Operators and Matrices
Volume 2, Number 4 (2008), 543–553

INVERSE PROBLEMS FOR STURM–LIOUVILLE OPERATORS ON GRAPHS WITH A CYCLE

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(*communicated by F. Gesztesy*)

Abstract. An inverse spectral problem is studied for second-order differential operators on graphs with a cycle and with standard matching conditions in the internal vertex. A uniqueness theorem is proved, and a constructive procedure for the solution is provided.

1. Introduction

1.1. We study inverse spectral problems for Sturm-Liouville differential operators on graphs with a cycle. Inverse spectral problems consist in recovering operators from their spectral characteristics. The main results on inverse spectral problems on an *interval* are presented in the monographs [1, 5, 7, 9, 10, 15, 17, 18]. Differential operators on graphs (networks, trees) often appear in natural sciences and engineering (see [13, 14] and the references therein). Most of the works in this direction are devoted to the so-called direct problems of studying properties of the spectrum and the root functions for operators on graphs. Inverse spectral problems, because of their nonlinearity, are more difficult to investigate, and nowaday there are only a number of papers in this area. In particular, inverse spectral problems of recovering coefficients of differential operators on trees (i.e on graphs without cycles) were studied in [2, 4, 8, 19, 20] and other papers. However, there are no similar general results in the inverse problem theory for graphs having cycles.

In this paper, we give a formulation and obtain the solution of the inverse spectral problem for Sturm-Liouville operators on graphs with a cycle and with standard matching conditions in the internal vertex. We prove the corresponding uniqueness theorem and provide a constructive procedure for the solution of this class of inverse problems.

1.2. Consider a compact graph T in $\mathbf{R^m}$ with the set of vertices $V = \{v_0, \dots, v_r\}$, $r \geqslant 1$, and the set of edges $\mathscr{E} = \{e_0, \dots, e_r\}$, where $v_1, \dots, v_r$ are the boundary vertices, v_0 is the internal vertex, $e_j = [v_j, v_0]$, $j = \overline{1,r}$, $\bigcap\limits_{j=0}^{r} e_j = \{v_0\}$, and e_0 is a

Mathematics subject classification (2000): 34A55, 34B24, 47E05.

Keywords and phrases: Sturm-Liouville operators, graphs with a cycle, standard matching condititons, inverse spectral problems.

Paper OAM-02-34

cycle. Thus, the graph T has one cycle e_0 and one internal vertex v_0. Let T_j, $j=\overline{0,r}$, be the length of the edge e_j. Each edge $e_j \in \mathscr{E}$ is parameterized by the parameter $x_j \in [0,T_j]$. It is convenient for us to choose the following orientation: for $j=\overline{1,r}$, the vertex v_j corresponds to $x_j=0$, and the vertex v_0 corresponds to $x_j=T_j$; for $j=0$, both ends $x_0=+0$ and $x_0=T_0-0$ corespond to v_0.

An integrable function Y on T may be represented as $Y=\{y_j\}_{j=\overline{0,r}}$, where the function $y_j(x_j)$, $x_j \in [0,T_j]$, is defined on the edge e_j. Let $q=\{q_j\}_{j=\overline{0,r}}$ be an integrable real-valued function on T; q is called the potential. Consider the following differential equation on T:

$$-y_j''(x_j)+q_j(x_j)y_j(x_j)=\lambda y_j(x_j), \quad j=\overline{0,r}, \tag{1}$$

where λ is the spectral parameter, the functions y_j, y_j', $j=\overline{0,r}$, are absolutely continuous on $[0,T_j]$ and satisfy the following matching conditions in the internal vertex v_0:

$$\left.\begin{aligned} &y_j(T_j)=y_0(0), \quad j=\overline{0,r} \quad \text{(continuity condition)},\\ &\sum_{j=0}^{r} y_j'(T_j)=y_0'(0) \quad \text{(Kirchhoff's condition)}. \end{aligned}\right\} \tag{2}$$

Matching conditions (2) are called the standard conditions. In electrical circuits, (2) expresses Kirchhoff's law; in elastic string network, it expresses the balance of tension, and so on.

Let us consider the boundary value problem $B_0(q)$ on T for equation (1) with the matching conditions (2) and with the Dirichlet boundary conditions at the boundary vertices $v_1,\dots,v_r$:

$$y_j(0)=0, \quad j=\overline{1,r}.$$

Moreover, we also consider the boundary value problems $B_k(q)$, $k=\overline{1,r}$, for equation (1) with the matching conditions (2) and with the boundary conditions

$$y_k'(0)=0, \quad y_j(0)=0, \quad j=\overline{1,r}\setminus k.$$

We denote by $\Lambda_k := \{\lambda_{kn}\}_{n\geqslant 0}$ the eigenvalues (counting with multiplicities) of $B_k(q)$, $k=\overline{0,r}$.

