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ASYMPTOTIC PSEUDOMODES OF TOEPLITZ MATRICES

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(*communicated by L. Rodman*)

Abstract. Questions in probability and statistical physics lead to the problem of finding the eigenvectors associated with the extreme eigenvalues of Toeplitz matrices generated by Fisher-Hartwig symbols. We here simplify the problem and consider pseudomodes instead of eigenvectors. This replacement allows us to treat fairly general symbols, which are far beyond Fisher-Hartwig symbols. Our main result delivers a variety of concrete unit vectors x_n such that if $T_n(a)$ is the $n \times n$ truncation of the infinite Toeplitz matrix generated by a function $a \in L^1$ satisfying mild additional conditions and λ is in the range of this function, then $\|T_n(a)x_n - \lambda x_n\| \to 0$.

1. Introduction and main results

The $n \times n$ Toeplitz matrix $T_n(a)$ generated by a complex-valued function a belonging to $L^1 := L^1(0, 2\pi)$ is the matrix $(a_{j-k})_{j,k=1}^n$ constituted by the Fourier coefficients

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} a(\theta)e^{-ik\theta}d\theta \quad (k \in \mathbf{Z})$$

of the function a. For a real number number $\alpha \in (0, \frac{1}{2})$, put

$$\omega_\alpha(\theta) = |1 - e^{i\theta}|^{-2\alpha} = 2^{-2\alpha}\left|\sin\frac{\theta}{2}\right|^{-2\alpha}.$$

This function, which is a special so-called Fisher-Hartwig symbol, is in L^1 and its Fourier coefficients are

$$(\omega_\alpha)_k = \Gamma(1-2\alpha)\,\frac{\sin\pi\alpha}{\pi(|k|+\alpha)}\,\frac{\Gamma(|k|+1+\alpha)}{\Gamma(|k|+1-\alpha)} \sim \Gamma(1-2\alpha)\,\frac{\sin\pi\alpha}{\pi}\,\frac{1}{|k|^{1-2\alpha}},$$

where $x_k \sim y_k$ means that $x_k/y_k \to 1$. Clearly, ω_α is real-valued, even (after extension to a 2π-periodic function on $\mathbf{R}$), and $\min_\theta \omega_\alpha(\theta) = \omega_\alpha(\pi) = 2^{-2\alpha}$. The matrices $T_n(\omega_\alpha)$ are symmetric and positive definite. Let

$$\lambda_1(T_n(\omega_\alpha)) \leqslant \lambda_2(T_n(\omega_\alpha)) \leqslant \ldots \leqslant \lambda_n(T_n(\omega_\alpha))$$

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be the eigenvalues of $T_n(\omega_\alpha)$. It is well known that $\lambda_k(T_n(\omega_\alpha)) \to \omega_\alpha(\pi)$ as $n \to \infty$ for each fixed $k \geqslant 1$. Matlab shows that the normalized eigenvectors for $\lambda_k(T_n(\omega_\alpha))$ are very close to

$$\sqrt{\frac{2}{n+1}}\left((-1)^{j+1}\sin\frac{jk\pi}{n+1}\right)_{j=1}^{n} \tag{1}$$

(see Figures 1 and 2).

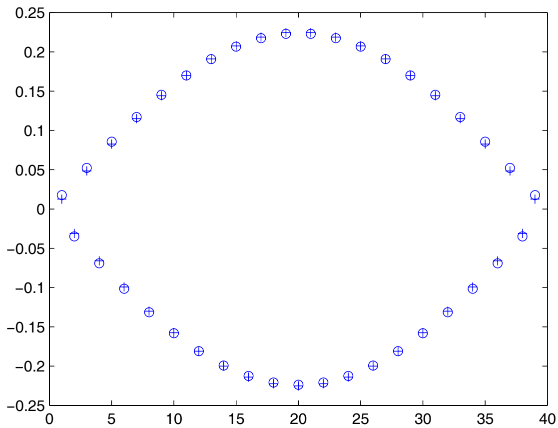

Figure 1. We see a normalized eigenvector for $\lambda_1(T_{39}(\omega_{1/4})) = 0.7074$ *(crosses) and the values of* $\sqrt{\frac{2}{40}}\,(-1)^{j+1}\sin\frac{\pi j}{40}$ *for* $j = 1, \ldots, 39$ *(circles).*

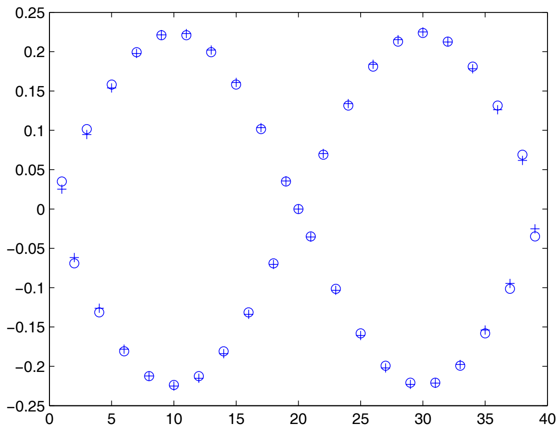

Figure 2. These are a normalized eigenvector for $\lambda_2(T_{39}(\omega_{1/4})) = 0.7082$ *(crosses) and the values of* $\sqrt{\frac{2}{40}}\,(-1)^{j+1}\sin\frac{2\pi j}{40}$ *for* $j = 1, \ldots, 39$ *(circles).*

The matrices $T_n(\omega_\alpha)$ are of interest for probabilists and statistical physicists. Let us first explain the connection with probability theory. Fractional Brownian motion (FBM for short) with Hurst index $H \in (0,1)$ is by definition the centered Gaussian process B_t^H, $t > 0$, with covariance function $\mathbb{E}[B_s^H B_t^H] = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$. It has $(H-\varepsilon)$-Hölder continuous trajectories for any $\varepsilon > 0$, so the process gets more and more irregular as $H \to 0$. The case $H = \frac{1}{2}$ corresponds to Brownian motion, which is Markov. Leaving aside this case, one has a non-Markovian Gaussian process with stationary increments, with very different properties depending on whether $H > \frac{1}{2}$ or $H < \frac{1}{2}$. A result known as the "invariance principle" (see the book by Samorodnitsky and Taqqu [14, Theorem 7.2.11]) states that, if $H > \frac{1}{2}$, then

$$N^{-H} \sum_{1 \leqslant j \leqslant [tN]} Y_j \to \frac{1}{\sqrt{H|2H-1|}} B_t^H \quad (0 \leqslant t \leqslant 1) \quad \text{as} \quad N \to \infty$$

(convergence of finite-dimensional distributions) if Y_j, $j \in \mathbb{Z}$, is any stationary sequence of centered Gaussian variables with a covariance function such that

$$\mathbb{E}[Y_j Y_k] =: r(|j-k|) \sim |k-j|^{2H-2} \quad \text{as} \quad |j-k| \to \infty.$$

Hence one may choose in particular the stationary covariance function associated with the above Toeplitz matrix $T_n(\omega_\alpha)$ for $\alpha = H - \frac{1}{2}$, namely, $\mathbb{E}[Y_j Y_k] = (\omega_{H-\frac{1}{2}})_{j-k}$. The same is true for $H < \frac{1}{2}$ provided that $\sum_{j\in\mathbb{Z}} \mathbb{E}[Y_0 Y_j] = \sum_{j\in\mathbb{Z}} r(|j|) = 0$, which is valid for $T_n(\omega_\alpha)$ (note that $\alpha \in (-\frac{1}{2}, 0)$ is negative in that case, which leads to a bounded Toeplitz operator). Knowing a quasi-exact diagonalization of the covariance matrix may help to compute the law of some functionals of FBM.

