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ESTIMATES OF INVERSES OF MULTIVARIABLE TOEPLITZ MATRICES

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Abstract. The Gohberg-Semencul formula provides a formula for the inverse of a Toeplitz matrix based on the entries in the first and last columns of the inverse, under certain nonsingularity conditions. In this paper we study similar formulas for multivariable Toeplitz matrices, and we show that in the positive definite case these expressions provide upper bounds for the inverse in the Loewner order. Some numerical experiments regarding the proximity of the estimate are included.

In this paper we prove the following multivariable generalization of the classical Gohberg-Semencul formula (see [6]; see also [5] for matrix valued generalizations). We first need some notation. As usual we let $\mathbb{N}_0, \mathbb{Z}, \mathbb{C}, \mathbb{T}, \mathbb{D}$ denote the sets of nonnegative integers, integers, complex numbers, complex numbers of modulus one, and complex numbers of modulus less than one, respectively. In addition, we denote $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$. Let $d \geqslant 1$. For $z = (z_1, \dots, z_d) \in \mathbb{C}^d$ and $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$, we let $z^k = z_1^{k_1} \cdots z_d^{k_d}$, where for negative k_i we have $z_i \neq 0$. When $n = (n_1, \dots, n_d) \in \mathbb{N}_0^d$ we let $\underline{n}$ denote the set $\underline{n} = \underline{n_1} \times \cdots \times \underline{n_d}$, where $\underline{p} = \{0, \dots, p\}$. When $P(z) = \sum_{k \in \underline{n}} P_k z^k$, we say that P is a polynomial *of degree at most* n. We say that P is *stable* when $P(z)$ is invertible for $z \in \overline{\mathbb{D}}^d$. With P we associate its *adjoint* P^*, which is given by $P^*(z) = \sum_{k \in \underline{n}} P_k^* z^{-k}$.

THEOREM 0.1. *Let $n \in \mathbb{N}_0^d$ and $P(z) = \sum_{k \in \underline{n}} P_k z^k$ and $R(z) = \sum_{k \in \underline{n}} R_k z^k$ be operator valued stable polynomials of degree at most n so that*

$$P(z)P(z)^* = R(z)^* R(z), z \in \mathbb{T}^d.$$

Put

$$F(z) = P(z)^{*-1} P(z)^{-1} = R(z)^{-1} R(z)^{*-1}, \tag{0.1}$$

and let $F(z) = \sum_{j \in \mathbb{Z}^d} F_j z^j$, $z \in \mathbb{T}^d$, be its Fourier expansion. Put $\Lambda = \underline{n} \setminus \{n\}$. Then

$$0 < [(F_{k-l})_{k,l \in \Lambda}]^{-1} \leqslant AA^* - B^*B, \tag{0.2}$$

where

$$A = (P_{k-l})_{k,l \in \Lambda}, \quad B = (R_{k-l})_{\substack{k \in n+\Lambda \\ l \in \Lambda}}$$

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and $P_k = R_k = 0$ *whenever* $k \notin \underline{n}$. *When* $d = 1$ *the second inequality in* (0.2) *is an equality.*

In Section 1 we will prove Theorem 0.1. In Section 2 we will perform some numerical experiments.

1. Proof of the main result

Let us start by illustrating Theorem 0.1 by specifying it in a low dimensional case.

EXAMPLE 1.1. Let $d = 2$ and $n_1 = n_2 = 2$. Thus $\Lambda = (\{0,1,2\} \times \{0,1,2\}) \setminus \{(2,2)\}$, which we will order lexicographically, giving

$$\Lambda = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1)\}.$$

Then

$$A = \begin{pmatrix} P_{00} & & & & & & & \\ P_{01} & P_{00} & & & & & & \\ P_{02} & P_{01} & P_{00} & & & & & \\ P_{10} & & & P_{00} & & & & \\ P_{11} & P_{10} & & P_{01} & P_{00} & & & \\ P_{12} & P_{11} & P_{10} & P_{02} & P_{01} & P_{00} & & \\ P_{20} & & & P_{10} & & & P_{00} & \\ P_{21} & P_{20} & & P_{11} & P_{10} & & P_{01} & P_{00} \end{pmatrix}$$