In contrast to the case of trees (see [19]), here the specification of the spectra Λ_k, $k=\overline{0,r}$ does not uniquely determine the potential, and we need an additional information. Let $S_j(x_j,\lambda)$, $C_j(x_j,\lambda)$, $j=\overline{0,r}$ be the solutions of equation (1) on the edge e_j with the initial conditions

$$S_j(0,\lambda)=C_j'(0,\lambda)=0, \quad S_j'(0,\lambda)=C_j(0,\lambda)=1.$$

For each fixed $x_j \in [0,T_j]$, the functions $S_j^{(\nu)}(x_j,\lambda)$, $C_j^{(\nu)}(x_j,\lambda)$, $j=\overline{0,r}$, $\nu=0,1$, are entire in λ of order $1/2$. Moreover,

$$\langle\, C_j(x_j,\lambda),\, S_j(x_j,\lambda)\rangle \equiv 1,$$

where $\langle y, z\rangle := yz' - y'z$ is the Wronskian of y and z. Denote $h(\lambda) := S_0(T_0, \lambda)$, $H(\lambda) := C_0(T_0, \lambda) - S_0'(T_0, \lambda)$. Let $\{\nu_n\}_{n\geqslant 1}$ be zeros of the entire function $h(\lambda)$, and put $\omega_n := \operatorname{sign} H(\nu_n)$, $\Omega = \{\omega_n\}_{n\geqslant 1}$. The inverse problem is formulated as follows.

INVERSE PROBLEM 1. Given Λ_k, $k = \overline{0, r}$ and Ω, construct the potential q on T.

Let us formulate the uniqueness theorem for the solution of Inverse Problem 1. For this purpose together with q we consider a potential $\tilde{q}$. Everywhere below if a symbol α denotes an object related to q, then $\tilde{\alpha}$ will denote the analogous object related to $\tilde{q}$. We recall that $r \geqslant 1$.

THEOREM 1. *If $\Lambda_k = \tilde{\Lambda}_k$, $k = \overline{0, r}$, and $\Omega = \tilde{\Omega}$, then $q = \tilde{q}$. Thus, the specification of Λ_k, $k = \overline{0, r}$ and Ω uniquely determines the potential q on T.*

This theorem will be proved in section 3. Moreover, we give a constructive procedure for the solution of Inverse Problem 1. In section 2 we introduce the main notions and prove some auxiliary propositions.

2. Auxiliary propositions

2.1. In this subsection we introduce the Weyl functions and the characteristic functions of the boundary value problems $B_k(q)$.

Fix $k = \overline{1, r}$. Let $\Phi_k = \{\Phi_{kj}\}_{j=\overline{0,r}}$, be the solution of equation (1) satisfying (2) and the boundary conditions

$$\Phi_{kj}(0, \lambda) = \delta_{jk}, \quad j = \overline{1, r}, \tag{3}$$

where δ_{jk} is the Kronecker symbol. Denote $M_k(\lambda) := \Phi_{kk}'(0, \lambda)$, $k = \overline{1, r}$. The function $M_k(\lambda)$ is called the *Weyl function* with respect to the boundary vertex v_k. Clearly,

$$\Phi_{kk}(x_k, \lambda) = C_k(x_k, \lambda) + M_k(\lambda) S_k(x_k, \lambda), \quad x_k \in [0, T_k], \quad k = \overline{1, r}, \tag{4}$$

and consequently, $\langle \Phi_{kk}(x_k, \lambda), S_k(x_k, \lambda)\rangle \equiv 1$. Denote $M_{kj}^1(\lambda) := \Phi_{kj}(0, \lambda)$, $M_{kj}^0(\lambda) := \Phi_{kj}'(0, \lambda)$. Then

$$\Phi_{kj}(x_j, \lambda) = M_{kj}^1(\lambda) C_j(x_j, \lambda) + M_{kj}^0(\lambda) S_j(x_j, \lambda), \quad x_j \in [0, T_j],\ j = \overline{0, r},\ k = \overline{1, r}. \tag{5}$$

