z \neq -\left(\frac{n\pi}{a}\right)^2, \tag{5.3}$$

$$\Gamma(0) = \frac{1}{a} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{5.4}$$

It satisfies (2.4) and (2.5) by

$$\Gamma(z) = -\tau G(0) + \tau(G(0) - G(z)) = \frac{1}{a} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \hat{\Gamma}_{(0)}(z).$$

For any

$$\psi = \psi_z + G(z)\Pi\Gamma_{\Pi,\Theta}(z)^{-1}\Pi\tau\psi_z \in \mathcal{D}(A_{\Pi,\Theta}) \subseteq H^2(0, a), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

one has

$$\rho\psi = \Pi\Gamma_{\Pi,\Theta}(z)^{-1}\Pi\tau\psi_z,$$

i.e.

$$\rho\psi \in \mathcal{R}(\Pi), \quad \Theta\rho\psi = \Pi(\tau\psi_z - \Gamma(z)\rho\psi).$$

Thus

$$\Pi\tau\psi = \Pi(\tau\psi_z - \Gamma(z)\rho\psi) = \Theta\rho\psi.$$

Since any $\psi \in H^2(0, a)$ can be decomposed as $\psi = (\psi - G(z)\rho\psi) + G(z)\rho\psi$ and $\psi - G(z)\rho\psi \in \mathcal{D}(A)$, a straightforward calculation then gives

$$A_{\Pi,\Theta} : \mathcal{D}(A_{\Pi,\Theta}) \subseteq L^2(0, a) \rightarrow L^2(0, a), \quad A_{\Pi,\Theta}\psi = \psi'',$$

$$\mathcal{D}(A_{\Pi,\Theta}) = \{\psi \in H^2(0, a) : \rho\psi \in \mathcal{R}(\Pi), \quad \Pi\tau\psi = \Theta\rho\psi\},$$

where $(\Pi, \Theta) \in \mathbf{E}(\mathfrak{h})$, $\mathfrak{h} = \mathbb{C}^2$. Thus the case $\Pi = 0$ reproduces A itself, the case $\Pi = 1$ gives the boundary conditions (here $\theta_{11}, \theta_{22} \in \mathbb{R}$, $\theta_{12} \in \mathbb{C}$)

$$\theta_{11} \psi(0+) - \psi'(0+) + \theta_{12} \psi(a-) = 0,$$

$$\bar{\theta}_{12} \psi(0+) + \theta_{22} \psi(a-) + \psi'(a-) = 0,$$

and the case $\Pi = w \otimes w$, $w \equiv (w_1, w_2)$ an unitary vector in \mathbb{C}^2 , gives the boundary conditions (here $\theta \in \mathbb{R}$)

$$w_2 \psi(0+) - w_1 \psi(a-) = 0,$$

$$\bar{w}_1 (\theta \psi(0+) - \psi'(0+)) + \bar{w}_2 (\theta \psi(a-) + \psi'(a-)) = 0.$$

The resolvent of $A_{\Pi,\Theta}$ is obtained by inserting the above expressions for $\Gamma(z)$, $G(z)$ and $G(\bar{z})^*$ into (2.7).

EXAMPLE 5.2. (The Laplacian on a bounded graph) Let

$$A = \oplus_{k=1}^n A_k : \oplus_{k=1}^n \mathcal{D}(A_k) \subseteq \oplus_{k=1}^n L^2(0, a_k) \rightarrow \oplus_{k=1}^n L^2(0, a_k),$$

$$A_k : \mathcal{D}(A_k) \subseteq L^2(0, a_k) \rightarrow L^2(0, a_k), \quad A_k \psi = \psi'',$$

$$\mathcal{D}(A_k) = \{\psi \in H^2(0, a_k) : \psi(0+) = \psi(a_k-) = 0\},$$

$$\tau = \oplus_{k=1}^n \tau_k : \oplus_{k=1}^n H^2(0, a_k) \rightarrow \mathbb{C}^{2n},$$

$$\tau_k : H^2(0, a_k) \rightarrow \mathbb{C}^2, \quad \tau_k \psi_k := (\psi'_k(0+), -\psi'_k(a_k-)).$$

One has

$$G(z) = \oplus_{k=1}^n G_k(z) : \mathbb{C}^{2n} \rightarrow \oplus_{k=1}^n L^2(0, a_k)$$

and

$$G(\bar{z})^* = \bigoplus_{k=1}^n G_k(\bar{z})^* : \bigoplus_{k=1}^n L^2(0, a_k) \rightarrow \mathbb{C}^{2n},$$

where $G_k(z)$ and $G_k(\bar{z})^*$ are given by (5.1) and (5.2) with $a = a_k$. Analogously

$$\Gamma(z) = \bigoplus_{k=1}^n \Gamma_k(z) : \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n},$$

where $\Gamma_k(z)$ is defined as in (5.3) (as in (5.4) when $z = 0$) with $a = a_k$.