As for physicists, they are interested in studying finite-size effects for Gaussian lattice models with long-range interactions. Namely, consider real-valued spins $\sigma(i), i \in \Lambda$ on a d-dimensional lattice $\Lambda \subset \mathbb{Z}^d$, and attach to each configuration $\{\sigma\} = (\sigma(i))_{i\in\Lambda} \in \mathbb{R}^\Lambda$ a Boltzmann weight proportional to $\exp -\beta Q(\{\sigma\})$, where $\beta > 0$ is the inverse of the temperature and Q is a quadratic form with a spectrum which is bounded below. The lattice is here considered to be infinite in $d-1$ dimensions, and finite with n layers in the dth direction. Assuming $d = 1$, this is equivalent to the above discretization of FBM if one sets $Q_n = (T_n(\omega_\alpha))^{-1}$ on $\Lambda = \{1, \ldots, n\}$. The matrix Q_n is no longer a Toeplitz matrix, but $(Q_n)_{i,j} \sim C|i-j|^{-1-2\alpha}$ as n and $|i-j|$ go to infinity with i, j staying close to the middle, that is, with $i/n, j/n \to \frac{1}{2}$; this is a consequence of an exact formula for Q_n which known as the Duduchava-Roch theorem; see [7, Prop. 2.2]. Alternatively, one may set $Q_n = T_n(\omega_{-\alpha})$, which gives an interaction depending only on the distance of the sites, but then of course the covariance matrix is no more stationary. In any case, physicists have been considering a Gaussian variant of this model (called "ferromagnetic spherical model" in the literature, see [3]) for $\alpha \in (0, \frac{1}{2})$, exhibiting a second-order phase transition at a positive critical temperature. Fine computations of finite-size effects have been obtained (see the book by Brankov, Danchev, and Tonchev [2]) for the partition function (free energy), the susceptibility (related to the integrated correlation function), the shift of the critical temperature, etc., relying on the non too physical "periodic boundary condition", which is more or less equivalent to replacing the Toeplitz matrix with its optimal circulant approximation. For the simplest computations, only the spectrum of the Toeplitz matrix is needed, so their results might be extended to the case of free boundary conditions (corresponding to the usual Toeplitz matrix) by taking into account corrections to Szegö's theorem on the asymptotic spectrum. But for the most interesting results, one needs to diagonalize the quadratic form Q_n. Note that different (including free) boundary conditions have been analyzed in the case of short-range interactions, where the Toeplitz matrix has only a finite number of non-zero diagonals and can be easily diagonalized; see [1]. Even in that simple case, finite-size effects depend strongly on the choice of boundary conditions.

This paper arose from the attempt to prove that (1) is indeed close to a kth eigenvector of $T_n(\omega_\alpha)$. We have not been able to achieve this goal, but in the course of our efforts we gained some insights that might be of independent interest.

Let $\{A_n\}_{n=1}^\infty$ be a sequence of $d(n)\times d(n)$ matrices. We think of A_n as a linear operator on $\mathbf{C}^{d(n)}$ with the ℓ^2 norm. The operator norm ($=$ spectral norm) of A_n is denoted by $\|A_n\|$. Fix a point $\lambda\in\mathbf{C}$. We call a sequence $\{x_n\}_{n=1}^\infty$ of nonzero vectors $x_n\in\mathbf{C}^{d(n)}$ an *asymptotic eigenvector* for λ if there exist two sequences $\{\lambda_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ such that

$$v_n\neq 0,\quad A_nv_n=\lambda_nv_n,\quad \lambda_n\to\lambda,\quad \left\|\frac{x_n}{\|x_n\|}-\frac{v_n}{\|v_n\|}\right\|\to 0,$$

and we refer to the sequence $\{x_n\}_{n=1}^\infty$ as an *asymptotic pseudomode* for λ if

$$\frac{\|A_nx_n-\lambda x_n\|}{\|x_n\|}\to 0.$$

Frequently we simply say that x_n itself is an asymptotic eigenvector or an asymptotic pseudomode. Trefethen and Embree's book [17] is the standard reference to this topic.

If $\|A_n\|\leqslant M<\infty$ for all n, then

$$\frac{\|A_nx_n-\lambda x_n\|}{\|x_n\|}\leqslant M\left\|\frac{x_n}{\|x_n\|}-\frac{v_n}{\|v_n\|}\right\|+|\lambda|\left\|\frac{x_n}{\|x_n\|}-\frac{v_n}{\|v_n\|}\right\|$$

and hence asymptotic eigenvectors are automatically asymptotic pseudomodes. This is no longer true if $\limsup\|A_n\|=\infty$ (see Proposition 2.1 below). Furthermore, independently of whether $\|A_n\|$ remains bounded or not, asymptotic pseudomodes need not to be asymptotic eigenvectors (Theorems 1.1 and 1.2 provide us with plenty of examples). Since $\|T_n(\omega_\alpha)\|\sim C_\alpha n^{2\alpha}$ with some constant C_α (see [8]), it follows that for $T_n(\omega_\alpha)$ ($0<\alpha<1/2$) the notions of asymptotic eigenvectors and asymptotic pseudomodes are two completely different concepts: an asymptotic eigenvector is not necessarily an asymptotic pseudomode and vice versa.

We denote by $C^{1+\gamma}[0,\pi]$ the set of all continuously differentiable functions on $[0,\pi]$ whose derivative satisfies a Hölder condition with the exponent γ and we let $\mathscr{R}[0,\pi]$ stand for the set of all Riemann integrable functions on $[0,\pi]$.

THEOREM 1.1. *Let $a \in L^1$ and suppose a is continuous in an open neighborhood of π. If $f \in C^{1+\gamma}[0,\pi]$ for some $\gamma > 0$, then*

$$\left(\sqrt{\frac{2}{n+1}} (-1)^{j+1} f\left(\frac{j\pi}{n+1}\right) \right)_{j=1}^{n}$$

is an asymptotic pseudomode of $T_n(a)$ for $\lambda = a(\pi)$.

This theorem implies that the vectors (1) are asymptotic pseudomodes of $T_n(\omega_\alpha)$ for $\lambda = \omega_\alpha(\pi)$. The following theorem shows that if we make stronger assumptions on a, then we may relax the requirements for f.