In particular, $M_{kk}^1(\lambda) = 1$, $M_{kk}^0(\lambda) = M_k(\lambda)$. Substituting (5) into (2) and (3) we obtain a linear algebraic system s_k with respect to $M_{kj}^\nu(\lambda)$, $\nu = 0, 1$, $j = \overline{0, r}$. The determinant $\Delta_0(\lambda)$ of s_k does not depend on k and has the form

$$\Delta_0(\lambda) = (d(\lambda) - 2) \prod_{j=1}^{r} S_j(T_j, \lambda) + D(\lambda) h(\lambda), \tag{6}$$

where

$$d(\lambda) = C_0(T_0, \lambda) + S_0'(T_0, \lambda), \tag{7}$$

$$D(\lambda)=\sum_{i=1}^{r}S_i'(T_i,\lambda)\prod_{j=1,j\neq i}^{r}S_j(T_j,\lambda)=\prod_{j=1}^{r}S_j(T_j,\lambda)\sum_{i=1}^{r}\frac{S_i'(T_i,\lambda)}{S_i(T_i,\lambda)}. \tag{8}$$

The function $\Delta_0(\lambda)$ is entire in λ of order $1/2$, and its zeros coincide with the eigenvalues of the boundary value problem $B_0(q)$. Solving the algebraic system s_k we get by Cramer's rule: $M_{kj}^s(\lambda)=\Delta_{kj}^s(\lambda)/\Delta_0(\lambda)$, $s=0,1$, $j=\overline{0,r}$, where the determinant $\Delta_{kj}^s(\lambda)$ is obtained from $\Delta_0(\lambda)$ by the replacement of the column which corresponds to $M_{kj}^s(\lambda)$ with the column of free terms. In particular,

$$M_k(\lambda)=-\frac{\Delta_k(\lambda)}{\Delta_0(\lambda)},\quad k=\overline{1,r}, \tag{9}$$

where

$$\Delta_k(\lambda)=(d(\lambda)-2)C_k(T_k,\lambda)\prod_{j=1,j\neq k}^{r}S_j(T_j,\lambda)+D_k(\lambda)h(\lambda), \tag{10}$$

$$D_k(\lambda)=C_k'(T_k,\lambda)\prod_{j=1,j\neq k}^{r}S_j(T_j,\lambda)+C_k(T_k,\lambda)\sum_{i=1,\,i\neq k}^{r}S_i'(T_i,\lambda)\prod_{j=1,j\neq i,k}^{r}S_j(T_j,\lambda). \tag{11}$$

We note that $\Delta_k(\lambda)$ is obtained from $\Delta_0(\lambda)$ by the replacement of $S_k^{(\nu)}(T_k,\lambda)$, $\nu=0,1$, with $C_k^{(\nu)}(T_k,\lambda)$, $\nu=0,1$. The function $\Delta_k(\lambda)$ is entire in λ of order $1/2$, and its zeros coincide with the eigenvalues of the boundary value problem $B_k(q)$. The functions $\Delta_k(\lambda)$, $k=\overline{0,r}$, are called the characteristic functions for the boundary value problems $B_k(q)$.

2.2. In this subsection we study the asymptotic behavior of solutions of equation (1). Let $\lambda=\rho^2$, $\operatorname{Im}\rho\geqslant 0$. Denote $\Lambda:=\{\rho:\ \operatorname{Im}\rho\geqslant 0\}$, $\Lambda^\delta:=\{\rho:\ \arg\rho\in[\delta,\pi-\delta]\}$. It is known (see [12]) that for each fixed $j=\overline{0,r}$ on the edge e_j, there exists a fundamental system of solutions of equation (1) $\{e_{j1}(x_j,\rho),e_{j2}(x_j,\rho)\}$, $x_j\in[0,T_j]$, $\rho\in\Lambda$, $|\rho|\geqslant\rho^*$ with the properties:
1) the functions $e_{js}^{(\nu)}(x_j,\rho)$, $\nu=0,1$, are continuous for $x_j\in[0,T_j]$, $\rho\in\Lambda$, $|\rho|\geqslant\rho^*$;
2) for each $x_j\in[0,T_j]$, the functions $e_{js}^{(\nu)}(x_j,\rho)$, $\nu=0,1$, are analytic for $\operatorname{Im}\rho>0$, $|\rho|>\rho^*$;
3) uniformly in $x_j\in[0,T_j]$, the following asymptotical formulae hold

$$e_{j1}^{(\nu)}(x_j,\rho)=(i\rho)^\nu\exp(i\rho x_j)[1],\ e_{j2}^{(\nu)}(x_j,\rho)=(-i\rho)^\nu\exp(-i\rho x_j)[1],\ \rho\in\Lambda,\ |\rho|\to\infty, \tag{12}$$

where $[1]=1+O(\rho^{-1})$.