and

$$B = \begin{pmatrix} R_{22} & R_{21} & R_{20} & R_{12} & R_{11} & R_{10} & R_{02} & R_{01} \\ & R_{22} & R_{21} & & R_{12} & R_{11} & & R_{02} \\ & & R_{22} & & & R_{12} & & \\ & & & R_{22} & R_{21} & R_{20} & R_{12} & R_{11} \\ & & & & R_{22} & R_{21} & & R_{12} \\ & & & & & R_{22} & & \\ & & & & & & R_{22} & R_{21} \\ & & & & & & & R_{22} \end{pmatrix}$$

If we let

$$(F_{k-l})_{k,l\in\Lambda}$$

$$= \begin{pmatrix} 0.1532 & 0.0186 & 0.0005 & 0.0084 & 0.0155 & 0.0231 & 0.0118 & -0.0079 \\ 0.0186 & 0.1532 & 0.0186 & 0.0005 & 0.0084 & 0.0155 & 0.0017 & 0.0118 \\ 0.0005 & 0.0186 & 0.1532 & 0.0016 & 0.0005 & 0.0084 & 0.0002 & 0.0017 \\ 0.0084 & 0.0005 & 0.0016 & 0.1532 & 0.0186 & 0.0005 & 0.0084 & 0.0155 \\ 0.0155 & 0.0084 & 0.0005 & 0.0186 & 0.1532 & 0.0186 & 0.0005 & 0.0084 \\ 0.0231 & 0.0155 & 0.0084 & 0.0005 & 0.0186 & 0.1532 & 0.0016 & 0.0005 \\ 0.0118 & 0.0017 & 0.0002 & 0.0084 & 0.0005 & 0.0016 & 0.1532 & 0.0186 \\ -0.0079 & 0.0118 & 0.0017 & 0.0155 & 0.0084 & 0.0005 & 0.0186 & 0.1532 \end{pmatrix}$$

then $AA^* - B^*B - [(F_{k-l})_{k,l\in\Lambda}]^{-1}$ equals the positive semidefinite matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1193 & 0.0566 & 0 & -0.0166 & 0.0954 & 0 & 0 \\ 0 & 0.0566 & 0.2564 & 0 & 0.0003 & 0.0339 & 0 & 0 \\ 0 & 0 & 0 & 0.0813 & -0.0441 & 0 & -0.0249 & -0.0309 \\ 0 & -0.0166 & 0.0003 & -0.0441 & 0.0494 & -0.0129 & 0.0528 & 0.0262 \\ 0 & 0.0954 & 0.0339 & 0 & -0.0129 & 0.0795 & 0 & 0 \\ 0 & 0 & 0 & -0.0249 & 0.0528 & 0 & 0.1575 & -0.0183 \\ 0 & 0 & 0 & -0.0309 & 0.0262 & 0 & -0.0183 & 0.0622 \end{pmatrix}.$$

The zeros in the matrix are no coincidence. Some zeros are easily explained using the main result in [2]. We will explain this further when we provide a new proof for one of the directions of that result (see Corollary 1.6).

We now need a few lemmas.

LEMMA 1.2. *Assume that the operator matrix* $(A_{ij})_{i,j=1}^2 : \mathscr{H}_1 \oplus \mathscr{H}_2 \to \mathscr{H}_1 \oplus \mathscr{H}_2$ *and the operator* A_{22} *are invertible. Then* $S = A_{11} - A_{12}A_{22}^{-1}A_{21}$ *is invertible and*

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} S^{-1} & * \\ * & * \end{pmatrix}. \tag{1.3}$$

Proof. Follows directly from the factorization

$$\begin{pmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{pmatrix} = \begin{pmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{pmatrix}. \tag{1.4}$$

□

We will be interested in looking at equation (1.3) in the following way: suppose that we have identified the inverse of a block matrix, and we are interested in the inverse of the (1,1) block. Then taking a Schur complement in the inverse of the complete block matrix gives us a formula for the inverse of this (1,1) block. Below is one way one can use this observation.

