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DETERMINANT REPRESENTATIONS OF APPELL POLYNOMIAL SEQUENCES

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Abstract. In this paper, we develop a new representation for Appell polynomial sequence via determinants. In addition, we demonstrate a direct application of this determinant formula (representation) for Appell sequence by deriving a determinant formula for a power series of a function $f(t)$ in terms of the power series of its algebraic inverse function $1/f(t)$.

1. Introduction

Appell polynomial sequences are connected with numerous problems of applied mathematics, theoretical physics, chemistry, approximation theory and several other mathematical branches. In the past few decades there has been an interest in Appell polynomials. Di Bucchianico recently summarized and documented more than five hundred old and new findings related to study on Appell polynomial sequences in [1]. One attention of such study is to find a novel representation for Appell polynomials. For instance, in [3], D. Lehmer illustrated six different approaches to representing Bernoulli polynomial sequence, which is one of Appell polynomial sequences. In this paper, we would like to develop a new representation for Appell polynomial sequences by using determinants.

2. Preliminaries

The following definition of Appell polynomials is not the traditional one in [4], but it is convenient for our approach.

DEFINITION 1. A polynomial sequence $\{a_n(x)\}$ is an Appell polynomial sequence associated with $g(t)$ if it satisfies the following differential equation

$$a_n'(x) = na_{n-1}(x), \quad n = 1, 2, 3, \ldots, \tag{1}$$

with the initial condition $a_n(0) = (\frac{1}{g(t)})^{(n)}|_{t=0}$.

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To avoid any unnecessary confusion, we use $f^{(k)}$ to stand for the kth order derivative of f and use f^k to represent the kth power of f in the entire paper. In addition, $f^{(0)} = f$ and $f^0 = 1$.

To prove our main theorem, we need to consider the $(n+1) \times (n+1)$ Leibniz matrix introduced by [2]

$$\mathscr{L}_n[f(t)] = \begin{bmatrix} f(t) & 0 & 0 & \cdots & 0 \\ \frac{f'(t)}{1!} & f(t) & 0 & \cdots & 0 \\ \frac{f''(t)}{2!} & \frac{f'(t)}{1!} & f(t) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{f^{(n)}(t)}{n!} & \frac{f^{(n-1)}(t)}{(n-1)!} & \frac{f^{(n-2)}(t)}{(n-2)!} & \cdots & f(t) \end{bmatrix}_{(n+1)\times(n+1)}. \tag{2}$$

Let $\mathbf{e}_{k,g(t)} = [0, 0, \cdots, 0, \frac{g(t)}{0!}, \frac{g'(t)}{1!}, \ldots, \frac{g^{(n-k)}(t)}{(n-k)!}]^T \in \mathscr{R}^{(n+1)}$. Then we can obtain the following lemma.

LEMMA 1.

$$\mathscr{L}_n[f(t)]\mathbf{e}_{k,g(t)} = \mathbf{e}_{k,f(t)g(t)}. \tag{3}$$

Proof. The ith entry of the product of $\mathscr{L}_n[f(t)]\mathbf{e}_{k,g(t)}$ is

$$\sum_{j=k}^{i} \frac{f^{(i-j)}(t)}{(i-j)!} \frac{g^{(j-k)}(t)}{(j-k)!}.$$

For $i \geqslant k$, let $m = j-k$, then the ith entry of the product of $\mathscr{L}_n[f(t)]\mathbf{e}_{k,g(t)}$ is

$$\begin{aligned} \sum_{m=0}^{i-k} \frac{f^{(i-m-k)}(t)g^{(m)}(t)}{(i-m-k)!m!} &= \frac{1}{(i-k)!} \sum_{m=0}^{i-k} \binom{i-k}{m} f^{(i-m-k)}(t)g^{(m)}(t) \\ &= \frac{1}{(i-k)!}(f(t)g(t))^{(i-k)}. \end{aligned} \tag{4}$$

For $i < k$, the ith entry of the product of $\mathscr{L}_n[f(t)]\mathbf{e}_{k,g(t)}$ is zero. This completes the proof of the Lemma. □