Fix $k=\overline{1,r}$. One has

$$\Phi_{kj}(x_j,\lambda)=A_{kj}^1(\rho)e_{j1}(x_j,\rho)+A_{kj}^0(\rho)e_{j2}(x_j,\rho),\quad x_j\in[0,T_j]. \tag{13}$$

Substituting (13) into (2) and (3) we obtain a linear algebraic system s_k^0 with respect to $A_{kj}^\nu(\lambda)$, $\nu=0,1$, $j=\overline{0,r}$. The determinant $\delta_0(\rho)$ of s_k^0 does not depend on k and has the form

$$\delta_0(\rho)=2(-2i\rho)^r\Delta_0(\lambda),\quad \rho\in\Lambda.$$

Moreover,

$$\delta_0(\rho) = (r+2)\exp\Big(-i\rho\sum_{j=0}^{r} T_j\Big)[1], \quad \rho \in \Lambda^\delta,\ |\rho| \in \infty. \tag{14}$$

Solving the algebraic system s_k^0 by Cramer's rule and using (12) and (14), we get

$$A_{kk}^1(\rho) = [1],\ A_{kk}^0(\rho) = -\frac{r}{r+2}\exp(2i\rho T_k)[1], \quad \rho \in \Lambda^\delta,\ |\rho| \in \infty.$$

Together with (12) and (13) this yields for each fixed $x_k \in [0, T_k)$:

$$\Phi_{kk}^{(\nu)}(x_k, \lambda) = (i\rho)^\nu \exp(i\rho x_k)[1], \quad \rho \in \Lambda^\delta,\ |\rho| \in \infty. \tag{15}$$

In particular, $M_k(\lambda) = (i\rho)[1],\ \rho \in \Lambda^\delta,\ |\rho| \in \infty.$ Moreover, uniformly in $x_j \in [0, T_j]$,

$$S_j^{(\nu)}(x_j, \lambda) = \frac{1}{2i\rho}\Big((i\rho)^\nu \exp(i\rho x_j)[1] - (-i\rho)^\nu \exp(-i\rho x_j)[1]\Big),\ \rho \in \Lambda,\ |\rho| \to \infty, \tag{16}$$

$$C_j^{(\nu)}(x_j, \lambda) = \frac{1}{2}\Big((i\rho)^\nu \exp(i\rho x_j)[1] + (-i\rho)^\nu \exp(-i\rho x_j)[1]\Big),\ \rho \in \Lambda,\ |\rho| \to \infty. \tag{17}$$

2.3. In this subsection we study properties of the spectra and the characteristic functions of $B_k(q)$. Let $\lambda_{kn}^0 = (\rho_{nk}^0)^2,\ k = \overline{0, r},$ be the eigenvalues of the boundary value problems $B_k(0)$ with the zero potential, and let $\Delta_k^0(\lambda)$ be the characteristic functions of $B_k(0)$. According to (6)-(8) and (10)-(11) we have

$$\Delta_0^0(\lambda) = 2(\cos\rho T_0 - 1)\prod_{j=1}^{r}\frac{\sin\rho T_j}{\rho} + \frac{\sin\rho T_0}{\rho}\sum_{i=1}^{r}\cos\rho T_i \prod_{j=1,\, j\neq i}^{r}\frac{\sin\rho T_j}{\rho}, \tag{18}$$

$$\begin{aligned}\Delta_k^0(\lambda) &= 2(\cos\rho T_0 - 1)\cos\rho T_k \prod_{j=1,\, j\neq k}^{r}\frac{\sin\rho T_j}{\rho} \\ &+ \frac{\sin\rho T_0}{\rho}\Big((-\rho\sin\rho T_k)\prod_{j=1,\, j\neq k}^{r}\frac{\sin\rho T_j}{\rho} + \cos\rho T_k \sum_{i=1,\, i\neq k}^{r}\cos\rho T_i \prod_{j=1,\, j\neq i,k}^{r}\frac{\sin\rho T_j}{\rho}\Big).\end{aligned} \tag{19}$$

Let $\tau := \operatorname{Im}\rho.$ It follows from (6)-(8), (10)-(11), (16) and (17) that for $|\rho| \to \infty$,

$$\left.\begin{aligned}\Delta_0(\lambda) &= \Delta_0^0(\lambda) + O\Big(\rho^{-r-1}\exp\Big(|\tau|\sum_{j=0}^{r} T_j\Big)\Big), \\ \Delta_k(\lambda) &= \Delta_k^0(\lambda) + O\Big(\rho^{-r}\exp\Big(|\tau|\sum_{j=0}^{r} T_j\Big)\Big), \quad k = \overline{1, r}.\end{aligned}\right\} \tag{20}$$