Proceeding as in the previous example one has

$$A_{\Pi, \Theta} : \mathcal{D}(A_{\Pi, \Theta}) \subseteq \bigoplus_{k=1}^n L^2(0, a_k) \rightarrow \bigoplus_{k=1}^n L^2(0, a_k),$$

$$A_{\Pi, \Theta}(\psi_1, \dots, \psi_n) = (\psi_1'', \dots, \psi_n''),$$

$$\mathcal{D}(A_{\Pi, \Theta}) = \{ \Psi \equiv (\psi_1, \dots, \psi_n) \in \bigoplus_{k=1}^n H^2(0, a_k) : \rho\Psi \in \mathcal{R}(\Pi), \quad \Pi\tau\Psi = \Theta\rho\Psi \},$$

where $(\Pi, \Theta) \in \mathbf{E}(\mathfrak{h})$, $\mathfrak{h} = \mathbb{C}^{2n}$, and

$$\rho = \bigoplus_{k=1}^n \rho_k : \bigoplus_{k=1}^n H^2(0, a_k) \rightarrow \mathbb{C}^{2n},$$

$$\rho_k : H^2(0, a_k) \rightarrow \mathbb{C}^2, \quad \rho_k \psi_k := (\psi_k(0+), \psi_k(a_k-)).$$

Moreover

$$\begin{aligned} (-A_{\Pi, \Theta} + z)^{-1} &= \bigoplus_{k=1}^n (-A_k + z)^{-1} \\ &+ (\bigoplus_{k=1}^n G_k(z))\Pi(\Theta + \Pi(\bigoplus_{k=1}^n \Gamma_k(z))\Pi)^{-1}\Pi(\bigoplus_{k=1}^n G_k(\bar{z})^*). \end{aligned}$$

The self-adjoint operator $A_{\Pi, \Theta}$ describes the Laplacian on a bounded graph with n edges, the k -th edge being identified with the segment $[0, a_k]$. By a similar construction it is possible to define the Laplacian on a graph with unbounded external lines. The boundary conditions $\Pi\tau\Psi = \Theta\rho\Psi$ specify the connectivity of the graph. For such a kind of operators, in the case the parametrisation is given by a couple of $n \times n$ matrices satisfying (4.1) and (4.2), see [17] and see [2] for the corresponding resolvent formula. In [20], Theorem 6, it was shown that such a parametrization in terms of a couple of matrices can be re-phrased in a way that coincides with our one (no resolvent formula was provided there).

EXAMPLE 5.3. (The Laplacian with n point interactions) Let

$$A : H^2(\mathbb{R}^3) \subseteq L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3), \quad A\psi = \Delta\psi,$$

$$\tau : H^2(\mathbb{R}^3) \rightarrow \mathbb{C}^n, \quad \tau\phi \equiv (\psi(y_1), \dots, \psi(y_n)),$$

where $y_k \in \mathbb{R}^3$, $1 \leq k \leq n$. Here $H^2(\mathbb{R}^3) \subset C_b(\mathbb{R}^3)$ denotes the usual Sobolev-Hilbert space of square integrable functions with square integrable second order (distributional) partial derivatives. Thus

$$S : \mathcal{D}(S) \subset L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3), \quad S\psi = \Delta\psi,$$

$$\mathcal{D}(S) := \{ \psi \in H^2(\mathbb{R}^3) : \psi(y_k) = 0, 1 \leq k \leq n \}.$$

Since the kernel of the resolvent of Δ is given by

$$(-\Delta + z)^{-1}(x_1, x_1) = \frac{e^{-\sqrt{z}|x_1-x_2|}}{4\pi|x_1-x_2|}, \quad \operatorname{Re}\sqrt{z} > 0,$$

one has, if $\zeta \equiv (\zeta_1, \dots, \zeta_n)$,

$$G(z) : \mathbb{C}^n \rightarrow L^2(\mathbb{R}^3), \quad [G(z)\zeta](x) = \sum_{k=1}^n \frac{e^{-\sqrt{z}|x-y_k|}}{4\pi|x-y_k|} \zeta_k \tag{5.5}$$

and

$$G(\bar{z})^* : L^2(\mathbb{R}^3) \rightarrow \mathbb{C}^n, \quad G(\bar{z})^* \equiv (G(\bar{z})^*_1, \dots, G(\bar{z})^*_n), \tag{5.6}$$

$$G(\bar{z})^*_k \psi := \int_{\mathbb{R}^3} \frac{e^{-\sqrt{\bar{z}}|x-y_k|}}{4\pi|x-y_k|} \psi(x) dx.$$