THEOREM 1.2. *Let a be in $L^\infty := L^\infty(0,2\pi)$ and suppose a is continuous in an open neighborhood of π. If $f \in \mathscr{R}[0,\pi]$, then*

$$\left(\sqrt{\frac{2}{n+1}} (-1)^{j+1} f\left(\frac{j\pi}{n+1}\right) \right)_{j=1}^{n}$$

is an asymptotic pseudomode of $T_n(a)$ for $\lambda = a(\pi)$.

Theorems 1.1 and 1.2 concern individual pseudomodes. A result on the collective behavior of asymptotic pseudomodes is in [19]. Let

$$U_n = \frac{1}{\sqrt{n}} \left(e^{2\pi ijk/n} \right)_{j,k=0}^{n-1}$$

be the Fourier matrix and denote by $U_n \mathbf{e}_k$ the kth column of U_n. Zamarashkin and Tyrtyshnikov [19] observed that if $a \in L^2 := L^2(0,2\pi)$, then

$$\sum_{k=0}^{n-1} \|T_n(a)U_n\mathbf{e}_k - \lambda_{n-k+1}(C_n(a))U_n\mathbf{e}_k\|^2 = o(n) \tag{2}$$

where $\operatorname{diag}(\lambda_0(C_n(a))), \ldots, \lambda_{n-1}(C_n(a)) := U_n C_n(a) U_n^*$ and $C_n(a)$ is the optimal circulant matrix for $T_n(a)$, that is, the uniquely determined circulant matrix X for which the Frobenius norm of $T_n(a) - X$ is minimal. They also stated that (2) does not necessarily hold for $a \in L^1$, but that if $a \in L^1$, then for each $\varepsilon > 0$ the number of $k \in \{0,1,\ldots,n-1\}$ for which

$$\min_\lambda \|T_n(a)U_n\mathbf{e}_k - \lambda U_n\mathbf{e}_k\| \geqslant \varepsilon$$

is $o(n)$. We here prove the following.

THEOREM 1.3. *Let $a \in L^1$ and suppose the 2π-periodic extension of a is continuous in an open neighborhood of $\theta_0 = 2\pi(1-\beta) \in [0,2\pi]$. Then*

$$\|T_n(a)U_n\mathbf{e}_k - a(\theta_0)U_n\mathbf{e}_k\| \to 0$$

whenever $n \to \infty$ and $k = \beta n + O(1)$. In other terms, $U_n\mathbf{e}_{\beta n+O(1)}$ is an asymptotic pseudomode of $T_n(a)$ for $\lambda = a(2\pi(1-\beta))$.

While (2), after division by n, may be regarded as a result on convergence in the mean, Theorem 1.3 may be viewed as a result on pointwise convergence.

2. Additional remarks

Here is the simple observation we mentioned in Section 1.

PROPOSITION 2.1. *Let $\{A_n\}$ be a sequence of matrices such that $\|A_n\| \to \infty$. Suppose $\lambda \in \mathbf{C}$ is a limiting point of the spectra of A_n, that is, there exist λ_n and v_n such that $\|v_n\| = 1$, $A_n v_n = \lambda_n v_n$, and $\lambda_n \to \lambda$. Then the sequence $\{A_n\}$ has asymptotic eigenvectors for λ which are not asymptotic pseudomodes for λ.*

Proof. There are y_n such that $\|y_n\| = 1$ and $\|A_n y_n\| = \|A_n\|$. Put $\varepsilon_n = 1/\sqrt{\|A_n\|}$ and $x_n = v_n + \varepsilon_n y_n$. Since $\|x_n\| \to 1$ and $\|x_n - v_n\| = \varepsilon_n \to 0$, the sequence $\{x_n\}$ is an asymptotic eigenvector for λ. On the other hand,

$$A_n x_n - \lambda x_n = A_n(v_n + \varepsilon_n y_n) - \lambda(v_n + \varepsilon_n y_n) = \varepsilon_n A_n y_n - (\lambda - \lambda_n) v_n - \lambda \varepsilon_n y_n,$$

whence $\|A_n x_n - \lambda x_n\| \geqslant \varepsilon_n \|A_n\| - |\lambda - \lambda_n| - |\lambda| \varepsilon_n \to \infty$. □

The following is obvious.

PROPOSITION 2.2. *Let $\{A_n\}$ be a sequence of matrices and $\lambda \in \mathbf{C}$. Suppose $A_n - \lambda I$ is invertible for all n. Then $\|(A_n - \lambda I)^{-1}\| \geqslant \alpha_n$ for all n if and only if there exists a sequence $\{x_n\}$ such that $\|A_n x_n - \lambda x_n\| / \|x_n\| \leqslant 1/\alpha_n$.*

From the work of Kac, Murdock, Szegö [10], Widom [18], Parter [11], [12], and Serra [15], [16] it is known that there exist constants $c_1, c_2 \in (0, \infty)$ such that

$$\frac{c_1}{n^2} \leqslant \lambda_1(T_n(\omega_\alpha)) - \omega_\alpha(\pi) \leqslant \frac{c_2}{n^2}$$

for all n. Since $T_n(\omega_\alpha)$ is selfadjoint, this implies that

$$\frac{n^2}{c_2} \leqslant \|(T_n(\omega_\alpha) - \omega_\alpha(\pi) I)^{-1}\| \leqslant \frac{n^2}{c_1}.$$

Thus, Proposition 2.2 reveals that there exist asymptotic pseudomodes x_n such that

$$\frac{\|T_n(\omega_\alpha) x_n - \omega_\alpha(\pi) x_n\|}{\|x_n\|} \tag{3}$$

does not exceed c_2/n^2, but that there is no asymptotic pseudomode x_n for which (3) is $o(1/n^2)$.

For a vector $x = (x_0, \ldots, x_{n-1}) \in \mathbf{C}^n$, we define the trigonometric polynomial Fx by

$$(Fx)(\theta) = \sum_{\ell=0}^{n-1} x_\ell e^{i\ell\theta} \quad (\theta \in \mathbf{R}). \tag{4}$$

Clearly, the jth component of $T_n(a)x$ equals the jth Fourier coefficient of the product of a and Fx,

$$(T_n(a)x)_j = \frac{1}{2\pi}\int_0^{2\pi} a(\theta)(Fx)(\theta)e^{-ij\theta}d\theta \quad (j = 0, \dots, n-1). \tag{5}$$

The following proposition is a simple application of (5) and reveals that $T_n(a)$ has asymptotic pseudomodes that are completely different from those of Theorems 1.1 and 1.2.

PROPOSITION 2.3. *Let $a \in L^1$ and suppose $|a(\theta) - a(\pi)| = O(|\theta - \pi|)$ as $\theta \to \pi$. Then the vectors x_n given by*

$$(x_n)_j = (-1)^j \binom{n-1}{j} \quad (j = 0, \dots, n-1)$$

are an asymptotic pseudomode of $T_n(a)$ for $\lambda = a(\pi)$.