LEMMA 1.3. *Let lower/upper and upper/lower factorizations of the inverse of a block matrix be given, as follows:*

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}^{-1} = \begin{pmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{pmatrix} \tag{1.5}$$

$$= \begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{pmatrix} \begin{pmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{pmatrix}, \tag{1.6}$$

and suppose that R_{22} *and* T_{22} *are invertible. Then*

$$B_{11} = P_{11}Q_{11} - R_{12}T_{21}. \tag{1.7}$$

Proof. Apply Lemma 1.2 to the equality

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}^{-1} = \begin{pmatrix} P_{11}Q_{11} & R_{12}T_{22} \\ R_{22}T_{21} & R_{22}T_{22} \end{pmatrix}.$$

□

COROLLARY 1.4. *Consider a positive definite operator matrix $(B_{ij})_{i,j=1}^3$ of which the lower/upper and upper/lower block Cholesky factorizations of its inverse are given, as follows:*

$$\begin{aligned} [(B_{ij})_{i,j=1}^3]^{-1} &= \begin{pmatrix} P_{11} & 0 & 0 \\ P_{21} & P_{22} & 0 \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \begin{pmatrix} P_{11}^* & P_{21}^* & P_{31}^* \\ 0 & P_{22}^* & P_{32}^* \\ 0 & 0 & P_{33}^* \end{pmatrix} &(1.8)\\ &= \begin{pmatrix} R_{11}^* & R_{21}^* & R_{31}^* \\ 0 & R_{22}^* & R_{32}^* \\ 0 & 0 & R_{33}^* \end{pmatrix} \begin{pmatrix} R_{11} & 0 & 0 \\ R_{21} & R_{22} & 0 \\ R_{31} & R_{32} & R_{33} \end{pmatrix}, &(1.9)\end{aligned}$$

with R_{22} and R_{33} invertible. Then

$$B_{11}^{-1} = P_{11}P_{11}^* - R_{21}^*R_{21} - R_{31}^*R_{31} \leqslant P_{11}P_{11}^* - R_{31}^*R_{31}. \tag{1.10}$$

Proof. By Lemma 1.3 we have that

$$\begin{aligned} B_{11}^{-1} &= P_{11}P_{11}^* - \begin{pmatrix} R_{21}^* & R_{31}^* \end{pmatrix} \begin{pmatrix} R_{21} \\ R_{31} \end{pmatrix} \\ &= P_{11}P_{11}^* - R_{21}^*R_{21} - R_{31}^*R_{31} \leqslant P_{11}P_{11}^* - R_{31}^*R_{31}. \end{aligned} \tag{1.11}$$

□

Before we prove the main result, we need to introduce some notation. Let $\mathscr{H}$ be a Hilbert space and let $\mathscr{B}(\mathscr{H})$ denote the Banach space of bounded linear operators on $\mathscr{H}$. We let $L_\infty = L_\infty(\mathbb{T}^d; \mathscr{B}(\mathscr{H}))$ denote the Lebesgue space of essentially bounded $\mathscr{B}(\mathscr{H})$-valued measurable functions on $\mathbb{T}^d$, and we let $L_2 = L_2(\mathbb{T}^d; \mathscr{H})$ and $H_2 = H_2(\mathbb{T}^d; \mathscr{H})$ denote the Lebesgue and Hardy space of square integrable $\mathscr{H}$-valued functions on $\mathbb{T}^d$, respectively. As usual we view H_2 as a subspace of L_2. For $L(z) = \sum_{i \in \mathbb{Z}^d} L_i z^i \in L_\infty$ we will consider its multiplication operator $M_L : L_2 \to L_2$ given by