By (2.2) the k -th component of $(z-w)G(\bar{w})^*G(z)\zeta$ is

$$\begin{aligned} (z-w)(G(\bar{w})^*G(z)\zeta)_k &= (\tau(G(w)) - G(z))\zeta_k \\ &= \lim_{x \rightarrow y_k} \frac{e^{-\sqrt{w}|x-y_k|} - e^{-\sqrt{z}|x-y_k|}}{4\pi|x-y_k|} \zeta_k + \sum_{j \neq k} \left(\frac{e^{-\sqrt{w}|y_k-y_j|}}{4\pi|y_k-y_j|} - \frac{e^{-\sqrt{z}|y_k-y_j|}}{4\pi|y_k-y_j|} \right) \zeta_j \end{aligned}$$

so that, according to (2.4), we can take $\Gamma(z) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ to be represented by the matrix with components

$$\Gamma_{kj}(z) = \begin{cases} \frac{\sqrt{z}}{4\pi} & k = j \\ -\frac{e^{-\sqrt{z}|y_k-y_j|}}{4\pi|y_k-y_j|} & k \neq j. \end{cases}$$

Note that we can alternatively define $\Gamma(z)$ by

$$\Gamma(z) := \hat{\Theta} + \hat{\tau}(G(0) - G(z)), \tag{5.7}$$

where

$$G(0) : \mathbb{C}^n \rightarrow L^2_{loc}(\mathbb{R}^3), \quad [G(0)\zeta](x) := \sum_{k=1}^n \frac{\zeta_k}{4\pi|x-y_k|}, \tag{5.8}$$

$\hat{\tau}$ is the extension of τ to $H^2_{loc}(\mathbb{R}^3) \subset C_b(\mathbb{R}^3)$ and the symmetric operator $\hat{\Theta}$ is represented by the matrix with components

$$\hat{\Theta}_{kj} = \begin{cases} 0 & k = j \\ -\frac{1}{4\pi|y_k-y_j|} & k \neq j. \end{cases}$$

Given, according to Theorem 2.1,

$$\psi = \psi_z + G(z)\zeta_\psi \in D(\Delta_{\Pi, \Theta}), \quad \zeta_\psi := \Pi \Gamma_{\Pi, \Theta}(z)^{-1} \Pi \tau \psi_z \quad z \in \mathbb{C} \setminus \mathbb{R},$$

one has, by using (5.7),

$$\begin{aligned} \Pi \hat{\tau}(\psi - G(0)\zeta_\psi) &= \Pi \tau \psi_z + \Pi(\hat{\Theta} - \Gamma(z))\Pi \zeta_\psi \\ &= \Pi \tau \psi_z - \Gamma_{\Pi, \Theta}(z)\zeta_\psi + \Pi \hat{\Theta} \Pi \zeta_\psi + \Theta \zeta_\psi. \end{aligned}$$

This gives $\Pi \hat{\tau}_0 \psi = \Theta \zeta_\psi$, where the renormalised trace operator $\hat{\tau}_0$ is defined by

$$\hat{\tau}_0 : \mathcal{D}(\Delta_{\Pi, \Theta}) \rightarrow \mathbb{C}^n, \quad \hat{\tau}_0 \psi := \hat{\tau}(\psi - G(0)\zeta_\psi) - \hat{\Theta}\zeta_\psi. \quad (5.9)$$

Note that such a definition says that the k -th component of $\hat{\tau}_0 \psi$ is given by

$$(\hat{\tau}_0 \psi)_k = \lim_{x \rightarrow y_k} \left(\psi(x) - \frac{1}{4\pi} \frac{\zeta_k}{|x - y_k|} \right), \quad 1 \leq k \leq n,$$

where here ζ_k denotes the k -th component of ζ_ψ . By Theorem 2.1 one has

$$\begin{aligned} \Delta_{\Pi, \Theta} \psi &= \Delta \psi_z + zG(z)\zeta_\psi \\ &= \Delta(\psi - G(0)\zeta_\psi) + zG(z)\zeta_\psi - \Delta(G(z) - G(0))\zeta_\psi \\ &= \Delta(\psi - G(0)\zeta_\psi). \end{aligned}$$

In conclusion, for any $(\Pi, \Theta) \in \mathbf{E}(\mathfrak{h})$, $\mathfrak{h} = \mathbb{C}^n$, one has

$$\begin{aligned} \Delta_{\Pi, \Theta} : \mathcal{D}(\Delta_{\Pi, \Theta}) &\subseteq L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3), \quad \Delta_{\Pi, \Theta} \psi := \Delta \psi_0, \\ \mathcal{D}(\Delta_{\Pi, \Theta}) &:= \{ \psi \in L^2(\mathbb{R}^3) : \psi = \psi_0 + G(0)\zeta_\psi, \\ &\psi_0 \in H_{loc}^2(\mathbb{R}^3), \quad \zeta_\psi \in \mathcal{R}(\Pi), \quad \Pi \hat{\tau}_0 \psi = \Theta \zeta_\psi \} \end{aligned}$$

and

$$(-\Delta_{\Pi, \Theta} + z)^{-1} = (-\Delta + z)^{-1} + G(z)\Pi(\Theta + \Pi\Gamma(z)\Pi)^{-1}\Pi G(\bar{z})^*.$$

The case $\Pi = 1$, Θ diagonal, reproduces the self-adjoint extensions appearing in [1] and references therein. For the general case (when the parametrisation is given by a couple of $n \times n$ matrices satisfying (4.1) and (4.2)) see [24].