Proof. We have

$$(Fx_n)(\theta) = (1 - e^{i\theta})^{n-1} = \left(-2i\sin\frac{\theta}{2}\right)^{n-1} e^{i\theta(n-1)/2}.$$

This implies that

$$\|x_n\|^2 = \frac{1}{2\pi}\int_0^{2\pi}\left|2\sin\frac{\theta}{2}\right|^{2n-2} d\theta = \frac{2^{2n-2}}{\sqrt{\pi}}\frac{\Gamma(n-1/2)}{\Gamma(n)} \sim \frac{2^{2n-2}}{\sqrt{\pi n}},$$

which, incidentally, can also be obtained from

$$\|x_n\|^2 = \sum_{j=0}^{n-1}\binom{n-1}{j}^2 = \binom{2n-2}{n-1} \sim \frac{2^{2n-2}}{\sqrt{\pi n}}.$$

Formula (5) yields

$$\delta_j := \frac{(T_n(a)x_n - a(\pi)x_n)_j}{\|x_n\|} = \frac{1}{2\pi\|x_n\|}\int_0^{2\pi}[a(\theta) - a(\pi)](Fx_n)(\theta)e^{-ij\theta}d\theta.$$

Thus,

$$|\delta_j| \leqslant C_1\frac{n^{1/4}}{2^{n-1}}\int_0^{2\pi}|a(\theta) - a(\pi)|2^{n-1}\left|\sin\frac{\theta}{2}\right|^{n-1} d\theta.$$

By assumption, there are a $\mu \in (0, \pi/2)$ and a finite constant K such that

$$|a(\theta) - a(\pi)| \leqslant K|\theta - \pi| \leqslant C_2K\left|\cos\frac{\theta}{2}\right|$$

for $\theta \in (\pi-\mu,\pi+\mu)$, whence

$$\begin{aligned} n^{1/4}\int_{|\theta-\pi|<\mu} |a(\theta)-a(\pi)| \left|\sin\frac{\theta}{2}\right|^{n-1} d\theta &\leqslant n^{1/4}\int_{|\theta-\pi|<\mu}\left|\cos\frac{\theta}{2}\right| \left|\sin\frac{\theta}{2}\right|^{n-1} d\theta \\ = n^{1/4}\int_{|x|<\mu}\left|\sin\frac{x}{2}\right| \left|\cos\frac{x}{2}\right|^{n-1} dx &\leqslant 2n^{1/4}\int_0^{\pi/2} \sin\frac{x}{2}\left(\cos\frac{x}{2}\right)^{n-1} dx \\ = -4n^{1/4}\frac{1}{n}\left(\cos\frac{x}{2}\right)^n\Big|_0^{\pi/2} &< \frac{4}{n^{3/4}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &n^{1/4}\int_{|\theta-\pi|>\mu} |a(\theta)-a(\pi)| \left|\sin\frac{\theta}{2}\right|^{n-1} d\theta \\ &\leqslant n^{1/4}\left(\sin\frac{\mu}{2}\right)^{n-1}\int_{|\theta-\pi|>\mu}|a(\theta)-a(\pi)|d\theta = O\left(\frac{1}{n^{3/4}}\right). \end{aligned}$$

Consequently,

$$\|\delta\|^2 = \sum_{j=0}^{n-1}|\delta_j|^2 = O\left(n\frac{1}{n^{6/4}}\right) = O\left(\frac{1}{n^{1/2}}\right) = o(1).$$

□

REMARK 2.4. Let $T(a) = (a_{j-k})_{j,k=1}^{\infty}$ be the infinite Toeplitz matrix generated by a. This matrix induces a bounded operator on $\ell^2 := \ell^2(\mathbf{N})$ if and only if $a \in L^\infty$. Suppose, for simplicity, a is the restriction to $[0,2\pi]$ of a continuous and 2π-periodic function on $\mathbf{R}$. Then the range $\mathscr{R}(a)$ of a is a closed, continuous, and naturally oriented curve in the plane. For $\lambda \in \mathbf{C}\setminus\mathscr{R}(a)$, denote by $\operatorname{wind}(a,\lambda)$ the winding number of $\mathscr{R}(a)$ about λ. If $\operatorname{wind}(a,\lambda)=0$, then $\{T_n(a)\}$ does not have asymptotic pseudomodes, because then, by a classical result of Gohberg and Feldman [9], $\|(T_n(a)-\lambda I)^{-1}\| = O(1)$. Let $\operatorname{wind}(a,\lambda) = -m < 0$. We then can write $a(\theta)-\lambda = b(\theta)e^{-im\theta}$ and the operator $T(b)$ can be shown to be invertible on ℓ^2. Put

$$u_j = T^{-1}(b)\mathbf{e}_j \quad (j=1,\ldots,m)$$

where $\mathbf{e}_j \in \ell^2$ is the sequence whose jth term is 1 and the remaining terms of which are zero. One can show that $u_1,\ldots,u_m$ form a basis in the null space of $T(a)-\lambda I$. Let finally $P_n : \ell^2 \to \mathbf{C}^n$ be projection onto the first n coordinates. In [6], it was proved that a sequence $\{x_n\}$ of vectors $x_n \in \mathbf{C}^n$ is an asymptotic pseudomode of $\{T_n(a)\}$ for λ if and only if there exist $c_1^{(n)},\ldots,c_m^{(n)} \in \mathbf{C}$ and $z_n \in \mathbf{C}^n$ such that

$$\frac{x_n}{\|x_n\|} = c_1^{(n)}u_1+\ldots+c_m^{(n)}u_m+z_n, \qquad \sup_{n\geqslant 1,1\leqslant j\leqslant m}|c_j^{(n)}| < \infty, \qquad \lim_{n\to\infty}\|z_n\| = 0.$$

Paper [6] also contains a characterization of all asymptotic pseudomodes of $\{T_n(a)\}$ for points λ with $\operatorname{wind}(a,\lambda) = m > 0$.

Reichel and Trefethen [13] were probably the first to observe that if a is a trigonometric polynomial, $\lambda \in \mathbf{C}\setminus\mathscr{R}(a)$, and $\operatorname{wind}(a,\lambda) \neq 0$, then $\|(T_n(a)-\lambda I)^{-1}\|$

increases exponentially fast and hence, by Proposition 2.2, there exist asymptotic pseudomodes x_n such that $\|T_n(a)x_n - \lambda x_n\|/\|x_n\|$ decays exponentially fast. See also Theorem 7.2 of [17]. Under the assumption that $\lambda \in \mathbf{C} \setminus \mathscr{R}(a)$ and $\operatorname{wind}(a,\lambda) \neq 0$, the growth of the norms $\|(T_n(a) - \lambda I)^{-1}\|$ for piecewise continuous and general continuous functions a is studied in [4] and [5], respectively.

In connection with all these results, the contribution of this paper to the topic is that we deliver concrete pseudomodes of $\{T_n(a)\}$ for points $\lambda \in \mathscr{R}(a)$.