$$(M_L(f))(z) = L(z)f(z).$$

The Toeplitz operator $T_L : H_2 \to H_2$ is defined as the compression of M_L to H_2. For $\Lambda \subset \mathbb{Z}^d$ we let S_Λ denote the subspace $\{F \in L_2 \ : \ F(z) = \sum_{k \in \Lambda} F_k z^k\}$ of L_2 consisting of those functions with Fourier support in Λ. In addition, we let P_Λ denote the orthogonal projection onto S_Λ. So, for instance, $P_{\mathbb{N}_0^d}$ is the orthogonal projection onto H_2 and $T_L = P_{\mathbb{N}_0^d} M_L P_{\mathbb{N}_0^d}^*$.

Proof of Theorem 0.1. Clearly we have that $M_{F^{-1}} = M_P M_{P^*} = M_{R^*} M_R$. With respect to the decomposition $L_2 = H_2^\perp \oplus H_2$ we get that

$$M_F = \begin{pmatrix} * & * \\ * & T_F \end{pmatrix}, M_P = \begin{pmatrix} * & 0 \\ * & T_P \end{pmatrix}, M_{P^{-1}} = \begin{pmatrix} * & 0 \\ * & T_{P^{-1}} \end{pmatrix}, \tag{1.12}$$

$$M_R = \begin{pmatrix} * & 0 \\ * & T_R \end{pmatrix}, M_{R^{-1}} = \begin{pmatrix} * & 0 \\ * & T_{R^{-1}} \end{pmatrix}, \tag{1.13}$$

where we used that $M_{P^{\pm 1}}[H_2] \subset H_2$ and $M_{R^{\pm 1}}[H_2] \subset H_2$ which follows as $P^{\pm 1}$ and $R^{\pm 1}$ are analytic in $\mathbb{D}^d$. It now follows that $T_F = (T_P)^{*-1}(T_P)^{-1}$ and thus

$$(T_F)^{-1} = T_P (T_P)^*. \tag{1.14}$$

Next, decompose $H_2 = S_\Lambda \oplus S_\Theta \oplus S_{n+\mathbb{N}_0^d}$, where $\Lambda = \underline{n} \setminus \{n\}$ and $\Theta = \mathbb{N}_0^d \setminus (\Lambda \cup (n + \mathbb{N}_0^d))$, and write T_P and T_R with respect to this decomposition:

$$T_P = \begin{pmatrix} P_{11} & & \\ P_{21} & P_{22} & \\ P_{31} & P_{32} & P_{33} \end{pmatrix}, T_R = \begin{pmatrix} R_{11} & & \\ R_{21} & R_{22} & \\ R_{31} & R_{32} & R_{33} \end{pmatrix}. \tag{1.15}$$

As the Fourier support of P and R lies in $\underline{n}$, and as $P(z)P(z)^* = R(z)^* R(z)$ on $\mathbb{T}^d$, it is not hard to show that

$$T_P T_P^* P^*_{n+\mathbb{N}_0^d} = T_R^* T_R P^*_{n+\mathbb{N}_0^d}, \tag{1.16}$$

which yields that

$$P_{31}P_{31}^* + P_{32}P_{32}^* + P_{33}P_{33}^* = R_{33}^* R_{33}, P_{21}P_{31}^* + P_{22}P_{32}^* = R_{32}^* R_{33}, P_{11}P_{31}^* = R_{31}^* R_{11}.$$