Using (18)-(20), by the well-known method (see, for example, [3]), one can obtain the following properties of the characteristic functions $\Delta_k(\lambda)$ and the eigenvalues Λ_k of the boundary value problems $B_k(q),\ k = \overline{0, r}.$

1) For $\rho \in \Lambda,\ |\rho| \to \infty,$

$$\Delta_0(\lambda) = O\Big(|\rho|^{-r} \exp\Big(|\tau| \sum_{j=0}^{r} T_j\Big)\Big),$$

$$\Delta_k(\lambda) = O\Big(|\rho|^{1-r} \exp\Big(|\tau| \sum_{j=0}^{r} T_j\Big)\Big), \quad k = \overline{1, r}.$$

2) There exist $h > 0,\ C_h > 0$ such that

$$|\Delta_0(\lambda)| \geqslant C_h |\rho|^{-r} \exp\Big(|\tau| \sum_{j=0}^{r} T_j\Big),$$

$$|\Delta_k(\lambda)| \geqslant C_h |\rho|^{1-r} \exp\Big(|\tau| \sum_{j=0}^{r} T_j\Big), \quad k = \overline{1, r},$$

for $|\tau| \geqslant h.$ Hence, the eigenvalues $\lambda_{kn} = \rho_{nk}^2$ lie in the domain $|\mathrm{Im}\,\rho| < h.$

3) The number $N_{\xi k}$ of zeros of $\Delta_k(\lambda)$ in the rectangle $\Pi_\xi = \{\rho : \ |\mathrm{Im}\,\rho| \leqslant h,\ \mathrm{Re}\,\rho \in [\xi, \xi + 1]\}$ is bounded with respect to $\xi.$

4) Denote $G_\delta = \{\rho : \ |\rho - \rho_{0n}| \geqslant \delta \ \ \forall n \geqslant 0\},\ \delta > 0.$ Then

$$|\Delta_0(\lambda)| \geqslant C_\delta |\rho|^{-r} \exp\Big(|\tau| \sum_{j=0}^{r} T_j\Big), \quad \rho \in G_\delta.$$

5) There exist numbers $r_N \to \infty$ such that for sufficiently small $\delta > 0,$ the circles $|\rho| = r_N$ lie in G_δ for all $N.$

6) For $n \to \infty,$

$$\rho_{nk} = \rho_{nk}^0 + O\Big(\frac{1}{\rho_{nk}^0}\Big).$$

2.4. In this subsection the reconstruction of the characteristic functions from their zeros is studied. Denote

$$\lambda_{kn}^{01} = \begin{cases} \lambda_{kn}^0 & \text{if} \quad \lambda_{kn}^0 \neq 0, \\ 1 & \text{if} \quad \lambda_{kn}^0 = 0. \end{cases} \tag{21}$$

By Hadamard's factorization theorem [6, p.289],

$$\Delta_k^0(\lambda) = A_k^0 \prod_{n=0}^{\infty} \frac{\lambda_{kn}^0 - \lambda}{\lambda_{kn}^{01}}\,, \tag{22}$$

where

$$A_k^0 = \frac{(-1)^{s_k}}{s_k!} \Big(\frac{\partial^{s_k}}{\partial \lambda^{s_k}}\, \Delta_k^0(\lambda)\Big)_{|\lambda=0}\,, \tag{23}$$

and $s_k \geqslant 0$ is the multiplicity of the zero eigenvalue.

Let us show that

$$\Delta_k(\lambda) = A_k^0 \prod_{n=0}^{\infty} \frac{\lambda_{kn} - \lambda}{\lambda_{kn}^{01}} . \tag{24}$$

Indeed, by Hadamard's factorization theorem,

$$\Delta_k(\lambda) = A_k \prod_{n=0}^{\infty} \frac{\lambda_{kn} - \lambda}{\lambda_{kn}^{1}} , \tag{25}$$

where $A_k \neq 0$ is a constant, and

$$\lambda_{kn}^1 = \begin{cases} \lambda_{kn} & \text{if} \quad \lambda_{kn} \neq 0, \\ 1 & \text{if} \quad \lambda_{kn} = 0. \end{cases}$$