3. Exponentials as pseudomodes

We now start with the proof of our main results. For a function $f \in \mathscr{R}[0,\pi]$, we denote by $V_n f$ the vector in $\mathbf{C}^n$ given by

$$V_n f = \left(\sqrt{\frac{2}{n+1}} (-1)^{j+1} f \left(\frac{j\pi}{n+1} \right) \right)_{j=1}^{n}.$$

Obviously,

$$\lim_{n\to\infty} \|V_n f\|^2 = \frac{2}{\pi} \int_0^{\pi} |f(x)|^2 dx. \tag{6}$$

Throughout what follows, $e_k(x) := e^{ikx}$. Recall that F is defined by (4).

LEMMA 3.1. *We have*

$$(FV_n e_k)(\theta) = -\sqrt{\frac{2}{n+1}}\, e^{-i\theta} e^{i\frac{n+1}{2}(\theta-\theta_n)} \frac{\sin \frac{n}{2}(\theta-\theta_n)}{\sin \frac{1}{2}(\theta-\theta_n)}$$

with $\theta_n = \pi - \frac{k\pi}{n+1}$.

Proof. This follows by straightforward computation:

$$\begin{aligned}
\sqrt{\frac{n+1}{2}}\, e^{i\theta}(FV_n e_k)(\theta) &= \sqrt{\frac{n+1}{2}} \sum_{\ell=1}^{n} (V_n e_k)_\ell e^{i\ell\theta} = \sum_{\ell=1}^{n} (-1)^{\ell+1} e^{\frac{ik\ell\pi}{n+1}} e^{i\ell\theta} \\
&= -\sum_{\ell=1}^{n} e^{i\ell\left(\pi + \frac{k\pi}{n+1} + \theta\right)} = -\sum_{\ell=1}^{n} e^{i\ell\left(\theta - \pi + \frac{k\pi}{n+1}\right)} = -\sum_{\ell=1}^{n} e^{i\ell(\theta-\theta_n)} \\
&= -e^{i(\theta-\theta_n)} \frac{1-e^{in(\theta-\theta_n)}}{1-e^{i(\theta-\theta_n)}} = -e^{i(\theta-\theta_n)} e^{i\frac{n}{2}(\theta-\theta_n)} e^{-i\frac{1}{2}(\theta-\theta_n)} \frac{\sin\frac{n}{2}(\theta-\theta_n)}{\sin\frac{1}{2}(\theta-\theta_n)}.
\end{aligned}$$

□

Our proof of Theorems 1.1 and 1.2 is based on the following theorem in the case $\eta = 0$. The case $\eta \neq 0$ is needed in the proof of Theorem 1.3.

THEOREM 3.2. *Let $a \in L^1$ be continuous in an open neighborhood of π. For a real number η, define a^η by $a^\eta(\theta) = a(\theta - \eta)$. Given any $\varepsilon > 0$, there exist $\eta_0 = \eta_0(a,\varepsilon)$, $n_0 = n_0(a,\varepsilon) \geqslant 1$, and $\delta_0 = \delta_0(a,\varepsilon) > 0$ such that*

$$\|T_n(a^\eta)V_n e_k - a^\eta(\pi)V_n e_k\| < \varepsilon$$

whenever $|\eta| \leqslant \eta_0$, $n \geqslant n_0$, and $|k|/n \leqslant \delta_0$.

Proof. Suppose a is continuous on $I_d := (\pi - d, \pi + d)$. Fix a number $\sigma > 0$. Then there exist functions $b \in C[0, 2\pi]$ and $c \in L^1$ depending only on a and σ such that

$$a = b + c, \quad b|I_d = a|I_d, \quad \|c\|_1 < \sigma,$$

where $\|\cdot\|_1$ is the L^1 norm. With $\theta_n = \pi - \frac{k\pi}{n+1}$,

$$\|T_n(a^\eta - a^\eta(\theta_n))V_n e_k\|^2 \leqslant 2\|T_n(b^\eta - a^\eta(\theta_n))V_n e_k\|^2 + 2\|T_n(c^\eta)V_n e_k\|^2. \qquad (7)$$

For the first term on the right we have

$$\|T_n(b^\eta - a^\eta(\theta_n))V_n e_k\|^2 \leqslant \frac{1}{2\pi}\int_0^{2\pi} |b^\eta(\theta) - a^\eta(\theta_n)|^2 |(FV_n e_k)(\theta)|^2 d\theta.$$

Now fix $\tau > 0$. Then there is a $\mu = \mu(a, \tau) > 0$ such that $\mu < d$ and $|a^\eta(\theta) - a^\eta(\pi)| < \tau/2$ for $|\eta| < \mu$ and $\theta \in I_\mu := (\pi - \mu, \pi + \mu)$. Assume

$$|\eta| < \mu \quad \text{and} \quad \frac{|k|\pi}{n+1} < \frac{\mu}{2}. \qquad (8)$$

Let first $|\theta - \pi| \geqslant \mu$. From (8) we obtain that $|\theta - \theta_n| > \mu/2$, and by periodicity we may assume that $|\theta - \theta_n| < \pi$. Thus, $\mu/4 < |\theta - \theta_n|/2 < \pi/2$ and Lemma 3.1 therefore gives

$$|(FV_n e_k)(\theta)| = \sqrt{\frac{2}{n+1}} \left| \frac{\sin \frac{n}{2}(\theta - \theta_n)}{\sin \frac{1}{2}(\theta - \theta_n)} \right| \leqslant \sqrt{\frac{2}{n}} \frac{1}{\sin(\mu/4)}. \qquad (9)$$

It follows that

$$\frac{1}{2\pi}\int_{|\theta-\pi| \geqslant \mu} |b^\eta(\theta) - a^\eta(\theta_n)|^2 |(FV_n e_k)(\theta)|^2 d\theta \leqslant \frac{4\,\|b^\eta\|_\infty^2}{2\pi} \frac{2}{n} \frac{2\pi}{\sin^2(\mu/4)} = \frac{8\,\|b\|_\infty^2}{n\,\sin^2(\mu/4)},$$

where $\|\cdot\|_\infty$ is the norm in L^∞. Now let $|\theta - \pi| < \mu$. Then $b^\eta(\theta) = a^\eta(\theta)$. Consider the intervals

$$H_j := \left(\theta_n + \frac{2(j-1)\pi}{n}, \theta_n + \frac{2j\pi}{n}\right), \quad j \in \mathbf{Z}.$$

From (8) we infer that $\theta_n \in I_\mu$ and hence

$$|b^\eta(\theta) - a^\eta(\theta_n)| \leqslant |a^\eta(\theta) - a^\eta(\pi)| + |a^\eta(\pi) - a^\eta(\theta_n)| < \frac{\tau}{2} + \frac{\tau}{2} = \tau.$$

Consequently,

$$\frac{1}{2\pi}\int_{|\theta-\pi|<\mu} |b^\eta(\theta) - a^\eta(\theta_n)|^2 |(FV_n e_k)(\theta)|^2 d\theta \leqslant \frac{\tau^2}{2\pi} \sum_{H_j \cap I_\mu \neq \emptyset} \int_{H_j} |(FV_n e_k)(\theta)|^2 d\theta. \qquad (10)$$