EXAMPLE 5.4. (The Laplacian with n point interactions, the vector-valued case) The previous example can be generalised by taking

$$\begin{aligned} A : H^2(\mathbb{R}^3; \mathfrak{f}) &\subseteq L^2(\mathbb{R}^3; \mathfrak{f}) \rightarrow L^2(\mathbb{R}^3; \mathfrak{f}), \\ A\psi &:= \Delta\psi + B\psi, \end{aligned} \quad (5.10)$$

where B is a symmetric operator in the d -dimensional Hilbert space \mathfrak{f} , and

$$\tau : H^2(\mathbb{R}^3; \mathfrak{f}) \rightarrow \mathfrak{h}, \quad \mathfrak{h} = \bigoplus_{k=1}^n \mathfrak{f}, \quad \tau\phi \equiv (\psi(y_1), \dots, \psi(y_n)), \dots$$

By using the unitary isomorphisms

$$\mathfrak{f} \simeq \mathbb{C}^d, \quad \bigoplus_{k=1}^n \mathfrak{f} \simeq \bigoplus_{i=1}^d \mathbb{C}^n, \quad L^2(\mathbb{R}^3; \mathfrak{f}) \simeq \bigoplus_{i=1}^d L^2(\mathbb{R}^3)$$

induced by the orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_d$ made of the normalised eigenvectors of B , and denoting by b_1, \dots, b_d the corresponding eigenvalues, one has

$$\begin{aligned} G(z) &= \bigoplus_{i=1}^d G_i(z) : \bigoplus_{i=1}^d \mathbb{C}^n \rightarrow \bigoplus_{i=1}^d L^2(\mathbb{R}^3), \\ G(\bar{z})^* &= \bigoplus_{i=1}^d G_i(\bar{z})^* : \bigoplus_{i=1}^d L^2(\mathbb{R}^3) \rightarrow \bigoplus_{i=1}^d \mathbb{C}^n. \end{aligned}$$

Here

$$G_i(z) : \mathbb{C}^n \rightarrow L^2(\mathbb{R}^3)$$

and

$$G_i(\bar{z})^* : L^2(\mathbb{R}^3) \rightarrow \mathbb{C}^n$$

are defined by evaluating the operators $G(\cdot)$ and $G(\cdot)^*$ given in (5.5) and (5.6) at $z - b_i$. Analogously

$$\Gamma(z) = \oplus_{i=1}^d \Gamma_i(z) : \oplus_{i=1}^d \mathbb{C}^n \rightarrow \oplus_{i=1}^d \mathbb{C}^n,$$

where $\Gamma_i(z)$ is represented by the matrix with components

$$\Gamma_{i,kj}(z) = \begin{cases} \frac{\sqrt{z-b_i}}{4\pi} & k = j \\ -\frac{e^{-\sqrt{z-b_i}|y_k-y_j|}}{4\pi|y_k-y_j|} & k \neq j. \end{cases}$$

Proceeding as in the previous example one has, for any $(\Pi, \Theta) \in \mathbf{E}(\mathfrak{h})$, $\mathfrak{h} = \oplus_{i=1}^n \mathfrak{f} \simeq \oplus_{i=1}^d \mathbb{C}^n \simeq \mathbb{C}^{nd}$,

$$\begin{aligned} A_{\Pi,\Theta} : \mathcal{D}(A_{\Pi,\Theta}) \subseteq L^2(\mathbb{R}^3; \mathfrak{f}) &\rightarrow L^2(\mathbb{R}^3; \mathfrak{f}), & A_{\Pi,\Theta} \psi &:= \Delta \psi_0 + B \psi_0, \\ \mathcal{D}(A_{\Pi,\Theta}) &:= \{ \psi \in L^2(\mathbb{R}^3; \mathfrak{f}) : \psi = \psi_0 + (\oplus_{i=1}^d G(0)) \zeta_\psi, \\ \psi_0 &\in H_{loc}^2(\mathbb{R}^3; \mathfrak{f}), \quad \zeta_\psi \in \mathcal{R}(\Pi), \quad \Pi \hat{\tau}_0 \psi = \Theta \zeta_\psi \}, \end{aligned}$$

$$\begin{aligned} (-A_{\Pi,\Theta} + z)^{-1} &= (-A + z)^{-1} \\ &+ (\oplus_{i=1}^d G_i(z)) \Pi (\Theta + \Pi (\oplus_{i=1}^d \Gamma_i(z)) \Pi)^{-1} \Pi (\oplus_{i=1}^d G_i(\bar{z})^*), \end{aligned}$$

where $G(0)$ is defined in (5.8) and $\hat{\tau}_0$ is here defined component-wise through (5.9) by writing $\psi = \psi_1 \mathbf{e}_1 + \dots + \psi_d \mathbf{e}_d$.