Let $\mathscr{N}_n$ be the following $n \times n$ block matrix defined by

$$\mathscr{N}_n = [\,\alpha \quad | \quad \beta\,],$$

where $\alpha = [g'(t)/1! \quad g''(t)/2! \quad \ldots \quad g^{(n)}(t)/n!]^T$ and

$$\beta = \begin{bmatrix} g(t) & 0 & 0 & \cdots & 0 \\ \frac{g'(t)}{1!} & g(t) & 0 & \cdots & 0 \\ \frac{g''(t)}{2!} & \frac{g'(t)}{1!} & g(t) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{g^{(n-2)}(t)}{(n-2)!} & \frac{g^{(n-3)}(t)}{(n-3)!} & \frac{g^{(n-4)}(t)}{(n-4)!} & \cdots & g(t) \\ \frac{g^{(n-1)}(t)}{(n-1)!} & \frac{g^{(n-2)}(t)}{(n-2)!} & \frac{g^{(n-3)}(t)}{(n-3)!} & \cdots & \frac{g'(t)}{1!} \end{bmatrix}_{(n)\times(n-1)}.$$

β is the $n\times(n-1)$ matrix obtained by deleting the last column from $\mathscr{L}_{n-1}[g(t)]$. We have the following Lemma

LEMMA 2.

$$\det[\mathscr{N}_n]=(-1)^n g^{n+1}(t)(1/g(t))^{(n)}/n!. \tag{5}$$

Proof. Using Lemma 1 yields

$$\mathscr{L}_{n-1}\left[\frac{1}{g(t)}\right]\mathscr{N}_n=\left[\ \mathscr{L}_{n-1}[\tfrac{1}{g(t)}]\alpha \quad | \quad \mathscr{L}_{n-1}[\tfrac{1}{g(t)}]\beta\ \right], \tag{6}$$

where $\mathscr{L}_{n-1}\left[\frac{1}{g(t)}\right]\beta=\begin{bmatrix} I_{n-1}\\ --\\ 0\end{bmatrix}$ and I_{n-1} is the $(n-1)\times(n-1)$ identity matrix. Thus, we have

$$\det\left[\mathscr{L}_{n-1}\left[\frac{1}{g(t)}\right]\mathscr{N}_n\right]=\left(\frac{1}{g(t)}\right)^n\det[\mathscr{N}_n]=(-1)^{n-1}\omega,$$

where ω is the nth entry of $\mathscr{L}_{n-1}[\frac{1}{g(t)}]\alpha$.

Therefore,

$$\det[\mathscr{N}_n]=(-1)^{n-1}g^n(t)\omega. \tag{7}$$

Next, let us evaluate ω

$$\begin{aligned}
\omega &= \left[\begin{matrix}\frac{(1/g(t))^{(n-1)}}{(n-1)!} & \frac{(1/g(t))^{(n-2)}}{(n-2)!} & \cdots & \frac{(1/g(t))}{0!}\end{matrix}\right]\left[\begin{matrix}\frac{g'(t)}{1!} & \frac{g''(t)}{2!} & \cdots & \frac{g^{(n)}(t)}{(n)!}\end{matrix}\right]^T\\
&= \sum_{k=1}^{n}\frac{(1/g(t))^{(n-k)}}{(n-k)!}\frac{g^{(k)}(t)}{k!}\\
&= \frac{1}{n!}\sum_{k=1}^{n}\binom{n}{k}(1/g(t))^{(n-k)}g^{(k)}(t) \qquad (8)\\
&= \frac{1}{n!}\left((g(t)/g(t))^{(n)}-\binom{n}{0}(1/g(t))^{(n)}g(t)^{(0)}\right).
\end{aligned}$$

Noting

$$(g(t)/g(t))^{(n)}=(1)^{(n)}=\sum_{k=0}^{n}\binom{n}{k}(1/g(t))^{(n-k)}g^{(k)}(t)=0,$$

for $n\geqslant 1$, leads

$$\sum_{k=1}^{n}\binom{n}{k}(1/g(t))^{(n-k)}g^{(k)}(t)=-g(t)\left(\frac{1}{g(t)}\right)^{(n)}. \tag{9}$$

Combining Eqs. (8) and (9) we obtain $\omega=\frac{-1}{n!}g(t)(\frac{1}{g(t)})^{(n)}$. Noting Eq. (7), we have

$$\det[\mathscr{N}_n]=(-1)^n g^{n+1}(t)(1/g(t))^{(n)}/n!.$$

This completes the proof of Lemma 2. □

3. Main theorem

After the load of definition and Lemmas, we are ready to develop our main theorem, which provides an alternative representation of Appell polynomial sequence.