If $H_j \cap I_\mu \neq \emptyset$, then necessarily

$$\pi - \mu < \theta_n + \frac{2j\pi}{n} \quad \text{and} \quad \theta_n + \frac{2(j-1)\pi}{n} < \pi + \mu,$$

which, by (8), means that

$$-\frac{3\mu}{4\pi}\,n < j < \frac{3\mu}{4\pi}\,n+1.$$

Obviously, we may a priori assume that $3\mu/(4\pi) < 1/4$ and $n > 2$, so that in (10) we have only to deal with terms for which $|j| < n/2$. Since $|\sin(nx)/\sin x| \leqslant n$ for $|x| \leqslant \pi/2$, we see from Lemma 3.1 that

$$|(FV_n e_k)(\theta)| \leqslant \sqrt{\frac{2}{n+1}}\,n < \sqrt{2n} \quad \text{for} \quad \theta \in H_{-1} \cup H_0.$$

If $j > 1$ and $\theta \in H_j$, then

$$0 < \frac{(j-1)\pi}{n} < \frac{\theta-\theta_n}{2} < \frac{j\pi}{n} < \frac{\pi}{2},$$

and using that $\sin x > (2/\pi)x$ for $0 < x < \pi/2$ we get from Lemma 3.1 that

$$|(FV_n e_k)(\theta)| \leqslant \sqrt{\frac{2}{n+1}}\,\frac{1}{\sin(\theta-\theta_n)/2} < \sqrt{\frac{2}{n+1}}\,\frac{\pi}{2}\,\frac{n}{(j-1)\pi} < \frac{1}{j-1}\sqrt{\frac{n}{2}}.$$

Analogously we obtain that

$$|(FV_n e_k)(\theta)| < \frac{1}{|j|}\sqrt{\frac{n}{2}}$$

for $\theta \in H_j$ and $j < 0$. Thus, the right-hand side of (10) is at most

$$\frac{\tau^2}{2\pi}\left(2n\cdot\frac{2\pi}{n} + 2n\cdot\frac{2\pi}{n} + \left(\sum_{j>1}\frac{n}{2}\,\frac{1}{(j-1)^2}\right)\frac{2\pi}{n} + \left(\sum_{j<0}\frac{n}{2}\,\frac{1}{|j|^2}\right)\frac{2\pi}{n}\right)$$

which is $(4+\pi^2/6)\tau^2 < 6\tau^2$.

We now turn to the second term on the right of (7). By (5),

$$|(T_n(c^\eta)V_n e_k)_j| \leqslant \frac{1}{2\pi}\int_{|\theta-\pi|>d} |c^\eta(\theta)|\,|(FV_n e_k)(\theta)|d\theta.$$

But if $|\theta-\pi| > d$ then, by (8), $|\theta-\theta_n| > d-\mu/2 > d/2$ and in the same way we proved estimate (9) we now get

$$|(T_n(c^\eta)V_n e_k)_j| \leqslant \sqrt{\frac{2}{n}}\,\frac{1}{\sin(d/4)}\,\frac{\|c^\eta\|_1}{2\pi} < \sqrt{\frac{2}{n}}\,\frac{1}{\sin(d/4)}\,\frac{\sigma}{2\pi},$$

where we used that $\|c^\eta\|_1 = \|c\|_1$. This gives

$$\|T_n(c^\eta)V_n e_k\|^2 = 2\pi\sum_{j=1}^{n}|(T_n(c^\eta)V_n e_k)_j|^2 < \frac{1}{\pi}\,\frac{\sigma^2}{\sin^2(d/4)}.$$

In summary, under assumption (8) the right-hand side of (7) does not exceed

$$\frac{16\|b\|_\infty^2}{n\,\sin^2(\mu/4)} + 12\tau^2 + \frac{2\sigma^2}{\pi\sin^2(d/4)}. \tag{11}$$

Now choose $\sigma > 0$ so that the third term in (11) is smaller than $\varepsilon^2/12$. Put $\tau = \varepsilon/12$ and $\eta_0 = \mu$. Then the second term in (11) is $\varepsilon^2/12$. Clearly, there are n_0 and δ_0 such that if $n \geqslant n_0$ and $|k|/n \leqslant \delta_0$, then the second assumption in (8) is satisfied and the first term in (11) is less than $\varepsilon^2/12$. Thus, for $|\eta| < \eta_0$, $n \geqslant n_0$, and $|k|/n \leqslant \delta_0$ we have $\|T_n(a^\eta)V_n - a^\eta(\theta_n)V_n e_k\| < \varepsilon/2$. It follows that

$$\|T_n(a^\eta)V_n - a^\eta(\pi)V_n e_k\| < \frac{\varepsilon}{2} + |a^\eta(\theta_n) - a^\eta(\pi)|\,\|V_n e_k\|, \tag{12}$$

and since $|a^\eta(\theta_n) - a^\eta(\pi)| < \tau/2 = \varepsilon/24$ and $\|V_n e_k\| = \sqrt{(2n)/(n+1)} < 2$, we arrive at the conclusion that (12) is smaller than ε. □

The following corollary is weaker than Theorem 1.1 but is actually all we need to conclude that (1) is an asymptotic pseudomode of $T_n(\omega_\alpha)$ for $\lambda = \omega_\alpha(\pi)$ or, more generally, of $T_n(a)$ for $\lambda = a(\pi)$ if $a \in L^1$ is continuous in an open neighborhood of π.

COROLLARY 3.3. *If $a \in L^1$ is continuous in an open neighborhood of π and f is an arbitrary trigonometric polynomial, then $V_n f$ is an asymptotic pseudomode of $T_n(a)$ for $\lambda = a(\pi)$.*

Proof. Let $f = \sum_{k=-m}^m c_k e_k$ be a trigonometric polynomial. From Theorem 3.2 we infer that

$$\|T_n(a)V_n f - a(\pi)V_n f\| \leqslant \sum_{k=-m}^{m} |c_k|\,\|T_n(a)V_n e_k - a(\pi)V_n e_k\| = o(1)$$

as $n \to \infty$, which together with (6) implies the assertion. □

Proof of Theorem 1.2. We denote by $\|\cdot\|_2$ the norm in $L^2(0,\pi)$. Given $\varepsilon > 0$, there exists a trigonometric polynomial $p = \sum_{k=-m}^m c_k e_k$ such that

$$2\,\|a\|_\infty \|f - p\|_2 < \frac{\varepsilon}{2}. \tag{13}$$

Clearly, $\|T_n(a)V_n f - a(\pi)V_n f\|$ does not exceed

$$\|T_n(a)V_n p - a(\pi)V_n p\| + (\|T_n(a)\| + |a(\pi)|)\|V_n(f - p)\|. \tag{14}$$