Thus we can factor $T_P T_P^*$ as

$$T_P T_P^* = \begin{pmatrix} \tilde{R}_{11}^* & \tilde{R}_{21}^* & R_{31}^* \\ & \tilde{R}_{22}^* & R_{32}^* \\ & & R_{33}^* \end{pmatrix} \begin{pmatrix} \tilde{R}_{11} & & \\ \tilde{R}_{21} & \tilde{R}_{22} & \\ R_{31} & R_{32} & R_{33} \end{pmatrix}, \tag{1.17}$$

for some $\tilde{R}_{11}, \tilde{R}_{21}$ and $\tilde{R}_{22}$. Combining now (1.14) with the two factorizations of $T_P T_P^*$ given via (1.17), we get by Corollary 1.4 that

$$[(F_{k-l})_{k,l \in \Lambda}]^{-1} \leqslant P_{11}P_{11}^* - R_{31}R_{31}^*.$$

This proves the claim. □

A more detailed analysis of where $T_P T_P^*$ and $T_R^* T_R$ coincide, other than indicated in (1.16), gives an alternative way to prove one direction of Theorem 2.4.1 (the direction (i) $\to$ (iii)) in [2] (see also [3] for the operator valued case). As we will see in the proof, the argument works only in the case of two variables ($d = 2$).

THEOREM 1.5. [2] *Let $d = 2$ and let n, P, R, F and Λ be as in Theorem 0.1. Then*

$$[(F_{k-l})_{k,l \in \Lambda}]^{-1}$$

has zeroes in location $(k,l) = ((k_1,k_2),(l_1,l_2))$ where $(k_1,l_2) = (n_1,n_2)$ or where $(k_2,l_1) = (n_2,n_1)$.

Proof. Let us decompose H_2 as

$$H_2 = S_{\underline{n_1-1}\times\underline{n_2-1}} \oplus S_{\underline{n_1-1}\times\{n_2\}} \oplus S_{\{n_1\}\times\underline{n_2-1}} \oplus S_{n_1+\mathbb{N}\times\underline{n_2-1}} \oplus S_{\underline{n_1-1}\times n_2+\mathbb{N}} \oplus S_{n+\mathbb{N}_0^2}. \quad (1.18)$$

Writing $T_PT_P^* - T_R^*T_R$ with respect to this decomposition we get that this operator is of the form

$$T_PT_P^* - T_R^*T_R = \begin{pmatrix} * & * & * & * & * & 0 \\ * & * & 0 & 0 & * & 0 \\ * & 0 & * & * & 0 & 0 \\ * & 0 & * & * & 0 & 0 \\ * & * & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For instance, to explain the zeroes in the $(2,3)$ and $(2,4)$ positions, note that

$$P_{\underline{n_1-1}\times\{n_2\}} T_PT_P^* P^*_{(n_1+\mathbb{N}_0)\times\underline{n_2-1}} = P_{\underline{n_1-1}\times\{n_2\}} T_R^*T_R P^*_{(n_1+\mathbb{N}_0)\times\underline{n_2-1}}.$$

The fact that the last column is zero (and by symmetry, the last row) comes from observation (1.16). As a general observation, notice that if operator matrices G and H coincide on certain locations, then so will the Schur complement expressions $G - KL^{-1}K^*$ and $H - KL^{-1}K^*$. Therefore, taking the Schur complement in $T_PT_P^*$ and $T_R^*T_R$ with respect to the last row and column (which is where they completely coincide), we see that the resulting operators Σ_P and Σ_R, respectively, satisfy

$$\Sigma_P - \Sigma_R = \begin{pmatrix} * & * & * & * & * \\ * & * & 0 & 0 & * \\ * & 0 & * & * & 0 \\ * & 0 & * & * & 0 \\ * & * & 0 & 0 & * \end{pmatrix}. \quad (1.19)$$