By the unitary isomorphism $L^2(\mathbb{R}^3) \otimes \mathfrak{f} \simeq L^2(\mathbb{R}^3; \mathfrak{f})$ given by $\psi \otimes \zeta \mapsto \psi \zeta$, which transforms $\Delta \otimes 1 + 1 \otimes B$ into A defined in (5.10), and by taking $\mathfrak{f} = \otimes_{k=1}^n \mathbb{C}^2 \simeq \mathbb{C}^{2^n}$, this example reproduces (for a particular choice of B) the self-adjoint extensions given in [6] describing systems made of a spin-less quantum particle and an array of n spin 1/2 (there the parametrisation is given by a couple of $n2^n \times n2^n$ matrices satisfying (4.1) and (4.2)).

EXAMPLE 5.5. (The Laplacian on a bounded domain) This last example is the extension of Example 5.1 to n dimensions. The main difference is due to the infinite dimensionality of \mathfrak{h} , i.e. the defect indices of S are infinite in this case. This requires the use of some not trivial analytic results which we entirely take from [13] and [21]. However, apart from these technical issues, we follow the same path as in the much simpler Example 5.1. This leads to the reproduction of the results obtained (for general strongly elliptic operators) in [13], Chapter III, about the complete classification in terms of boundary conditions of the self-adjoint extensions of the minimal Laplacian on a bounded domain. The study of boundary value problems by means of self-adjoint extensions goes back to [8] and was further developed in [32].

Given $\Omega \subset \mathbb{R}^n$, $n > 1$, a bounded open set with a boundary $\partial\Omega$ which is a smooth embedded sub-manifold (these hypotheses could be weakened), $H^m(\Omega)$ denotes the usual Sobolev-Hilbert space of functions on Ω with square integrable partial (distributional) derivatives of any order $k \leq m$ and $H^s(\partial\Omega)$, s real, denotes the fractional

order Sobolev-Hilbert space defined, since here $\partial\Omega$ can be made a smooth compact Riemannian manifold, as the completion of $C^\infty(\partial\Omega)$ with respect of the scalar product

$$\langle f, g \rangle_{H^s(\partial\Omega)} := \langle f, (-\Delta_{LB} + 1)^s g \rangle_{L^2(\partial\Omega)}.$$

Here the self-adjoint operator Δ_{LB} is the Laplace-Beltrami operator in $L^2(\partial\Omega)$. With such a definition $(-\Delta_{LB} + 1)^{s/2}$ can be extended to a unitary map, which we denote by the same symbol,

$$(-\Delta_{LB} + 1)^{s/2} : H^r(\partial\Omega) \rightarrow H^{r-s}(\partial\Omega).$$

For successive notational convenience we pose

$$\Lambda := (-\Delta_{LB} + 1)^{1/2} : H^s(\partial\Omega) \rightarrow H^{s-1}(\partial\Omega), \quad \Sigma := \Lambda^{-1}.$$

The continuous and surjective linear operator

$$\gamma : H^2(\Omega) \rightarrow H^{3/2}(\partial\Omega) \oplus H^{1/2}(\partial\Omega), \quad \gamma\phi := (\rho\phi, \tau\phi),$$

is defined (see e.g. [21], Chapter 1, Section 8.2) as the unique bounded extension of

$$\tilde{\gamma} : C^\infty(\bar{\Omega}) \rightarrow C^\infty(\partial\Omega) \times C^\infty(\partial\Omega), \quad \tilde{\gamma}\phi := (\tilde{\rho}\phi, \tilde{\tau}\phi),$$

where

$$\tilde{\rho}\phi(x) := \phi(x), \quad \tilde{\tau}\phi(x) := n(x) \cdot \nabla\phi(x) \equiv \frac{\partial\phi}{\partial n}(x), \quad x \in \partial\Omega,$$

and n denotes the inner normal vector on $\partial\Omega$. By Green's formula the linear operator γ can be further extended (see [21], Chapter 2, Section 6.5) to a continuous map

$$\hat{\gamma} : \mathcal{D}(\Delta_{max}) \rightarrow H^{-1/2}(\partial\Omega) \oplus H^{-3/2}(\partial\Omega), \quad \hat{\gamma}\phi = (\hat{\rho}\phi, \hat{\tau}\phi),$$

where

$$\mathcal{D}(\Delta_{max}) := \{\phi \in L^2(\Omega) : \Delta\phi \in L^2(\Omega)\}.$$