It follows from (22) and (25) that

$$\frac{\Delta_k(\lambda)}{\Delta_k^0(\lambda)} = \frac{A_k}{A_k^0} \prod_{n=0}^{\infty} \frac{\lambda_{kn}^{01}}{\lambda_{kn}^1} \prod_{n=0}^{\infty} \Big(1 + \frac{\lambda_{kn} - \lambda_{kn}^0}{\lambda_{kn}^0 - \lambda}\Big).$$

Using properties of the characteristic functions and the eigenvalues one gets for negative λ:

$$\lim_{\lambda \to -\infty} \frac{\Delta_k(\lambda)}{\Delta_k^0(\lambda)} = 1, \qquad \lim_{\lambda \to -\infty} \prod_{n=0}^{\infty} \Big(1 + \frac{\lambda_{kn} - \lambda_{kn}^0}{\lambda_{kn}^0 - \lambda}\Big) = 1,$$

and consequently,

$$A_k = A_k^0 \prod_{n=0}^{\infty} \frac{\lambda_{kn}^1}{\lambda_{kn}^{01}} .$$

Substituting this relation into (25) we arrive at (24).

Thus, the specification of the spectrum $\Lambda_k = \{\lambda_{kn}\}_{n \geqslant 0}$ uniquely determines the characteristic function $\Delta_k(\lambda)$ by (24) where A_k^0 and $\{\lambda_{kn}^{01}\}$ are defined by (21), (23), (18) and (19).

3. Solution of Inverse Problem 1

In this section we provide a constructive procedure for the solution of Inverse Problem 1 and prove its uniqueness. First we consider auxiliary inverse problems.

3.1. Fix $k = \overline{1,r}$, and consider the following auxiliary inverse problem on the edge e_k, which is called IP(k).

IP(k). Given $M_k(\lambda)$, construct $q_k(x_k)$, $x_k \in [0, T_k]$.

In IP(k) we construct the potential only on the edge e_k, but the Weyl function $M_k(\lambda)$ brings a global information from the whole graph. In other words, IP(k) is not a local inverse problem related to the edge e_k. Let us prove the uniqueness theorem for the solution of IP(k).

THEOREM 2. *If $M_k(\lambda) = \tilde{M}_k(\lambda)$, then $q_k(x_k) = \tilde{q}_k(x_k)$ a.e. on $[0, T_k]$. Thus, the specification of the Weyl function M_k uniquely determines the potential q_k on the edge e_k.*

Proof. Let us define the matrix $P^k(x_k,\lambda) = [P^k_{js}(x_k,\lambda)]_{j,s=1,2}$ by the formula

$$P^k(x_k,\lambda)\begin{bmatrix} \tilde{\Phi}_{kk}(x_k,\lambda) & \tilde{S}_k(x_k,\lambda) \\ \tilde{\Phi}'_{kk}(x_k,\lambda) & \tilde{S}'_k(x_k,\lambda) \end{bmatrix} = \begin{bmatrix} \Phi_{kk}(x_k,\lambda) & S_k(x_k,\lambda) \\ \Phi'_{kk}(x_k,\lambda) & S'_k(x_k,\lambda) \end{bmatrix}. \tag{26}$$

Then (26) yields

$$\left.\begin{aligned} \Phi_{kk}(x_k,\lambda) &= P^k_{11}(x_k,\lambda)\tilde{\Phi}_{kk}(x_k,\lambda) + P^k_{12}(x_k,\lambda)\tilde{\Phi}'_{kk}(x_k,\lambda), \\ S_k(x_k,\lambda) &= P^k_{11}(x_k,\lambda)\tilde{S}_k(x_k,\lambda) + P^k_{12}(x_k,\lambda)\tilde{S}'_k(x_k,\lambda). \end{aligned}\right\} \tag{27}$$

Since $\langle \Phi_{kk}(x_k,\lambda),\, S_k(x_k,\lambda)\rangle \equiv 1,$ one has

$$P^k_{1s}(x_k,\lambda) = (-1)^s\Big(\Phi_{kk}(x_k,\lambda)\tilde{S}_k^{(2-s)}(x_k,\lambda) - \tilde{\Phi}_{kk}^{(2-s)}(x_k,\lambda)S_k(x_k,\lambda)\Big). \tag{28}$$

It follows from (15), (16) and (28) that

$$P^k_{1s}(x_k,\lambda) = \delta_{1s} + O(\rho^{-1}), \quad \rho \in \Lambda^\delta,\ |\rho| \to \infty,\ x_k \in (0, T_k]. \tag{29}$$