By Corollary 3.3, the first term in (14) is smaller than $\varepsilon/2$ if n is large enough. Since $\|T_n(a)\| \leqslant \|a\|_\infty$ and $|a(\pi)| \leqslant \|a\|_\infty$ and since, by (6),

$$\|V_n(f - p)\| \to \sqrt{\frac{2}{\pi}}\,\|f - p\|_2 < \|f - p\|_2,$$

we obtain from (13) that the second term in (14) does not exceed $\varepsilon/2$ for all sufficiently large n. Thus, $\|T_n(a)V_n f - a(\pi)V_n f\| \to 0$. As, again by (6), $\|V_n f\| \to \sqrt{2/\pi}\,\|f\|_2$, it follows that $V_n f$ is an asymptotic pseudomode for $\lambda = a(\pi)$. □

REMARK 3.4. Working with intervals like the H_j in the proof of Theorem 3.2 and taking into account that

$$\sum_{j=1}^{[n/2]} \left(\cos \frac{j\pi}{n+1}\right)^{2n} = O(\sqrt{n}\,),$$

one can modify the proof of Proposition 2.3 to show that Proposition 2.3 is also true under the hypothesis that $a \in L^1$ and that a is continuous in an open neighborhood of π.

4. Unbounded symbols

In this section we prove Theorem 1.1. The following observation is very simple but we find worth it to be stated as a separate proposition.

PROPOSITION 4.1. *If $a \in L^1$ then $\|T_n(a)\| = o(n)$ as $n \to \infty$.*

Proof. The norm of a Toeplitz matrix one diagonal of which consists of ones and the remaining diagonals of which are zero is 1. Consequently,

$$\frac{\|T_n(a)\|}{n} \leqslant \frac{1}{n} \sum_{j=-(n-1)}^{n-1} |a_j|. \tag{15}$$

Since $|a_j| \to 0$ as $|j| \to \infty$ by the Riemann-Lebesgue theorem, the successive arithmetic means of these numbers and thus also (15) go to zero as well. □

Proof of Theorem 1.1. Continue the function f from $[0,\pi]$ to a 2π-periodic function g in $C^{1+\gamma}$ on all of $\mathbf{R}$. Clearly, $V_nf = V_ng$. For each $m \geqslant 1$ there exists a trigonometric polynomial

$$p_m(\theta) = \sum_{k=-m}^{m} p_k^{(m)} e^{ik\theta}$$

such that $\|g - p_m\|_\infty \leqslant D/m^{1+\gamma}$, where D is a finite constant depending only on f and $\|\cdot\|_\infty$ is the L^∞ norm on $[0,2\pi]$ (see, e.g., Theorem 13.6 or Theorem 13.14 of Chapter III of [20]). The Fourier coefficients of g admit the estimate

$$\begin{aligned} |g_j| &= \left|\int_0^{2\pi} (g(\theta) - p_{|j|-1}(\theta))e^{-ij\theta}d\theta\right| \leqslant \int_0^{2\pi} |g(\theta) - p_{|j|-1}(\theta)|d\theta \\ &\leqslant \frac{2\pi D}{(|j|-1)^{1+\gamma}} \leqslant \frac{C}{|j|^{1+\gamma}} \quad (|j| \geqslant 2). \end{aligned} \tag{16}$$

Let σ_μ be the μth Fejér-Cèsaro mean of the Fourier series of g,

$$\sigma_\mu(\theta) = \sum_{|k| \leqslant \mu} \left(1 - \frac{|k|}{\mu+1}\right) g_k e^{ik\theta}. \tag{17}$$

Theorem 13.5 of Chapter III of [20] tells us that if we let

$$\tau_m = 2\sigma_{2m-1} - \sigma_{m-1}, \tag{18}$$

then

$$\|g - \tau_m\|_\infty \leqslant \frac{4D}{m^{1+\gamma}}. \tag{19}$$

We can write

$$\tau_m(\theta) = \sum_{|j|\leqslant 2m-1} \tau_j^{(m)} e^{ij\theta},$$

and from (16) to (18) we obtain that

$$\begin{aligned}
&\sum_{|j|\leqslant 2m-1} |\tau_j^{(m)}| \\
&\quad = \sum_{|j|\leqslant m-1} \left|2\left(1-\frac{|j|}{2m}\right) - \left(1-\frac{|j|}{m}\right)\right| |g_j| + \sum_{m\leqslant|j|\leqslant 2m-1} \left|2\left(1-\frac{|j|}{2m}\right)\right| |g_j| \\
&\quad < 2\sum_{j=-\infty}^{\infty} |g_j| =: E.
\end{aligned}$$

Fix $\varepsilon > 0$. The number $\|T_n(a)V_nf - a(\pi)V_nf\|$ is at most

$$\|T_n(a)V_n\tau_m - a(\pi)V_n\tau_m\| + (\|T_n(a)\| + |a(\pi)|)\,\|V_n(f - \tau_m)\|. \tag{20}$$

The first term in (20) has the upper bound

$$\sum_{|j|\leqslant 2m-1} |\tau_j^{(m)}|\, \|T_n(a)V_ne_j - a(\pi)V_ne_j\|. \tag{21}$$

Now put $m = [n^{1/(1+\gamma)}]$. Since $(2m-1)/n \to 0$ as $n \to \infty$, Theorem 3.2 implies that there is an n_1 such that

$$\|T_n(a)V_ne_j - a(\pi)V_ne_j\| < \frac{\varepsilon}{2E}$$

whenever $n \geqslant n_1$ and $|j| \leqslant 2m-1$. Thus, if $n \geqslant n_1$ then (21) does not exceed

$$\frac{\varepsilon}{2E} \sum_{|j|\leqslant 2m-1} |\tau_j^{(m)}| < \frac{\varepsilon}{2}.$$

By virtue of (19),

$$\begin{aligned}
|(V_n(f-\tau_m))_j| &= \sqrt{\frac{2}{n+1}} \left| f\left(\frac{j\pi}{n+1}\right) - \tau_m\left(\frac{j\pi}{n+1}\right)\right| \\
&= \sqrt{\frac{2}{n+1}} \left| g\left(\frac{j\pi}{n+1}\right) - \tau_m\left(\frac{j\pi}{n+1}\right)\right| \leqslant \sqrt{\frac{2}{n+1}}\,\frac{4D}{m^{1+\gamma}}.
\end{aligned}$$

This gives

$$\|V_n(f-\tau_m)\|^2 \leqslant n\,\frac{2}{n+1}\,\frac{16\,D^2}{m^{2(1+\gamma)}} = O\left(\frac{1}{n^2}\right).$$

Combining the last estimate with Proposition 4.1 we see that the second term in (20) is

$$(o(n)+|a(\pi)|)\,O\left(\frac{1}{n}\right)=o(1)$$

and thus smaller than $\varepsilon/2$ for $n \geqslant n_2$. Consequently, if $n \geqslant \max(n_1,n_2)$, then (20) is less than ε. In summary, we proved that (20) goes to zero as $n \to \infty$, which together with (6) yields the assertion. □

Using translation invariance, we can now easily construct asymptotic pseudomodes at all "regular" values of the generating function of a Toeplitz matrix.