The matrix $[(F_{k-l})_{k,l\in\Lambda}]^{-1}$ is the Schur complement of $T_PT_P^*$ supported in Λ, which is the same as the Schur complement in Σ_P supported in the first three rows and columns (as $\Lambda = (\underline{n_1-1}\times\underline{n_2-1}) \cup (\underline{n_1-1}\times\{n_2\}) \cup (\{n_1\}\times\underline{n_2-1})$). Using Lemma 2.1 in [1] one gets that $\Sigma_R = G^*G$ where

$$G = (R_{k-l})_{k,l\in\mathbb{N}_0^2\setminus n+\mathbb{N}_0^2}.$$

With respect to the decomposition (1.18) we have that Σ_R is of the form

$$\Sigma_R = \begin{pmatrix} * & * & * & * & * \\ * & * & 0 & 0 & * \\ * & 0 & * & * & 0 \\ * & 0 & * & * & 0 \\ * & * & 0 & 0 & * \end{pmatrix}.$$

By (1.19) it follows that Σ_P must have the same form. But now, if we take the Schur complement in Σ_P supported in the first three rows and columns, we get that

$$[(F_{k-l})_{k,l\in\Lambda}]^{-1} = \begin{pmatrix} * & * & * \\ * & * & 0 \\ * & 0 & * \end{pmatrix} - \begin{pmatrix} * & * \\ 0 & * \\ * & 0 \end{pmatrix} \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}^{-1} \begin{pmatrix} * & 0 & * \\ * & * & 0 \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & 0 \\ * & 0 & * \end{pmatrix}.$$

This proves the result. □

COROLLARY 1.6. *Let $d=2$ and let A,B and F be as in Theorem 0.1. Then*

$$AA^* - B^*B - [(F_{k-l})_{k,l\in\Lambda}]^{-1}$$

has zeros in the locations $(k,l) = ((k_1,k_2),(l_1,l_2))$ where $k=0$ or $l=0$ or $(k_1,l_2)=(n_1,n_2)$ or $(k_2,l_1)=(n_2,n_1)$.

Proof. As AA^*, B^*B and $[(F_{k-l})_{k,l\in\Lambda}]^{-1}$ all have zeros in positions $(k,l) = ((k_1,k_2),(l_1,l_2))$ where $(k_1,l_2)=(n_1,n_2)$ or $(k_2,l_1)=(n_2,n_1)$, then so does $AA^* - B^*B - [(F_{k-l})_{k,l\in\Lambda}]^{-1}$. It therefore remains to show that $AA^* - B^*B - [(F_{k-l})_{k,l\in\Lambda}]^{-1}$ has zeros in locations (k,l) where k or l is zero. We focus on the case when $l=0$. We will show this by showing that $[(F_{k-l})_{k,l\in\Lambda}]^{-1}$ and AA^*-B^*B coincide in the first column ($l=0$). It follows from the equality (0.1). Indeed, as $F(z)P(z) = P(z)^{*-1}$ and $P(z)$ is stable, we have that

$$\sum_{l\in\underline{n}} F_{-l}P_l = P_0^{*-1}, \ \sum_{l\in\underline{n}} F_{k-l}P_l = 0, \ k\in\underline{n}\setminus\{0\}. \tag{1.20}$$

In addition, since $F(z)R(z)^* = R(z)^{-1}$, we also have that

$$\sum_{l\in\underline{n}} F_lR_l^* = R_0^{-1}\ , \sum_{l\in\underline{n}} F_{l-k}R_l^* = 0, \ k\in-\underline{n}\setminus\{0\}. \tag{1.21}$$

Replacing l by $n-l$, and k by $n-k$ in (1.21) we obtain

$$\sum_{l\in\underline{n}} F_{n-l}R_{n-l}^* = R_0^{-1}, \ \sum_{l\in\underline{n}} F_{k-l}R_{n-l}^* = 0, \ k\in\Lambda. \tag{1.22}$$