Let us remark that $H^2(\Omega)$ is strictly contained in $\mathcal{D}(\Delta_{max})$ when $n > 1$; by elliptic regularity one has (see [13], Proposition III 5.2, [21], Chapter 2, Section 7.3)

$$H^2(\Omega) = \left\{ \phi \in \mathcal{D}(\Delta_{max}) : \hat{\rho}\phi \in H^{3/2}(\partial\Omega) \right\} \quad (5.11)$$

$$= \left\{ \phi \in \mathcal{D}(\Delta_{max}) : \hat{\tau}\phi \in H^{1/2}(\partial\Omega) \right\}. \quad (5.12)$$

Now let A be the self-adjoint operator in $L^2(\Omega)$ given by the Dirichlet Laplacian

$$A : \mathcal{D}(A) \subseteq L^2(\Omega) \rightarrow L^2(\Omega) \quad A\psi \equiv \Delta^D\psi = \Delta\psi,$$

$$\mathcal{D}(\Delta^D) = \{\psi \in H^2(\Omega) : \rho\psi = 0\}$$

and let $\tau : \mathcal{H}_A \rightarrow \mathfrak{h}$, with $\mathfrak{h} = H^{1/2}(\partial\Omega)$, be the normal derivative operator along $\partial\Omega$ defined above. Thus we are looking for all self-adjoint extensions of the symmetric operator given by the minimal Laplacian

$$S : \mathcal{D}(S) \subseteq L^2(\Omega) \rightarrow L^2(\Omega), \quad S\psi \equiv \Delta_{min}\psi := \Delta\psi,$$

$$\mathcal{D}(S) \equiv \mathcal{D}(\Delta_{min}) \equiv H_0^2(\Omega) := \{ \psi \in H^2(\Omega) : \rho\psi = \tau\psi = 0 \} .$$

Note that by defining the maximal Laplacian Δ_{max} as the distributional Laplacian restricted to $\mathcal{D}(\Delta_{max})$, one has $\Delta_{max} = (\Delta_{min})^*$.

By the definition of $G(0)$ one has, for any $h \in H^1(\partial\Omega)$ and for any $\psi \in L^2(\Omega)$,

$$\langle G(0)h, \psi \rangle_{L^2(\Omega)} = -\langle \Lambda h, \tau(\Delta^D)^{-1}\psi \rangle_{L^2(\partial\Omega)} .$$

Therefore

$$[G(0)h](x) = - \int_{\partial\Omega} \Lambda h(y) \frac{\partial}{\partial n} g(x, y) d\sigma(y) ,$$

where g is the Dirichlet Green function of Ω for the Laplacian, and so

$$G(0) = K\Lambda ,$$

where K denotes the Poisson operator, i.e. $K : H^{-1/2}(\partial\Omega) \rightarrow \mathcal{D}(\Delta_{max})$ is the continuous linear operator (see e.g. [21], Chapter 2, Section 6) which solves the Dirichlet boundary value problem

$$\Delta K h = 0 , \tag{5.13}$$

$$\hat{\rho} K h = h . \tag{5.14}$$

By (2.2) one has

$$G(z) = G(0) - z(-\Delta^D + z)^{-1}G(0) = -\Delta^D(-\Delta^D + z)^{-1}K\Lambda \tag{5.15}$$

and so $G(z)h \in \mathcal{D}(\Delta_{max})$, $h \in H^{1/2}(\partial\Omega)$, solves the Dirichlet boundary value problem

$$\Delta G(z)h = zG(z)h ,$$

$$\hat{\rho} G(z)h = \Lambda h .$$

Thus

$$\mathcal{R}(G(z)) \cap \mathcal{D}(\Delta^D) = \{0\}$$

and condition (2.1) is satisfied.

Now, according to (2.6), we define the bounded linear operator

$$\Gamma(z) : H^{1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega) ,$$

$$\Gamma(z) := \hat{\Gamma}_{(0)}(z) \equiv \tau(G(0) - G(z)) = z\tau(-\Delta^D + z)^{-1}K\Lambda . \tag{5.16}$$

Since $\mathcal{D}(\Delta_{\Pi, \Theta}^D) \subseteq \mathcal{D}(\Delta_{max})$, the trace operators $\hat{\rho}$ and $\hat{\tau}$ act on $\mathcal{D}(\Delta_{\Pi, \Theta}^D)$. Thus for any $\psi \in \mathcal{D}(\Delta_{\Pi, \Theta}^D)$, which according to Theorem 2.1 can be written as

$$\psi = \psi_z + G(z)\Pi\Gamma_{\Pi, \Theta}(z)^{-1}\Pi\tau\psi_z \in , \quad z \in \mathbb{C} \setminus \mathbb{R} ,$$

one has

$$\hat{\rho}\psi = \Lambda\Pi\Gamma_{\Pi, \Theta}(z)^{-1}\Pi\tau\psi_z , \tag{5.17}$$

i.e.