THEOREM 4.2. *Let $a \in L^1$ and suppose the 2π-periodic extension of a is continuous in an open neighborhood neighborhood of $\theta_0 \in \mathbf{R}$. Assume also that at least one of the following holds:* (a) *$a \in L^\infty$ and $f \in \mathscr{R}[0,\pi]$,* (b) *$f \in C^{1+\gamma}[0,\pi]$ for some $\gamma > 0$. Then the sequence $\{V_{n,\theta_0}f\}$ given by*

$$V_{n,\theta_0}f=\left(\sqrt{\frac{2}{n+1}}\,e^{-i(j+1)\theta_0}f\left(\frac{j\pi}{n+1}\right)\right)_{j=1}^{n}$$

is an asymptotic pseudomode of $\{T_n(a)\}$ for $\lambda = a(\theta_0)$.

Proof. We have $V_{n,\theta_0}f = D_{\theta_0}V_nf$ where D_{θ_0} is the unitary diagonal matrix

$$D_{\theta_0}=\operatorname{diag}\left(e^{i(j+1)(\pi-\theta_0)}\right)_{j=1}^{n}.$$

Consequently,

$$\begin{aligned}\|T_n(a)V_{n,\theta_0}f-a(\theta_0)V_{n,\theta_0}f\| &= \|T_n(a)D_{\theta_0}V_nf-a(\theta_0)D_{\theta_0}V_nf\| \\ &= \|D_{\theta_0}^{-1}T_n(a)D_{\theta_0}V_nf-a(\theta_0)V_nf\|. \qquad (22)\end{aligned}$$

It can be readily verified that $D_{\theta_0}^{-1}T_n(a)D_{\theta_0} = T_n(a^{\pi-\theta_0})$ where $a^{\pi-\theta_0}(\theta) = a(\theta+\theta_0-\pi)$. Since $a^{\pi-\theta_0}(\pi) = a(\theta_0)$, we see that (22) is $\|T_n(a^{\pi-\theta_0})V_nf - a^{\pi-\theta_0}(\pi)V_nf\|$. Theorems 1.1 and 1.2 therefore imply that $V_{n,\theta_0}f$ is an asymptotic pseudomode for $a(\theta_0)$. □

5. Pseudomodes from inside the Fourier basis

In this section we prove Theorem 1.3.

Recall that $\theta_0 = 2\pi(1-\beta)$. A sequence of integers of the form $\beta n + O(1)$ can be written as $\beta n + k_n + r_n$ where k_n are integers satisfying $|k_n| \leqslant M$ and $r_n \in [0,1)$. We have to prove that

$$\|T_n(a)U_n\mathbf{e}_{\beta n+k_n+r_n}-a(\theta_0)U_n\mathbf{e}_{\beta n+k_n+r_n}\|\to 0.$$

For $j = 0,\dots,n-1$,

$$(U_n\mathbf{e}_{\beta n+k_n+r_n})_j=\frac{1}{\sqrt{n}}\,e^{\frac{2\pi i}{n}(\beta n+k_n+r_n)j}=\frac{1}{\sqrt{n}}\,e^{\frac{2\pi i}{n}\left(n-\frac{n\theta_0}{2\pi}+k_n+r_n\right)j}=\frac{1}{\sqrt{n}}\,e^{-ij\eta_n}e^{\frac{2\pi i}{n}k_nj}$$

with $\eta_n = \theta_0 - \frac{2\pi r_n}{n}$. Define $a^{\pi-\eta_n}$ by $a^{\pi-\eta_n}(\theta) = a(\theta+\eta_n-\pi)$ as in Theorem 3.2

and in the proof of Theorem 4.2. From Theorem 3.2 we deduce that

$$\|T_n(a^{\pi-\eta_n})V_n e_{2k_n} - a^{\pi-\eta_n}(\pi)V_n e_{2k_n}\| \to 0,$$

which, with D_{η_n} as in the proof of Theorem 4.2, gives

$$\|T_n(a)D_{\eta_n}V_n e_{2k_n} - a(\eta_n)D_{\eta_n}V_n e_{2k_n}\| \to 0.$$

Since $\|D_{\eta_n}V_n e_{2k_n}\| \to \sqrt{2}$ and a is continuous at θ_0, it follows that

$$\|T_n(a)D_{\eta_n}V_n e_{2k_n} - a(\theta_0)D_{\eta_n}V_n e_{2k_n}\| \to 0. \tag{23}$$

We have

$$D_{\eta_n}V_n e_{2k_n} = \left(\sqrt{\frac{2}{n+1}}\, e^{-i(j+2)\eta_n} e^{\frac{2\pi i(j+1)k_n}{n+1}}\right)_{j=0}^{n-1}.$$

Hence $D_{\eta_n}V_n e_{2k_n} = \sqrt{2n/(n+1)}\, e^{-2i\eta_n} x_n$ with

$$x_n := \left(\frac{1}{\sqrt{n}}\, e^{-ij\eta_n} e^{\frac{2\pi i(j+1)k_n}{n+1}}\right)_{j=0}^{n-1}.$$

From (23) we obtain that $\|T_n(a)x_n - a(\theta_0)x_n\| \to 0$. As

$$\begin{aligned} &\Big| \|T_n(a)U_n\mathbf{e}_{\beta n+k_n+r_n} - a(\theta_0)U_n\mathbf{e}_{\beta n+k_n+r_n}\| - \|T_n(a)x_n - a(\theta_0)x_n\| \Big| \\ &\qquad \leqslant \|T_n(a) - a(\theta_0)I\|\, \|U_n\mathbf{e}_{\beta n+k_n+r_n} - x_n\|, \end{aligned} \tag{24}$$

it suffices to show that the right-hand side of (24) goes to zero. The estimation

$$\begin{aligned} \|U_n\mathbf{e}_{\beta n+k_n+r_n} - x_n\|^2 &= \frac{1}{n}\sum_{j=1}^{n-1}\left|e^{\frac{2\pi ijk_n}{n}} - e^{\frac{2\pi i(j+1)k_n}{n+1}}\right|^2 \\ &= \frac{4}{n}\sum_{j=0}^{n-1}\sin^2\left(\pi k_n\left(\frac{j}{n} - \frac{j+1}{n+1}\right)\right) \\ &= \frac{4}{n}\sum_{j=0}^{n-1}\sin^2\frac{\pi k_n(n-j)}{n(n+1)} \\ &\leqslant \frac{4}{n}\, n\, \sin^2\frac{\pi k_n}{n+1} \leqslant \frac{4\pi^2 k_n^2}{(n+1)^2} \end{aligned}$$

shows that $\|U_n\mathbf{e}_{\beta n+k_n+r_n} - x_n\| = O(1/n)$. By Proposition 4.1, $\|T_n(a) - a(\theta_0)I\| = o(n)$. As the product of these two is $o(1)$ we arrive at the conclusion that the right-hand side of (24) goes to zero. $\square$

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