THEOREM 1. *Let* $\{a_n(x)\}$ *be an Appell polynomial sequence defined by the generating function*

$$\frac{e^{xt}}{g(t)} = \sum_{n=0}^{\infty} \frac{a_n(x)t^n}{n!}. \tag{10}$$

Then, for $n = 0, 1, 2, , \ldots,$

$$a_n(x) = (-1)^n n! g_0^{-n-1} \det \begin{bmatrix} 1 & g_0 & 0 & 0 & \ldots & 0 \\ \frac{x}{1!} & g_1 & g_0 & 0 & \cdots & 0 \\ \frac{x^2}{2!} & g_2 & g_1 & g_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{x^{n-1}}{(n-1)!} & g_{n-1} & g_{n-2} & g_{n-3} & \cdots & g_0 \\ \frac{x^n}{n!} & g_n & g_{n-1} & g_{n-2} & \cdots & g_1 \end{bmatrix}_{(n+1)\times(n+1)}, \tag{11}$$

where $g(t) = \sum_{n=0}^{\infty} g_n t^n$ *, i.e.,* $g_n = g^{(n)}(t)/n!|_{t=0}$.

Proof. To show that $\{a_n(x)\}$ is an Appell polynomial sequence with respect to a function $g(t)$, we need to prove that $\{a_n(x)\}$ satisfies the differential equation Eq. (1) with corresponding initial condition, i.e., $a_n'(x) = na_{n-1}(x), \quad n = 1, 2, 3, \ldots,$ with the initial condition $a_n(0) = (\frac{1}{g(t)})^{(n)}|_{t=0}$.
Differentiating Eq. (11) leads

$$a_n'(x) = (-1)^n n! g_0^{-n-1} \det \begin{bmatrix} 0 & g_0 & 0 & 0 & \ldots & 0 \\ 1 & g_1 & g_0 & 0 & \cdots & 0 \\ \frac{x}{1!} & g_2 & g_1 & g_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{x^{n-2}}{(n-2)!} & g_{n-1} & g_{n-2} & g_{n-3} & \cdots & g_0 \\ \frac{x^{n-1}}{(n-1)!} & g_n & g_{n-1} & g_{n-2} & \cdots & g_1 \end{bmatrix}_{(n+1)\times(n+1)}. \tag{12}$$

Expanding the determinant in Eq. (12) along the first row yields

$$\begin{aligned} a_n'(x) &= n(-1)^{n-1}(n-1)! g_0^{-n} \det \begin{bmatrix} 1 & g_0 & 0 & 0 & \ldots & 0 \\ \frac{x}{1!} & g_1 & g_0 & 0 & \cdots & 0 \\ \frac{x^2}{2!} & g_2 & g_1 & g_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{x^{n-2}}{(n-2)!} & g_{n-2} & g_{n-3} & g_{n-4} & \cdots & g_0 \\ \frac{x^{n-1}}{(n-1)!} & g_{n-1} & g_{n-2} & g_{n-3} & \cdots & g_1 \end{bmatrix}_{n\times n} \quad (13) \\ &= na_{n-1}(x). \end{aligned}$$