Now (1.20) implies that $\sum_{l\in\Lambda} F_{-l}P_l = P_0^{*-1} - F_{-n}P_n$, and thus

$$\sum_{l\in\Lambda} F_{-l}P_lP_0^* = I - F_{-n}P_nP_0^*. \tag{1.23}$$

Equation (1.22) with $k=0$ and multiplied on the right with R_n implies that

$$\sum_{l\in\Lambda} F_{-l}R_{n-l}^*R_n = -F_{-n}R_0^*R_n. \tag{1.24}$$

Using that $P(z)P(z)^* = R(z)^*R(z)$, and thus $P_nP_0^* = R_0^*R_n$, we get by combining (1.23) and (1.24) that

$$\sum_{l\in\Lambda} F_{-l}P_lP_0^* - \sum_{l\in\Lambda} F_{-l}R_{n-l}^*R_n = I. \tag{1.25}$$

Next, combining (1.20) and (1.22), we get that

$$\sum_{l\in\Lambda} F_{k-l}P_lP_0^* - \sum_{l\in\Lambda} F_{k-l}R_{n-l}^*R_n = F_{n-l}P_nP_0^* - F_{n-l}R_0^*R_n = 0, \ k\in\Lambda\setminus\{0\}. \tag{1.26}$$

Combining (1.25) and (1.26) yields that

$$[(F_{k-l})_{k,l\in\Lambda}](AA^* - B^*B)$$

has a first column equal to $\begin{pmatrix} I & 0 & \cdots & 0 \end{pmatrix}^T$, and thus the first columns of $[(F_{k-l})_{k,l\in\Lambda}]^{-1}$ and $AA^* - B^*B$ coincide. This proves the claim. □

2. Numerical results

The result in Theorem 0.1 may lead to effective algorithms for dealing with multilevel positive definite matrices. We have performed some numerical experiments where the off-diagonal entries of the symmetric multilevel Toeplitz matrix were chosen to be random numbers in the interval $[-1, 1]$, and the main diagonal entry was chosen so that 1 is the smallest eigenvalue of the matrix. Below the results are shown.

size(T)	$\lambda_{max}(T^{-1} - AA^* + B^*B)/\lambda_{max}(T^{-1})$	cond(T)	cond($(AA^* - B^*B)T$)
24	0.1563	4.5806	1.4535
24	0.1415	4.3732	1.3832
24	0.1644	4.2670	1.5566
99	0.2507	8.0674	2.9742
99	0.2347	9.1402	2.6518
99	0.2284	5.8057	2.1937
399	0.3223	21.8879	8.3612
399	0.2403	6.8317	2.5388
399	0.1469	3.6809	1.5269
1599	0.2881	37.2075	11.1203
1599	0.1968	6.7081	2.1329
1599	0.2055	6.6974	2.0156

One sees that the maximal eigenvalue of $T^{-1} - (AA^* - B^*B)$ ranges between 0.14 and 0.33 times the maximial eigenvalue of T^{-1}, and the condition number of $(AA^* - B^*B)T$ is in all cases less than half the condition number of T. In the three variable case we get similar results:

size(T)	$\lambda_{max}(T^{-1} - AA^* + B^*B)/\lambda_{max}(T)$	cond(T)	cond($(AA^* - B^*B)T$)
26	0.1151	3.8545	1.3411
26	0.1662	3.8994	1.5386
26	0.2367	12.2489	2.8928
124	0.2208	4.2367	2.0020
124	0.1755	3.3766	1.6656
124	0.2222	4.1797	1.8494

It seems that these are encouraging signs that the Gohberg-Semencul expression can be used in finding a fast algorithm for linear systems involving multivariable Toeplitz matrices. We plan to pursue this in a future publication. Let us mention that in [9] an approximation algorithm using matrices of so-called low tensor rank is proposed for this multivariable setting, and they state initial "encouraging" results. Other papers have indicated negative results, such as [10] where it was shown that any circulant-like preconditioner for multivariable Toeplitz matrices is not superlinear (this result was later generalized for other classes of preconditioners; see [11]). One of the main reasons for the lack of fast algorithms for the multivariable Toeplitz setting has been that the highly useful Gohberg-Semencul and related formulas (see [6], [7], [8], [12], [4]) for the inverse matrices have only been established in the one-variable case.

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