$$\Sigma\hat{\rho}\psi \in \mathcal{D}(\Theta) \subseteq \mathcal{R}(\Pi) , \quad \Theta\Sigma\hat{\rho}\psi = \Pi(\tau\psi_z - \Gamma(z)\Sigma\hat{\rho}\psi) , \tag{5.18}$$

and

$$\hat{\tau}\psi = \tau\psi_z - \Gamma(z)\Sigma\hat{\rho}\psi + \hat{\tau}G(0)\Sigma\hat{\rho}\psi. \tag{5.19}$$

Such relations (5.17)-(5.19) show that for any $\psi \in \mathcal{D}(\Delta_{\Pi,\Theta}^D)$ the regularised trace operator $\hat{\tau}_0$ defined by

$$\hat{\tau}_0\psi := \hat{\tau}(\psi - G(0)\Sigma\hat{\rho}\psi)$$

is $H^{1/2}(\partial\Omega)$ -valued and the boundary condition

$$\Pi\hat{\tau}_0\psi = \Theta\Sigma\hat{\rho}\psi$$

holds true. By elliptic regularity one can define $\hat{\tau}_0$ on a larger domain: by (5.11) one has

$$\mathcal{D}(\Delta_{max}) \cap \mathcal{K}(\hat{\rho}) = \mathcal{D}(\Delta^D)$$

and so, for any $\psi \in \mathcal{D}(\Delta_{max})$, $\psi - G(z)\Sigma\hat{\rho}\psi$ belongs to $\mathcal{D}(\Delta^D)$. Thus (see [13], Theorem III 1.2)

$$\begin{aligned} \hat{\tau}_0 : \mathcal{D}(\Delta_{max}) &\rightarrow H^{1/2}(\partial\Omega), \\ \hat{\tau}_0\psi &= \tau(\psi - G(0)\Sigma\hat{\rho}\psi) = \tau(\psi - K\hat{\rho}\psi) = \hat{\tau}\psi - P\hat{\rho}\psi, \end{aligned}$$

where the bounded linear operator P , known as the Dirichlet-to-Neumann operator over $\partial\Omega$, is given by (see e.g. [13], Theorem III 1.1)

$$P : H^{-1/2}(\partial\Omega) \rightarrow H^{-3/2}(\partial\Omega), \quad P := \hat{\tau}K.$$

Since

$$\psi - G(0)\Sigma\hat{\rho}\psi - (\Delta^D)^{-1}\Delta\psi \in \mathcal{K}(\Delta^D) = \{0\},$$

alternatively one can define $\hat{\tau}_0$ by (see [13], Theorem III 1.2)

$$\hat{\tau}_0 : \mathcal{D}(\Delta_{max}) \rightarrow H^{1/2}(\partial\Omega), \quad \hat{\tau}_0 := \tau(\Delta^D)^{-1}\Delta.$$

Since any $\psi \in \mathcal{D}(\Delta_{max})$ can be decomposed as

$$\psi = (\psi - G(z)\Sigma\hat{\rho}\psi) + G(z)\Sigma\hat{\rho}\psi$$

and

$$\begin{aligned} \Delta_{\Pi,\Theta}^D\psi &= \Delta^D\psi_z + zG(z)\Sigma\hat{\rho}\psi \\ &= \Delta\psi + (-\Delta + z)G(z)\Sigma\hat{\rho}\psi \\ &= \Delta\psi, \end{aligned}$$

in conclusion by Theorem 2.1 one obtains, for any $(\Pi, \Theta) \in \mathbf{E}(\mathfrak{h})$, $\mathfrak{h} = H^{1/2}(\partial\Omega)$,

$$\Delta_{\Pi,\Theta}^D : \mathcal{D}(\Delta_{\Pi,\Theta}^D) \subseteq L^2(\Omega) \rightarrow L^2(\Omega), \quad \Delta_{\Pi,\Theta}^D\psi = \Delta\psi,$$

$$\mathcal{D}(\Delta_{\Pi,\Theta}^D) = \{\psi \in \mathcal{D}(\Delta_{max}) : \Sigma\hat{\rho}\psi \in \mathcal{D}(\Theta), \quad \Pi\hat{\tau}_0\psi = \Theta\Sigma\hat{\rho}\psi\}$$

and

$$(-\Delta_{\Pi,\Theta}^D + z)^{-1} = (-\Delta^D + z)^{-1} + G(z)\Pi(\Theta + \Pi\Gamma(z)\Pi)^{-1}\Pi G(z)^*,$$

where $G(z)$ and $\Gamma(z)$ are given in (5.15) and (5.16) respectively.