According to (4) and (28),

$$P^k_{1s}(x_k,\lambda) = (-1)^s\Big(\Big(C_k(x_k,\lambda)\tilde{S}_k^{(2-s)}(x_k,\lambda) - \tilde{C}_k^{(2-s)}(x_k,\lambda)S_k(x_k,\lambda)\Big)$$

$$+(M_k(\lambda) - \tilde{M}_k(\lambda))S_k(x_k,\lambda)\tilde{S}_k^{(2-s)}(x_k,\lambda)\Big).$$

Since $M_k(\lambda) = \tilde{M}_k(\lambda)$, it follows that for each fixed x_k, the functions $P^k_{1s}(x_k,\lambda)$ are entire in λ of order $1/2$. Together with (29) this yields $P^k_{11}(x_k,\lambda) \equiv 1$, $P^k_{12}(x_k,\lambda) \equiv 0$. Substituting these relations into (27) we get $\Phi_{kk}(x_k,\lambda) \equiv \tilde{\Phi}_{kk}(x_k,\lambda)$ and $S_k(x_k,\lambda) \equiv \tilde{S}_k(x_k,\lambda)$ for all x_k and λ, and consequently, $q_k(x_k) = \tilde{q}_k(x_k)$ a.e. on $[0, T_k]$. □

Using the method of spectral mappings [17] for the Sturm-Liouville operator on the edge e_k one can get a constructive procedure for the solution of the inverse problem IP(k). Here we only explain ideas briefly; for details and proofs see [17]. Take $\tilde{q} = 0$. Then $\tilde{S}_k(x_k,\lambda) = \frac{\sin\rho x_k}{\rho}$. Fix $k = \overline{1,r}$. Denote $\lambda' = \min\limits_{l\geqslant 0}(\lambda_{0l}, \tilde{\lambda}_{0l})$ and take a fixed $\delta > 0$. In the λ - plane we consider the contour γ (with counterclockwise circuit) of the form $\gamma = \gamma^+ \cup \gamma^- \cup \gamma'$, where $\gamma^\pm = \{\lambda :\ \pm\mathrm{Im}\,\lambda = \delta;\, \mathrm{Re}\,\lambda \geqslant \lambda'\}$, $\gamma' = \{\lambda :\ \lambda - \lambda' = \delta\exp(i\alpha),\ \alpha \in (\pi/2, 3\pi/2)\}$. For each fixed $x_k \in [0, T_k]$, the function $S_k(x_k,\lambda)$ is the unique solution of the following linear integral equation

$$S_k(x_k,\lambda) = \tilde{S}_k(x_k,\lambda) + \frac{1}{2\pi i}\int_\gamma \tilde{D}_k(x_k,\lambda,\mu)S_k(x_k,\mu)\,d\mu, \tag{30}$$

where $\tilde{D}_k(x,\lambda,\mu) = \int_0^x \tilde{S}_k(t,\lambda)\tilde{S}_k(t,\mu)\hat{M}_k(\mu)\,dt$, $\hat{M}_k(\mu) := M_k(\mu) - \tilde{M}_k(\mu)$. The potential q_k on the edge e_k can be constructed from the solution of the integral equation (30) via the formula

$$q_k(x_k) = \frac{1}{2\pi i}\int_\gamma (S_k(x_k,\lambda)\tilde{S}_k(x_k,\lambda))'\hat{M}_k(\lambda)\,d\lambda$$

or by the formula $q_k(x_k) = \lambda + S_k''(x_k,\lambda)/S_k(x_k,\lambda)$. It is also possible to construct the potential from the discrete spectral data. For this purpose one can calculate the contour integral in (30) by the residue theorem and transform the integral equation (30) to the equation in a space of sequences; for details see [17].

3.2. Consider the following auxiliary inverse problem on the edge e_0, which is called IP(0).

IP(0). Given $d(\lambda), h(\lambda), \Omega$, construct $q_0(x_0)$, $x_0 \in [0,T_0]$.

This inverse problem were studied in [11, 16] and other papers. For convenience of the readers we describe here the solution of IP(0). We remind that

$$d(\lambda) = C_0(T_0,\lambda) + S_0'(T_0,\lambda), \quad H(\lambda) = C_0(T_0,\lambda) - S_0'(T_0,\lambda), \quad \omega_n = \operatorname{sign} H(\nu_n),$$

where $\{\nu_n\}_{n\geqslant 1}$ are zeros of $h(\lambda)$. Clearly,

$$S_0'(T_0,\nu_n) = (d(\nu_n) - H(\nu_n))/2. \tag{31}$$

Since $\langle C_0(T_0,\lambda), S_0(T_0,\lambda)\rangle \equiv 1$, it follows that

$$H^2(\lambda) - d^2(\lambda) = -4(1 + C_0'(T_0,\lambda)h(\lambda)),$$

and consequently,

$$H(\nu_n) = \omega_n\sqrt{d^2(\nu_n) - 4}. \tag{32}$$

Denote $\alpha_n := \int_0^{T_0} S_0^2(t,\nu_n)\,dt$. Then (see [9, 10])

$$\alpha_n = \dot{h}(\nu_n)S_0'(T_0,\nu_n), \quad \dot{h}(\lambda) := \frac{dh(\lambda)}{d\lambda}. \tag{33}$$