Setting $x=0$ in Eq. (11) we have

$$
\begin{aligned}
a_n(0) &= (-1)^n n! g_0^{-n-1} \det \begin{bmatrix} 1 & g_0 & 0 & 0 & \cdots & 0 \\ 0 & g_1 & g_0 & 0 & \cdots & 0 \\ 0 & g_2 & g_1 & g_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & g_{n-1} & g_{n-2} & g_{n-3} & \cdots & g_0 \\ 0 & g_n & g_{n-1} & g_{n-2} & \cdots & g_1 \end{bmatrix}_{(n+1)\times(n+1)} \\
&= (-1)^n n! g_0^{-n-1} \det \begin{bmatrix} g_1 & g_0 & 0 & \cdots & 0 & 0 \\ g_2 & g_1 & g_0 & \cdots & 0 & 0 \\ g_3 & g_2 & g_1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ g_{n-1} & g_{n-2} & g_{n-3} & \cdots & g_1 & g_0 \\ g_n & g_{n-1} & g_{n-2} & \cdots & g_2 & g_1 \end{bmatrix}_{n\times n}. \qquad (14)
\end{aligned}
$$

Since $g_n = g^{(n)}(t)/n!|_{t=0}$, for $n=0,1,2,\ldots$, Lemma 2 and Eq. (14) suggests

$$a_n(0) = (-1)^n n! g_0^{-n-1} (-1)^n g^{n+1}(t)(1/g(t))^{(n)}/n!|_{t=0} = (1/g(t))^{(n)}|_{t=0}.$$

By the uniqueness of solutions of the system of ODEs and Definition 1, we know $\{a_n(x)\}$ is the Appell polynomial sequence associated with $g(t)$. This completes the proof of the theorem. □

The immediate consequences of Theorem 1 are novel formulas for some well-known Appell polynomial sequences. Bernoulli polynomial sequence $\{\mathscr{B}_n(x)\}$ is the Appell polynomial sequence associated with $g(t) = \frac{e^t-1}{t} = \sum_{k=1}^{\infty} \frac{1}{k!} t^{k-1}$. Therefore, $g_k = \frac{1}{(k+1)!}$. Thus, by Theorem 1, we have

COROLLARY 1. *For* $n=0,1,2,\ldots,$

$$\mathscr{B}_n(x) = (-1)^n n! \det \begin{bmatrix} 1 & \frac{1}{1!} & 0 & 0 & \cdots & 0 \\ \frac{x}{1!} & \frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0 \\ \frac{x^2}{2!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{x^{n-1}}{(n-1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \cdots & \frac{1}{1!} \\ \frac{x^n}{n!} & \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{2!} \end{bmatrix}_{(n+1)\times(n+1)}. \qquad (15)$$

From Corollary 1 we obtain a new representation formula for Bernoulli numbers $\{\mathscr{B}_n\}$. Since $\{\mathscr{B}_n(0)\} = \{\mathscr{B}_n\}$, we have

$$\mathscr{B}_n = \mathscr{B}_n(0) = (-1)^n n! \det \begin{bmatrix} 1 & \frac{1}{1!} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2!} & \frac{1}{1!} & 0 & \cdots & 0 \\ 0 & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \cdots & \frac{1}{1!} \\ 0 & \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{2!} \end{bmatrix}_{(n+1)\times(n+1)}. \tag{16}$$

Similarly, Euler polynomial sequences $\{\mathscr{E}_n(x)\}$ is the Appell polynomial sequence associated with $g(t) = \frac{1}{2}(e^t + 1) = 1 + 1/2\sum_{k=1}^{\infty} \frac{1}{k!}t^k$. Therefore, $g_k = \frac{1}{2}\frac{1}{k!}$, for $k = 1, 2, \dots,$ and $g_0 = 1$. Thus, by Theorem 1, we have

COROLLARY 2. *For* $n = 0, 1, 2, \dots,$

$$\mathscr{E}_n(x) = (-1)^n n! \det \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ \frac{x}{1!} & \frac{1}{2\cdot 1!} & 1 & 0 & \cdots & 0 \\ \frac{x^2}{2!} & \frac{1}{2\cdot 2!} & \frac{1}{2\cdot 1!} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{x^{n-1}}{(n-1)!} & \frac{1}{2\cdot(n-1)!} & \frac{1}{2\cdot(n-2)!} & \frac{1}{2\cdot(n-3)!} & \cdots & 1 \\ \frac{x^n}{n!} & \frac{1}{2\cdot n!} & \frac{1}{2\cdot(n-1)!} & \frac{1}{2\cdot(n-2)!} & \cdots & \frac{1}{2\cdot 1!} \end{bmatrix}_{(n+1)\times(n+1)}. \tag{17}$$