Since $\Sigma : H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$ is unitary and Π is an orthogonal projection in $H^{1/2}(\partial\Omega)$, we can re-parametrise the extensions $\Delta_{\Pi,\Theta}^D$ by the couple (X, L) , where $X \subseteq H^{-1/2}(\partial\Omega)$ is a closed subspace and L is self-adjoint from X to its dual (with respect to the $L^2(\partial\Omega)$ pairing) X^* . In this case one has

$$\mathcal{D}(\Delta_{X,L}^D) = \{\psi \in \mathcal{D}(\Delta_{max}) : \hat{\rho}\psi \in \mathcal{D}(L), \quad \hat{\tau}_0\psi|_X = L\hat{\rho}\psi\},$$

where the boundary condition $\hat{\tau}_0\psi|_X = L\hat{\rho}\psi$ means

$$\forall f \in X \cap L^2(\partial\Omega), \quad \langle \hat{\tau}_0\psi, f \rangle_{L^2(\partial\Omega)} = \langle L\hat{\rho}\psi, f \rangle_{L^2(\partial\Omega)}.$$

This alternative description reproduces the one obtained (for a general strongly elliptic operator) in [13], Theorem III 4.1 (also see [14], [15] and references therein).

The usual Robin-like boundary conditions can be recovered in the following way: since $G(0)h$ solves (5.13)-(5.14), if $h \in H^{5/2}(\partial\Omega)$ then $G(0)h \in H^2(\Omega)$ by elliptic regularity (see (5.11)) and so we can define the unbounded operator in $H^{1/2}(\partial\Omega)$

$$\Theta_0 : H^{5/2}(\partial\Omega) \subseteq H^{1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega),$$

$$\Theta_0 := -\tau G(0) = -\tau K \Lambda = -P \Lambda.$$

Θ_0 is symmetric since P is $L^2(\partial\Omega)$ -symmetric by Green's formula and so, given any $L^2(\partial\Omega)$ -symmetric bounded linear operator

$$B : H^{3/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega),$$

we can define the unbounded symmetric operator

$$\Theta_B : H^{5/2}(\partial\Omega) \subseteq H^{1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega),$$

$$\Theta_B := \Theta_0 + B \Lambda = (-P + B) \Lambda.$$

Since Λ^2 is self-adjoint and $(-P + B)\Sigma$ is bounded, one has

$$\Theta_B^* = \Lambda^2((-P + B)\Sigma)^* \supseteq \Theta_B.$$

Thus $((-P + B)\Sigma)^*$ coincides with $\Sigma^2(-P + B)\Lambda$ on $H^{5/2}(\partial\Omega)$ and therefore Θ_B is self-adjoint if and only if

$$\{\phi \in H^{-1/2}(\partial\Omega) : (-P + B)\phi \in H^{1/2}(\partial\Omega)\} \subseteq H^{3/2}(\partial\Omega). \tag{5.20}$$

Here, by a slight abuse of notation, we used the same symbol B to denote the bounded extension of B given by the operator on $H^{-1/2}(\partial\Omega)$ to $H^{-3/2}(\partial\Omega)$ obtained by considering its adjoint (with respect to the $L^2(\partial\Omega)$ pairing). By elliptic regularity one has (see [13], Theorem III 5.4)

$$\{\phi \in H^{-1/2}(\partial\Omega) : P\phi \in H^s(\partial\Omega)\} \subseteq H^{s+1}(\partial\Omega).$$

Thus (5.20) holds true (by an iterative argument) when B maps $H^s(\partial\Omega)$ to $H^{s-1+\epsilon}(\partial\Omega)$ for any $s \in [-\frac{1}{2}, \frac{3}{2}]$ and for some $\epsilon > 0$ (see [13], Chapter III, Section 6, for a similar kind of results). So B could be a pseudo-differential operator of order $1 - \epsilon$, in

particular the multiplication by a (sufficiently regular) function. The case in which $\epsilon = 0$ is more delicate and a direct analysis is required: (5.20) holds true when $-P + B$ is an elliptic pseudo-differential operator of order one; this can be checked by studying its symbol (see [15], Corollary 9.34, for the case in which $A = -\Delta + 1$, B is differential of order one and $\Omega = \mathbb{R}_+^n$).

By taking $\Pi = 1$, $\Theta = \Theta_B$, one then obtains the self-adjoint extension

$$\Delta_B : \mathcal{D}(\Delta_B) \subseteq L^2(\Omega) \rightarrow L^2(\Omega), \quad \Delta_B \psi := \Delta \psi,$$

with domain (the second expression being consequence of elliptic regularity, see (5.11))

$$\begin{aligned} \mathcal{D}(\Delta_B) &:= \left\{ \psi \in \mathcal{D}(\Delta_{\max}) : \hat{\rho}\psi \in H^{3/2}(\partial\Omega), \hat{\tau}\psi = B\hat{\rho}\psi \right\} \\ &\equiv \left\{ \psi \in H^2(\Omega) : \tau\psi = B\rho\psi \right\}. \end{aligned}$$

We refer to [13], Chapter III, Section 6, for a detailed study of the properties of $-P + B$ and their relations with properties of Δ_B .

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