The data $\{\nu_n,\alpha_n\}_{n\geqslant 1}$ are called the spectral data for the potential q_0. It is known (see [7, 9, 10, 15]) that the function q_0 can be uniquely constructed from the given spectral data $\{\nu_n,\alpha_n\}_{n\geqslant 1}$. Thus, IP(0) has been solved, and the following theorem is valid.

THEOREM 3. *The specification of $d(\lambda), h(\lambda), \Omega$ uniquely determines the potential $q_0(x_0)$ on $[0,T_0]$. The function q_0 can be constructed by the following algorithm.*

ALGORITHM 1. Given $d(\lambda), h(\lambda), \Omega$.
1) Find $\{\nu_n\}_{n\geqslant 1}$ as the zeros of $h(\lambda)$.
2) Calculate $H(\nu_n)$ by (32).
3) Find $S_0'(T_0,\nu_n)$ by (31).
4) Calculate $\{\alpha_n\}_{n\geqslant 1}$ using (33).

5) Construct q_0 from the given spectral data $\{\nu_n, \alpha_n\}_{n\geqslant 1}$ by solving the classical inverse Sturm-Liouville problem.

3.3. Let us go on to the solution of Inverse Problem 1. First we give the proof of Theorem 1. Assume that $\Lambda_k = \tilde{\Lambda}_k,\ k = \overline{0,r},$ and $\Omega = \tilde{\Omega}.$ Then, according to the results of subsection 2.4, one has

$$\Delta_k(\lambda) \equiv \tilde{\Delta}_k(\lambda), \quad k = \overline{0,r}.$$

By virtue of (9) this yields

$$M_k(\lambda) \equiv \tilde{M}_k(\lambda), \quad k = \overline{1,r}.$$

Applying Theorem 2 for each fixed $k = \overline{1,r},$ we obtain

$$q_k(x_k) = \tilde{q}_k(x_k) \text{ a.e. on } [0, T_k], \quad k = \overline{1,r},$$

and consequently,

$$C_k(x_k,\lambda) \equiv \tilde{C}_k(x_k,\lambda),\ S_k(x_k,\lambda) \equiv \tilde{S}_k(x_k,\lambda), \quad k = \overline{1,r}, \quad x_k \in [0, T_k].$$

Taking (6) and (10) into account we deduce

$$d(\lambda) = \tilde{d}(\lambda), \quad h(\lambda) = \tilde{h}(\lambda).$$

Since $\Omega = \tilde{\Omega},$ it follows from Theorem 3 that

$$q_0(x_0) = \tilde{q}_0(x_0) \text{ a.e. on } [0, T_0],$$

and Theorem 1 is proved.

The solution of Inverse Problem 1 can be constructed by the following algorithm.

ALGORITHM 2. Given $\Lambda_k,\ k = \overline{0,r}$ and $\Omega.$

1) Construct $\Delta_k(\lambda),\ k = \overline{0,r}$ by (24) where A_k^0 and $\{\lambda_{kn}^{01}\}$ are defined by (21), (23), (18) and (19).
2) Find $M_k(\lambda),\ k = \overline{1,r}$ via (9).
3) For each fixed $k = \overline{1,r},$ solve the inverse problem IP(k) and find $q_k(x_k),\ x_k \in [0, T_k]$ on the edge e_k.
4) For each fixed $k = \overline{1,r},$ construct $C_k(x_k,\lambda), S_k(x_k,\lambda),\ x_k \in [0, T_k].$
5) Calculate $d(\lambda)$ and $h(\lambda)$ using (6) and (10).
6) Construct $q_0(x_0),\ x_0 \in [0, T_0]$ from $d(\lambda), h(\lambda), \Omega$ using Algorithm 1.

Acknowledgement. This research was supported in part by Grants 07-01-00003 and 07-01-92000-NSC-a of Russian Foundation for Basic Research and Taiwan National Science Council.

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(Received May 12, 2008)

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