Actually, Theorem 1 provides a nice algorithm for finding power series of a function $f(t)$ if the power series of the algebraic inverse function $1/f(t)$ is known. We summarize it as following:

COROLLARY 3. *Let* $g(t)$ *be the algebraic inverse function of* $f(t)$, *i.e.,* $g(t) = 1/f(t)$. *If* $g(t)$ *has power series* $g(t) = \sum_{k=0}^{\infty} g_k t^k$, *then the power series of* $f(t)$ *is*

$$\sum_{k=0}^{\infty} \frac{(-1)^k m_k}{g_0^{k+1}} t^k, \tag{18}$$

where

$$m_k = \det \begin{bmatrix} 1 & g_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & g_1 & g_0 & 0 & \cdots & 0 & 0 \\ 0 & g_2 & g_1 & g_0 & \cdots & 0 & 0 \\ 0 & g_3 & g_2 & g_1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & g_{k-1} & g_{k-2} & g_{k-3} & \cdots & g_1 & g_0 \\ 0 & g_k & g_{k-1} & g_{k-2} & \cdots & g_2 & g_1 \end{bmatrix}_{(k+1)\times(k+1)}, \quad \textit{for} \quad k = 0, 1, 2, 3, \cdots. \tag{19}$$

Proof. Considering Theorem 1 and setting $x = 0$ in Eqs. (10) and (11) yields

$$f(t) = \frac{1}{g(t)} = \sum_{n=0}^{\infty} \frac{a_n(0)t^n}{n!}, \tag{20}$$

where

$$\begin{aligned} a_n(0) &= (-1)^n n! g_0^{-n-1} \det \begin{bmatrix} 1 & g_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & g_1 & g_0 & 0 & \cdots & 0 & 0 \\ 0 & g_2 & g_1 & g_0 & \cdots & 0 & 0 \\ 0 & g_3 & g_2 & g_1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & g_{n-1} & g_{n-2} & g_{n-3} & \cdots & g_1 & g_0 \\ 0 & g_n & g_{n-1} & g_{n-2} & \cdots & g_2 & g_1 \end{bmatrix}_{(n+1)\times(n+1)} \\ &= (-1)^n n! g_0^{-n-1} m_n. \end{aligned} \tag{21}$$

Replacing $a_n(0)$ in Eq. (20) by $(-1)^n n! g_0^{-n-1} m_n$ leads the result in Corollary 3. □

A useful application of Corollary 3 is to find the determinant expression of a well-known sequence. We illustrate the idea in the following example.

EXAMPLE 1. *It is well known that Euler sequence* $\{\mathscr{E}_n\} = \{1, 0, -1, 0, 5, 0, -61, 0, 1385, \ldots\}$ *has its exponential generating function* $f(t) = secht = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{\mathscr{E}_n}{n!} t^n$. *Then,* $g(t) = 1/f(t) = \frac{e^t + e^{-t}}{2} = \sum_{k=0}^{\infty} \frac{1+(-1)^k}{2(k)!} t^k$. *Therefore,* $g_k = \frac{1+(-1)^k}{2(k)!}$. *Thus, by Corollary 3, we have, for* $n = 0, 1, 2, \ldots,$

$$\mathscr{E}_k / k! = (-1)^k m_k, \qquad \mathscr{E}_k = (-1)^k k! m_k \tag{22}$$

where m_k *is*

$$m_k = \det \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2!} & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{2!} & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \frac{1+(-1)^{k-1}}{2(k-1)!} & \frac{1+(-1)^{k-2}}{2(k-2)!} & \frac{1+(-1)^{k-3}}{2(k-3)!} & \cdots & 0 & 1 \\ 0 & \frac{1+(-1)^k}{2(k)!} & \frac{1+(-1)^{k-1}}{2(k-1)!} & \frac{1+(-1)^{k-2}}{2(k-2)!} & \cdots & \frac{1}{2!} & 0 \end{bmatrix}_{(k+1)\times(k+1)}. \tag{23}$$

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